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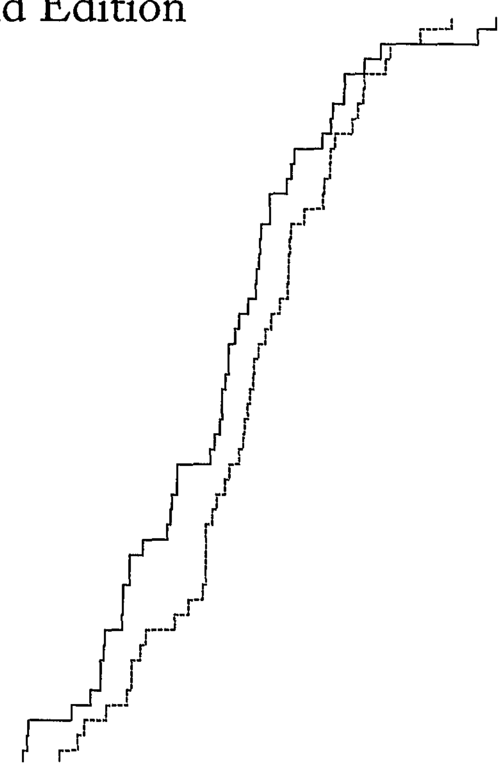
Statistical Methods in Experimental Physics

2nd Edition



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CONTENTS

Preface to the Second Edition v
Preface to the First Edition vii

Chapter 1. Introduction 1

1.1. Outline 1
1.2. Language 2
1.3. Two Philosophies 3
1.4. Notation 4

Chapter 2. Basic Concepts in Probability 9

2.1. Definitions of Probability 9
2.1.1. Mathematical probability 10
2.1.2. Frequentist probability 10
2.1.3. Bayesian probability 11
2.2. Properties of Probability 12
2.2.1. Addition law for sets of elementary events 12
2.2.2. Conditional probability and independence 13
2.2.3. Example of the addition law: scanning efficiency 14
2.2.4. Bayes theorem for discrete events 15
2.2.5. Bayesian use of Bayes theorem 16
2.2.6. Random variable 17
2.3. Continuous Random Variables 18
2.3.1. Probability density function 19
2.3.2. Change of variable 20

2.3.3. Cumulative, marginal and conditional distributions 21
2.3.4. Bayes theorem for continuous variables 22
2.3.5. Bayesian use of Bayes theorem for continuous variables . . . 22
2.4. Properties of Distributions 24
2.4.1. Expectation, mean and variance 24
2.4.2. Covariance and correlation 26
2.4.3. Linear functions of random variables 28
2.4.4. Ratio of random variables 30
2.4.5. Approximate variance formulae 32
2.4.6. Moments 33
2.5. Characteristic Function 34
2.5.1. Definition and properties 34
2.5.2. Cumulants 37
2.5.3. Probability generating function 38
2.5.4. Sums of a random number of random variables 39
2.5.5. Invariant measures 41

Chapter 3. Convergence and the Law of Large Numbers 43

3.1. The Tchebycheff Theorem and Its Corollary 43
3.1.1. Tchebycheff theorem 43
3.1.2. Bienaymé–Tchebycheff inequality 44
3.2. Convergence 45
3.2.1. Convergence in distribution 45
3.2.2. The Paul Levy theorem 46
3.2.3. Convergence in probability 46
3.2.4. Stronger types of convergence 47
3.3. The Law of Large Numbers 47
3.3.1. Monte Carlo integration 48
3.3.2. The Central Limit theorem 49
3.3.3. Example: Gaussian (Normal) random number generator . . 51

Chapter 4. Probability Distributions 53

4.1. Discrete Distributions 53
4.1.1. Binomial distribution 53
4.1.2. Multinomial distribution 56
4.1.3. Poisson distribution 57
4.1.4. Compound Poisson distribution 60
4.1.5. Geometric distribution 62

4.1.6. Negative binomial distribution 63
4.2. Continuous Distributions 64
4.2.1. Normal one-dimensional (univariate Gaussian) 64
4.2.2. Normal many-dimensional (multivariate Gaussian) 67
4.2.3. Chi-square distribution 70
4.2.4. Student's t-distribution 73
4.2.5. Fisher–Snedecor F and Z distributions 77
4.2.6. Uniform distribution 79
4.2.7. Triangular distribution 79
4.2.8. Beta distribution 80
4.2.9. Exponential distribution 82
4.2.10. Gamma distribution 83
4.2.11. Cauchy, or Breit–Wigner, distribution 84
4.2.12. Log-Normal distribution 85
4.2.13. Extreme value distribution 87
4.2.14. Weibull distribution 89
4.2.15. Double exponential distribution 89
4.2.16. Asymptotic relationships between distributions 90
4.3. Handling of Real Life Distributions 91
4.3.1. General applicability of the Normal distribution 91
4.3.2. Johnson empirical distributions 92
4.3.3. Truncation 93
4.3.4. Experimental resolution 94
4.3.5. Examples of variable experimental resolution 95

Chapter 5. Information 99

5.1. Basic Concepts 100
5.1.1. Likelihood function 100
5.1.2. Statistic 100
5.2. Information of R.A. Fisher 101
5.2.1. Definition of information 101
5.2.2. Properties of information 101
5.3. Sufficient Statistics 103
5.3.1. Sufficiency 103
5.3.2. Examples 104
5.3.3. Minimal sufficient statistics 105
5.3.4. Darmois theorem 106
5.4. Information and Sufficiency 108

5.5. Example of Experimental Design 109

Chapter 6. Decision Theory 111

6.1. Basic Concepts in Decision Theory 112
6.1.1. Subjective probability, Bayesian approach 112
6.1.2. Definitions and terminology 113
6.2. Choice of Decision Rules 114
6.2.1. Classical choice: pre-ordering rules 114
6.2.2. Bayesian choice 115
6.2.3. Minimax decisions 116
6.3. Decision-theoretic Approach to Classical Problems 117
6.3.1. Point estimation 117
6.3.2. Interval estimation 118
6.3.3. Tests of hypotheses 118
6.4. Examples: Adjustment of an Apparatus 121
6.4.1. Adjustment given an estimate of the apparatus performance 121
6.4.2. Adjustment with estimation of the optimum adjustment . . 123
6.5. Conclusion: Indeterminacy in Classical and Bayesian Decisions . . 124

Chapter 7. Theory of Estimators 127

7.1. Basic Concepts in Estimation 127
7.1.1. Consistency and convergence 128
7.1.2. Bias and consistency 129
7.2. Usual Methods of Constructing Consistent Estimators 130
7.2.1. The moments method 131
7.2.2. Implicitly defined estimators 132
7.2.3. The maximum likelihood method 135
7.2.4. Least squares methods 137
7.3. Asymptotic Distributions of Estimates 139
7.3.1. Asymptotic Normality 139
7.3.2. Asymptotic expansion of moments of estimates 141
7.3.3. Asymptotic bias and variance of the usual estimators 144
7.4. Information and the Precision of an Estimator 146
7.4.1. Lower bounds for the variance — Cramér–Rao inequality . . 147
7.4.2. Efficiency and minimum variance 149
7.4.3. Cramér–Rao inequality for several parameters 151
7.4.4. The Gauss–Markov theorem 152
7.4.5. Asymptotic efficiency 153

7.5. Bayesian Inference 154
7.5.1. Choice of prior density 154
7.5.2. Bayesian inference about the Poisson parameter 156
7.5.3. Priors closed under sampling 157
7.5.4. Bayesian inference about the mean, when the variance is known 157
7.5.5. Bayesian inference about the variance, when the mean is known 159
7.5.6. Bayesian inference about the mean and the variance 161
7.5.7. Summary of Bayesian inference for Normal parameters . . . 162

Chapter 8. Point Estimation in Practice 163

8.1. Choice of Estimator 163
8.1.1. Desirable properties of estimators 164
8.1.2. Compromise between statistical merits 165
8.1.3. Cures to obtain simplicity 166
8.1.4. Economic considerations 168
8.2. The Method of Moments 170
8.2.1. Orthogonal functions 170
8.2.2. Comparison of likelihood and moments methods 172
8.3. The Maximum Likelihood Method 173
8.3.1. Summary of properties of maximum likelihood 173
8.3.2. Example: determination of the lifetime of a particle in a restricted volume 175
8.3.3. Academic example of a poor maximum likelihood estimate . 177
8.3.4. Constrained parameters in maximum likelihood 179
8.4. The Least Squares Method (Chi-Square) 182
8.4.1. The linear model 183
8.4.2. The polynomial model 185
8.4.3. Constrained parameters in the linear model 186
8.4.4. Normally distributed data in nonlinear models 190
8.4.5. Estimation from histograms; comparison of likelihood and least squares methods 191
8.5. Weights and Detection Efficiency 193
8.5.1. Ideal method maximum likelihood 194
8.5.2. Approximate method for handling weights 196
8.5.3. Exclusion of events with large weight 199
8.5.4. Least squares method 201

8.6. Reduction of Bias 204
8.6.1. Exact distribution of the estimate known 204
8.6.2. Exact distribution of the estimate unknown 206
8.7. Robust (Distribution-free) Estimation 207
8.7.1. Robust estimation of the centre of a distribution 208
8.7.2. Trimming and Winsorization 210
8.7.3. Generalized p^{th}-power norms 211
8.7.4. Estimates of location for asymmetric distributions 213

Chapter 9. Interval Estimation 215

9.1. Normally distributed data 216
9.1.1. Confidence intervals for the mean 216
9.1.2. Confidence intervals for several parameters 218
9.1.3. Interpretation of the covariance matrix 223
9.2. The General Case in One Dimension 225
9.2.1. Confidence intervals and belts 225
9.2.2. Upper limits, lower limits and flip-flopping 227
9.2.3. Unphysical values and empty intervals 229
9.2.4. The unified approach 229
9.2.5. Confidence intervals for discrete data 231
9.3. Use of the Likelihood Function 233
9.3.1. Parabolic log-likelihood function 233
9.3.2. Non-parabolic log-likelihood functions 234
9.3.3. Profile likelihood regions in many parameters 236
9.4. Use of Asymptotic Approximations 238
9.4.1. Asymptotic Normality of the maximum likelihood estimate . 238
9.4.2. Asymptotic Normality of $\partial \ln L/\partial\theta$ 238
9.4.3. $\partial L/\partial\theta$ confidence regions in many parameters 240
9.4.4. Finite sample behaviour of three general methods of interval estimation 240
9.5. Summary: Confidence Intervals and the Ensemble 246
9.6. The Bayesian Approach 248
9.6.1. Confidence intervals and credible intervals 249
9.6.2. Summary: Bayesian or frequentist intervals? 250

Chapter 10. Test of Hypotheses 253

10.1. Formulation of a Test 254
10.1.1. Basic concepts in testing 254

10.1.2. Example: Separation of two classes of events 255
10.2. Comparison of Tests 257
10.2.1. Power 257
10.2.2. Consistency 259
10.2.3. Bias 260
10.2.4. Choice of tests 261
10.3. Test of Simple Hypotheses 263
10.3.1. The Neyman–Pearson test 263
10.3.2. Example: Normal theory test versus sign test 264
10.4. Tests of Composite Hypotheses 266
10.4.1. Existence of a uniformly most powerful test for the exponential family 267
10.4.2. One- and two-sided tests 268
10.4.3. Maximizing local power 269
10.5. Likelihood Ratio Test 270
10.5.1. Test statistic 270
10.5.2. Asymptotic distribution for continuous families of hypotheses 271
10.5.3. Asymptotic power for continuous families of hypotheses . . 273
10.5.4. Examples 274
10.5.5. Small sample behaviour 279
10.5.6. Example of separate families of hypotheses 282
10.5.7. General methods for testing separate families 285
10.6. Tests and Decision Theory 287
10.6.1. Bayesian choice between families of distributions 287
10.6.2. Sequential tests for optimum number of observations . . . 292
10.6.3. Sequential probability ratio test for a continuous family of hypotheses 297
10.7. Summary of Optimal Tests 298

Chapter 11. Goodness-of-Fit Tests 299

11.1. GOF Testing: From the Test Statistic to the P-value 299
11.2. Pearson's Chi-square Test for Histograms 301
11.2.1. Moments of the Pearson statistic 302
11.2.2. Chi-square test with estimation of parameters 303
11.2.3. Choosing optimal bin size 304
11.3. Other Tests on Binned Data 308
11.3.1. Runs test 308

11.3.2. Empty cell test, order statistics 309
11.3.3. Neyman–Barton smooth test 311
11.4. Tests Free of Binning 313
11.4.1. Smirnov–Cramér–von Mises test 314
11.4.2. Kolmogorov test 316
11.4.3. More refined tests based on the EDF 317
11.4.4. Use of the likelihood function 317
11.5. Applications 318
11.5.1. Observation of a fine structure 318
11.5.2. Combining independent estimates 323
11.5.3. Comparing distributions 327
11.6. Combining Independent Tests 330
11.6.1. Independence of tests 330
11.6.2. Significance level of the combined test 331

References 335

Subject Index 341