Particle Accelerators
1975, Vol. 7, pp. 41–47

SOLENOID-LENS EFFECTS IN BEAM-TRANSPORT EQUATIONS†

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(*Received March 17, 1975; in final form June 13, 1975*)

Beam-transport equations governing the transport of low-energy charged-particle beams are modified to include the effects of magnetic (solenoid) lenses. The rms beam-transport equations are modified for the case in which the beam entering the magnetic lens has the same particle distribution in both x and y (the transverse coordinates) and the beam is symmetric in both coordinate directions. An alternative treatment is presented for the case where space-charge forces can be neglected.

1 INTRODUCTION

Analysis of the transport of low-energy charged-particle beams frequently makes use of the governing differential equations for beam transport derived by Emigh,[1] Sacherer,[2] or Kapchinsky and Vladimirsky.[3] These formulations include space-charge effects and the effects of linear, external focussing elements. The effects of magnetic (solenoid) lenses are not included in the formalisms, however, and it is desirable to include them. This paper develops the appropriate modifications to the rms beam-transport equations for the case in which the beam entering the magnetic lens has the same particle distribution in both x and y, the transverse coordinates. The beam is further assumed to be symmetric in each coordinate direction. We will follow the notation of Ref. 1 for the most part.

For those instances in which space-charge forces can be neglected, an alternative treatment is presented, which is based on the direct integration of the equations of motion and requires only the assumption that the incoming beam is symmetric in each coordinate direction.

2 BEAM-TRANSPORT EQUATIONS

We briefly summarize here, for the sake of completeness, the derivation of the rms transport equations. The rms value squared of the x-position of all the particles at position $z(t)$§ within an interval Δz (unspecified, but large enough to have a statistically significant number N of particles in it) is defined to be

$$R_x^2(t) = \frac{1}{N}\sum_{j=1}^{N} 4x_j^2(t), \qquad (1)$$

and the rms velocity squared is defined as:

$$V_x^2(t) = \frac{1}{N}\sum_{j=1}^{N} 4\dot{x}_j^2(t). \qquad (2)$$

The factor 4 is introduced to make R_x for a uniform density distribution correspond to the physical boundary of the distribution.

Differentiating Eq. (1) twice yields the governing differential equation for $R_x(t)$:

$$\ddot{R}_x - \frac{(R_x^2 V_x^2 - R_x^2 \dot{R}_x^2)}{R_x^3} - \frac{4}{NR_x}\sum_{j=1}^{N} x_j \ddot{x}_j = 0. \qquad (3)$$

The last term of this equation can be written as the sum of two terms, namely an internal (space-charge) force term and an external force term. Reference 1 gives expressions for these various terms for pulsed beams and for dc beams, and for various beam-line components such as quadrupole magnets, bending magnets, and accelerating columns. In this work we will derive the expressions necessary to include magnetic (solenoid) lenses.

† Work performed under the auspices of the U.S. Energy Research and Development Administration.

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§ The particles are followed as a function of time.

The rms emittance squared, occurring in the second term of Eq. (3) is given by

$$E_x^2 = R_x^2 V_x^2 - R_x^2 \dot{R}_x^2, \tag{4}$$

and can be related to an area in $x - \dot{x}$ space that is constant for most cases, but not for motion in a solenoid, as will be shown.

Similar equations to Eqs. (1) through (4) for the y direction clearly exist, it being necessary only to change the letter x into y.

3 THE SOLENOID LENS

Figure 1 shows the magnetic field assumed to be produced by the lens. Figure 2 shows the variation of the longitudinal component of the field, which is assumed to rise quickly from 0 to a value B near $z = 0$, whence it remains constant for a distance

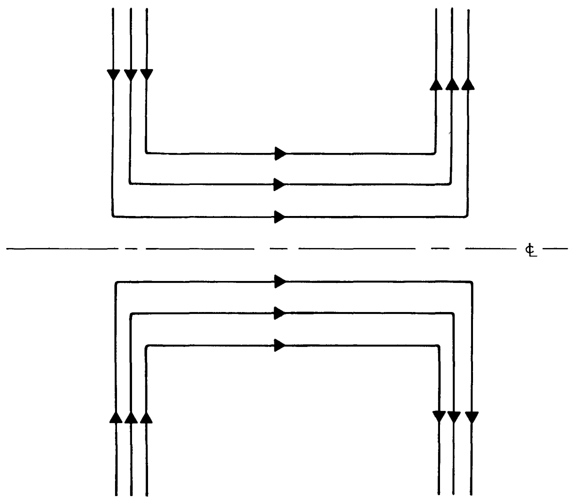

FIGURE 1 Assumed magnetic-field distribution.

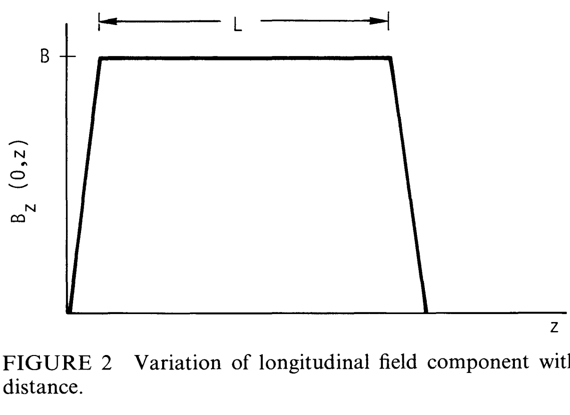

FIGURE 2 Variation of longitudinal field component with distance.

L and then decreases quickly to 0. The radial magnetic field is nonzero only at the ends. If the particles of the beam have no angular momentum about the z axis before entering the solenoid (i.e., in cylindrical polar coordinates, $\dot{\theta} = 0$) then by Busch's theorem[4] they will have an angular velocity by

$$\dot{\theta} = -\frac{qB}{2m}, \tag{5}$$

where q is the charge and m is the mass of a beam particle. (Rationalized mks units will be used throughout this work.)

Consider then the contribution to the last term of Eq. (3) produced by the longitudinal magnetic field:

$$\frac{4}{NR_x} \sum_j x_j (\ddot{x}_j)_{\text{solenoid}} = \frac{4}{NR_x} \sum_j x_j \frac{q}{m} \dot{y}_j B. \tag{6}$$

Expressing the x and y coordinates in terms of cylindrical polar coordinates gives

$$x_j = r_j(t) \cos \theta_j(t), \tag{7a}$$

$$y_j = r_j(t) \sin \theta_j(t), \tag{7b}$$

so that the transverse speeds are given by

$$\begin{aligned} \dot{x}_j &= \dot{r}_j \cos \theta_j - r_j \dot{\theta}_j \sin \theta_j \\ &= \dot{r}_j \cos \theta_j + \frac{qB}{2m} r_j \sin \theta_j, \end{aligned} \tag{8a}$$

$$\dot{y}_j = \dot{r}_j \sin \theta_j - \frac{qB}{2m} r_j \cos \theta_j. \tag{8b}$$

Inserting Eqs. (8b) and (7a) into Eq. (6) gives, for the term of interest,

$$\begin{aligned} &\frac{4}{NR_x} \sum_j x_j (\ddot{x}_j)_{\text{solenoid}} \\ &= \frac{4}{NR_x} \frac{qB}{m} \sum_j \left(r_j \dot{r}_j \cos \theta_j \sin \theta_j - \frac{qB}{2m} x_j^2 \right). \end{aligned} \tag{9}$$

Consider the subset of particles having a given value of the product $r_j \dot{r}_j$, within a specified small interval. These particles will, under the assumptions of equal, symmetric x and y distributions, be randomly located in azimuth and hence will yield zero for the average of the product $\sin \theta_j \cos \theta_j$, averaged over the subset. Thus only the second term on the right-hand side of Eq. (9) will survive, and it is equal to $-(qB/m)^2 R_x/2$.

With this development, Eq. (3) can be written

$$\ddot{R}_x - \frac{E_x^2}{R_x^3} + \frac{1}{2}\left(\frac{qB}{m}\right)^2 R_x - \frac{4}{NR_x}\sum x_j(\ddot{x}_j)_{\text{space charge}} = 0, \quad (10)$$

with a similar equation for the y values. In Refs. 1 through 3 it is shown that the last term in this equation has, for circularly symmetric beams, the value $-(qI/2\pi\varepsilon_0 mv)R^{-1}$ where $R = R_x = R_y$, I is the beam current, and v is the longitudinal speed of the beam particles.

4 END EFFECTS

The angular velocity possessed by a particle while in the solenoid is the result of the particle's having passed through a region in which the radial component of the magnetic field, B_r, differs from zero, yielding a torque about the z axis. We first evaluate the value of B_r for the assumed field geometry by drawing a Gaussian pill box straddling the end region of the solenoid, as shown in Figure 3. The flux integral of the magnetic field over the pill box is then

$$2\pi r\Delta z B_r + \pi r^2 B_z = 0,$$

yielding†

$$B_r\Delta z = -\frac{r}{2}B. \quad (11)$$

In traversing the end region, the particles are subjected to an impulsive force, which will change the values of $\dot{x}$ and $\dot{y}$ for a given particle, without changing the x and y values. The longitudinal speed v is assumed constant throughout the interaction with the solenoid (it actually decreases upon entry and increases upon exit); the impulse force is given by

$$\mathbf{F}_{\text{impulse}} = -\mathbf{i}qvB_r\sin\theta + \mathbf{j}qvB_r\cos\theta. \quad (12)$$

The term corresponding to an impulse opposite to the direction of motion has been omitted; it is proportional to r^2/ρ^2, where $\rho \equiv (mv/qB)$ is the bending radius of the particles in the field of the solenoid.

† In a cylindrically symmetric system, with B_z not a function of r, $\nabla\cdot\mathbf{B} = 0$ gives $B_r = -\frac{1}{2}r(\partial B_z/\partial z)$. Integration of this result through the end region gives the result (11) independently of the assumed z dependence at the end field.

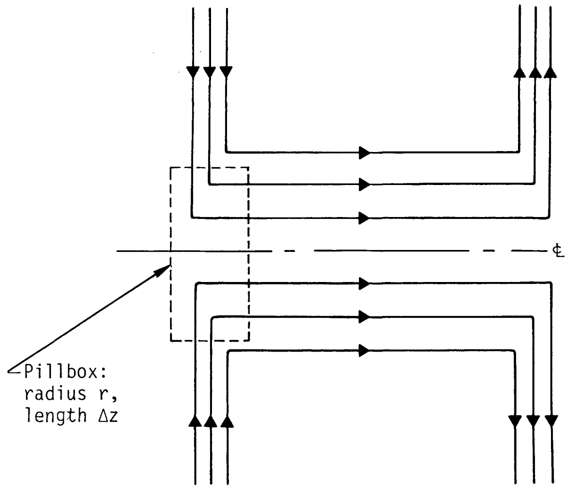

FIGURE 3 Geometry for calculation of the radial component of the magnetic field.

For the x motion of particle j, then, we have the equation

$$\ddot{x}_j = \frac{qvB}{2m\Delta z} r_j \sin\theta_j,$$

so that, integrating over the (brief) time interval $\Delta t = \Delta z/v$, we obtain for the change in $\dot{x}$:

$$\Delta\dot{x}_j = \frac{qB}{2m} y_j, \quad (13)$$

while for the motion in the y direction we have the result

$$\Delta\dot{y}_j = -\frac{qB}{2m} x_j. \quad (14)$$

Equations (13) and (14) are valid for particles entering the solenoid; upon leaving the solenoid similar equations are obtained, the only difference being that the algebraic signs of the right-hand sides are reversed.

Since the emittance is related to the amount of energy in the transverse motion, and since the transverse speeds have been changed upon entry into the solenoid, it is logical to expect that the emittances have been changed. This expectation can be verified by calculating the emittance just after entry (+ means "after entry," − means "before entry"):

$$R_{x+}^2 = R_{x-}^2, \text{ since positions are not changed.} \quad (15)$$

$$V_{x+}^2 = \frac{4}{N}\sum \dot{x}_{j+}^2 = \frac{4}{N}\sum\left(\dot{x}_{j-} + \frac{qB}{2m}y_j\right)^2. \quad (16)$$

Now $\dot{x}_{j-} = \dot{r}_{j-} \cos\theta_j$, since before entry $\dot{\theta}_j = 0$, so that the cross term that appears in the squared expression under the summation can be written as $(qB/m)\dot{r}_j r_j \cos\theta_j \sin\theta_j$, which will average to zero as explained in the text following Eq. (9). Therefore we have

$$V_{x+}^2 = V_{x-}^2 + \left(\frac{qB}{2m}\right)^2 R_y^2, \tag{17}$$

for the change in the mean square velocity upon entering the solenoid.

The second term in the expression for the emittance, Eq. (4), involves the product $R_x\dot{R}_x$. We have for this product after entry

$$(R_x\dot{R}_x)_+ = \frac{4}{N}\sum x_{j+}\dot{x}_{j+} = \frac{4}{N}\sum x_j\left(\dot{x}_{j-} + \frac{qB}{2m}\,y_j\right).$$

The term $\sum x_j y_j$ vanishes due to the assumed symmetry of the beam in each coordinate direction. Thus we have,

$$(R_x\dot{R}_x)_+ = (R_x\dot{R}_x)_-.$$

Then the emittance squared just after entry is given by

$$\begin{aligned} E_{x+}^2 &= (R_x^2 V_x^2 - R_x^2\dot{R}_x^2)_+ \\ &= R_{x-}^2\left(V_{x-}^2 + \left(\frac{qB}{2m}\right)^2 R_{y-}^2\right) - (R_x\dot{R}_x)_-^2 \\ &= E_{x-}^2 + \left(\frac{qB}{2m}\right)^2 R_x^2 R_y^2. \quad \text{(entry)} \end{aligned} \tag{18}$$

Upon leaving the solenoid similar considerations yield the result

$$E_{x+}^2 = E_{x-}^2 - \left(\frac{qB}{2m}\right)^2 R_x^2 R_y^2. \quad \text{(exit)} \tag{19}$$

In Eqs. (18) and (19) it must be understood that the product $R_x^2 R_y^2$ is a function of z and must be evaluated at the appropriate end of the solenoid.

5 THE TIME DEPENDENCE OF THE EMITTANCE

That the emittance changes value upon entry and exit from the solenoid is simply a reflection of the fact that the emittance as defined by Eq. (4) is not a constant of the motion in the presence of a solenoidal field. In this section we will discover that there exists a constant of the motion that reduces to the emittance when the solenoidal field vanishes. Starting with Eq. (16) it can readily be shown that

$$E_x\dot{E}_x = \frac{16}{N^2}\left(\sum x_j^2 \sum \dot{x}_j\ddot{x}_j - \sum x_j\dot{x}_j \sum x_j\ddot{x}_j\right). \tag{20}$$

But inside the solenoid we have

$$\ddot{x}_j = \frac{qB}{m}\,\dot{y}_j.$$

Hence the term $\sum \dot{x}_j\ddot{x}_j$ will become proportional to $\sum \dot{x}_j\dot{y}_j$, which will vanish because of the assumed identity of distributions in the x and y directions.

The term $\sum x_j\ddot{x}_j$ is equal to $(qB/m)\sum x_j\dot{y}_j$, which is calculated in the following way. From Eq. (8b) we have

$$\sum x_j\dot{y}_j = \sum r_j\cos\theta_j\left(\dot{r}_j\sin\theta_j - \frac{qB}{2m}\,r_j\cos\theta_j\right).$$

Again we observe that the first term averages to zero, so that,

$$\frac{qB}{m}\sum x_j\dot{y}_j = -\frac{1}{2}\left(\frac{qB}{m}\right)^2 \sum x_j^2.$$

Thus Eq. (20) becomes,

$$\begin{aligned} E_x\dot{E}_x &= \frac{16}{N^2}\frac{1}{2}\left(\frac{qB}{m}\right)^2 \sum x_j\dot{x}_j \sum x_j^2 \\ &= \frac{1}{2}\left(\frac{qB}{m}\right)^2 (R_x\dot{R}_x)R_x^2. \end{aligned} \tag{21}$$

This last equation may also be written as,

$$\frac{\mathrm{d}E_x^2}{\mathrm{d}t} = \left(\frac{qB}{2m}\right)^2 \frac{\mathrm{d}}{\mathrm{d}t}R_x^4,$$

which implies that the quantity $E_x^2 - (qB/2m)^2 R_x^4$ is a constant of the motion. The value of this constant is obtained from Eq. (18) (recall, $R_y = R_x$) and is seen to be simply the emittance squared of the beam prior to entry into the solenoid.† That is, we may write,

$$E_x^2(t) = E_x^2(0) + \left(\frac{qB}{2m}\right)^2 R_x^4(t), \tag{22}$$

where $E_x(0)$ is the emittance of the beam just before entering the solenoid.

† Lee[5] has derived this result for relativistic, cylindrically symmetric beams. The result is not dependent on the details of the variation of B_z with z.

6 THE OVERALL PICTURE

In this section we summarize the various bits and pieces required for calculating beam transport. In the derivations employed above it was necessary to assume equal symmetric distributions in x and y, and so in this section we can drop the x and y subscripts.

The rms size squared of the beam obeys the equation [cf. Eqs. (10) and (22)].

$$\ddot{R} - \frac{E^2(0)}{R^3} + \left(\frac{qB}{2m}\right)^2 R - \frac{qI}{2\pi\varepsilon_0 mv}\frac{1}{R} = 0, \quad (23)$$

where $E^2(0)$ is the initial emittance of the beam.

Equation (23) can be written in the more conventional form with z as the independent variable by making the substitution $t = z/v$, with the results being (the prime indicates differentiation with respect to z):

$$R'' - \frac{\varepsilon^2(0)}{R^3} + \frac{1}{4\rho^2}R - \frac{q}{m2\pi\varepsilon_0}\frac{I}{v^3}\frac{1}{R} = 0, \quad (24)$$

where $\varepsilon^2 = E^2/v^2$ is the conventional normalized emittance, i.e., $(x - x'$ phase-space area$)^2/\pi^2$.

Although Eqs. (23) and (24) have been derived assuming an especially simple $B_z(z)$, they are correct for any z variation; thus the equations may be integrated right through the end regions.

7 AN ALTERNATIVE APPROACH

When space-charge forces can be neglected, we can integrate the equations of motion for a given particle and determine its position and velocity upon leaving the solenoid in terms of the values of these quantities upon entrance. Thus, given the statistical characterization of the beam at one end of the solenoid, it is possible to deduce the values of these statistical parameters at the other end.

We assume the same magnetic-field distribution as before, i.e., uniform B_z over the length L, and sharp ends. From Busch's theorem a particle will have its azimuth changed, upon traversing the length of the solenoid, according to,

$$\theta(L) = \theta(0) - \frac{qB}{2m}\frac{L}{v}, \\ = \theta(0) - \frac{L}{2\rho}. \quad (25)$$

We again assume that the longitudinal speed v is constant, in which case the equation of motion for the radius of a particle is[6]

$$r'' + \left(\frac{qB_z}{2mv}\right)^2 r = 0,$$

or, for constant $B_z = B$, we have

$$r'' + \frac{1}{4\rho^2}r = 0. \quad (26)$$

The solution of this last equation gives $r(z = L)$ in terms of $r(z = 0)$ and $r'(z = 0)$:

$$r(L) = r(0)\cos\frac{L}{2\rho} + 2\rho r'(0)\sin\frac{L}{2\rho}, \quad (27)$$

which gives us, upon differentiation, an expression for $r'(z = L)$:

$$r'(L) = -\frac{1}{2\rho}r(0)\sin\frac{L}{2\rho} + r'(0)\cos\frac{L}{2\rho}. \quad (28)$$

(From this point on, the argument indication $z = 0$ will generally be suppressed, while $z = L$ will always be indicated.)

The last two equations, together with Eq. (25) allow us to calculate the x and y coordinates of a particle, as well as the values of x' and y'. Denoting the quantity $L/2\rho$ by Φ, these quantities are given by

$$\begin{aligned} x(L) &= r(L)\cos\theta(L) \\ &= [r(0)\cos\Phi + 2\rho r'(0)\sin\Phi]\cos(\theta(0) - \Phi), \\ &= (r\cos\Phi + 2\rho r'\sin\Phi) \\ &\quad \times (\cos\theta\cos\Phi + \sin\theta\sin\Phi) \\ &= x\cos^2\Phi + y\cos\Phi\sin\Phi \\ &\quad + 2\rho x'\sin\Phi\cos\Phi + 2\rho y'\sin^2\Phi, \end{aligned} \quad (29)$$

$$\begin{aligned} y(L) &= r(L)\sin\theta(L) \\ &= y\cos^2\Phi - x\cos\Phi\sin\Phi \\ &\quad + 2\rho y'\sin\Phi\cos\Phi - 2\rho x'\sin^2\Phi, \end{aligned} \quad (30)$$

$$\begin{aligned} x'(L) &= r'(L)\cos\theta(L) \\ &= x'\cos^2\Phi + y'\cos\Phi\sin\Phi \\ &\quad - \frac{1}{2\rho}x\sin\Phi\cos\Phi - \frac{1}{2\rho}y\sin^2\Phi, \end{aligned} \quad (31)$$

$$\begin{aligned} y'(L) &= r'(L)\sin\theta(L) \\ &= y'\cos^2\Phi - x'\cos\Phi\sin\Phi \\ &\quad - \frac{1}{2\rho}y\sin\Phi\cos\Phi + \frac{1}{2\rho}x\sin^2\Phi. \end{aligned} \quad (32)$$

Dealing with these four expressions clearly involves great quantities of simple algebra; the intent here is to point out the key features of the manipulations without going into all the details.

In order to calculate the rms beam size squared, $R_x^2(L)$, we need to evaluate $\sum x_j^2(L)$. That is, for each particle we need to square the expression (29) and sum the result over all particles in the ensemble of interest. In this squared expression there occur, among other terms, terms involving $\sum x_j y_j$, $\sum x_j' y_j'$, and $\sum (x_j y_j' + x_j' y_j)$. The first two of these sums vanish due to the assumed symmetry of the distributions in each coordinate direction in the incoming beam, while the third sum is the derivative of the first and therefore also vanishes. One should note that once the beam is inside the solenoid these sums would only vanish if the incoming beam had the same distributions in x and y.

With these observations it is tedious but straightforward to verify that

$$\begin{aligned} R_x^2(L) = {} & \cos^2 \Phi \, (R_x^2 \cos^2 \Phi + R_y^2 \sin^2 \Phi) \\ & + 4\rho^2 \sin^2 \Phi (U_x^2 \cos^2 \Phi + U_y^2 \sin^2 \Phi) \\ & + 4\rho \sin \Phi \cos \Phi (R_x R_x' \cos^2 \Phi \\ & + R_y R_y' \sin^2 \Phi), \end{aligned} \tag{33}$$

where $U_x^2(0) = (4/N) \sum (x_j')^2$. The expression corresponding to Eq. (33) for the beam size in the y direction is obtained from Eq. (33) by replacing all x's by y's and vice versa.

The derivative of the rms beam size at the exit of the solenoid is obtained by calculating the sum $\sum x_j(L) x_j'(L)$. The result is

$$\begin{aligned} R_x(L) R_x'(L) = {} & (\cos^2 \Phi - \sin^2 \Phi) \\ & \times (R_x R_x' \cos^2 \Phi + R_y R_y' \sin^2 \Phi) \\ & - \frac{1}{2\rho} \cos \Phi \sin \Phi \\ & \times (R_x^2 \cos^2 \Phi + R_y^2 \sin^2 \Phi) \\ & + 2\rho \cos \Phi \sin \Phi \\ & \times (U_x^2 \cos^2 \Phi + U_y^2 \sin^2 \Phi). \end{aligned} \tag{34}$$

Equations (33) and (34), together with their y counterparts, allow one to establish the values of R_x, R_x', R_y, and R_y' at the end of the solenoid. The only other parameters required to specify the beam for further beam-transport calculations are the emittances.

For the emittances we need to evaluate $U_x^2(L) = (4/N) \sum (x_j'(L))^2$ and the corresponding y parameter. In exactly the same fashion as was done for R_x^2, we square Eq. (31) for each particle and sum over all particles. The result is

$$\begin{aligned} U_x^2(L) = {} & \cos^2 \Phi \, (U_x^2 \cos^2 \Phi + U_y^2 \sin^2 \Phi) \\ & + \frac{1}{4\rho^2} \sin^2 \Phi (R_x^2 \cos^2 \Phi + R_y^2 \sin^2 \Phi) \\ & - \frac{1}{\rho} \cos \Phi \sin \Phi (R_x R_x' \cos^2 \Phi \\ & + R_y R_y' \sin^2 \Phi). \end{aligned} \tag{35}$$

Another whole page of compact algebra using Eqs. (33) through (35) brings us to the result

$$\varepsilon_x^2(L) \equiv (R_x^2 U_x^2 - R_x^2 R_x'^2)_L = r^2 u^2 - (rr')^2, \tag{36}$$

where

$$r^2 = R_x^2 \cos^2 \Phi + R_y^2 \sin^2 \Phi, \tag{37a}$$

$$u^2 = U_x^2 \cos^2 \Phi + U_y^2 \sin^2 \Phi, \tag{37b}$$

and

$$rr' = R_x R_x' \cos^2 \Phi + R_y R_y' \sin^2 \Phi. \tag{37c}$$

From Eq. (36) it is readily seen that if the x and y distributions are identical then $\varepsilon_x^2(L) = \varepsilon_x^2(0)$, i.e., the emittance is conserved. It is also readily verified that if $\Phi = \pi/2$ then the x and y emittances are interchanged.

Having passed the beam through the solenoid, we can then continue the rms beam-transport integration by using Eqs. (33), (34), and (36). However, if the beam is not round (identical x and y distributions) the solenoid will have rotated the spatial pattern of the beam and, for Φ not a multiple of $\pi/2$, the principal axes of the spatial distribution will not lie along the coordinate axes. The latter observation only affects the space-charge term in the beam-envelope equation and may or may not be significant in any calculation.

Finally, to make contact with the previous calculations relating to emittance within the solenoid, we point out first that, on the inside, Eq. (31) becomes, since θ is a function of position,

$$x'(z) = r'(z) \cos \theta(z) + \frac{1}{2\rho} r(z) \sin \theta(z).$$

Tracing the added term through all the manipulations above gives the result, for a round beam,

$$\begin{aligned}\varepsilon_x^2(z) &= \varepsilon_x^2(0) + \frac{1}{4\rho^2}\,[R^2\cos^2\Phi(z) \\ &\quad + 4\rho^2U^2\sin^2\Phi(z) \\ &\quad + 2\rho RR'\cos\Phi(z)\sin\Phi(z)]^2 \\ &= \varepsilon_x^2(0) + \frac{R^4(z)}{4\rho^2}.\end{aligned}$$

This last equality follows from Eq. (33), evaluated for a round beam. Thus we have arrived at the result (22) by another means.

8 OBSERVATIONS

In this final section we restrict ourselves to round beams, for which we can drop the x and y subscripts. We rewrite Eqs. (33) and (34) to obtain

$$\begin{aligned}R^2(L) &= R^2\cos^2\Phi + 4\rho^2U^2\sin^2\Phi \\ &\quad + 4\rho RR'\sin\Phi\cos\Phi,\end{aligned} \tag{38}$$

and

$$\begin{aligned}R(L)R'(L) &= RR'(\cos^2\Phi - \sin^2\Phi) \\ &\quad - \frac{R^2}{2\rho}\cos\Phi\sin\Phi \\ &\quad + 2\rho U^2\cos\Phi\sin\Phi.\end{aligned} \tag{39}$$

The first of these two equations can be put into a more suggestive form by using the relation $U^2 = R'^2 + \varepsilon^2/R^2$ [cf. Eq. (36)]:

$$R^2(L) = (R\cos\Phi + 2\rho R'\sin\Phi)^2 + \frac{4\rho^2\varepsilon^2}{R^2}\sin^2\Phi. \tag{40}$$

This equation says, among other things, that if $R' = 0$ upon entering the solenoid, then the smallest value that $R(L)$ can have occurs for $\Phi = \pi/2$, for which $R(L)$ has the value $2\rho\varepsilon/R$. In this circumstance, since Eq. (39) gives us, for $\Phi = \pi/2$, $R(L)R'(L) = -R(0)R'(0)$, we observe that the beam exits with $R' = 0$. Thus a $\Phi = \pi/2$ solenoid acts upon a beam at a waist or anti-waist simply as a beam-size reducer.

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