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RANDOM ERRORS IN THE MAGNETIC FIELD OF SUPERCONDUCTING DIPOLES AND QUADRUPOLES†

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Random errors in the magnetic field of superconducting magnets are likely to be larger than those found in conventional warm magnets. The random error multipoles introduced by random errors in the position of the current blocks are computed for cosine dipoles and quadrupoles and for window-frame dipoles. Analytical results are compared with computed results, and the computed results are compared with the results of measurements made on models of the cosine and window-frame magnets.

I INTRODUCTION

Random errors in the magnetic field of superconducting magnets are likely to be larger than those found in conventional warm magnets.[1] In superconducting magnets, the current coils are often not hidden by the iron pole face, and are relatively close to the region where the good field is required. This is particularly true for circular magnets with a cosine current distribution and for rectangular window-frame magnets. Random errors in the positions of the current blocks introduce random multipoles in the field of the magnets. These random multipoles can excite the nonlinear imperfection resonances which appear to be of considerable importance in storage accelerators having intense beams with long lifetimes.

This paper computes the random error multipoles in cosine magnets and in window-frame magnets, which are introduced by random errors in the current block positions.

II COSINE DIPOLES AND QUADRUPOLES

One method of constructing a cosine dipole is shown in Figure 1, in which the cosine distribution is approximated by current blocks, each of which carries a current which is proportional to the cosine of azimuthal angle. The random error multipole fields introduced by errors in the current block position in a cosine dipole or quadrupole may be computed analytically. This was done by Parzen,[2] Randle,[3] and Ries.[4] The error multipoles have also been found more exactly by numerical computations by Dahl[5] and by Parzen and Jellett.

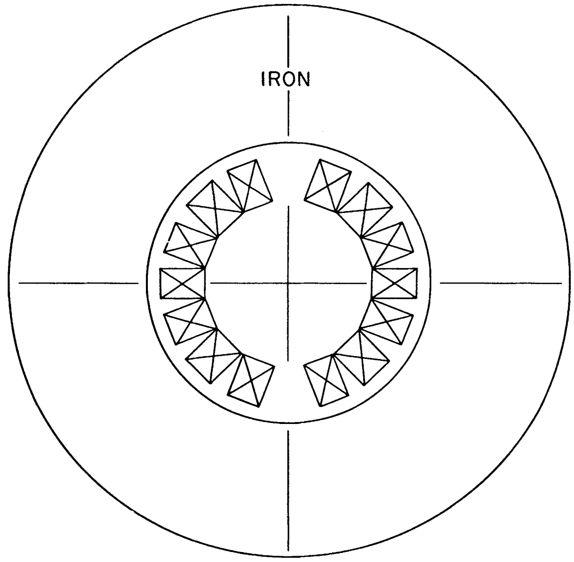

FIGURE 1 Geometry of a superconducting dipole magnet with a cosine current distribution.

In the case of the cosine dipole, the rms value of the random error multipoles introduced by an rms error in the position of the current block is given by (see Section IV for derivation)

$$\Delta b_n = \left(\frac{2}{N_b}\right)^{1/2} \frac{n+1}{R^{n+1}} \times \varepsilon \, \frac{[1 + (R/\rho)^{2(n+1)} - \delta_{n0}(R/\rho)^{n+1}]^{1/2}}{1 + (R/\rho)} \tag{2.1}$$

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where the error field ΔB_z on the median plane is written as

$$\Delta B_z = B_0(\Delta b_0 + \Delta b_1 x + \Delta b_2 x^2 + \cdots),$$

B_0 is the unperturbed field of the dipole, R is the average radial position of the current blocks, N_b is the total number of current blocks in the magnet, ε is the rms error in the position of the current blocks, $\rho = a^2/R$ and a is the inner radius of the iron shield.

The result Eq. (2.1) assumes that the current blocks have zero radial thickness. The multipoles in the radial field, Δa_n, where

$$B_r = B_0(\Delta a_0 + \Delta a_1 x + \Delta a_2 x^2 + \cdots)$$

are roughly equal to the Δb_n given by Eq. (2.1) except for $n = 0$ where $-\delta_{n0}$ is replaced by δ_{n0}.

The result Eq. (2.1) may be generalized to quadrupoles, sextupoles, etc., and is given by (see Section IV for derivation)

$$\Delta b_n = \left(\frac{2}{N_b}\right)^{1/2} \frac{n+1}{R^{n+1-m}} \times \varepsilon \frac{[1 + (R/\rho)^{2(n+1)} - \delta_{nm}(R/\rho)^{n+1}]^{1/2}}{1 + (R/\rho)^{m+1}} \tag{2.2}$$

where $m = 0$ for a dipole, $m = 1$ for a quadrupole, $m = 2$ for a sextupole, etc. The result is the same for Δa_n except $-\delta_{nm}$ is replaced by δ_{nm}.

The accuracy of the theoretical result for the random multipoles as given by Eq. (2.2) may be estimated by comparing this result with the computer results for a dipole magnet which has 6 blocks per quadrant which are 1.905 cm thick, whose inner radius is 6 cm, and for a random coil position error of 0.005 cm. The inner radius of the iron shield is 8.25 cm. The analytical and computer results are given in Table I.

The rms random error multipole present in a quadrupole are also given in Table I for a quadrupole which has 3 blocks per octant which are 1.905 cm thick, which has an aperture of 12 cm, and with an iron shield inner radius of 8.25 cm. The rms random error in the current block positions is 0.005 cm. Error multipoles are introduced in both the vertical component of the field and the radial component of the field in the median plane, and the error multipoles are approximately the same for both components.

The accuracy of the analytical result for the random multipole present in a quadrupole may be estimated by comparing this result with the computer results found by Parzen and Jellett. Both computer and analytical results are listed in Table I. The quadrupole multipoles are defined by writing the error field as

$$\Delta B_z = B_1(\Delta b_0 + \Delta b_1 x + \Delta b_2 x^2 + \cdots),$$

where B_1 is the unperturbed gradient of the quadrupole.

One may note that the error multipoles present in the quadrupoles are larger than those present in the dipole by the factor $R(1 + R/\rho)/[1 + (R/\rho)^2]$, R being the average radius of the coils. This factor is about 10 for the 12-cm aperture

TABLE I

The random multipoles generated by a random error in the current block positions of 0.005 cm for a cosine dipole and a cosine quadrupole each having an aperture diameter of 12 cm, and whose current blocks are 1.905 cm thick radially. Analytic and computed results are given. The a_n and b_n are almost equal.

Dipole				Quadrupole			
n	Δb_n Analytic	Δb_n Computed	Units	n	Δb_n Analytic	Δb_n Computed	Units
0	1.660	1.772	10^{-4} cm^0	0	1.356	1.307	10^{-3} cm^0
1	4.016	3.874	10^{-5} cm^{-1}	1	3.280	4.019	10^{-4} cm
2	8.330	8.091	10^{-6} cm^{-2}	2	9.345	7.008	10^{-5} cm^{-1}
3	1.572	1.537	10^{-6} cm^{-3}	3	1.283	1.302	10^{-5} cm^{-2}
4	2.812	2.739	10^{-7} cm^{-4}	4	2.295	2.340	10^{-6} cm^{-3}
5	4.835	4.919	10^{-8} cm^{-5}	5	3.953	4.036	10^{-7} cm^{-4}
6	8.110	8.194	10^{-9} cm^{-6}	6	6.629	6.716	10^{-8} cm^{-5}
7	1.333	1.319	10^{-9} cm^{-7}	7	1.090	1.270	10^{-8} cm^{-6}
8	2.158	2.270	10^{-10} cm^{-8}	8	1.780	1.906	10^{-9} cm^{-7}
9	3.449	3.640	10^{-11} cm^{-9}				

magnets computed in Table I. Thus in an accelerator whose dipoles are powered to 40 kG, and whose quadrupoles go to 10 kG/cm, and for which the combined length of the dipoles in a cell is 9 times larger than the combined length of the quadrupoles, one finds that the error multipoles in the quadrupoles are about one-third as effective as the error multipoles in the dipoles in disturbing the particle orbits.

There are several other ways to approximate a cosine current distribution besides the one shown in Figure 1. One can expect that Eq. (2.2) should be roughly valid for any method of approximating the cosine current distribution with an error of a factor of 2 or 3.

Displacement of the Current Coils as a Whole

A particular kind of random error which is not included in the above occurs when all the current blocks in the cosine current winding of a magnet are all displaced by the same amount.

One can show (see Section IV) that a horizontal displacement, Δx, of the entire coil will introduce the multipoles Δb_{m-1}, and Δb_{m+1}, where $m = 0$ for a dipole magnet, $m = 1$ for a quadrupole magnet, etc. A vertical displacement, Δy, introduces the multipoles Δa_{m-1} and Δa_{m+1}. A horizontal displacement Δx produces

$$\Delta b_{m-1} = -m\,\Delta x \frac{1}{1 + (R/\rho)^{m+1}}, \tag{2.3a}$$

$$\Delta b_{m+1} = (m + 2)\frac{\Delta x}{R^2}\left(\frac{R}{\rho}\right)^{m+2}\frac{1}{1 + (R/\rho)^{m+1}}, \tag{2.3b}$$

$$\rho = a^2/R, \tag{2.3c}$$

where R is the average radius of the current winding, and ρ is the radius to the image current, where a is the inner radius of the iron shield. Equations (2.3) are derived for a cosine winding of zero thickness inside an iron shield of infinite permeability. The same result in magnitude is found for the Δa_{m-1} and Δa_{m+1} introduced by a vertical displacement Δy.

The random multipoles in a 12-cm aperture dipole and quadrupole which are generated by a random error in the position of the coil as a whole of 0.005 cm are given in Table II. Both the analytical results from Eqs. (2.3) and exact computer results are shown. The error in the analytic result is about 10%.

III WINDOW-FRAME DIPOLE MAGNETS

In computing the random error multipoles, one has to make some assumption as to how the current-carrying conductors can move. Figure 2 shows a window-frame magnet which has 4 current blocks. Actually, each of the 4 current blocks is made up of many conductors. In this calculation, it is assumed that each of the 4 current blocks can have random errors in their positions. This will give a larger error field than what is obtained if one assumes that all the conductors that make up each of the 4 current blocks would move in a random way.

The calculations given below are for a window-frame magnet with a 10 cm × 10 cm aperture, whose 4 current blocks have a rms random error in their position of $\varepsilon = 0.005$ cm. The error multipoles introduced by moving one of the 4 current blocks shown in Figure 2 in either the vertical or horizontal direction was found using the GRACY

TABLE II

The random multipoles generated by a random error of 0.005 cm in the position of the current coil as a whole for a cosine dipole and a cosine quadrupole each having an aperture diameter of 12 cm, a circular iron shield with an inner radius of 8.25 cm, and whose current blocks are 1.905-cm thick radially. Analytic and computed results are given. The results for the a_n are the same as for the b_n.

Dipole				Quadrupole			
n	Δb_n Analytic	Δb_n Computed	Units	n	Δb_n Analytic	Δb_n Computed	Units
0	0.007	0.000	10^{-4} cm^0	0	−3.513	−3.962	10^{-3} cm^1
1	5.540	6.693	10^{-5} cm^{-1}	1	0.005	0.000	10^{-4} cm^0
2	0.031	0.000	10^{-6} cm^{-2}	2	6.425	8.583	10^{-5} cm^{-1}
3	0.000	0.000	10^{-7} cm^{-3}	3	0.007	0.000	10^{-5} cm^{-2}

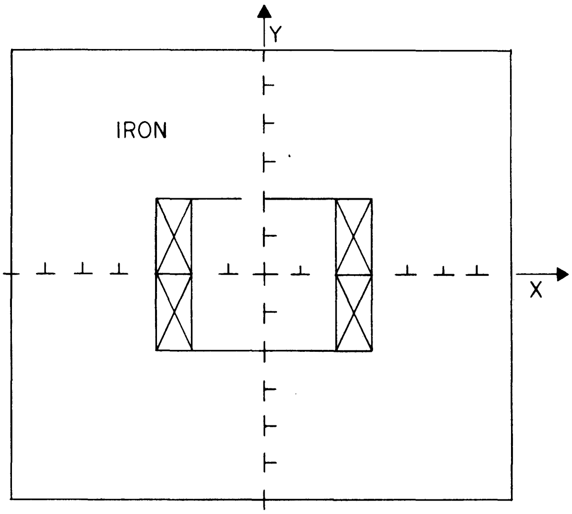

FIGURE 2 Geometry of a window-frame dipole magnet.

magnet program. An error in the position of a current block destroys the median plane symmetry and the error field introduced has both vertical and horizontal components in the median plane.

The rms multipoles introduced by a random rms error of $\varepsilon = 0.005$ cm in the position of all 4 current blocks in either the horizontal or vertical direction is given in Table III. Table III also gives the error multipoles introduced by moving the current block in the upper right corner by the amount $\Delta x = -0.005$ cm, and $\Delta y = -0.005$ cm. These results are for a window-frame magnet with a 10×10 cm aperture and whose coils are 2.54 cm thick.

The results given in Table III may be scaled to different values of the rms error in the coil positions, ε, since the results are linear in ε, and to window-frame magnets with different square apertures since the multipoles scale like R^{n+1}, where R is the width of the aperture.

The rms results given in the first two columns of Table III are computed by assuming each of the four coils can move randomly in the x and y directions with a rms error of 0.005 cm. In an ideal window-frame magnet, the coils appear constrained so as not to be able to move in certain directions. However, in an actual magnet there are usually spaces between the coils and the iron so that they can move. The rms result given in Table I is thus twice the square root of the sum of squares of the multipoles generated by the displacements $\Delta x = 0.005$ cm and $\Delta y = 0.005$ cm which are also given in Table III.

IV COMPARISON WITH EXPERIMENT

Experimental measurements have been made of the random error multipoles present in two identical cosine magnets,[8] the 8-cm models ISA I and ISA II, and in two identical window-frame magnets,[9] which are part of the 8° bend system. These measurements can be compared with the results of theoretical calculations of the random error multipole present in cosine magnets,[2,5] and in window-

TABLE III

Random error multipoles in the median plane field of a window-frame magnet with a 10 cm × 10 cm aperture and 2.54 cm thick coils. The rms multipoles are for a random error in each of the 4 coils of $\varepsilon = 0.005$ cm. The $\Delta x = -0.005$ cm and $\Delta y = -0.005$ cm columns give the multipoles generated by the indicated displacement of the coil in the upper right corner. The Δb_n are the multipoles in the B_z field, and Δa_n are the multipoles in the B_r field.

	rms		$\Delta x = -0.005$ cm		$\Delta y = -0.005$ cm		
n	Δb_n	Δa_n	Δb_n	Δa_n	Δb_n	Δa_n	Units
0	0.504	2.418	0.000	0.478	0.252	−1.11	10^{-4} cm^{0}
1	0.320	0.728	−0.000	0.159	0.160	−0.327	10^{-4} cm^{-1}
2	0.103	0.109	0.002	0.031	0.052	−0.048	10^{-4} cm^{-2}
3	2.07	1.30	−0.000	0.045	1.03	−0.384	10^{-6} cm^{-3}
4	0.322	0.245	−0.005	0.122	0.161	−0.012	10^{-6} cm^{-4}
5	4.20	3.59	−0.000	1.67	2.106	0.176	10^{-8} cm^{-5}
6	0.521	0.670	0.04	0.33	0.26	0.0	10^{-8} cm^{-6}
7	8.8	5.8	−0.000	2.9	4.4	0.0	10^{-10} cm^{-7}
8	2.3	1.2	0.000	0.6	1.2	0.0	10^{-11} cm^{-8}

frame magnets[7] due to a random error in the position of the current blocks of 0.005 cm.

This comparison is shown in Figure 3 where the error multipoles in the radial component of the magnetic field in the median plane, Δa_n, are plotted against n. In order to make the comparison easier, the results for the cosine magnets, which have a radius to the inner coil surface of 4 cm, were scaled to those of a cosine magnet with a radius of 5 cm. In the 8° window-frame magnets, the inner surface of the coil is 5 cm from the center.

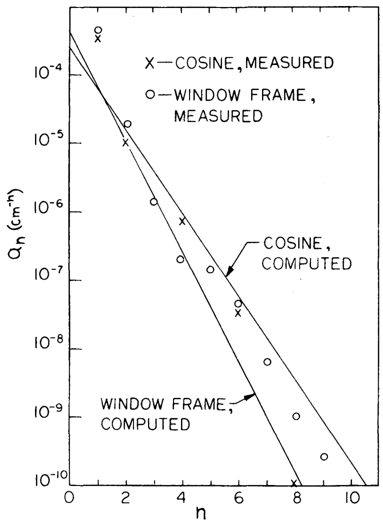

FIGURE 3 A comparison of the measured and computed random error multipoles in a cosine dipole and a window-frame dipole.

The solid lines in Figure 3 are the computed results for the cosine and window-frame magnets. They indicate that the higher multipoles present in the window-frame are considerably smaller than those of the cosine magnet. This can be understood by realizing that the $\Delta a_n \sim 1/R^{n+1}$ where R is some effective radius of the coils, and that the effective R of the window-frame magnet is $\sqrt{2}$ times larger than the effective R of the cosine magnet. The computed results should be treated with caution, since the assumption of the calculation for the window-frame magnet that each of the 4 current blocks moves as a whole is probably not valid and is worse for the higher multipoles.

The experimentally measured results are also shown in Figure 3 as crosses for the cosine magnet and circles for the window-frame magnet. The experimental results for the Δa_n vary with the level of field excitation of the dipoles, and are somewhat different for the two models of each magnet. Also, in some cases, the origin of the Δa_n is thought to be known and these may not be considered as due to random errors. The experimental results shown on the graph are the largest values measured for the Δa_n, as one changes the field level and for the two models. This gives a pessimistic estimate of the multipoles present.

The agreement of the measurement with theory seems reasonable. One can expect rather large deviations since the theoretical result is an rms result, and a particular measurement can give results differing considerably from the rms result. With the limited experimental results available, one might conclude that the random error multipoles present for the cosine and window-frame dipoles are about equal, and this is roughly true even for the higher multipoles.

Derivation of Analytical Results

This section gives a brief summary of the derivation of the results given in Section III. Further details are given in Refs. 2–4.

We start from the following expression for the two-dimensional vector potential when no iron is present.

$$A(r,\theta) = -\frac{\mu}{2\pi}\int dS' \ln R\, j(r',\theta'), \tag{4.1a}$$

$$R = \{r'^2 - 2rr'\cos(\theta-\theta') + r^2\}^{1/2} \tag{4.1b}$$

where $\mu = 4\pi/10$ and $dS' = r'\,dr'\,d\theta'$.

Using the expansion

$$\ln\{1 - 2x\cos\theta + x^2\}^{1/2} = -\sum_{k=1}^{\infty}\frac{x^k}{k}\cos k\theta, \tag{4.2}$$

we can find for small r near the center of the magnet the multipole expansion

$$A(r,\theta) = \sum_{k=1}^{\infty}(A_k\cos k\theta + C_k\sin k\theta)r^k, \tag{4.3}$$

where

$$A_k = \frac{\mu}{2\pi k}\int dS\,\frac{1}{r^k}\,j(r,\theta)\cos k\theta, \tag{4.4}$$

and a similar expression for C_k obtained by replacing $\cos k\theta$ by $\sin k\theta$.

If the current carrying conductors are all within a circular iron shield with inner radius a, and with infinite permeability, then one can show[2] that the expression (4.4) for the multipole A_k is replaced by

$$A_k = \frac{\mu}{2\pi k}\int \mathrm{d}S\left[\frac{1}{r^k} + \left(\frac{r}{a^2}\right)^k\right] j(r, \theta)\cos k\theta. \quad (4.5)$$

A current-carrying filament located at r, θ and carrying a current I will produce the multipoles

$$A_k = \frac{\mu I}{2\pi k}\left[\frac{1}{r^k} + \frac{1}{\rho^k}\right]\cos k\theta, \quad (4.6)$$

where $\rho = a^2/r$ and a displacement of the filament by Δr, $\Delta\theta$ will cause the multipoles to change by ΔA_k.

$$\Delta A_k = \frac{\mu I}{2\pi}\left\{-\frac{1}{r}\left[\frac{1}{r^k} + \frac{1}{\rho^k}\right]\sin k\theta\, r\,\Delta\theta - \frac{1}{r}\left[\frac{1}{r^k} - \frac{1}{\rho^k}\right]\cos k\theta\,\Delta r\right\}, \quad (4.7)$$

with a similar expression for ΔC_k.

Random Displacements

If there are M such filaments each situated at r_n, θ_n carrying the current I_n, undergoing displacements Δr_n and $r_n\,\Delta\theta_n$ which each have the rms value of ε, then one finds for the rms change in ΔA_k

$$\overline{\Delta A_k^2} = \left(\frac{\mu}{2\pi}\right)^2 \varepsilon^2 \sum_n \frac{I_n^2}{r_n^2} \times \left[\left(\frac{1}{r^{2k}} + \frac{1}{\rho^{2k}}\right) - \cos 2k\theta\,\frac{1}{\rho^k r^k}\right], \quad (4.8)$$

where $\overline{\Delta A_k}$ is the rms value of ΔA_k and $\rho_n = a^2/r_n$.

For the simple case where one assumes that I_n varies like $I\cos k_0\theta$, and $r_n = R$ where $k_0 = 1$ for a dipole, $k_0 = 2$ for a quadrupole, etc., then one can approximate the sum over n by

$$\sum_n I_n^2 = I^2\frac{M}{2},$$

$$\sum_n I_n^2 \cos 2k\theta = \delta_{kk_0} I^2 \frac{M}{4},$$

and find that

$$\overline{\Delta A_k} = \frac{\mu I}{2\pi}\frac{\varepsilon}{R}\left(\frac{M}{2}\right)^{1/2}\left[\frac{1}{R^{2k}} + \frac{1}{\rho^{2k}} - \delta_{kk_0}\frac{1}{R^k\rho^k}\right]^{1/2}, \quad (4.9a)$$

where $\rho = a^2/R$.

A similar expression can be found for the rms change in ΔC_k,

$$\overline{\Delta C_k} = \frac{\mu I}{2\pi}\left(\frac{\varepsilon}{R}\right)\left(\frac{M}{2}\right)^{1/2}\left[\frac{1}{R^{2k}} + \frac{1}{\rho^{2k}} + \delta_{kk_0}\frac{1}{R^k\rho^k}\right]^{1/2}. \quad (4.9b)$$

For the $\cos k_0\theta$ winding, the unperturbed multipoles are given by Eq. (4.1) as

$$A_{k_0} = \frac{\mu I}{2\pi k_0}\left[\frac{1}{R^{k_0}} + \frac{1}{\rho^{k_0}}\right]\frac{M}{2}. \quad (4.10)$$

Using Eqs. (4.9) and (4.10) for the multipoles in the vector potential, one can find the perturbed multipoles in the field as

$$\Delta b_n = \left(\frac{2}{M}\right)^{1/2}\frac{(n+1)\varepsilon}{R^{k+1-k_0}} \times \frac{[1 + (R/\rho)^{2k} - \delta_{kk_0}(R/\rho)^k]^{1/2}}{[1 + (R/\rho)^{k_0}]} \quad (4.11a)$$

where $k = n + 1$. A similar expression can be derived for the Δa_n, the error multipole in the radial field in the median plane

$$\Delta a_n = \left(\frac{2}{M}\right)^{1/2}\frac{(n+1)\varepsilon}{R^{k+1-k_0}} \times \frac{[1 + (R/\rho)^{2k} + \delta_{kk_0}(R/\rho)^k]^{1/2}}{[1 + (R/\rho)^{k_0}]}. \quad (4.11b)$$

Displacement of the Coil as a whole

If all the M conductors are displaced a distance Δx, then $\Delta r_n = \Delta x\cos\theta_n$, and $r_n\,\Delta\theta_n = -\Delta x\sin\theta_n$, and from Eq. (4.7) we find

$$\Delta A_k = -\frac{\mu\,\Delta x}{2\pi}\sum_n \frac{I_n}{r_n} \times \left\{\frac{1}{r_n^k}\cos(k-1)\theta_n - \frac{1}{\rho_n^k}\cos(k+1)\theta_n\right\}. \quad (4.12)$$

For the simple case, where the conductors all lie on the circle $r_n = R$, and $I_n = I\cos k_0\theta$, then Eq. (4.12) gives only two nonzero multipoles

$$\Delta A_{k_0-1} = -\frac{\mu I}{2\pi}\frac{\Delta x}{R}\frac{1}{R^{k_0-1}}\cdot\frac{M}{2}, \quad (4.13a)$$

$$\Delta A_{k_0+1} = \frac{\mu I}{2\pi}\frac{\Delta x}{R}\frac{1}{\rho^{k_0+1}}\frac{M}{2}. \quad (4.13b)$$

Using Eq. (4.10) for the unperturbed field A_{k_0} one finds the multipoles in the perturbed field

$$\Delta b_{m-1} = -\Delta x \, m \frac{1}{1 + (R/\rho)^{m+1}}, \tag{4.14a}$$

$$\Delta b_{m+1} = \frac{\Delta x}{R^2}(m+2)\left(\frac{R}{\rho}\right)^{m+2} \frac{1}{1 + (R/\rho)^{m+1}}, \tag{4.14b}$$

where $m = k_0 - 1$.

A similar analysis shows that a displacement Δy of the entire coil produces Δa_{m-1} and Δa_{m+1} which are equal in magnitude to the Δb_{m-1} and Δb_{m+1} caused by the Δx displacement.

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