

X. STATISTICAL COMMUNICATION THEORY

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RESEARCH OBJECTIVES

This group is interested in a variety of problems in statistical communication theory. Current research is primarily concerned with: development of a level-selector tube, crosscorrelation functions under nonlinear transformation, and the saddlepoint method of integration.

1. The level-selector tube described in the Quarterly Progress Report of January 15, 1956, page 107, is being developed with the primary objective of applying to filtering and prediction problems of current interest the nonlinear theory reported in "A Theory of Nonlinear Systems" by A. G. Bose, Technical Report 309, May 15, 1956. Other applications of the tube include its use in the measurement of probability densities at high frequencies and its use as a function generator.

2. The relationship between the crosscorrelation functions of two time functions before and after nonlinear transformation is of considerable theoretical and practical importance. For a certain class of time functions, any nonlinear no-memory transformation leaves the crosscorrelation function unchanged except for a scale factor. A necessary and sufficient condition for the invariance property to hold has been formulated and proved. This condition generalizes past results on the invariance property. Further study is being made. The application of this property to the improvement of correlation measurement techniques is also being investigated.

3. In many communication problems inverse Laplace transformations are difficult to obtain. One method that offers a straightforward means of estimating time functions from frequency functions is the saddlepoint method. Aside from being a powerful tool in transient analysis, the method is important in obtaining probability densities from characteristic functions, and power-density spectra from autocorrelation functions. Work is being done on (a) a redevelopment of the method with the objective of making it better known and appreciated by engineers, (b) an investigation of methods for reducing the approximation errors, (c) the exploitation of certain complex-plane interpretations of time functions which saddlepoint techniques make possible.

4. The project on signal theory will soon be completed. This is a study of the possibility of achieving a mathematical description of physical signals which embodies, more realistically than the usual functional representation, our limitations in performing measurements.

5. Another project that is nearly completed is the project on analytic nonlinear systems.

Y. W. Lee

A. THE INVARIANCE PROPERTY OF CORRELATION FUNCTIONS UNDER NONLINEAR TRANSFORMATIONS

Further work on the invariance property (1) has yielded the following results and generalizations. Some frequently encountered examples of separable processes are: (a) any wave alternating between two amplitudes a and b , randomly or otherwise; (b) any frequency-modulated or phase-modulated wave in which the (stationary) modulation is independent of the carrier phase; (c) any carrier-suppressed

amplitude-modulated wave in which the (stationary) modulation is separable and independent of the carrier phase. Note that, although the modulation is required to be a separable process for case c, the modulation for case b is completely arbitrary.

Continuing with the investigation of the characteristics of the separable class, we find that when two independent separable processes, $x(t)$ and $y(t)$, are added, the sufficient conditions for the sum process to be separable are either

$$\rho_x(\tau) = \rho_y(\tau) \quad (1a)$$

or

$$[f_x(\xi)]^{1/\sigma_x^2} = [f_y(\xi)]^{1/\sigma_y^2} \quad (1b)$$

and one of these two conditions is necessary for the sum process to be separable. We have restricted ourselves to independent processes, for, otherwise, we would need a fourth-order joint probability density function of the two processes $x(t)$ and $y(t)$. We do not want to make any assumptions about the higher-order probability density functions.

If we add N independent separable processes, ($N > 2$), necessary conditions seem to be very difficult to obtain. However, two sufficient conditions are

$$\rho_1(\tau) = \rho_2(\tau) = \dots = \rho_N(\tau) \quad (2a)$$

or

$$[f_1(\xi)]^{1/\sigma_1^2} = [f_2(\xi)]^{1/\sigma_2^2} = \dots = [f_N(\xi)]^{1/\sigma_N^2} \quad (2b)$$

Thus, for example, the sum of any number of independent squared gaussian processes (zero means), with identical spectra, is a separable process. Note also that, in general, sums of separable processes tend to become nonseparable. This will be discussed more fully later.

If we multiply two independent separable processes $x(t)$ and $y(t)$, the product process is always separable if both initial processes have zero means. Here no mention is made of spectra or statistics – only of zero means. The necessity for the two processes to have zero means has not been demonstrated but it is felt to be true in the general case. The answer hinges on the linear dependence or independence of the correlation functions of each process. As a start towards demonstrating that zero means are necessary, we have been able to show that if one process has a zero mean and the product process is separable, then the other process must also have a zero mean.

It follows that the product of any number of zero-mean independent separable processes is always separable. (Independence is required for products for the same reason that was stated for sums.) Thus, for example, the product of a zero-mean gaussian

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process with a sine wave of arbitrary frequency is a separable process. Also, the product of any number of sine waves of different frequencies is always separable.

The present results are a generalization of some of Luce's (2) work which was based on more stringent assumptions. More general results relating to those of Luce will be stated later.

We can show that any process belonging to Barrett and Lampard's (3) class Λ is always a separable process; the converse is not necessarily true. Therefore the class of separable processes is more general than class Λ .

In connection with Brown's (4) work we find that if

$$a_{1n}(\tau) = d_n a_{11}(\tau) = d_n \rho(\tau) \quad \text{for all } n, \tau$$

the process is always separable. The converse is true only if $a_{1n}(\tau)$ exists for all n . This is equivalent to requiring that all moments of the process exist. We have also tacitly assumed that the second-order probability density function can be expanded in a series and the $a_{1n}(\tau)$ can be determined for all n . This is, indeed, a formidable task. Note how much simpler it is to test separability. We can also show that if the process is separable, the sequence $\{d_n\}$ satisfies the relation, $d_n = 0$, $n \neq 1$, since here we are talking about one process instead of two, as Brown does. This is Brown's class Λ^* . Some additional results of Brown that relate to Booton's (5, 6, 7) equivalent-gain networks are best discussed with reference to Booton's work.

In Booton's equivalent-gain networks, a nonlinear no-memory device is replaced by a linear no-memory device chosen so that the mean-square difference between the two outputs for a particular type of input is minimized. We then find that separable processes are necessary and sufficient in order for the error caused by the linearization to be uncorrelated with the input for any nonlinear device. Also, neglecting the error term from the linear approximation can be shown to be equivalent to neglecting the autocorrelation of the error in the nonlinear device output autocorrelation function for separable processes.

If we generalize the linear approximation to a constant plus a linear term, we obtain the results that have been discussed and, in addition, we find that the constant of proportionality involved in the invariance property for the particular nonlinear device is the equivalent ac gain (or linear term) of the approximative linear network for separable processes. Thus we have an interesting connection between the invariance property and the equivalent linear networks of Booton.

In best-estimate procedure of a signal in noise (additive), given a sample value of the signal plus noise, we know that the best estimate in the mean-square sense is the mean of the signal conditioned on the value of the signal plus noise. For separable processes, the best estimate takes a very simple form, which involves only the unpredicted estimate, the correlation function, and the mean of the process. In the case of prediction without noise we also find that the best estimate involves only $\rho(\tau)$, μ , and

the value of the signal at the present time for separable processes. The error involved is then easily evaluated in terms of $\rho(\tau)$, μ , and the signal power. We are dealing here exclusively with the case in which only one sample of a signal is given, and not the complete past.

In Wiener's (8) optimum linear networks we find that for separable processes the optimum network takes a very simple form if the desired output is a no-memory distortion of the input. For nonseparable processes, no such simple result holds. More generally, the nonlinear part of the desired output is best approximated by a simple linear network for separable processes.

The use of separable processes for the measurement of linear network transfer functions is also possible with fairly simple correlation equipment (no multiplier).

By looking at Eqs. 1a and 1b, we see that it is easy to construct nonseparable processes. However, a more interesting result is that many processes cannot be classified as separable or nonseparable because of the lack of information. For instance, all of the following processes are indeterminate, as far as separability is concerned:

- | | | |
|--|---|-----|
| (a) separable process through linear network | } | (3) |
| (b) sum of process and process delayed | | |
| (c) sum of dependent waves | | |
| (d) product of dependent waves | | |
| (e) inverse of separable process. | | |

The first four processes are indeterminate because of the lack of knowledge of properties of higher-order statistics. (The first, indeed, requires a knowledge of all orders of input probability density functions.) The last process is merely a matter of coincidence that depends on the particular second-order probability density function.

There are two obvious directions to follow for generalization of the separable class of processes. The first is to generalize to processes that are "separable of degree n ." For example,

$$g(x_2, \tau) = \sum_{k=1}^n g_k^{(1)}(x_2) g_k^{(2)}(\tau)$$

where g is defined as

$$g(x_2, \tau) = \int_{-\infty}^{\infty} (x_1 - \mu) p(x_1, x_2; \tau) dx_1$$

It is possible to show that there are processes that belong to each and every degree of $n \geq 1$. We can also demonstrate that some processes require $n = \infty$. Hence, a wide variety of processes are evidently possible. By considering sums of independent separable processes, we can obtain arbitrary n merely by contradicting Eqs. 2a and 2b.

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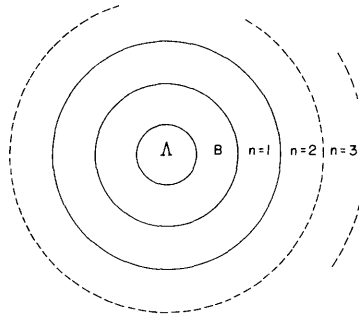


Fig. X-1. Classes of separable processes.

In fact, we can show that, in general, adding two separable independent processes of degrees m and n gives a process separable of degree $m + n$. (The degree is always $\leq m + n$.) Uses for these processes are unknown; their existence is only pointed out to demonstrate the wide variety of possible processes.

A diagram of the types of possible probability density functions is shown in Fig. X-1. The area included in each circle represents a particular class of processes, and includes all of the other circles (classes) of smaller radii (generality). There is an infinity of such circles, a process existing in each and every circle. The smallest circle, Λ , is Barrett and Lampard's class, B is Brown's class, and the circle of index n is a class separable of degree n . Note that the class $n = 1$ is the class that satisfies the invariance property. The classes for $n > 1$ do not.

The second generalization is to consider g_3 (and possibly higher orders):

$$g_3(x_2, x_3; \tau_1, \tau_2) = \int (x_1 - \mu) p(x_1, x_2, x_3; \tau_1, \tau_2) dx_1 \quad (4)$$

where $p(x_1, x_2, x_3; \tau_1, \tau_2)$ is the third-order probability density function of the process. Here also, there are cases that are separable of various degrees. Thus, in general, we would have to investigate processes separable of order m and degree n . Again, this is pointed out for its existence rather than for its use. However, it must be noted that in order to answer some of the questions raised by list 3, we need to consider some higher-order properties such as Eq. 4. For example, we need some results on the fourth-order statistics to answer problems c and d in list 3. These investigations are beyond the scope of the present work but are possible useful generalizations.

Extensions to nonstationary processes and time-variant devices are relatively straightforward.

A. H. Nuttall

[See following page for references.]

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References

1. A. H. Nuttall, Invariance of correlation functions under nonlinear transformations, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., Oct. 15, 1957, p. 63.
2. R. D. Luce, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., April 15, 1953, p. 37.
3. J. F. Barrett and D. G. Lampard, An expansion for some second-order probability distributions and its applications to noise problems, Trans. IRE, vol. PGIT-1, no. 1, pp. 10-15 (March 1955).
4. J. Brown, On a crosscorrelation property for stationary random processes, Trans. IRE, vol. PGIT-3, no. 1, pp. 28-31 (March 1957).
5. R. C. Booton, Jr., Nonlinear control systems with statistical inputs, Report 61, Dynamic Analysis and Control Laboratory, M.I.T., March 1, 1952.
6. R. C. Booton, Jr., M. V. Mathews, and W. W. Seifert, Nonlinear servomechanisms with random inputs, Report 70, Dynamic Analysis and Control Laboratory, M.I.T., Aug. 20, 1953.
7. R. C. Booton, Jr., The analysis of nonlinear control systems with random inputs, Proc. Symposium on Nonlinear Circuit Analysis II, Polytechnic Institute of Brooklyn, April 23-24, 1953, pp. 369-391.
8. N. Wiener, Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications (The Technology Press - John Wiley and Sons, Inc., New York, 1949).

B. SADDLEPOINT METHOD OF INTEGRATION

1. Introduction

Many problems that arise in network transient analysis yield frequency-domain solutions with time transformations that are either complicated or do not exist in closed form. Such solutions are often valueless to the engineer, since they do not allow him to estimate system behavior as a function of either parameter values or time. Since important classes of problems (e.g., waveguide and matched-filter transient responses) yield frequency-domain solutions of this kind, there have been several attempts to formulate general methods of approximating inverse transformations.

A means of approximate integration in the complex plane, known as the "extended saddlepoint method" (or "method of steepest descent") has been used by several authors to obtain solutions to specific problems (1, 2, 3). More recently, attempts have been made to extend the method to general transient analysis (4, 5, 6). In particular, Dr. M. V. Cerrillo's papers have presented the essential elements of saddlepoint integration.

The saddlepoint method of integration, as extended by these authors, gives the engineer a tool for rapidly estimating and interpreting integrals of the type

$$g(x) = \frac{1}{2\pi j} \int_{\gamma_y} F(y) e^{w(x,y)} dy$$

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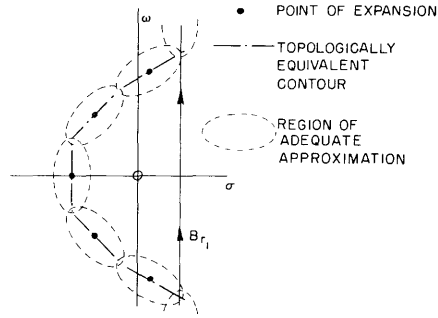


Fig. X-2. Br_1 and a topologically equivalent contour.

directly in the complex s -plane. The frequency with which this integral is used in analyzing transients, obtaining probability distributions from characteristic functions, and determining autocorrelation functions from power spectra makes the method an extremely valuable one. However, its mathematical development has tended to obscure the method's simplicity of application. Moreover, there has been little published work (7) either on methods of reducing the approximation errors or on complex-plane interpretations.

In view of these facts, three topics that are related to the saddlepoint method of transient analysis have been investigated:

1. Redevelopment of the method, which emphasizes the concepts used in the approximation procedure, illustrated by examples of its application to problems. A discussion of the relationship between the time and complex frequency behavior of a function has been included in this work.

2. Consideration of methods of reducing the approximation errors. Several methods of error control have been investigated, and two of them have been formulated. The objective has been the development of error-control techniques that can be readily applied to common problems, rather than the development of completely general procedures.

3. Exploitation of the complex-plane interpretations in problems of "equivalence" between distributed and lumped systems. It appears possible to describe this equivalence in terms of one of the parameters of the approximation. One example has been prepared, but no general study of this topic has been undertaken.

2. Elements of the Saddlepoint Method of Transient Analysis

In Laplace transform theory the operation $\mathcal{L}^{-1} [G(s)] = g(t)$ is defined by

$$g(t) = \frac{1}{2\pi j} \int_{Br_1} G(s) e^{st} ds \quad (1)$$

where Br_1 is the contour of integration shown in Fig. X-2. If we replace $G(s)$ by

$$G(s) = F(s) e^{\phi(s)} \quad (2)$$

then Eq. 1 takes the form

$$g(t) = \frac{1}{2\pi j} \int_{Br_1} F(s) e^{st+\phi(s)} ds \quad (3)$$

In order to approximate $g(t)$, we wish to choose a contour of integration, γ_s , which is topologically equivalent to Br_1 , but along which an approximate integration can be readily performed.

An obvious way to perform the approximate integration is to choose a set of points in the s -plane, expand both $F(s)$ and $[st + \phi(s)]$ about the points in power series, and then integrate along a contour, topologically equivalent to Br_1 , which passes through the points in such a way that it always lies in regions of adequate approximations, as shown in Fig. X-2. The number of points and the number of terms that are needed to approximate $F(s)$ and $[st + \phi(s)]$ in the neighborhoods of the points depends upon the choice of locations for the points. In the saddlepoint method of approximate integration we attempt to locate these points in such a way that a minimum region of accurate approximation is required about each point.

A small region of accurate approximation about a point of expansion can be used if we so choose the point, and its associated contour, that $\exp[st + \phi(s)]$ behaves along the contour as

$$e^{A(t)} e^{-r^n} \quad (4)$$

where r is arc length. In this case $\exp[st + \phi(s)]$ acts as a weighting function in Eq. 3, and concentrates the contributions to the integral in the immediate vicinity of the point.

Since $st + \phi(s)$ can be expanded in a Taylor series about any point, s_c , of analyticity, we can say that

$$st + \phi(s) = s_c t + \phi(s_c) + [t + \phi'(s_c)] (s - s_c) + \frac{\phi''(s_c)}{2} (s - s_c)^2 + \dots$$

in the vicinity of s_c . We cannot establish a contour that passes through s_c so that Eq. 4 is satisfied if $[t + \phi'(s_c)] \neq 0$, since only one line radiates from s_c along which $\{[st + \phi(s)] - [s_c t + \phi(s_c)]\}$ is real and negative when the coefficient of $(s - s_c)$ is nonzero. The requirement

$$t + \phi'(s_c) = 0 \quad (5)$$

must, therefore, establish the locations of the desired points of expansion. Equation 5 defines the saddlepoint locations of $[st + \phi(s)]$.

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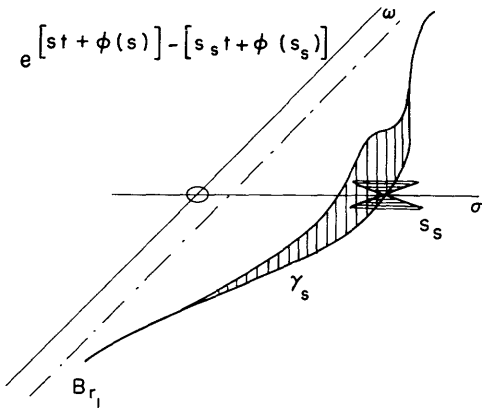


Fig. X-3. Behavior of $\exp\{[st + \phi(s)] - [s_s t + \phi(s_s)]\}$ along lines of steepest descent from s_s .

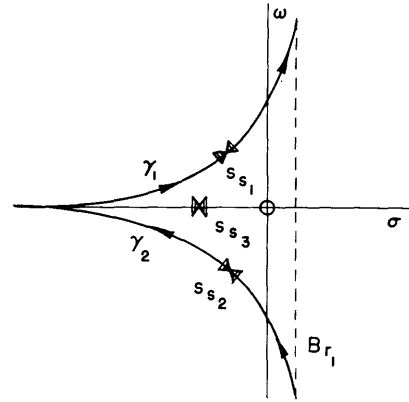


Fig. X-4. Primary and secondary saddlepoints in a typical configuration.

If we call the desired points of expansion s_s , they are defined by

$$s_s = \phi'^{-1}(t) \quad (6)$$

The contours that pass through these points along which $\{[st + \phi(s)] - [s_s t + \phi(s_s)]\}$ is real and negative are called "lines of steepest descent." An example of the case in which Eq. 6 defines only one saddlepoint is shown in Fig. X-3.

Naturally, not all of the saddlepoints defined by Eq. 6, with their associated lines of steepest descent, will always be needed in the establishment of a topologically equivalent contour. For instance, in Fig. X-4 we need only compute the contributions from integration through the regions about s_{s1} and s_{s2} , since γ_1 and γ_2 form a contour topologically equivalent to Br_1 . We then refer to s_{s1} and s_{s2} as primary saddlepoints.

Once we have established the primary saddlepoints, we can approximate the behavior of $st + \phi(s)$ in their vicinity, integrate the approximate functions along the approximate lines of steepest descent from each saddlepoint, and then sum these contributions to obtain $g^*(t)$.

In the simplest case we assume that $|\phi''(s_s)| \gg |\phi'''(s_s)|$, so that

$$st + \phi(s) \approx s_s t + \phi(s_s) + \frac{\phi''(s_s)}{2!} (s - s_s)^2 \quad (7)$$

This implies that the contour γ_s^* is a straight line through the saddlepoint, and

$$e^{[st + \phi(s)] - [s_s t + \phi(s_s)]} = e^{-\left|\frac{\phi''(s_s)}{2!}\right| r^2}$$

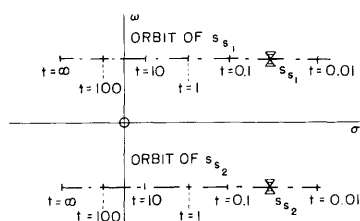


Fig. X-5. Orbits of two first-order saddlepoints.

along γ_S^* . In the example of Fig. X-3 this approximation involves a change of the contour of integration to one tangent to γ_S at s_S and an idealization of the bell shape of the exponential term along that contour.

Finally, we note that Eq. 6 defines saddlepoint locations that are functions of t . The primary saddlepoints follow loci in the s -plane as t varies. These loci are referred to as saddlepoint orbits, an example of which is given in Fig. X-5.

If we assume that $|\phi''(s_S)| \gg |\phi'''(s_S)|$, the solution for the integral along the lines of steepest descent from a saddlepoint is easily obtained. The solution will depend upon the behavior of $F(s)$ in the vicinity of the saddlepoint. The three basic types of behavior in the vicinity of s_S are:

- | | | |
|-----|--|---------|
| (a) | $F(s) \approx F(s_S)$ | NEUTRAL |
| (b) | $F(s) \approx F'(s_S) (s - s_a)$ | ZERO |
| (c) | $F(s) \approx \frac{\text{Res } s_k}{s - s_k}$ | POLAR |

For these behaviors, $g_i^*(t)$, the integral through a particular saddlepoint s_{s_i} , is

$$a. \quad g_i^*(t) = \frac{F(s_S) e^{s_{s_i} t + \phi(s_{s_i})}}{\sqrt{2 \pi \phi''(s_{s_i})}} \quad (8)$$

if $F(s)$ varies slowly in the vicinity of s_{s_i} .

$$b. \quad g_i^*(t) = \frac{F'(s_S) (s_S - s_a)}{\sqrt{2 \pi \phi''(s_S)}} e^{s_S t + \phi(s_S)} \quad (9)$$

if $F(s)$ has a first-order zero at s_a which is close to s_{s_i} .

$$c. \quad g_i^*(t) = R_k e^{s_k t + \phi(s_k)} \left\{ \frac{1}{2} \left(1 + \text{erf} \frac{X}{2} \right) \right\} \quad (10)$$

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where

$$\frac{X}{2} = \frac{t + \phi'(s_k)}{\sqrt{2\phi''(s_k)}} \quad (11)$$

if $F(s)$ has a first-order pole at s_k , which is close to s_i .

3. Example of Saddlepoint Integration

As an example of saddlepoint integration, consider the function

$$G(s) = \frac{1}{s+a} \quad (12)$$

Rewriting it, so that a $\phi(s)$ exists, we obtain

$$G(s) = F(s) e^{\phi(s)} = e^{-\ln(s+a)} \quad (13)$$

The saddlepoints are determined by

$$\begin{aligned} t + \phi'(s) = 0 &\Rightarrow t = \frac{-d}{ds} \phi(s) = \frac{1}{s+a} \\ \therefore s_s &= \frac{1}{t} - a \end{aligned} \quad (14)$$

In this case there is only one saddlepoint, whose orbit is the real s -axis between ∞ and $-a$. It can be shown that the lines of steepest descent from the saddlepoint form a contour that is topologically equivalent to Br_1 ; hence we may use saddlepoint integration. Since $F(s) = 1$, form 8 is applicable. Thus

$$\begin{aligned} \phi(s_s) &= -\ln(s_s + a) = -\ln \frac{1}{t} = \ln(t) \\ \phi''(s_s) &= \frac{1}{(s_s + a)^2} = t^2 \\ \therefore g^*(t) &= \frac{1 e^{\left(\frac{1}{t} - a\right)t + \ln t}}{\sqrt{2\pi t^2}} = \frac{t}{t} \frac{e}{\sqrt{2\pi}} e^{-at} \\ g^*(t) &\approx 1.08 e^{-at} \end{aligned} \quad (15)$$

The actual inverse transform is e^{-at} , which Eq. 15 closely approximates.

4. Error-Reduction Techniques

Although we have only sketched the procedure used in obtaining saddlepoint approximations, two sources of error are apparent. By assuming that Eq. 7 is an adequate

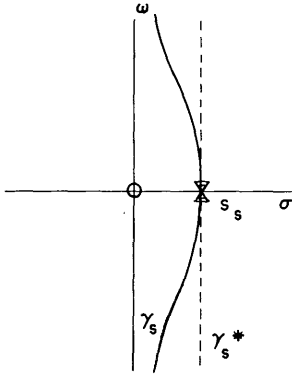


Fig. X-6. Actual and approximate lines of steepest descent from a first-order saddlepoint.

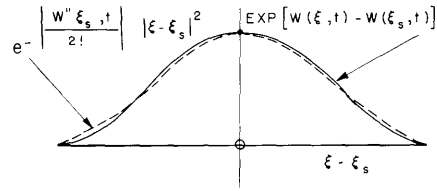


Fig. X-7. Actual and approximate behavior of $\exp[w(\xi, t) - w(\xi_s, t)]$ along γ_s .

approximation of $[st + \phi(s)]$, we imply that

(a) The lines of steepest descent from s_s can be approximated by a straight line through s_s , tangent to them at that point.

(b) $\exp[st + \phi(s)] - [s_s t + \phi(s_s)] = \exp(-k|s - s_s|^2)$ along these approximate lines of steepest descent.

Unless both of these approximations are valid, errors are introduced into the solution. There are methods of dealing with each of these approximation errors, which, although they are not completely general, work in a large number of practical problems.

Consider the first approximation. If the actual lines of steepest descent are curved, as shown in Fig. X-6, the function $e^{st + \phi(s) - [s_s t + \phi(s_s)]}$ is no longer real along γ_s^* . Its approximation by $\exp(-kr^2)$, or even by the exact behavior along γ_s , will not yield the proper contour integral, since $ds|_{\gamma_s^*} \neq ds|_{\gamma_s}$. However, suppose that it is possible to find a transformation of coordinates $s = f(\xi)$, so that in the ξ -plane the lines of steepest descent are approximately straight. In the transformed coordinate system it would then be possible to approximate

$$g(t) = \int_{\gamma_\xi} H(\xi) e^{w(\xi, t)} d\xi \quad (16)$$

more closely, if the second assumption is valid, since $d\xi|_{\gamma_\xi} = d\xi|_{\gamma_\xi^*}$.

This process of changing coordinates, so that integral 16 has associated with it lines of steepest descent from primary saddlepoints that are approximately straight in the region of interest is called the "coordinate transformation method of error control."

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There exists no specific technique for selecting the proper transformation for a particular problem. However, once the actual lines of steepest descent in the s-plane are known, reference to Kober's "Dictionary of Conformal Representations," or to a similar source, will usually suggest the appropriate choice.

Relationships for the integral through a saddlepoint of $w(\xi, t)$ can be derived in a similar manner. This has already been done (4).

Assuming that the actual lines of steepest descent and the approximate lines lie close to each other, we are still left with the second assumption. We must adequately approximate $\exp[w(\xi, t) - w(\xi_s, t)]$ along γ_{ξ}^* , in order to keep the error in $g^*(t)$ small. The two functions, $\exp[w(\xi, t) - w(\xi_s, t)]$ and $\exp\left[\left|\frac{w''(\xi_s, t)}{2!}\right| \cdot \left|\xi - \xi_s\right|^2\right]$, might vary, as shown in Fig. X-7, along $\gamma_{\xi}^* \approx \gamma_{\xi}$.

The most obvious approach to this problem of representation would be to use more terms in the power-series approximation of $w(\xi, t)$ in the region about ξ_s . However, in order to ensure close approximation of the actual behavior at large distances from ξ_s , many terms may have to be kept. Also, we already know the integral associated with the behavior of $\exp(-kr^2)$. So, instead of approximating $w(\xi, t)$ by a power series, we shall approximate $\exp[w(\xi, t) - w(\xi_s, t)]$ by

$$e^{w(\xi, t) - w(\xi_s, t)} = \sum_{\nu=1}^n A_{\nu} e^{-k_{\nu} |\xi - \xi_{\nu}|^2} \quad (17)$$

along γ_{ξ}^* (i. e., we approximate $\exp[w(\xi, t) - w(\xi_s, t)]$ by a sum of "normal" curves with properly selected "means" and "variances"). Substituting Eq. 17 in Eq. 18, and interchanging variables, we obtain

$$g^*(t) = \sum_{\nu=1}^n \frac{1}{2\pi j} \int_{\gamma_{\xi}^*} A_{\nu} H(\xi) e^{-k_{\nu} |\xi - \xi_{\nu}|^2} \quad (18)$$

Each term in this series represents a saddlepoint, located at ξ_{ν} , where $H(\xi)$ has been replaced by $A_{\nu} H(\xi)$ for the purpose of evaluating the integral. The general saddlepoint integral is already known (Eqs. 8, 9, and 10), so we need only sum these solutions to obtain $g^*(t)$.

Each of the saddlepoints that is introduced in the approximation improvement procedure is called a "satellite saddlepoint," since its existence is a mathematical fiction and it always lies on the lines of steepest descent from the actual primary saddlepoint. The method of error reduction is referred to as "satellite saddlepoint error control."

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[See following page for references.]

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References

1. M. V. Cerrillo, Transient phenomena in waveguides, Technical Report 33, Research Laboratory of Electronics, M.I.T., Jan. 3, 1948.
2. J. D. Pearson, The transient motion of sound waves in tubes, Quart. J. Mech. Appl. Math., Vol. VI, Pt. 3, p. 313 (1953).
3. A. Sommerfeld, Über die Ausbreitung der Wellen in der drahtlosen Telegraphie, Ann. Physik, Series 4, Vol. 28, pp. 665-736 (1909).
4. M. V. Cerrillo, An elementary introduction to the theory of the saddlepoint method of integration, Technical Report 55:2a, Research Laboratory of Electronics, M.I.T., May 3, 1950.
5. P. C. Clemmow, Some extensions of the method of steepest descent, Quart. J. Mech. Appl. Math., Vol. III, Pt. 2, p. 241 (1950).
6. B. L. Van der Waerden, Appl. Sci. Res., Vol. B2, p. 33 (1951).
7. Most of the material on error control discussed here is based on the unpublished work of Dr. M. V. Cerrillo.

C. EXPANSIONS FOR SOME SECOND-ORDER PROBABILITY DISTRIBUTIONS

Barrett and Lampard (1) have shown that certain second-order probability density distributions may be expanded in a single series of orthogonal polynomials. This class of distributions, denoted by Λ , include some important distributions that occur in physical problems. It also has been demonstrated (1) that distributions in this class satisfy the invariance property (2). For example, the second-order gaussian probability distribution and the second-order distribution of a sine wave are both in Λ . In the following discussion the expansions for two additional distributions in Λ , and two related expansions are presented. The notation used follows that of Barrett and Lampard (1). The proofs are omitted here.

Distributions in Λ are characterized by the general form

$$p(x_1, x_2; \tau) = p(x_1) p(x_2) \sum_{n=0}^{\infty} a_n(\tau) \theta_n(x_1) \theta_n(x_2) \quad (1)$$

The gaussian second-order probability distribution can be written

$$p(x_1, x_2; \tau) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x_1^2 + x_2^2}{2\sigma^2}\right) \sum_{n=0}^{\infty} \frac{[\rho(\tau)]^n H_n(x_1/\sigma) H_n(x_2/\sigma)}{n!} \quad (2)$$

where $H_n(z)$ are Hermite polynomials.

If a stationary random process with the second-order distribution given by Eq. 2 is passed through a full-wave square-law device, it can be shown that the output

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distribution is given by

$$p(y_1, y_2; \tau) = \frac{(y_1 y_2)^{-1/2}}{2\pi\sigma^2} \exp\left(-\frac{y_1 + y_2}{2\sigma^2}\right) \sum_{n=0}^{\infty} [\rho(\tau)]^{2n} \frac{(n!)^2}{2^{2n}} \cdot L_n^{-1/2}(y_1/2\sigma^2) L_n^{-1/2}(y_2/2\sigma^2) \quad (3)$$

when $y_1, y_2 > 0$; otherwise $p(y_1, y_2; \tau) = 0$. The orthogonal polynomials

$$L_i^a(z) = \frac{e^z z^{-a}}{i!} \frac{d^i}{dz^i} (e^{-z} z^{i+a}) \quad (4)$$

are the associated Laguerre polynomials. Comparing Eq. 3 with the general form given in Eq. 1, we have

$$p(y) = \left. \begin{array}{l} \frac{1}{\sqrt{2\pi y} \sigma} \exp\left(-\frac{y}{2\sigma^2}\right) \quad \text{for } y \geq 0 \\ 0 \quad \text{for } y < 0 \end{array} \right\} \quad (5)$$

and

$$\left. \begin{array}{l} a_n(\tau) = [\rho(\tau)]^{2n} \\ \theta_n(y) = \frac{n! 2^n}{\sqrt{(2n)!}} L_n^{-1/2}(y/2\sigma^2) \end{array} \right\} \quad (6)$$

Thus $p(y_1, y_2; \tau)$ is in Λ .

The expansion for the second-order distribution of a sine wave of constant amplitude p is given (see ref. 1) by

$$p(x_1, x_2; \tau) = \frac{1}{\pi^2} \left(p^2 - x_1^2\right)^{-1/2} \left(p^2 - x_2^2\right)^{-1/2} \sum_{n=0}^{\infty} \epsilon_n T_n(x_1/p) T_n(x_2/p) \cos n \omega \tau \quad (7)$$

when $\frac{x_1^2}{p^2} < 1$, $\frac{x_2^2}{p^2} < 1$; otherwise $p(x_1, x_2; \tau) = 0$. $T_n(x)$ are the Tchebycheff polynomials

of the first kind defined by

$$T_n(x) = \cos\{n \cos^{-1} x\} \quad (8)$$

and

$$\epsilon_n = \left. \begin{array}{l} 1 \quad \text{for } n = 0 \\ 2 \quad \text{for } n = 1, 2, \dots \end{array} \right\} \quad (9)$$

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If a process with this distribution, see Eq. 7, is passed through a full-wave square-law device, the output second-order distribution can be shown to have the form

$$p(y_1, y_2; \tau) = \frac{1}{4\pi^2} (y_1 y_2)^{-1/2} (p^2 - y_1)^{-1/2} (p^2 - y_2)^{-1/2} \cdot \sum_{n=0}^{\infty} \epsilon_n \cos 2n\omega\tau T_{2n}(\sqrt{y_1}/p) T_{2n}(\sqrt{y_2}/p) \quad (10)$$

for $0 \leq y_1 \leq p^2$ and $0 \leq y_2 \leq p^2$; otherwise $p(y_1, y_2; \tau) = 0$, and

$$p(y) = \left. \begin{array}{l} \frac{y^{-1/2}}{2\pi} (p^2 - y)^{-1/2} \quad 0 \leq y \leq p^2 \\ 0 \quad \text{elsewhere} \end{array} \right\} \quad (11)$$

and

$$\left. \begin{array}{l} \theta_n(y) = T_{2n}(\sqrt{y}) \\ a_n(\tau) = \cos 2n\omega\tau \end{array} \right\} \quad (12)$$

This distribution is also in Λ .

A full-wave linear rectification of the gaussian process and the sine wave each produces a waveform with a second-order distribution that is expandable in a single series of orthogonal polynomials. However, these distributions are not in Λ , nor do they satisfy the invariance property. Nevertheless, the expansions still appear useful, since they separate into products of functions of the three variables, y_1 , y_2 , and τ , with the first-order distributions as a common factor.

The full-wave rectification, $y = |x|$, of the gaussian distribution gives the expansion

$$p(y_1, y_2; \tau) = \frac{2}{\pi\sigma^2} \exp\left(-\frac{1}{2} \left[\frac{y_1^2 + y_2^2}{\sigma^2} \right]\right) \sum_{n=0}^{\infty} \frac{[\rho(\tau)]^{2n} H_{2n}(y_1/\sigma) H_{2n}(y_2/\sigma)}{(2n)!} \quad (13)$$

for $y_1, y_2 \geq 0$; otherwise $p(y_1, y_2; \tau) = 0$. The first-order distribution associated with Eq. 13 is

$$p(y) = \left. \begin{array}{l} \frac{2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right) \quad y \geq 0 \\ 0 \quad y < 0 \end{array} \right\} \quad (14)$$

The full-wave rectification, $y = |x|$, of the sine wave generates a process whose

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second-order distribution is given by

$$p(y_1, y_2; \tau) = \frac{4}{\pi} (p^2 - y_1^2)^{-1/2} (p^2 - y_2^2)^{-1/2} \cdot \sum_{n=0}^{\infty} \epsilon_n T_{2n}(y_1/p) T_{2n}(y_2/p) \cos 2n\omega\tau \quad (15)$$

when $0 \leq y_1 \leq p$, and $0 \leq y_2 \leq p$; otherwise $p(y_1, y_2; \tau) = 0$. The first-order distribution associated with Eq. 15 is

$$p(y) = \left. \begin{array}{ll} \frac{2}{\pi} (p^2 - y^2)^{-1/2} & 0 \leq y \leq p \\ 0 & \text{elsewhere} \end{array} \right\} \quad (16)$$

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References

1. J. F. Barrett and D. G. Lampard, An expansion for some second-order probability distributions and its application to noise problems, Trans. IRE, vol. PGIT-1, pp. 10-15 (March 1955).
2. A. H. Nuttall, Invariance of correlation functions under nonlinear transformations, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., Jan. 15, 1957, p. 61.