II. MICROWAVE GASEOUS DISCHARGES*

Prof. S. C. Brown	R. R. Brown II	J. J. McCarthy
Prof. W. P. Allis	C. D. Buntschuh	R. G. Meyerand, Jr.
Prof. D. O. Akhurst	S. Frankenthal	W. J. Mulligan
Dr. G. Bekefi	R. B. Hall	Judith L. Shaver
Dr. S. J. Buchsbaum	J. L. Hirshfield	D. R. Whitehouse
	W. R. Kittredge	

A. DIAMAGNETISM OF A LONG CYLINDRICAL PLASMA

The diffusion of a plasma that is contained in an infinitely long cylindrical tube in an axial magnetic field B_o can be solved exactly in the ambipolar limit; that is, for

$$n_+ - n_- << n_- \tag{1}$$

The subscript "plus" in Eq. 1 refers to ions, and the subscript "minus" to electrons. The result will show that the plasma is diamagnetic (see Eq. 17). To measure the diamagnetism we can change the applied magnetic field B_o (or the plasma density) either continuously ($\dot{B}_o$) or discontinuously (ΔB_o) and observe the flux change ($\dot{\phi}$ or $\Delta\phi$) through the cylinder. However, changing magnetic fields induce currents in the plasma. We wish to know when these induced currents will mask the diamagnetic effect. There must be a maximum rate of change $j\omega = \dot{B}_o/B_o$, or minimum relaxation time Δt, beyond which the conductivity rather than the diamagnetism of the plasma is measured. It is this limit that is sought.

We shall assume that the motions of both particles are of the diffusive rather than of the free-fall type; that is

$$\omega < \nu_{c_\ell} \tag{2}$$

where the collision frequency ν_{c_ℓ} is the smaller of ν_{c+} and ν_{c-}. The plasma equations can then be written

$$\Gamma_- - \mu_- B \times \Gamma_- = -\nabla D_- n_- - E\mu_- n_- \tag{3}$$

$$\Gamma_+ + \mu_+ B \times \Gamma_+ = -\nabla D_+ n_+ + E\mu_+ n_+ \tag{4}$$

$$\nabla \cdot \Gamma_\pm = \nu n_- \qquad \nabla \cdot E = \frac{(n_+ - n_-)e}{\epsilon_o} \tag{5}$$

$$\nabla \times H = (\Gamma_+ - \Gamma_-)e \qquad \nabla \times E = -\dot{B} \tag{6}$$

where Γ is the particle current, D the free diffusion coefficient, μ the mobility, E the

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electric field, and ν the net rate of generation of electrons. Equations 3 and 4 need not be separated into dc and ac components because, with the assumption of Eq. 2, the dc and ac mobilities are the same. Experimentally, frequencies are low and densities are high, and as a result the displacement current has been neglected. Under the ambipolar condition (Eq. 1) we can replace n_+ and n_- by n everywhere except in Poisson's equation. This equation is only used in the determination of the difference $(n_+ - n_-)$. The dimensionless product $\mu B = \omega_b/\nu_c$, which always appears, is the ratio of the cyclotron angular frequency to the collision frequency. It is proportional to B/p_o, with p_o the reduced pressure of the gas that was originally present.

Equations 3 and 4 can be combined to give

$$\mu_- \boldsymbol{\Gamma}_+ + \mu_+ \boldsymbol{\Gamma}_- + \mu_+\mu_- B \times (\boldsymbol{\Gamma}_+ - \boldsymbol{\Gamma}_-) = -\nabla(\mu_- D_+ + \mu_+ D_-)n \tag{7}$$

$$\boldsymbol{\Gamma}_+ - \boldsymbol{\Gamma}_- + B \times (\mu_+\boldsymbol{\Gamma}_+ + \mu_-\boldsymbol{\Gamma}_-) = \nabla(D_- n_- - D_+ n_+) + E(\mu_+ + \mu_-)n \tag{8}$$

We now specialize these equations for our particular problem, in which we assume the symmetry of an infinite cylinder; that is,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \theta} = 0 \tag{9}$$

whence conservation of charge and Eq. 1 require that

$$\boldsymbol{\Gamma}_{+r} = \boldsymbol{\Gamma}_{-r} \tag{10}$$

Equations 7 and 8 yield

$$\boldsymbol{\Gamma}_r = -\frac{\partial}{\partial r} D_a n + \mu_a B(\boldsymbol{\Gamma}_{+\theta} - \boldsymbol{\Gamma}_{-\theta}) = -\frac{\mu_a}{e}\frac{\partial}{\partial r}\left[ne(T_+ + T_-) + \frac{B^2}{2\mu_o}\right] \tag{11}$$

$$\frac{J_\theta}{e} = \boldsymbol{\Gamma}_{+\theta} - \boldsymbol{\Gamma}_{-\theta} = (\mu_+ + \mu_-)(E_\theta n - B\boldsymbol{\Gamma}_r) \tag{12}$$

where D_a is the ambipolar diffusion coefficient, and $\mu_a = \mu_+\mu_-/(\mu_+ + \mu_-)$; use has been made of the Einstein relation

$$T = \frac{D}{\mu} \tag{13}$$

which gives the temperature in electron-volts. Equation 11 shows that both the kinetic and magnetic pressures drive the radial particle flow, and in this form the radial flow appears to be independent of the azimuthal field E_θ. On the other hand, eliminating J_θ through the use of Eqs. 12 and 11 gives

$$(1 + \mu_+\mu_- B^2)\boldsymbol{\Gamma}_r = -\frac{\partial}{\partial r} D_a n + \mu_+\mu_- B E_\theta n \tag{14}$$

in which the usual diffusion coefficient across a magnetic field, $D_a/(1 + \mu_+\mu_- B^2)$, enters.

Neglecting E_θ and eliminating both Γ_r and J_θ from Eqs. 12, 11, and 8, we obtain the radial ambipolar field

$$E_r = \frac{\left(1 + \mu_-^2 B^2\right) \nabla D_+ n - \left(1 + \mu_+^2 B^2\right) \nabla D_- n}{\left(1 + \mu_+ \mu_- B^2\right)\left(\mu_+ + \mu_-\right) n} \tag{15}$$

Equation 15 reduces to

$$E_r = \frac{\nabla(\mu_+ T_+ - \mu_- T_-) n}{\mu_+ + \mu_-}$$

for low magnetic fields, and to

$$E_r = \frac{\nabla(\mu_- T_+ - \mu_+ T_-) n}{\mu_+ + \mu_-}$$

for high magnetic fields. These two expressions usually have opposite signs.

If we eliminate Γ_r from Eqs. 11 and 12, we obtain the circulating current J_θ.

$$\frac{\left(1 + \mu_+ \mu_- B^2\right) J_\theta}{e} = (\mu_+ + \mu_-) \left(B \frac{\partial D_a n}{\partial r} + E_\theta n\right) \tag{16}$$

By using Ampere's law, we can integrate Eq. 16 to yield

$$ne(T_+ + T_-) = \frac{B_o^2}{2\mu_o} \left[1 - \frac{B^2}{B_o^2} + \frac{\ln B_o^2/B^2}{\mu_+ \mu_- B_o^2}\right] + \int_0^r \frac{E_\theta ne}{\mu_a B} dr \tag{17}$$

in which B_o is the external magnetic field, and B is the internal field at a point where the electron density is n. The term in brackets shows that $B < B_o$; hence the plasma is diamagnetic.

It is interesting to note how the diamagnetic current J_θ is built up. The circular orbits of the particles arising from the magnetic field have a magnetic moment per unit volume

$$M = \frac{-ne(T_+ + T_-)}{B} \tag{18}$$

and hence there is a diamagnetic Amperian (or fictitious) current

$$J_M = \frac{\partial}{\partial r} \frac{ne^2(T_+ + T_-)}{B} \tag{19}$$

However, since B is not uniform, the orbits are not circular but epicyclic. There is a real current that runs counter to J_M and produces a net diamagnetic current

$$J_N = \frac{e}{B} \frac{\partial ne(T_+ + T_-)}{\partial r} \tag{20}$$

This current is responsible for the first two terms in the bracket of Eq. 17 and gives the diamagnetism if there are no collisions and no radial flow, as can be seen by integrating Eq. 11 with $\Gamma_r = 0$. The introduction of collisions produces a radial flow across the magnetic field. This induces a paramagnetic current that further reduces the diamagnetism and contributes the third term in the bracket. All three terms constitute the "true" diamagnetism. If the collisions become very frequent, $\mu_\pm \to 0$, the diamagnetism goes to zero, as is well known for electrons in thermal equilibrium.

The last term in Eq. 17 is attributable to the electric field induced by a changing magnetic field.

$$E_\theta = \frac{-1}{r} \int_0^r \dot{B} r \, dr = \frac{-\dot{\bar{B}} r}{2} \tag{21}$$

Assuming that the electrons and ions are distributed according to the normal diffusion mode with diffusion length Λ and assuming that B is uniform, we find that

$$\int_0^R \frac{ne}{\mu_a B} \frac{dr}{r} \int_0^r \dot{B} r \, dr = 0.83 \frac{\Lambda^2}{D_a} \frac{\dot{\bar{B}}}{B} n_o e(T_+ + T_-) \tag{22}$$

where n_o is the density on the axis. As a result, in order to measure the diamagnetism without having to correct for the induced current, we must have

$$\frac{\dot{\bar{B}}}{B} < \frac{D_a}{\Lambda^2} < \nu_c \tag{23}$$

That is, the logarithmic change in the magnetic field must be slower than the diffusion frequency in the absence of a magnetic field. The second inequality is required for the mean free path to be less than the diffusion length, and also for diffusion theory to hold. The solution of Eq. 17 under condition 23 is shown in Fig. II-1.

The changing magnetic field has an effect on the radial flow, as well as on the angular current. From Eqs. 14 and 21 we derive the radial velocity

$$v_r = \frac{dr}{dt} = \frac{-1}{1 + \mu_+\mu_- B^2} \frac{1}{n} \frac{\partial}{\partial r} D_a n - \frac{\mu_+\mu_- B\dot{\bar{B}} r}{2\left(1 + \mu_+\mu_- B^2\right)} \tag{24}$$

The first term gives the velocity of diffusion of the plasma across the magnetic field. The second, as we shall show, is somewhat less than the velocity of the lines of magnetic force. If the diffusion velocity is larger than the velocity of the lines, the plasma will adjust itself to the boundary conditions at the edge of the plasma; otherwise, it

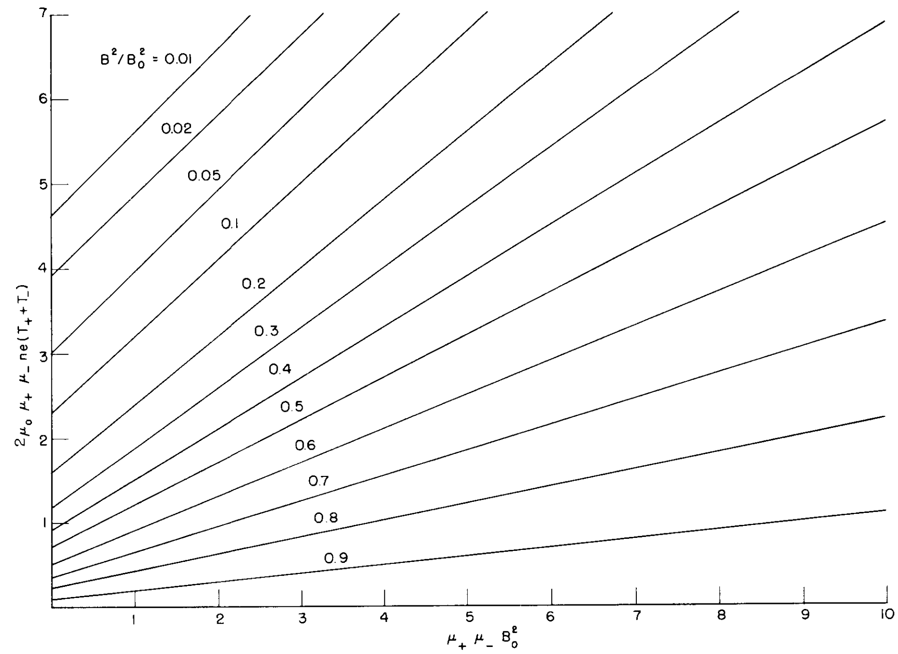

Fig. II-1. Variation of the plasma diamagnetism with the applied magnetic field.

cannot do so. It is interesting to see just what the plasma does when we neglect the diffusion velocity in Eq. 24 and integrate the rest of the equation. The result of this procedure is

$$r^4\left(1 + \mu_+\mu_-B^2\right) = \text{constant} \tag{25}$$

If $\mu_+\mu_-B^2 >> 1$, this means that Br^2 is a constant, which, in turn, means that the plasma moves in and out with the lines of force. If $\mu_+\mu_-B^2 << 1$, r is a constant, and the plasma is fixed in space. Thus Eq. 25 defines the amount of slipping between the plasma and the magnetic field in excess of that caused by diffusion.

Let $A = \pi r_o^2$ be the cross section of the plasma. If the compression of the plasma by the motion of the field is adiabatic, $ne(T_+ + T_-)A^\gamma$ is constant, and if the compression produces only two-dimensional heating, $\gamma = 2$. Under these conditions and with $\mu_+\mu_-B^2 >> 1$, it is impossible to observe diamagnetism. Assuming a uniform plasma and multiplying Eq. 17 by A^2, we have

$$ne(T_+ + T_-)A^2 = \frac{B^2A^2}{2\mu_o}\left[\frac{B_o^{\ 2}}{B^2} - 1 + \frac{\ln B_o^{\ 2}/B^2}{\mu_+\mu_-B^2}\right] \tag{26}$$

The left-hand side is constant and so is the factor in front of the right-hand side. The logarithmic term is small. Hence B_o/B is constant with changes in B_o.

The flux through a coil of area A_c around the plasma is

$$\phi = AB + (A_c - A)B_o \tag{27}$$

By Eq. 25, AB is constant; and, from the proportionality of B to B_o, AB_o is also constant. Hence all changes in ϕ are changes in A_cB_o and, therefore, the changes in ϕ are the same as they would be if the plasma were absent. The compression of the plasma exactly compensates for the diamagnetism under the aforementioned assumptions. Thus both the compression and the induced current defeat attempts to measure diamagnetism unless condition 23 is satisfied.

If experimental conditions are such that inequality 2 is not satisfied, a similar analysis can be carried through. The two main results expressed in Eqs. 23 and 25 remain the same, except that the dc mobilities have to be replaced by the complex ac mobilities. As a result, condition 23 is much harder to satisfy, and the in-and-out motion of the plasma is reduced.

In a weak plasma the diamagnetic effect is small, and Eq. 17 can be solved for δB, where $B = B_o + \delta B$. This gives

$$\delta B = \frac{-\mu_+\mu_- B_o}{1 + \mu_+\mu_- B_o^2}\, \mu_o n e(T_+ + T_-) \tag{28}$$

The emf across a measuring coil is

$$\text{emf} = N\mu_+\mu_- \frac{\left(1 - \mu_+\mu_- B_o^2\right)}{\left(1 + \mu_+\mu_- B_o^2\right)^2}\, \mu_o e(T_+ + T_-) \int_{\substack{\text{area of}\\ \text{plasma}}} n\, \dot{B}_o\, dA \tag{29}$$

where N is the number of turns on the coil.

W. P. Allis, S. J. Buchsbaum

B. MICROWAVE MEASUREMENTS OF PLASMAS IN MAGNETIC FIELDS

This report summarizes the microwave methods that are used to measure electron densities in a narrow plasma column that is placed coaxially in a cylindrical microwave cavity. Permeating the plasma is an axial uniform magnetic field B_o. The microwave conductivity of a plasma at low degrees of ionization, and for low electron temperatures, is a tensor given by

$$\overset{\leftrightarrow}{\sigma} = \frac{ne^2}{2m\omega} \begin{pmatrix} \ell + r & -j(\ell - r) & 0 \\ j(\ell - r) & \ell + r & 0 \\ 0 & 0 & 2p \end{pmatrix} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & 0 \\ \sigma_{\theta r} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} \tag{1}$$

$$\left.\begin{matrix}\ell \\ r\end{matrix}\right\} = \frac{1}{\nu_c + j(\omega \pm \omega_b)} \qquad p = \frac{1}{\nu_c + j\omega}$$

where n is the electron density; ν_c, the electron collision frequency for momentum transfer (assumed constant); ω, the microwave radian frequency; and ω_b, the cyclotron frequency ($\omega_b = eB_o/m$).

The infinite number of modes of a cylindrical microwave cavity can be divided into a number of classes according to the manner in which the narrow plasma column interacts with the microwave field of each class of modes. There are three such classes. A quantitative discussion is presented for each class, but numerical examples are given only for those modes of each class that are actually used in the existing apparatus.

1. TM_{0m0} Class

The electric field of these modes has only an axial component. In the absence of the plasma, it is given by

$$E_z = E_o J_o \left(\frac{\chi_{om} r}{a}\right) \qquad E_r = E_\theta = 0 \tag{2}$$

where E_o is the field on the axis, a is the radius of the cavity, and χ_{om} is the m^{th} root of $J_o(x) = 0$. Since the microwave electric field is parallel to B_o, the ac motion of the electrons is generally not affected by the static magnetic field (for an exception, see Quarterly Progress Report of April 15, 1958, page 12). Consequently, the interaction of the plasma with the microwave field is fully described by the σ_{zz} part of the conductivity tensor. With the plasma present, the electric field is given by the wave equation

$$\frac{d^2E_z}{dr^2} + \frac{1}{r}\frac{dE_z}{dr} + k^2K_{zz}E_z = 0 \qquad 0 \leqslant r \leqslant R \tag{3a}$$

and

$$\frac{d^2E_z}{dr^2} + \frac{1}{r}\frac{dE_z}{dr} + k^2E_z = 0 \qquad R \leqslant r \leqslant a \tag{3b}$$

where R is the radius of the plasma; a is the radius of the cavity; $k = (2\pi)/\lambda = \omega/c$, with λ the resonant wavelength; and K_{zz} is the effective dielectric coefficient of the plasma given by

$$K_{zz} = 1 + \frac{\sigma_{zz}}{j\omega\epsilon_o} = 1 - \frac{ne^2}{m\epsilon_o\omega^2}\frac{\left(1 + j\nu_c/\omega\right)}{\left(1 + \nu_c^{\,2}/\omega^2\right)} = 1 - \frac{\omega_p^{\,2}}{\omega^2}\frac{\left(1 + j\nu_c/\omega\right)}{\left(1 + \nu_c^{\,2}/\omega^2\right)} \tag{4}$$

where ω_p is the plasma frequency. For a uniform plasma, the solution of Eqs. 3 is

$$E_z = E_o J_o\left(k\sqrt{K_{zz}}\,r\right) \qquad 0 \leqslant r \leqslant R \tag{4a}$$

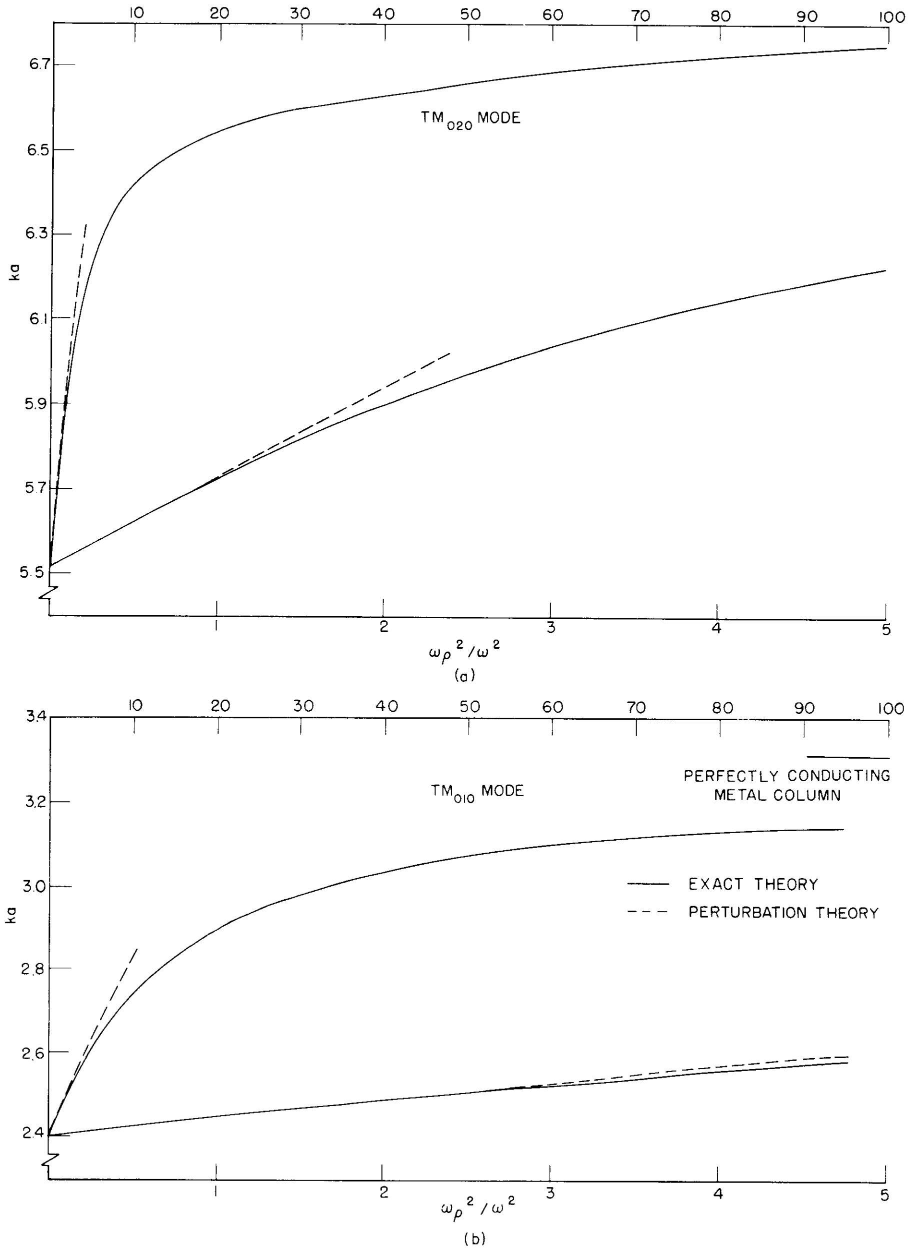

Fig. II-2. Resonant frequency of a cylindrical microwave cavity with a coaxial plasma column, $R/a = 1/10$. (a) TM_{020} mode; (b) TM_{010} mode.

$$E_z = AJ_o(kr) + BN_o(kr) \qquad R \leqslant r \leqslant a \tag{4b}$$

The field E_z and its derivative must be continuous at $r = R$, and E_z must vanish at $r = a$. These boundary conditions yield an equation for the resonant wavelength

$$\frac{CJ_o(kR) + N_o(kR)}{CJ_1(kR) + N_1(kR)} = \frac{J_o[(K_{zz})^{1/2}kR]}{(K_{zz})^{1/2}J_1[(K_{zz})^{1/2}kR]} \qquad C = -\frac{N_o(ka)}{J_o(ka)} \tag{5}$$

Equation 5 possesses an infinite number of roots which correspond to the infinite number of modes of TM_{0m0} class. It was solved numerically for its lowest two roots for the special case of $\nu_c = 0$ and $R/a = 1/10$. The solutions are plotted in Fig. II-2 in a form that can be used for a cavity of any radius.

When the electron density is low, the solution of Eq. 5 can be obtained from the well-known perturbation formula

$$\frac{\Delta f}{f} = \frac{1}{2} \frac{\left(1 + j\nu_c/\omega\right)}{\left(1 + \nu_c^2/\omega^2\right)} \frac{\int_{\text{plasma}} \left(\omega_p^2/\omega^2\right) E_o^2 \, dv}{\int_{\text{cavity}} E_o^2 \, dv}$$

$$= \frac{1}{2} \frac{J_o^2(\chi_{om}R/a) + J_1^2(\chi_{om}R/a)}{J_1^2(\chi_{om})} \left(\frac{R}{a}\right)^2 \frac{\omega_p^2}{\omega^2} \frac{\left(1 + j\nu_c/\omega\right)}{\left(1 + \nu_c^2/\omega^2\right)} \tag{6}$$

The range of validity of the perturbation formula, Eq. 6, is of interest. Perhaps, contrary to our expectation, it is not limited by the condition $\omega_p^2 \ll \omega^2$. This is because the electric field is perpendicular to density gradients and there is no ac charge separation (1, 2). The limiting condition is that the plasma skin depth be larger than the plasma radius. Since for $\nu_c = 0$ the skin depth δ is of the order of $\left[(\omega/c)\left(\omega_p^2/\omega^2 - 1\right)^{1/2}\right]^{-1}$, the value of $\left(\omega_p^2/\omega^2\right)$ at which the perturbation formula begins to depart from the exact solution of Eq. 5 should decrease with increasing frequency. This is evident from Fig. II-2 when the exact and the perturbation solutions for the TM_{010} and the TM_{020} modes are compared.

The TM_{0m0} class of modes is ideal for measuring low electron densities ($n < 10^{12}$ cm^{-3} for 6-cm radiation). At high electron densities, the frequency shifts become very large and are difficult to manage experimentally.

2. TM_{1mn} and TE_{1mn} Classes

When the radius of the plasma column is small and the integer m describing the mode is not too large, then in the plasma region the electric field of this class of modes can be considered as linearily polarized in the direction transverse to the static magnetic field B. A linearily polarized field can, in turn, be considered as composed of two circularly polarized fields rotating in opposite directions. Since the electrons in the plasma have a preferred direction of rotation about the lines of B_o, the plasma can be expected to interact strongly with the circularly polarized field that rotates with the electrons, and weakly with the field that rotates opposite to the motion of the electrons. As a consequence of the two types of interaction, the cavity with the plasma present resonates at two distinct frequencies for each mode in this class. This effect can be looked at in another manner. Each mode in this class is degenerate in frequency, since, for each mode, two orthonormal fields that will resonate at the same frequency can be established in the cavity at right angles to each other in the azimuthal plane. The non-isotropic plasma removes this degeneracy. It is this last approach that we shall adopt to obtain a quantitative expression for the shifted frequencies in the limit of low densities $\left(\omega_p^2/\omega^2 \ll 1\right)$. For greater generality, we consider two resonant modes whose frequencies are close to each other. Let the fields of the two modes associated with the empty cavity be E_1, H_1 and E_2, H_2, and their resonant frequencies ω_1 and ω_2, respectively. The fields satisfy the equations

$$\begin{aligned} \vec{\nabla}\times\vec{E}_1 &= -j\omega_1\mu_o\vec{H}_1 & \qquad \vec{\nabla}\times\vec{E}_2 &= -j\omega_2\mu_o H_2 \\ \vec{\nabla}\times\vec{H}_1 &= j\omega_1\epsilon_o\vec{E}_1 & \qquad \vec{\nabla}\times\vec{H}_2 &= j\omega_2\epsilon_o\vec{E}_2 \end{aligned} \tag{7}$$

In the presence of a low-density magnetized plasma the field in the cavity is assumed to be composed of a linear combination of the characteristic fields of the original modes

$$E = a_1E_1 + a_2E_2 \qquad\qquad H = b_1H_1 + b_2H_2 \tag{8}$$

The E and H satisfy

$$\nabla\times\vec{E} = -j\omega\mu_o\vec{H} \tag{9a}$$

$$\nabla\times\vec{H} = j\omega\epsilon_o\vec{E} + \overleftrightarrow{\sigma}\cdot\vec{E} \tag{9b}$$

where $\overleftrightarrow{\sigma}$ is the plasma conductivity tensor given by Eq. 1, and ω is the resonant frequency of the cavity with the plasma present. Using Eqs. 7 and 8 in Eq. 9, we obtain

$$\nabla\times(a_1E_1 + a_2E_2) = -j\mu_o(a_1\omega_1H_1 + a_2\omega_2H_2) = -j\mu_o\omega(b_1H_1 + b_2H_2) \tag{10a}$$

$$\nabla\times(b_1H_1 + b_2H_2) = j\epsilon_o(b_1\omega_1E_1 + b_2\omega_2E_2) = j\omega\epsilon_o(a_1E_1 + a_2E_2) + \overleftrightarrow{\sigma}\cdot\vec{E} \tag{10b}$$

Multiplying Eq. 10a by H_1^* and then by H_2^* (with the star denoting complex conjugate), integrating over the cavity volume, and using the fact that the characteristic fields are orthonormal, that is,

$$\int H_i \cdot H_j^* \, dv = \int E_i \cdot E_j^* \, dv = 0 \qquad i, j = 1, 2 \tag{11}$$

we obtain

$$b_1 = \frac{\omega_1}{\omega} a_1 \qquad b_2 = \frac{\omega_2}{\omega} a_2 \tag{12}$$

Similarly, multiplying Eq. 10b by E_1^* and then by E_2^*, integrating over the volume of the cavity, and making use of Eq. 12, we obtain

$$a_1 \frac{\omega_1^2}{\omega} \int E_1^2 \, dv = a_1 \omega \int E_1^2 \, dv + \frac{1}{j\epsilon_o} \int \vec{E}_1^* \cdot \overleftrightarrow{\sigma} \cdot \vec{E} \, dv \tag{13a}$$

and

$$a_2 \frac{\omega_2^2}{\omega} \int E_2^2 \, dv = a_2 \omega \int E_2^2 \, dv + \frac{1}{j\epsilon_o} \int \vec{E}_2^* \cdot \overleftrightarrow{\sigma} \cdot E \, dv \tag{13b}$$

Define

$$\frac{1}{j\omega\epsilon_o} \int E_j^* \cdot (\overleftrightarrow{\sigma} \cdot \vec{E}_i) \, dv = I_{ij} \tag{14}$$

then Eq. 13 reduces to

$$a_1 \left[\left(\frac{\omega_1^2}{\omega^2} - 1 \right) \int E_1^2 \, dv - I_{11} \right] - a_2 I_{21} = 0 \tag{15a}$$

$$- a_1 I_{12} + a_2 \left[\left(\frac{\omega_2^2}{\omega^2} - 1 \right) \int E_2^2 \, dv - I_{22} \right] = 0 \tag{15b}$$

The set of Eqs. 15 are linear homogeneous equations for the coefficients a_1 and a_2. A nontrivial solution will exist if and only if the determinant

$$D = \left\| \begin{matrix} \left[\left(\frac{\omega_1^2}{\omega^2} - 1 \right) \int E_1^2 \, dv - I_{11} \right] & - I_{21} \\ -I_{12} & \left[\left(\frac{\omega_2^2}{\omega^2} - 1 \right) \int E_2^2 \, dv - I_{22} \right] \end{matrix} \right\| \tag{16}$$

vanishes. The equation $D = 0$ is quadratic in ω^2 and yields two resonant frequencies that are shifted away from ω_1 and ω_2. If the original modes are degenerate, we have

$$\omega_1 = \omega_2 = \omega_o; \qquad \int E_1^{\,2}\, dv = \int E_2^{\,2}\, dv = \int E_o^{\,2}\, dv$$

and $I_{11} = I_{22}$. The equation $D = 0$ then becomes

$$\frac{\omega_o^2}{\omega^2} - 1 = \frac{I_{11} \pm (I_{12}\, I_{21})^{1/2}}{\int E_o dv} \tag{17}$$

The resonant frequency ω is double-valued; that is, the nonisotropic plasma has removed the original degeneracy of the two modes.

As an example, let us consider the effect of a magnetized plasma on the resonant frequency of a TM_{111} mode. The field of this mode in the absence of the plasma is given by

$$E_r = -\frac{k_3}{k} J_1'(k_1 r) \cos\theta \sin k_3 z \tag{18a}$$

$$E_\theta = \frac{k_3}{k} \frac{J_1(k_1 r)}{k_1 r} \sin\theta \sin k_3 z \tag{18b}$$

$$E_z = \frac{k_1}{k} J_1(k_1 r) \cos\theta \cos k_3 z \tag{18c}$$

where $k_1 = \chi_{11}/a$, $k_3 = \pi/L$, $k^2 = k_1^{\,2} + k_3^{\,2}$, $\chi_{11} = 3.832$ is the first root of $J_1(x) = 0$, a is the radius of the cavity, and L is its length. Using Eqs. 18 in Eq. 17, we obtain the complex frequency shift

$$\frac{\Delta f}{f} = \frac{1}{4J_o^{\,2}(\chi_{11})} \frac{k_3^{\,2}}{k^2} \left(\frac{R}{a}\right)^2 \frac{\omega_p^{\,2}}{\omega^2} \frac{1}{(1 \pm \omega_b/\omega) - j(\nu_c/\omega)} \tag{19}$$

where R is the radius of the plasma. In Eq. 19, terms of the order of $(R/a)^2$ were neglected as compared with unity. The real part of Eq. 19 gives the change in the resonant frequency of the cavity; twice the complex part yields the change in the $(1/Q)$ value of the cavity. Equation 19 gives two frequency shifts, $(\Delta f/f)_-$ and $(\Delta f/f)_+$, which correspond to the circularly polarized waves rotating with and against the electrons. The ratio $(\Delta f_-/\Delta f_+)$ is independent of density. When the pressure is low, so that $\nu_c < |\omega - \omega_b|$, it depends only on the magnetic field and is given by

$$\frac{\Delta f_-}{\Delta f_+} = \frac{\omega + \omega_b}{\omega - \omega_b} \tag{20}$$

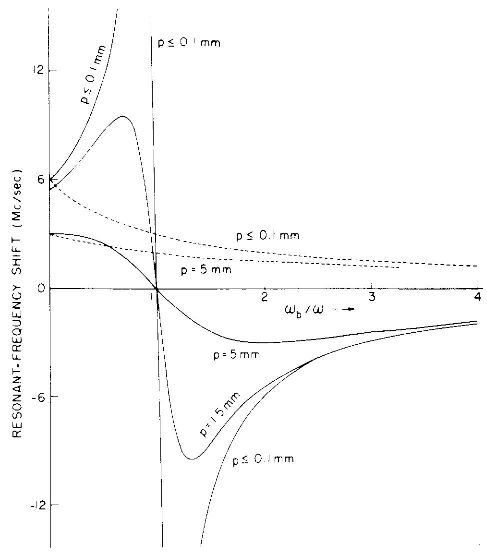

Fig. II-3. Resonant-frequency shift of TM_{111}-mode cylindrical cavity with a coaxial plasma column, $R/a = 1/10$, $f_o = 4400$ mc, density = 10^{10} cm^{-3}.
—— Field rotates with electrons.
----- Field rotates against electrons.

The validity of Eq. 19 is assured as long as it is experimentally verified that Eq. 20 holds. Note that $(\Delta f/f)_+$ is always positive, while $(\Delta f/f)_-$ is positive for $\omega_b < \omega$, and is negative for $\omega_b > \omega$. For finite ν_c, $(\Delta f/f)_-$ vanishes at cyclotron resonance; that is, at $\omega = \omega_b$. Figure II-3 exhibits the resonant frequency of the TM_{111} mode calculated for a particular cavity size.

3. TE_{0mn} Class

The modes in this class are ideal for the measurement of high-density plasmas in narrow columns in the absence of a magnetic field. This is so because the field of these modes in the absence of a plasma possesses only an azimuthal component.

$$E_{\theta_o} \approx J_1\left(\frac{\chi_{01} r}{a}\right) \sin\left(\frac{\pi z}{L}\right)$$

$$E_{r_o} = E_{z_o} = 0 \qquad (21)$$

The field is, therefore, perpendicular to the density gradient and does not induce any ac space charge. It has been shown (2) that for the lowest mode of this class, the TE_{011} mode, a perturbation formula similar to Eq. 6 is valid for values of ω_p^2/ω^2 as large as 10.

The presence of a static magnetic field complicates the situation in that the magnetic field causes radial currents and fields which, as we shall see, contribute to the frequency shift of the cavity when the density is high. An analysis similar to that used in the previous section indicates that the frequency shift in the presence of a magnetic field is given by

$$\frac{\Delta f}{f} = \frac{1}{2}\frac{j}{\omega\epsilon_o}\frac{\int_{\text{plasma}} E_o \cdot (\overleftrightarrow{\sigma} \cdot \vec{E})\, dv}{\int E_o \cdot E\, dv} \tag{22}$$

where E_o is the field in the absence of a plasma, and E is the field in the presence of a plasma. Equation 22 is correct for small $\Delta f/f$.

At low electron densities we can approximate E by E_o. Then $E_o \cdot (\overleftrightarrow{\sigma} \cdot \vec{E}) \approx \sigma_{\theta\theta} E_{\theta_o}^2$, where E_{θ_o} is given by Eq. 21, and $\sigma_{\theta\theta}$ is the second diagonal component of the conductivity tensor (Eq. 1). Equation 22 then yields for the lowest mode of the class (the TE_{011} mode), and for a uniform plasma of radius R,

$$\frac{\Delta f}{f} = \frac{1}{32}\frac{\chi_{01}^2}{J_o^2(\chi_{01})}\left(\frac{R}{a}\right)^4 \frac{\omega_p^2}{\omega}\left[\frac{1}{(\omega - \omega_b) - j\nu_c} + \frac{1}{(\omega + \omega_b) - j\nu_c}\right] \tag{23}$$

where $\chi_{11} = 3.832$ is the first root of $J_o'(x) = 0$, and in which terms of the order of $(R/a)^2$ were neglected as compared with unity. The real part of Eq. 23 gives the frequency shift of the cavity

$$\text{Re}\,\frac{\Delta f}{f} = \frac{1}{16}\frac{\chi_{01}^2}{J_o^2(\chi_{01})}\left(\frac{R}{a}\right)^4 \frac{\omega_p^2\left(\nu_c^2 + \omega^2 - \omega_b^2\right)}{\left[\nu_c^2 + (\omega + \omega_b)^2\right]\left[\nu_c^2 + (\omega - \omega_b)^2\right]} \tag{24}$$

Equation 24 exhibits an interesting property of this class of modes, namely, that the resonant frequency shift is zero; that is, the plasma is purely resistive (3), not at cyclotron resonance, but at $\nu_c^2 + \omega^2 = \omega_b^2$. The reason for this, which can be seen by examining the two terms in the square brackets of Eq. 23, is the result of mixing the effects of the two circularly polarized waves at each point in the plasma. Unlike the situation in the previous class of modes, the two waves are not separated by the plasma, and the resonant frequency of the cavity remains single-valued.

When the electron density is high, the approximation $E \approx E_o$ is no longer adequate, mainly because of the presence of induced radial fields in the plasma. These can be approximated as follows. From the continuity equation,

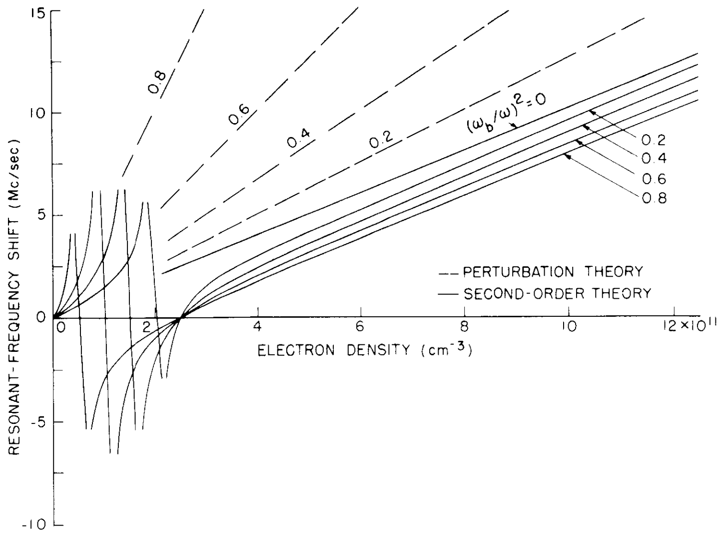

Fig. II-4. Frequency shift of TE_{011}-mode cylindrical cavity with a coaxial plasma column, $R/a = 1/10$, $f_o = 4500$ mc.

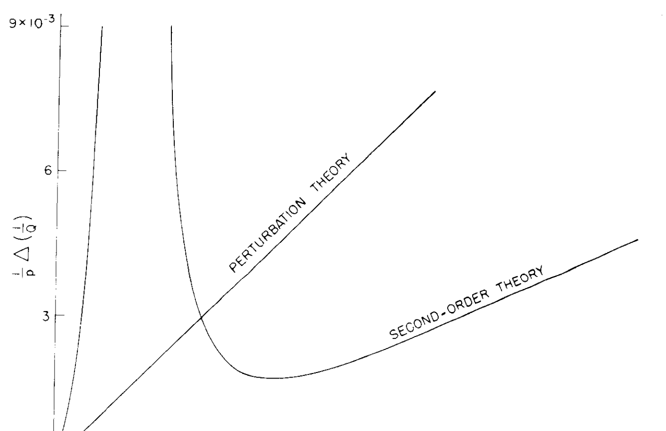

Fig. II-5. Q value of TE_{011}-mode cylindrical cavity with coaxial plasma column, $R/a = 1/10$, $f_o = 4500$ mc, $(\omega_b/\omega)^2 = 0.5$.

$$\frac{\partial\rho}{\partial t} + \nabla \cdot J = j\omega\rho + \nabla \cdot (\overleftrightarrow{\sigma} \cdot \vec{E}) = 0 \tag{25}$$

and from Poisson's equation,

$$\nabla \cdot E = \frac{\rho}{\epsilon_o} \tag{26}$$

we obtain

$$\nabla \cdot (\overleftrightarrow{K} \cdot \vec{E}) = 0 \tag{27}$$

where $\overleftrightarrow{K} = \overleftrightarrow{J} + \overleftrightarrow{\sigma}/j\omega\epsilon_o$ is the effective plasma tensor coefficient. In our problem we have azimuthal symmetry. We also assume that axial E-fields are negligible (which implies that the radial field is zero outside the plasma). Equation 27 then yields $E_r = -(K_{r\theta}/K_{rr})E_\theta$, and the expression $(E_o \cdot \overleftrightarrow{\sigma} \cdot \vec{E})$ becomes $(\sigma_{\theta\theta} - \sigma_{r\theta}K_{r\theta}/K_{rr})E_{\theta_o}E_\theta$, in which, again, we approximate E_θ by E_{θ_o}. Under these conditions, Eq. 22 yields

$$\frac{\Delta f}{f} = \frac{1}{16} \frac{\chi_{01}^2}{J_o^2(\chi_{01})} \left(\frac{R}{a}\right)^4 \eta \left[\frac{(1-\eta)(1-\beta^2-\gamma^2-\eta)+\beta^2(2-\eta)}{(1-\beta^2-\gamma^2-\eta)^2+\beta^2(2-\eta)^2} + j\beta\frac{(1-\eta)(2-\eta)-(1-\beta^2-\gamma^2-\eta)}{(1-\beta^2-\gamma^2-\eta)^2+\beta^2(2-\eta)^2}\right] \tag{28}$$

where $\eta = \omega_p^2/\omega^2$, $\beta = \nu_c/\omega$, and $\gamma = \omega_b/\omega$. The frequency shift obtained from Eq. 28 is plotted in Fig. II-4 and compared with that obtained from Eq. 24, in which radial fields are neglected. The two agree for $\eta \ll 1$ (Eq. 28 reduces to Eq. 24 for $\eta \ll 1$) but disagree violently for large η. A striking feature is the oscillation in the frequency shift. The magnitude of the resonance is the larger, the smaller ν_c is. Twice the imaginary part of Eq. 28 yields the change in the $(1/Q)$ value of the cavity. This is plotted in Fig. II-5 for a particular value of magnetic field. Here we have a resonance minimum in $\Delta(1/Q)$. In the vicinity of the resonance, the Q value of the cavity is so low that accurate measurements of the frequency shifts are difficult to obtain. As a result, a quantitative experimental verification of Eq. 28 has not yet been made, although a resonance in the Q value of the cavity was observed. Unfortunately, Eq. 28 can, at best, be only qualitatively correct, and then only in the limit of vanishing plasma radius. This is because our neglect of the axial E-field in the plasma and of the radial field outside the plasma is not compatible with the boundary conditions at the walls of the cavity. However, an exact solution of this problem is possible. It results in a transcendental equation for the complex resonant frequency of the cavity in the form of a 6×6 determinant which is now being processed on the IBM 704 computer.

S. J. Buchsbaum

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C. EFFECT OF ELECTRON TEMPERATURE ON PLASMA CONDUCTIVITY

The plasma conductivity tensor $\overleftrightarrow{\sigma}$

$$\overleftrightarrow{\sigma} = \frac{ne^2}{2m}\begin{pmatrix} \ell + r & -j(\ell+r) & 0 \\ j(\ell-r) & \ell + r & 0 \\ 0 & 0 & 2p \end{pmatrix} \tag{1}$$

$$\left.\begin{matrix}\ell \\ r\end{matrix}\right\} = \frac{1}{\nu_c + j(\omega \pm \omega_b)} \qquad p = \frac{1}{\nu_c + j\omega}$$

is correct only in the limit of infinite wavelength (as we shall see, this is equivalent to the case of zero electron temperature). This is so, because in the derivation (1) of Eq. 1 it was assumed that the E-field varied only as $\exp(j\omega t)$; that is, no account was taken of its wave nature. Consequently, it may be inadmissible to use Eq. 1 in Maxwell's equations for obtaining propagation constants of waves in plasma in magnetic fields for those waves whose phase velocity is much less than the velocity of light and also approaches the random velocity of the electrons.

For example, it is well known that the propagation constant k of a circularly polarized wave that propagates along the direction of a static magnetic field B_o in an infinite uniform plasma is given by

$$k_{\pm} = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_p^2}{\omega^2}\,\frac{1}{(\omega \pm \omega_b) - j\nu_c}\right] \tag{2}$$

where ω_p is the plasma frequency, ν_c is the electron collision frequency for momentum transfer, and ω_b is the cyclotron frequency ($\omega_b = eB/m$). The minus sign in the denominator is associated with the wave that rotates with the electrons, and the plus sign with the wave that rotates against the electrons. The phase velocity of the minus wave, as given by Eq. 2, is much smaller than the velocity of light when $\omega_b \geqslant \omega$ and $\omega_p > \omega$, and, under these conditions, the hot electrons should affect the propagation constant.

Our task, then, is to calculate the electron temperature-dependent plasma conductivity, $\overleftrightarrow{\sigma}_T$, by taking account of the wave nature of the field, and then using $\overleftrightarrow{\sigma}_T$, rather than Eq. 1, for deriving the propagation constant of the wave. It will become clear during the derivation that $\overleftrightarrow{\sigma}_T$ depends not only upon the magnitude of the propagation constant but also on its direction. For simplicity, in this report we shall only consider propagation along the static magnetic field. The beginning of the derivation is the Boltzmann equation for the electron distribution function $f(\vec{r}, \vec{v}, t)$

$$\frac{\partial f}{\partial t} + v \cdot \nabla f - \frac{e}{m}(E + v \times B) \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll} \tag{3}$$

We set

$$f = nf_o(v) + f_1(\vec{r}_1\vec{v}, t) \tag{4a}$$

$$\vec{E} = \vec{E}_1(r, t) \tag{4b}$$

$$\vec{B} = \vec{a}_3 B_o + \vec{B}_1(\vec{r}_1 t) \tag{4c}$$

substitute in Eq. 3, and linearize to obtain

$$\frac{\partial f_1}{\partial t} + v \cdot \nabla f_1 + \frac{e}{m}\vec{B}_o \cdot \vec{v} \times \frac{\partial f_1}{\partial \vec{v}} = \frac{ne}{m} E_1 \cdot \frac{\partial f_o}{\partial \vec{v}} - \nu_c f_1 \tag{5}$$

where n is the electron density, and ν_c is the electron collision frequency for momentum transfer. Equations 4 are now solved by a method analogous to that used by Bernstein (2). We set

$$\vec{E}_1(\vec{r}) = \vec{E}e^{j(\omega t - \vec{k} \cdot \vec{r})} \tag{6a}$$

and

$$f_1(\vec{v}, \vec{r}, t) = F(\vec{v})e^{j(\omega t - \vec{k} \cdot \vec{r})} \tag{6b}$$

and substitute Eqs. 6 in Eq. 5, using $\vec{k} = \vec{a}_3 k$ and changing variables to

$$\vec{v} = \vec{a}_1 w \cos \phi + \vec{a}_2 w \sin \phi + \vec{a}_3 u \tag{7}$$

Equation 5 then becomes

$$j(\omega - j\nu_c - uk)F + \omega_b \frac{\partial F}{\partial \phi} = \frac{ne}{m} E \cdot \frac{\partial f_o(v)}{\partial \vec{v}} \tag{8}$$

Its solution is

$$F = \frac{ne}{m\omega_b} E \cdot \int_{-\infty}^{\phi} \frac{\partial f_o(v')}{\partial (v')} \exp\left[-j\left(\frac{\omega - j\nu_c - uk}{\omega_b}\right)(\phi - \phi')\right] d\phi' \tag{9a}$$

$$F = \frac{ne}{m\omega_b} E \cdot \int_{-\infty}^{\phi} G \frac{\partial f_o(v')}{\partial v'} d\phi' \tag{9b}$$

where

$$G = \exp\left[-j\left(\frac{\omega - j\nu_c - uk}{\omega_b}\right)(\phi - \phi')\right] = \exp\left[-j\alpha(\phi - \phi')\right]$$

$$\alpha = \frac{(\omega - j\nu_c - uk)}{\omega_b} \tag{10}$$

Since the current J is given by

$$J = -e \int F\vec{v}d^3v \tag{11}$$

by setting

$$\vec{J} = \overleftrightarrow{\sigma}_T \cdot \vec{E} \tag{12}$$

we obtain the tensor conductivity

$$\overleftrightarrow{\sigma}_T = -\frac{ne^2}{m\omega_b} \int \vec{v}d^3v \int_{-\infty}^{\phi} \frac{\partial f_o(v')}{\partial \vec{v}'} G\, d\phi' \tag{13}$$

The conductivity $\overleftrightarrow{\sigma}_T$ obtains its tensor nature because of the juxtaposition of the two vectors $\vec{v}$ and $\partial f_o(\vec{v}')/v'$. The integrations indicated in Eq. 13 are considerably facilitated if $\overleftrightarrow{\sigma}_T$ is expressed in rotating coordinates

$$\frac{1}{\sqrt{2}}(x \pm jy) = \frac{1}{\sqrt{2}} r \exp(\pm j\theta); \; \frac{1}{\sqrt{2}}(v_x \pm jv_y) = \frac{1}{\sqrt{2}} \omega \exp(\pm j\phi)$$

through the unitary transformation*

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & j & 0 \\ 1 & -j & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \quad U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -j & j & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \tag{14}$$

---

*Applying this transformation to the conductivity of Eq. 1, we obtain

$$\sigma' = U\sigma U^{-1} = \frac{ne^2}{m} \begin{pmatrix} \ell & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & p \end{pmatrix}$$

from which Eq. 2 follows immediately, since

$$k_+^2 = \frac{\omega^2}{c^2}\left(1 + \frac{ne^2}{jm\omega\epsilon_o}\ell\right) \text{ and } k_-^2 = \frac{\omega^2}{c^2}\left(1 + \frac{ne^2}{jm\omega\epsilon_o} r\right)$$

Hence

$$\overleftrightarrow{\sigma}'_T = U\overleftrightarrow{\sigma}_T U^{-1} = -\frac{ne^2}{2\omega_b} \int U \cdot \vec{v} d^3v \int_{-\infty}^{\infty} G \frac{2f_o(\vec{v}')}{\partial \vec{v}'} \cdot U^{-1} \, dv' \tag{15}$$

To integrate Eq. 15 we note, first, that

$$U \cdot \vec{v} = \frac{w}{\sqrt{2}} \begin{pmatrix} e^{j\phi} \\ e^{-j\phi} \end{pmatrix} \tag{16a}$$

$$\frac{\partial f_o(v')}{\partial v'} \cdot U^{-1} = \frac{1}{\sqrt{2}} \frac{\partial f_o}{\partial w} \left(e^{-j\phi'}, \; e^{j\phi'} \right) \tag{16b}$$

If we use Eqs. 16 it can be easily shown that all off-diagonal terms in $\overleftrightarrow{\sigma}'_T$ vanish. Then we have

$$\overleftrightarrow{\sigma}'_T = \begin{pmatrix} \sigma'_{T_\ell} & 0 & 0 \\ 0 & \sigma'_{T_r} & 0 \\ 0 & 0 & \sigma_{T_{zz}} \end{pmatrix}$$

The component σ_{zz} is associated with longitudinal plasma-oscillation waves. It is not of direct interest here and will not be evaluated. The other two components are obtained from

$$\sigma'_{T_{\ell,r}} = -\frac{ne^2}{m\omega_b^2} \int_{-\infty}^{\infty} du \int_0^{\infty} \frac{\partial f_o}{\partial w} w^2 dw \int_0^{2\pi} d\phi \int_{-\infty}^{\phi} \exp[\pm j(\phi - \phi')] \exp[-j\alpha(\phi - \phi') \, d\phi'] \tag{17}$$

To compute the last two integrals in Eq. 17 we change variables to $\phi - \phi' = x$, $d\phi' = -dx$ to obtain

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\phi} \exp[\pm j(\phi - \phi')] \exp[-j(\phi - \phi')] \, d\phi' = 2\pi \int_0^{\infty} \exp[-j(\alpha \mp 1)x] \, dx \tag{18}$$

We take f_o to be Maxwellian

$$f_o = \left(\frac{m}{2\pi KT}\right)^{3/2} \exp\left(-\frac{mv^2}{2KT}\right) = \left(\frac{m}{2\pi KT}\right)^{3/2} \exp\left(-\frac{m(w^2 + u^2)}{2KT}\right) \tag{19}$$

The use of Eqs. 19 and 18 in Eq. 17 results in standard integrals over w and u which finally yield

$$\sigma_{T_{\ell, r}} = \frac{ne^2}{m\omega_b} \int_0^\infty \exp\left[-j\left(\frac{\omega - j\nu_c}{\omega_b} \pm 1\right) x - \frac{1}{2}k^2 r_b{}^2 x^2\right] dx \tag{20}$$

where r_b is the plasma Larmor orbit defined by

$$r_b = \frac{v_p}{\omega_b} = \left(\frac{KT}{m\omega_b}\right)^{1/2} \tag{21}$$

We note that as the electron temperature T goes to zero or as the wavelength goes to infinity, $\overleftrightarrow{\sigma}'_T$ reduces to $\overleftrightarrow{\sigma}'$, as it should. For finite temperatures, the integral in Eq. 20 converges as long as Re k > Im k. Equation 20 can be expressed in terms of incomplete complex error functions. For our purpose, however, we obtain only the first two terms of σ'_T in terms of (kr_b). Then

$$\sigma'_{T_{\ell, r}} = \frac{ne^2}{m[\nu_c + j(\omega \pm \omega_b)]} \left[1 + \frac{k^2 r_b{}^2}{\left[\frac{\omega - j\nu_c}{\omega_b} \pm 1\right]^2} + \dots\right] \tag{22}$$

in which conditions must be such that the second term in the brackets of Eq. 22 is smaller than unity.

To obtain a dispersion relation for the propagation constant, we use Eq. 20 (or Eq. 22 in Maxwell's equations). This yields

$$k_\pm{}^2 = \frac{\omega^2}{c^2} - j\frac{\omega^2}{c^2}\frac{\omega_p^2}{\omega\omega_b} \int_0^\infty \exp\left[-j\left(\frac{\omega - j\nu_c}{\omega_b} \pm 1\right) x - \frac{1}{2}k^2 r_b{}^2 x^2\right] dx \tag{23}$$

We plan to work on numerical solutions of Eq. 23.

S. J. Buchsbaum

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