DOI: 10.15393/j3.art.2020.6490

UDC 517.38, 512.6

S. M. Davarpanah, M. E. Omidvar, H. R. Moradi

ON SOME INEQUALITIES FOR τ-MEASURABLE OPERATORS

Abstract. This paper deals with the Choi's inequality for measurable operators affiliated with a given von Neumann algebra. Some Young and Cauchy-Schwarz type inequalities for τ-measurable operators are also given.

Key words: *von Neumann algebra, positive operator, noncommutative L_p-space, Young inequality, Cauchy-Schwarz inequality*

2010 Mathematical Subject Classification: *Primary 47A30, Secondary 47L05, 47L50*

1. Introduction and Preliminaries. Throughout the paper, we denote by $\mathcal{M}$ a semi-finite von Neumann algebra acting on the Hilbert space $\mathcal{H}$, with a normal faithful semi-finite trace τ. We denote the identity in $\mathcal{M}$ by $\mathbf{1}$ and let $\mathcal{P}$ denote the projection lattice of $\mathcal{M}$. We write $p \sim q$ for $p, q \in \mathcal{P}$ if $p = u^*u$ and $q = uu^*$ for some $u \in \mathcal{M}$. A closed densely defined linear operator x in $\mathcal{H}$ with the domain $D(x) \subseteq \mathcal{H}$ is said to be *affiliated with* $\mathcal{M}$ if $u^*xu = x$ for all unitary u that belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, then x is said to be τ-*measurable* if for every $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that $e(\mathcal{H}) \subseteq D(x)$ and $\tau(1 - e) < \varepsilon$. The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M},\tau)$, or, simply, $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closures of the algebraic sum and product; see [7]. A closed densely defined linear operator x admits a unique polar decomposition $x = u|x|$, where u is a partial isometry such that $u^*u = (\ker x)^{\perp}$ and $uu^* = \overline{\mathrm{im} x}$ (with $\mathrm{im} x = x(D(x))$. We call $r(x) = (\ker x)^{\perp}$ and $l(x) = \overline{\mathrm{im} x}$ the left and right supports of x, respectively. Thus, $l(x) \sim r(x)$. Note that $l(x)$ (resp., $r(x)$) is the least projection e such that $ex = x$ (resp., $xe = x$). If x is self-adjoint, then $r(x) = l(x)$. This common projection is then said to be the support of x and denoted by $s(x)$. For further details, we see [8].

Let $\mathcal{M}_+$ be the positive part of $\mathcal{M}$. Set $S_+(\mathcal{M}) = \{x \in \mathcal{M}_+ : \tau(s(x)) < \infty\}$ and let $S(\mathcal{M})$ be the linear span of $S_+(M)$. Let $0 < p < \infty$, the non-commutative L_p-space $L_p(\mathcal{M}, \tau)$ is the completion of $(S, \|\cdot\|_p)$, where $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty$ for each $x \in L_p(\mathcal{M}, \tau)$. In addition, we put $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$ and denote by $\|\cdot\|_p (= \|\cdot\|)$ the usual operator norm. It is well known that $L_p(\mathcal{M}, \tau)$ are Banach spaces under $\|\cdot\|_p$ for $1 \leqslant p < \infty$ and they have a lot of expected properties of classical L_p-spaces. Let x be a τ-measurable operator and $t > 0$. The "t-th singular number (or generalized s-number) of x" is defined by [5]

$$\mu_t(x) = \inf\{\|xe\| : e \in P, \tau(1-e) \leqslant t\}.$$

Recall that a linear map Φ is positive if $\Phi(X)$ is positive whenever X is positive. The celebrated Jensen inequality for operators [2] states that if X is a positive operator (self-adjointness is enough), Φ is a positive linear map, and f is an operator monotone on the interval $[0, \infty)$, then

$$\Phi(f(X)) \leqslant f(\Phi(X)). \tag{1}$$

In this paper, we prove the same result for measurable operators affiliated with a given von Neumann algebra. Furthermore, we use the technique of Zhao and Wu [11], via the notion of generalized singular numbers studied by Fack and Kosaki [5], to obtain generalizations of results in [11] for τ-measurable operators case. The obtained inequalities improve known results in [9]. In addition, Audenaert in [1] obtained that if X and Y are two $n \times n$ matrices and $0 \leqslant \nu \leqslant 1$, then for any unitarily invariant norm $\|\cdot\|_u$,

$$\|XY^*\|_u^2 \leqslant \|\nu X^*X + (1-\nu) Y^*Y\|_u \|(1-\nu) X^*X + \nu Y^*Y\|_u. \tag{2}$$

In the next section, we present a τ-measurable version of (2).

2. Main Theorems. We need the following lemma [3, Theorem 5]:

Lemma 1. *Let $\mathcal{M}$ be a von Neumann algebra on Hilbert space $\mathcal{H}$ and a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an operator monotone with respect to $\mathcal{M}$. Then $f(A) \leqslant f(B)$ for any pair of positive self-adjoint operators A, B affiliated with $\mathcal{M}$, such that $A \leqslant B$.*

We are ready to prove our promised extension of inequality (1).

Theorem 1. *Let Φ be a unital positive linear continuous map, $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an operator monotone function with respect to $\mathcal{M}$*

and $x \in S(\mathcal{M})$. *Then*

$$\Phi(f(x)) \leqslant f(\Phi(x)).$$

Proof. We use the same strategy as in [4, Corollary 3.2]. Put $x_n = x_{\chi([0,n])}$. It is clear that x_n is a increasing sequence of positive operators in $\mathcal{M}$ and converges nearly everywhere to x. Note that x_n commute with x for every n. So, the convergence nearly everywhere of the sequence x_n to x can be considered as in the commutative case. Therefore, for an operator monotone function f with respect to $\mathcal{M}$, and thus continuous on $\mathbb{R}^+$, the sequence $f(x_n)$ converges nearly everywhere to $f(x)$. By Lemma 1, since $x_n \leqslant x_{n+1} \leqslant \ldots \leqslant x$, we also have

$$f(x_n) \leqslant f(x_{n+1}) \leqslant \ldots \leqslant f(x),$$

and, since Φ is a positive linear continuous map,

$$\Phi(f(x_n)) \leqslant \Phi(f(x_{n+1})) \leqslant \ldots \leqslant \Phi(f(x)).$$

Consequently, $\Phi(f(x_n))$ converge nearly everywhere to $\Phi(f(x))$. On the other hand, for every x_n, by inequality (1), we have

$$\Phi(f(x_n)) \leqslant f(\Phi(x_n)) \leqslant f(\Phi(x)).$$

Tending $n \to \infty$, we obtain the desired inequality. □

The following result can be found in [10, Lemma 3.2].

Lemma 2. *Let* $x, y \in S(\mathcal{M})$ *and* $z \in \mathcal{M}$. *Then, for every* $r > 0$,

$$\||x^*zy|^r\|_p^2 \leqslant \||xx^*z|^r\|_p \||zyy^*|^r\|_p.$$

Theorem 2. *Let* $x, y \in S(\mathcal{M})$ *and* $z \in \mathcal{M}$. *Then, for every* $r > 0$,

$$g(s,t) = \left\||x^{1-t}zy^{1+s}|^r\right\|_p \left\||x^{1+t}zy^{1-s}|^r\right\|_p$$

is log-convex on $[-1,1] \times [-1,1]$, *hence is convex, and attains its minimum at* $(0,0)$.

Proof. The function g is continuous and $g(s,t) = g(-s,-t)$ $(s,t \in [0,1])$. Thus, it is enough to show that

$$g(s_1,t_1) \leqslant \frac{1}{2}\{g(s_1+s_2,t_1+t_2) + g(s_1-s_2,t_1-t_2)\}$$

where $s_1 \pm s_2, t_1 \pm t_2 \in [-1,1] \times [-1,1]$.

Applying Lemma 2,

$$\left\| |x^{1-t_1} z y^{1+s_1}|^r \right\|_2 = \left\| |x^{t_2} \left(x^{1-t_1-t_2} z y^{1+s_1-s_2} \right) y^{s_2}|^r \right\|_p \leqslant$$
$$\leqslant \left\{ \left\| |x^{1-(t_1-t_2)} z y^{1+(s_1-s_2)}|^r \right\|_p \left\| |x^{1-(t_1+t_2)} z y^{1+(s_1+s_2)}|^r \right\|_p \right\}^{\frac{1}{2}} \quad (3)$$

and

$$\left\| |x^{1+t_1} z y^{1-s_1}|^r \right\|_p = \left\| |x^{t_2} \left(x^{1+t_1-t_2} z y^{1-s_1-s_2} \right) y^{s_2}|^r \right\|_p \leqslant$$
$$\leqslant \left\{ \left\| |x^{1+(t_1+t_2)} z y^{1-(s_1+s_2)}|^r \right\|_p \left\| |x^{1+(t_1-t_2)} z y^{1-(s_1-s_2)}|^r \right\|_p \right\}^{\frac{1}{2}}. \quad (4)$$

Applying (3), (4), and the arithmetic-geometric mean inequality, we get

$$g(s_1, t_1) = \left\| |x^{1-t_1} z y^{1+s_1}|^r \right\|_2 \left\| |x^{1+t_1} z y^{1-s_1}|^r \right\|_2 \leqslant$$
$$\leqslant \left\{ g(s_1 + s_2, t_1 + t_2)\, g(s_1 - s_2, t_1 - t_2) \right\}^{\frac{1}{2}} \leqslant$$
$$\leqslant \frac{1}{2} \left[g(s_1 + s_2, t_1 + t_2) + g(s_1 - s_2, t_1 - t_2) \right].$$

The proof is completed. □

Using this observation, we give the following corollary.

Corollary. *Let $x, y \in S(\mathcal{M})$ and $z \in \mathcal{M}$. Then, for every $r > 0$,*

$$\left\| |x^{\frac{1}{2}} z y^{\frac{1}{2}}|^r \right\|_p^2 \leqslant \left\| |x^t z y^{1-s}|^r \right\|_p \left\| |x^{1-t} z y^s|^r \right\|_p \leqslant$$
$$\leqslant \max \left\{ \left\| |xz|^r \right\|_p \left\| |zy|^r \right\|_p, \left\| |xzy|^r \right\|_p \left\| |z|^r \right\|_p \right\}$$

where $0 \leqslant s, t \leqslant 1$.

Proof. If we replace s, t, x, y by $2s - 1, 2t - 1, x^{\frac{1}{2}}, y^{\frac{1}{2}}$, respectively, in Theorem 2, we see that the function $g(s,t) = \left\| |x^t z y^{1-s}|^r \right\|_p \left\| |x^{1-t} z y^s|^r \right\|_p$ is jointly convex on $[0,1] \times [0,1]$ and attains its minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$. Hence,

$$\left\| |x^{\frac{1}{2}} z y^{\frac{1}{2}}|^r \right\|_p^2 \leqslant \left\| |x^t z y^{1-s}|^r \right\|_p \left\| |x^{1-t} z y^s t|^r \right\|_p.$$

In addition, since the function g is continuous and convex on $[0,1] \times [0,1]$, it follows that g attains its maximum at the vertices of the square. Moreover, due to the symmetry, there are two possibilities for the maximum. □

The Corollary can be regarded as an extension of [10, Corollary 3.4].

In the following result, we present a τ-measurable version of the main result in [1]. We emphasize that the method of proof is completely different from the present proof in [6, Theorem 3.6].

Theorem 3. *Let x, y be two τ-measurable positive operators. Then*

$$\tau(xy)^2 \leqslant \tau(\nu\, x^2 + (1-\nu)y^2)\; \tau((1-\nu)x^2 + \nu\, y^2), \tag{5}$$

for $0 \leqslant \nu \leqslant 1$.

Proof. Note that the function $f(\nu) = \tau\left(x^\nu y^{1-\nu}\right)$ is log-convex. Consequently, the function

$$g(\nu) = f(\nu)f(1-\nu)$$

is log-convex. Since g is symmetric with respect to $\nu = \frac{1}{2}$, it follows that $f(1/2) \leqslant f(\nu)$. This means

$$\tau\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right) \leqslant \tau\left(x^\nu y^{1-\nu}\right)\; \tau\left(x^{1-\nu}y^\nu\right), \; -\infty < \nu < \infty.$$

Now, using Theorem 4, for $0 \leqslant \nu \leqslant 1$, we infer

$$\tau\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right) \leqslant \tau(\nu\, x + (1-\nu)y)\; \tau((1-\nu)x + \nu\, y).$$

Replacing x and y by their squares, we get the desired inequality. □

Note that inequality (5) interpolates between the arithmetic-geometric mean inequality and Cauchy-Schwarz inequality for τ-measurable operators. That is, for $\nu = 0$ we obtain the Cauchy-Schwarz type inequality

$$\tau(xy)^2 \leqslant \tau(x^2)\tau(y^2),$$

while we obtain the arithmetic-geometric mean inequality

$$\tau(xy) \leqslant \frac{1}{2}\tau(x^2 + y^2)$$

for $\nu = \frac{1}{2}$.

Recently, Shao in [?, Theorem 3.1] obtained a refinement of the Young inequality

$$\tau\left(x^\nu y^{1-\nu}\right) + r_0\Big(\tau(x)^{\frac{1}{2}} - \tau(y)^{\frac{1}{2}}\Big)^2 \leqslant \tau\left(\nu x + (1-\nu)\, y\right) \tag{6}$$

where $x, y \in L_1(\mathcal{M})$ are positive operators, and $r_0 = \min\{\nu, 1-\nu\}$ with $\nu \in (0,1)$. We close this paper by improving (6).

Theorem 4. *Let $x, y \in L_1(\mathcal{M})$ be positive operators and $\nu \in (0,1)$.*

1) *If* $0 < \nu \leqslant \frac{1}{2}$, *then*

$$r_0\Big((\tau(xy))^{\frac{1}{4}} - (\tau(x))^{\frac{1}{2}}\Big)^2 + \nu\Big((\tau(x))^{\frac{1}{2}} - (\tau(y))^{\frac{1}{2}}\Big)^2 + \tau\left(x^{1-\nu}y^{\nu}\right) \leqslant \leqslant \tau\left((1-\nu)x + \nu y\right). \tag{7}$$

2) *If* $\frac{1}{2} < \nu < 1$, *then*

$$r_0\Big((\tau(xy))^{\frac{1}{4}} - (\tau(x))^{\frac{1}{2}}\Big)^2 + (1-\nu)\Big((\tau(x))^{\frac{1}{2}} - (\tau(y))^{\frac{1}{2}}\Big)^2 + + \tau\left(x^{1-\nu}y^{\nu}\right) \leqslant \tau\left((1-\nu)x + \nu y\right). \tag{8}$$

Proof. We prove only (7) as (8) goes similarly. By [11, Lemma 1], we have

$$(1-\nu)\mu_t(x) + \nu\mu_t(y) \geqslant \geqslant r_0\Big(\mu_t(xy)^{\frac{1}{4}} - \mu_t(x)^{\frac{1}{2}}\Big)^2 + \nu\Big(\mu_t(x)^{\frac{1}{2}} - \mu_t(y)^{\frac{1}{2}}\Big)^2 + \mu_t(x)^{1-\nu}\mu_t(y)^{\nu},$$

where $r = \min\{\nu, 1-\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

Hence

$$\tau((1-\nu)x + \nu y) = (1-\nu)\tau(x) + \nu\tau(y) = = \int_0^\infty \Big[(1-\nu)\mu_t(x) + \nu\mu_t(y)\Big]dt \geqslant$$

$$\geqslant r_0\Big(\int_0^\infty \Big[\mu_t(xy)^{\frac{1}{2}} + \mu_t(x) - 2(\tau(xy))^{\frac{1}{4}}(\tau(x))^{\frac{1}{2}}\Big]dt\Big) +$$

$$+ \nu\Big(\int_0^\infty \Big[\mu_t(x) + \mu_t(y) - 2(\tau(x))^{\frac{1}{2}}(\tau(y))^{\frac{1}{2}}\Big]dt\Big) + \int_0^\infty \mu_t(x)^{1-\nu}\mu_t(y)^{\nu}dt \geqslant$$

$$\geqslant r_0\Bigg(\tau(x) + \int_0^\infty \mu_t\big((xy)^{\frac{1}{2}}\big)dt -$$

$$- 2\Big(\int_0^\infty \Big((\tau(xy))^{\frac{1}{4}}\Big)^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^\infty \Big((\tau(x))^{\frac{1}{2}}\Big)^2 dt\Big)^{\frac{1}{2}}\Bigg) +$$

$$+\nu\Bigg(\tau(x)+\tau(y)-2\Big(\int_0^\infty\Big(\big(\tau(x)\big)^{\frac{1}{2}}\Big)^{\frac{1}{2}}dt\Big)^2\Big(\int_0^\infty\Big(\big(\tau(y)\big)^{\frac{1}{2}}\Big)^2dt\Big)^{\frac{1}{2}}\Bigg)+$$

$$+\int_0^\infty \mu_t\left(x^{1-\nu}y^\nu\right)dt=$$

$$=r_0\Big(\big(\tau(xy)\big)^{\frac{1}{4}}-\big(\tau(x)\big)^2\Big)^2+\nu\Big(\big(\tau(x)\big)^{\frac{1}{2}}-\big(\tau(y)\big)^{\frac{1}{2}}\Big)^2+\tau\left(x^{1-\nu}y^\nu\right). \quad (9)$$

The proof is completed. □

Acknowledgment. The authors would like to express their hearty thanks to the referees for their valuable suggestions and comments for revising the manuscript. This work was financially supported by Islamic Azad University, Ferdows Branch.

References

[1] Audenaert K. *Interpolating between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities*, Oper. Matrices., 2015, vol 9, no. 2, pp. 475–479. DOI: https://doi.org/10.7153/oam-09-29.

[2] Choi M. D. *A Schwarz inequality for positive linear maps on C^*-algebras*, Illinois J. Math., 1974, vol. 18, no. 4, pp. 565–574. DOI: https://doi.org/10.1215/ijm/1256051007.

[3] Dinh T. H., Tikhonov O. E. *To the theory of operator monotone and operator convex functions*, Russian Maths., 2010, vol. 54, no. 3, pp. 7–11. DOI: https://doi.org/10.3103/S1066369X10030023.

[4] Dinh T. H. *Some inequalities for measurable operators*, Internat. J. Math. Anal., 2014, vol. 8, no. 22, pp. 1083–1087. DOI: http://dx.doi.org/10.12988/ijma.2014.39220.

[5] Fack T., Kosaki H. *Generalized s-numbers of τ-measurable operators*, Pacific J. Math., 1986, vol. 123, no. 2, pp. 269–300. DOI: https://doi.org/10.2140/pjm.1986.123.269.

[6] Han Y., Shao J. *Notes on two recent results of Audenaert*, Electron. J. Linear Algebra., 2016, vol. 31, pp. 147–155. DOI: https://doi.org/10.13001/1081-3810.3202.

[7] Nelson E. *Note on noncommutative integration*, J. Funct. Anal., 1974, vol. 15, no. 2, pp. 103–116. DOI: https://doi.org/10.1016/0022-1236(74)90014-7.

[8] Pisier G., Q. Xu Q. *Noncommutative L^p-spaces.* In: Handbook of the Geometry of Banach Spaces, 2003, Vol. 2, pp. 1459 – 1517, North-Holland, Amsterdam.

[9] Shao J. *On Young and Heinz inequalities for τ-measurable operators*, J. Math. Anal. Appl., 2014, vol. 414, no. 1, pp. 243–249. DOI: https://doi.org/10.1016/j.jmaa.2014.01.017.

[10] Shao J., Han Y. *Some convexity inequalities in noncommutative L_p-spaces*, J. Inequal. Appl., 2014, Article number: 385 (2014). DOI: https://doi.org/10.1186/1029-242X-2014-385.

[11] Zhao J., Wu J. *Operator inequalities involving improved Young and its reverse inequalities*, J. Math. Anal. Appl., 2015, vol. 421, no. 2, pp. 1779–1789. DOI: https://doi.org/10.1016/j.jmaa.2014.08.032.

Received June 15, 2019.
In revised form, August 24, 2019.
Accepted September 17, 2019.
Published online October 10, 2019.

S. M. Davarpanah
Department of Mathematics, Ferdows Branch, Islamic Azad University, Ferdows, Iran.
E-mail: davarpanah@ferdowsiau.ac.ir

M. E. Omidvar
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.
E-mail: erfanian@mshdiau.ac.ir

H. R. Moradi
Young Researchers and Elite Club, Mashhad Branch, Islamic Azad University, Mashhad, Iran.
E-mail: hrmoradi@mshdiau.ac.ir