VIII. STATISTICAL COMMUNICATION THEORY

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A. SHAPING CORRELATION FUNCTIONS WITH NONLINEAR NO-MEMORY NETWORKS

Consider the time-invariant nonlinear no-memory network f of Fig. VIII-1, with stationary input i(t) and output r(t). The output r(t) is given in terms of the input i(t) by

$$r(t) = f\left[i(t)\right]$$

The problem to be considered is: Given an input process i(t), does there exist a nonlinear no-memory network f that will produce any prescribed output autocorrelation function $\phi_d(\tau)$? If not, can we find a nonlinear no-memory network f that will give an approximation to the desired correlation $\phi_d(\tau)$?

The first question can be answered in the negative by means of a specific counter-example. Let us consider an input process i(t), of which the second-order probability density function belongs to the density-function class Λ (1).

$$p_i(x_1, x_2; \tau) = p_i(x_1)\, p_i(x_2) \sum_{n=0}^{\infty} a_n(\tau)\, \theta_n(x_1)\, \theta_n(x_2) \tag{1}$$

where $p_i(x)$ is the first-order probability density function of the input process. The n^{th}-order probability density function $p_i(x_1, \ldots, x_n; \tau_1, \ldots, \tau_{n-1})$ of the input is defined as

$$p_i(x_1, \ldots, x_n; \tau_1, \ldots, \tau_{n-1})\, dx_1 \ldots dx_n = \text{Probability}\{x_1 < i(t) \leq x_1 + dx_1; \ldots; x_n < i(t + \tau_{n-1}) \leq x_n + dx_n\}$$

The actual output autocorrelation function $\phi_o(\tau)$ is

$$\phi_o(\tau) = \overline{r(t)\, r(t+\tau)}$$

$$\phi_o(\tau) = \int\int y_1 y_2\, p_r(y_1, y_2; \tau)\, dy_1\, dy_2 \tag{2}$$

$$= \overline{f\left[i(t)\right] f\left[i(t+\tau)\right]}$$

$$= \int\int f(x_1)\, f(x_2)\, p_i(x_1, x_2; \tau)\, dx_1\, dx_2 \tag{3}$$

(All integrals are over the whole range of the variables.) $p_r(y_1, y_2; \tau)$ is the second-order probability density function of the output. Substituting Eq. 1 in Eq. 3, we obtain

$$\phi_o(\tau) = \sum_{n=0}^{\infty} c_n^2 a_n(\tau) \tag{4}$$

where

$$c_n = \int f(x)\, p_i(x)\, \theta_n(x)\, dx$$

Thus, for $p_i(x_1, x_2;\tau)$ in Λ, the output autocorrelation function $\phi_o(\tau)$ must be given by Eq. 4. Note that all of the coefficients of $\{a_n(\tau)\}$ in Eq. 4 are non-negative. But not all autocorrelation functions can be expanded in this form. This means that arbitrary correlation functions cannot be attained by nonlinear no-memory networks alone. The following specific example will verify the insufficiency of Eq. 4: Let the input be a gaussian process (with zero mean and unit variance, for the sake of convenience), and with autocorrelation

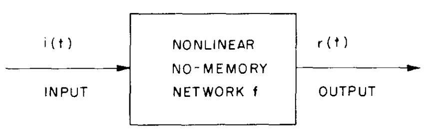

Fig. VIII-1. Nonlinear no-memory transformation.

$$\rho(\tau) = a_1(\tau) = e^{-|\tau|}$$

Then,

$$a_n(\tau) = [a_1(\tau)]^n = e^{-n|\tau|}$$

Now let us choose

$$\phi_d(\tau) = e^{-|\tau|} - \frac{1}{2} e^{-2|\tau|} \tag{5}$$

Then $\phi_d(\tau)$, as given in Eq. 5, may not be expanded as in Eq. 4, since

$$\lim_{\tau \to 0+} \phi_d'(\tau) = 0$$

and

$$\lim_{\tau \to 0+} \sum_{n=0}^{\infty} c_n^2 a_n'(\tau) = -\lim_{\tau \to 0+} \sum_{n=0}^{\infty} n\, c_n^2 e^{-n\tau} < 0$$

unless $c_n = 0$ for all n; however, this restriction makes it useless. Thus the slopes of $\phi_d(\tau)$ and $\sum_{n=0}^{\infty} c_n^2 a_n(\tau)$ could never be matched at $\tau = 0+$. Therefore, it is impossible to produce arbitrary autocorrelation functions from a given input process by means of nonlinear no-memory networks alone.

This result leads to another interesting conclusion: With only nonlinear no-memory networks, it is impossible to produce arbitrary second-order probability density functions from a given input process [with a given second-order probability density function $p_i(x_1, x_2;\tau)$]. For, if it were possible to do so, arbitrary autocorrelation functions could be produced, since the autocorrelation of the output is given in terms of the output second-order probability density function by Eq. 2.

Moreover, since the output second-order probability density function $p_r(y_1, y_2;\tau)$ is determined by the higher-order probability density functions of the output, it follows that: With only nonlinear no-memory networks, it is impossible to produce arbitrary n^{th}-order probability density functions, $n \geqslant 2$, from a given input process. This contrasts with the case $n = 1$, in which a nonlinear no-memory network can always be found to shape a given input first-order probability density function into a desired output first-order probability density function (except when the input probability density function contains impulses).

We shall now consider the second question posed at the beginning of the discussion. First, we refer to Eq. 3. Then, let f be expanded in a series, in which the coefficients $\{c_n\}$ are as yet undetermined:

$$f(x) = \sum_{n=0}^{\infty} c_n w_1(x)\, Q_n(x) \tag{6}$$

where $Q_n(x)$ is an n^{th}-order polynomial and $w_1(x)$ is an arbitrary weighting function, both of which are known. Then

$$\phi_o(\tau) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m\, c_n\, g_{mn}(\tau) \tag{7}$$

where

$$g_{mn}(\tau) = \int\int w_1(x_1)\, w_1(x_2)\, Q_m(x_1)\, Q_n(x_2)\, p_i(x_1, x_2;\tau)\, dx_1\, dx_2$$

is a known function. Thus, if $\phi_d(\tau)$ can be expressed approximately as a sum, as in Eq. 7,

$$\phi_d(\tau) \approx \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_m\, d_n\, g_{mn}(\tau) \tag{8}$$

then we choose

$$f(x) = \sum_{n=0}^{\infty} d_n \, w_1(x) \, Q_n(x)$$

to obtain the approximation to the desired correlation function $\phi_d(\tau)$. The determination of the constants for the approximation Eq. 8 is difficult, but the following procedure is one possibility: Define an error E_N

$$E_N = \int \left[\phi_d(\tau) - \sum_{m=0}^{N} \sum_{n=0}^{N} c_m \, c_n \, g_{mn}(\tau) \right]^2 w_2(\tau) \, d\tau$$

where $w_2(\tau)$ is an arbitrary weighting function. Choose c_N, after having chosen $c_o, \ldots, c_{N-1}$, to minimize E_N. In other words, a progressive determination of the c_n's leads to an approximation of $\phi_d(\tau)$. The approximation can be checked for any number of terms, and a judicious choice of $g_{mn}(\tau)$ could reduce the number of terms that is necessary for a fair approximation of $\phi_d(\tau)$.

A special case of Eq. 6 is

$$f(x) = \sum_{n=0}^{\infty} c_n \theta_n(x)$$

where $\{\theta_n(x)\}$ is the sequence of polynomials (1) associated with the first-order probability density function of the input i(t). Then, we have

$$\phi_o(\tau) = \sum_{n=0}^{\infty} c_n^2 \, a_n(\tau)$$

Therefore, if $\phi_d(\tau)$ is given approximately by

$$\phi_d(\tau) \approx \sum_{n=0}^{N} d_n \, a_n(\tau)$$

where $d_n \geqslant 0$, we can construct

$$f(x) = \sum_{n=0}^{\infty} \left(\pm d_n^{1/2} \right) \theta_n(x)$$

to give the desired approximation. The plus or minus signs are arbitrary and can be chosen for simplicity of synthesis of the nonlinear device.

Extensions to nonstationary inputs and time-varying devices are straightforward. Similarly, the problem of shaping crosscorrelation functions of two input processes by means of two nonlinear no-memory networks can be solved.

A. H. Nuttall

References

1. J. F. Barrett and D. G. Lampard, Trans. IRE, vol. IT-1, no. 1, pp. 10-15 (March 1955).

B. A THEORY OF SIGNALS

In the Quarterly Progress Report of April 15, 1957, page 73, we introduced the linear vector space $\mathcal{O}$ whose elements are finite numerical operators of the form

$$\Omega = \sum_{n=1}^{N} a_n E^{-t_n} \tag{1}$$

where E^{-t_n} is the shift operator defined by $E^{-t_n} f(t) = f(t - t_n)$. We also showed that $\mathcal{O}$ is a unitary space, since it is possible to define a norm for the elements of the space, as well as an inner, or dot, product that gives rise to this norm. With the notation

$$\phi_W(t) \equiv 2W \frac{\sin Wt}{Wt}$$

a possible norm $\|\Omega\|$ of an operator Ω defined as in Eq. 1 was found to be

$$\|\Omega\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\Omega\phi_W(t)]^2 \, dt = \bar{\Omega}\Omega\phi_W(0) \tag{2}$$

where $\bar{\Omega}$ is just Ω folded over, so as to sample forward instead of backward; i.e., if $\Omega = \Sigma a_n E^{-t_n}$, then $\bar{\Omega} = \Sigma a_n E^{+t_n}$. The corresponding inner product of two operators Ω_1 and Ω_2 is defined by

$$[\Omega_1, \Omega_2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega_1 \phi_W(t) \, \Omega_2 \phi_W(t) \, dt = \bar{\Omega}_1 \Omega_2 \phi_W(0) \tag{3}$$

In this report we shall study and use some of the properties of these concepts.

1. Properties of the Norm

The interesting thing about the norm defined above is that it provides a physically meaningful measure of distance between networks and it embodies some of our limitations

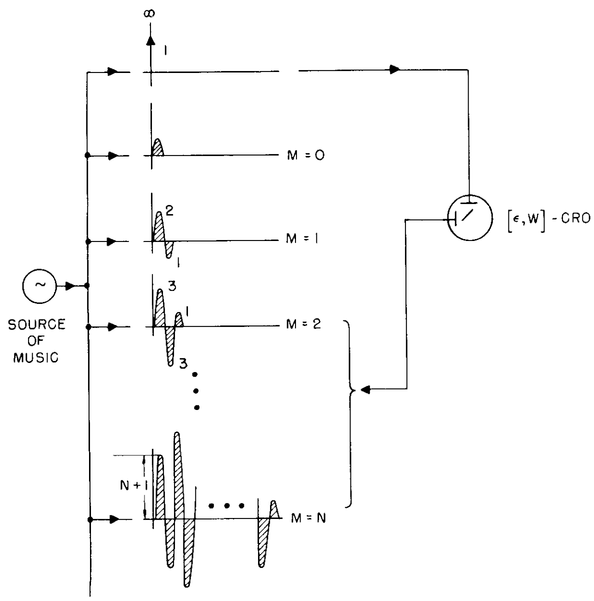

Fig. VIII-2. Pure transmission kernels of various orders excited by music. Beyond some sufficiently high order, outputs are indistinguishable on an [ε , W]-oscilloscope.

in performing measurements, and hence in specifying signals. For example, consider the following experiment, which was performed by Dr. M. V. Cerrillo and K. Joannou.

A set of M networks is prepared, with the impulse responses shown in Fig. VIII-2. These responses are Cerrillo's pure transmission kernels of orders 0, 1, . . . , M, given by

$$a_{k,m} = (-1)^m \prod_{p=0}^{m}{}' \left(\frac{\mu_p}{\mu_k - \mu_p} \right) \qquad k = 0, 1, \ldots, m$$

where $a_{k,m}$ is the area of the k^{th} window in the response of order m, with $k \leq m$. The window width, 2λ, is chosen appropriately small. Suppose that (as was actually done in the laboratory) these networks are all excited by an arbitrary band-limited source (music was used in the experiment) and that the outputs of the networks are compared with the input by placing them, one at a time, on the vertical and horizontal plates of a laboratory oscilloscope. It turned out that, for all networks of order higher than a sufficiently large N, the network outputs could not be distinguished from the input or from each other. This result indicates, first, that Cerrillo kernels of sufficiently high order are actually capable of producing pure transmission, as far as a CRO with finite resolution can tell, and second, that as far as such an oscilloscope can tell, there is no difference, in the operation performed, between any two networks of order greater than N. (This number N can be as small as zero if the window width is small enough, or the oscilloscope crude enough.)

To us, the interesting thing about this experiment is that although the impulse responses of the networks differ enormously, for $m \geq N$ the networks are indistinguishable in their mode of operation on music. That this is possible is not surprising, since the result obtains for a restricted (although large) class of sources and the oscilloscope has finite resolution. But it raises the question: How can we describe or explicitly exhibit in the time domain the fact that, in spite of their very different impulse responses, these networks are so similar in their mode of operation that their outputs are indistinguishable? The norm, as defined in Eq. 2, accomplishes this purpose.

To see how this comes about, we recall from the Quarterly Progress Report of April 15, 1957, page 73, that when Ω operates on any band-limited, integrable square function f(t), the output is bounded by

$$|\Omega f(t)| \leq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f^2\, dt\right)^{1/2} \|\Omega\| \tag{4}$$

which shows that, for all inputs whose energy into a one-ohm load is less than or equal to some fixed number K_o, the output of the network, $\Omega f(t)$, is small whenever $\|\Omega\|$ is small.

A sufficiently strong converse is also true: whenever the output $\Omega f(t)$ is small for <u>all</u> inputs with energy less than or equal to K_o, then $\|\Omega\|$ is also small. This results from the fact that Eq. 4 was obtained by means of the Schwartz inequality, and that it can be shown that there always exists one specific function f(t), belonging to the class of acceptable inputs, for which Eq. 4 becomes an equality for at least one value of t. This function, clearly the one that gives the tallest output, turns out to be $f(t) = \bar{\Omega}\phi_W(t)$. Therefore, if the output $|\Omega f(t)| < \epsilon$ for all inputs, then, in particular, for the maximal input, $|\Omega(\bar{\Omega}\phi_W(t))| < \epsilon$ for all values of t. Therefore, at t = 0, $\|\Omega\|^2 = |\Omega\bar{\Omega}\phi_W(0)| < \epsilon$. (It is always possible to define the classes of acceptable inputs and operators in such a way that $\bar{\Omega}\phi_W(t)$ will indeed be contained in the class of acceptable inputs.)

If, now, $\Omega = \Omega_1 - \Omega_2$, that is, if Ω is the difference between two specified operators Ω_1 and Ω_2, then the discussion indicates that (a) if the distance $\|\Omega_1 - \Omega_2\|$ is small, the difference in the outputs from Ω_1 and Ω_2, $|\Omega_1 f(t) - \Omega_2 f(t)|$, will be small for all acceptable inputs, and (b) if the difference in outputs is small for all acceptable inputs, then the distance $\|\Omega_1 - \Omega_2\|$ is small. Note that the impulse responses corresponding to Ω_1 and Ω_2 can, at the same time, be entirely different.

To check these results with the specific experiment described above, let T_n denote the vector corresponding to the pure transmission kernel of order n. Then it can be shown that the vectors T_n satisfy the difference equation

$$T_n - T_{n-1} = E^{-\lambda}\nabla^n \tag{5}$$

where ∇^n is the n^{th} backward difference operator (1), and the sampling spacing is 2λ. Equation 5 shows, incidentally, that

$$T_n = E^{-\lambda} \sum_{i=0}^{n} \nabla^i \tag{6}$$

We need not go any farther than this to see, for example, that for n large enough (or λ small enough), the operation of two successive kernels can be made indistinguishable. Using Eq. 4, we obtain

$$|(T_n - T_{n-1})\, f(t)| \leq \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f^2\, dt\right]^{1/2} \cdot \|T_n - T_{n-1}\| \leq K_o \|T_n - T_{n-1}\| \tag{7}$$

But

$$\|T_n - T_{n-1}\| = \|E^{-\lambda}\nabla^n\| = \|\nabla^n\|$$

and it can be shown that

$$\|\nabla^n\| \leq \left(\frac{2W}{2n+1}\right)^{1/2} (2\lambda W)^n$$

where W is the bandwidth of the input ensemble. As long as $\lambda < 1/2W$, it is clearly possible to choose n and λ so as to make the bound in Eq. 7 smaller than the resolution of any preassigned oscilloscope.

In terms of the metric established by our norm, the property that is common to the networks of Fig. VIII-2 (of order higher than some N) is that they are all contained within a hypersphere (in $\mathcal{O}$-space) of radius ϵ smaller than the resolution of the oscilloscope that is used for measuring.

2. Approximation Properties of Singular Networks

The problem we wish to study now is the relationship between the singular networks represented by the vectors Ω and smooth finite-memory networks, i.e., networks whose impulse responses are smooth and of finite duration. We shall proceed as follows: (a) determine what happens to the output when a singular response is shifted slightly, as in Fig. VIII-3a; (b) determine what happens with a combination of such shifts, as in Fig. VIII-3b; and (c) use this technique to build up pulses out of impulses, thus obtaining smooth networks or simple-function approximations to smooth networks.

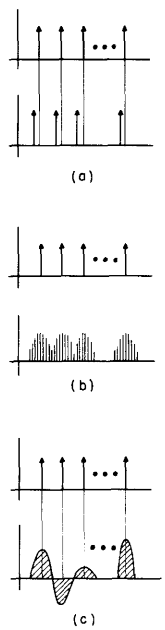

Fig. VIII-3. Alterations on singular impulse responses: making pulses out of impulses.

Notice, at the start, that to shift an impulse response by an amount λ, all that has to be done is to multiply the corresponding vector

$$\Omega = \Sigma_n a_n E^{-t_n} \tag{8}$$

by $E^{-\lambda}$; thus

$$E^{-\lambda}\Omega = \Omega E^{-\lambda} = \Sigma_n a_n E^{-(t_n+\lambda)} \tag{9}$$

is obtained. Denote $E^{-\lambda}\Omega$ by Ω_1. We want a bound on $|\Omega f(t) - \Omega_1 f(t)| = |(\Omega - \Omega_1) f(t)|$. Using Eq. 4, we obtain

$$|(\Omega - \Omega_1)f(t)| \leq \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f^2 \, dt\right]^{1/2} \times \|\Omega - \Omega_1\| \tag{10}$$

so that our problem is to determine the distance $\|\Omega - \Omega_1\|$ from the original vector to the new (shifted) vector Ω_1.

In terms of the inner product (Eq. 3) the distance is given by

$$\begin{aligned} \|\Omega - \Omega_1\|^2 &= [\Omega - \Omega_1, \Omega - \Omega_1] \\ &= [\Omega, \Omega] + [\Omega_1, \Omega_1] - [\Omega, \Omega_1] - [\Omega_1, \Omega] \end{aligned} \tag{11}$$

Since our norm is invariant under translations,

$$[\Omega_1, \Omega_1] = \|\Omega_1\|^2 = \|\Omega\|^2 = [\Omega, \Omega]$$

and, because of the symmetry of the inner product, the last two terms in Eq. 11 are equal, so that

$$\|\Omega - \Omega_1\|^2 = 2\left(\|\Omega\|^2 - [\Omega, \Omega_1]\right) \tag{12}$$

Now, it can be shown that

$$[\Omega, \Omega_1] = \frac{1}{2\pi} R_{\Omega\phi}(\lambda) \tag{13}$$

where $R_{\Omega\phi}(\lambda)$ is the autocorrelation of $\Omega \phi_W(t)$ evaluated at λ, and, therefore, that for the special case $\Omega_1 = \Omega$ (i.e., $\lambda = 0$),

$$\|\Omega\|^2 = \frac{1}{2\pi} R_{\Omega\phi}(0) \tag{14}$$

Using these results in Eq. 12, we have

$$\|\Omega - \Omega_1\|^2 = \frac{2}{2\pi} [R_{\Omega\phi}(0) - R_{\Omega\phi}(\lambda)] \tag{15}$$

We have shown (Quarterly Progress Report, April 15, 1955, p. 38) that the auto-correlation function $R(\tau)$ of any bandlimited process satisfies the inequality, $R(0) - R(\tau) \leq k\tau^2$, where $k \leq W^2 R(0)$, and W is the radian bandwidth of the process. Substitution in Eq. 15 yields

$$\|\Omega - \Omega_1\|^2 \leq \frac{2}{2\pi} R_{\Omega\phi}(0)\, W^2\lambda^2 = 2\|\Omega\|^2\, W^2\lambda^2 \tag{16}$$

the last step being obtained by using Eq. 14.

Using Eq. 16 in Eq. 10, and denoting the energy term by K_f, we have, finally, that the difference in output from an operator Ω and the same operator shifted by an amount λ is bounded by

$$|(\Omega - \Omega_1)\, f(t)| \leq \sqrt{2}\, K_f\, \|\Omega\|\, W|\lambda| \tag{17}$$

This is the desired first result. We shall use it now to build pulses out of impulses, at the same time controlling the error.

Suppose that the desired pulse shape is $p(\lambda)$ (origin of λ at each impulse) and that the given singular response is Ω. Consider, instead, the response $\Omega_i = a_i\Omega$ (a_i, a real number) and shift Ω_i by an amount λ_i. Then, from Eq. 17, we have $|(\Omega_i - E^{-\lambda_i}\Omega_i)\, f(t)| \leq \sqrt{2}\, K_f\, \|\Omega_i\|\, W|\lambda_i| = \sqrt{2}\, K_f \|\Omega\| \cdot |a_i\lambda_i|$, since $\|\Omega_i\| = |a_i| \cdot \|\Omega\|$. Suppose that this is done several times (i.e., for a sequence of values of i) and that the results are added:

$$\Sigma_i\, |a_i\Omega f(t) - a_i\Omega f(t - \lambda_i)| \leq \sqrt{2}\, K_f\, \|\Omega\|\, W\, \Sigma_i\, |\lambda_i a_i| \tag{18}$$

Since it is always true that $|\Sigma_i c_i| \leq \Sigma_i |c_i|$, Eq. 18 implies that

$$|\Sigma_i\, a_i\Omega f(t) - \Sigma_i\, a_i\Omega f(t - \lambda_i)| \leq \sqrt{2}\, K_f\, \|\Omega\|\, W\, \Sigma_i |\lambda_i a_i| \tag{19}$$

Now let $a_i = p(\lambda_i)\, \Delta\lambda$, so that any impulse shifted to the position λ_i has its area multiplied by a number a_i which is proportional to the desired pulse height at that point. Then Eq. 19 becomes

$$|\Omega f(t)\, [\Sigma_i\, p(\lambda_i)\, \Delta\lambda] - \Omega[\Sigma_i\, p(\lambda_i)\, f(t - \lambda_i)\, \Delta\lambda]| \leq \sqrt{2}\, K_f\, \|\Omega\|\, W\, \Sigma_i\, |\lambda_i\, p(\lambda_i)|\, \Delta\lambda$$

In the limit as $\Delta\lambda \to 0$, the sums tend to Riemann integrals and, since it can be shown that the inequality is preserved in the course of the limit process, we obtain

$$\left| \left[\int_{\Lambda} p(\lambda)\, d\lambda \right] \Omega f(t) - \Omega \left[\int_{\Lambda} p(\lambda)\, f(t-\lambda)\, d\lambda \right] \right| \leq \sqrt{2}\, K_f \, \|\Omega\| \, W \int_{\Lambda} |\lambda\, p(\lambda)|\, d\lambda \quad (20)$$

In Eq. 20, Λ is the base or support of the pulse $p(\lambda)$. This is the desired final result.

To interpret it, let us first normalize our pulses $p(\lambda)$ so that $\int_{\Lambda} p(\lambda)\, d\lambda = 1$. Then the first term on the left-hand side in Eq. 20 is just $\Omega f(t)$, which is the output from the singular network Ω (whose impulse response is shown in Fig. VIII-4a, in which it was assumed, for graphical simplicity, that the impulses are equally spaced). The second term

$$\Omega \int_{\Lambda} p(\lambda)\, f(t-\lambda)\, d\lambda \quad (21)$$

is just the output of the smooth network which is obtained from Ω by replacing all its impulses with pulses in such a way that corresponding pulses and impulses have the same area, as in Fig. VIII-4b. To see this, let

$$\Omega = \sum_{k=1}^{N} a_k \, E^{-t_k}$$

in which case the corresponding smooth response, denoted by $h(t)$, can be written

$$h(t) = \sum_{k} a_k \, p(t - t_k) = \left[\sum_{k} a_k \, E^{-t_k} \right] p(t) = \Omega p(t)$$

Comparing this with Eq. 21, we see that

$$\Omega \int_{\Lambda} p(\lambda)\, f(t-\lambda)\, d\lambda = \int_{\Lambda} [\Omega p(\lambda)]\, f(t-\lambda)\, d\lambda = \int_{0}^{T} h(\lambda)\, f(t-\lambda)\, d\lambda$$

is just the output from the smooth network $h(t)$.

If $p(\lambda)$ is a unidirectional normalized pulse (i.e., a pulse that does not change sign), and if it extends from $-\lambda_o$ to λ_o, then Eq. 20 becomes just

$$\left| \Omega f(t) - \Omega \int_{-\lambda_o}^{\lambda_o} p(\lambda)\, f(t-\lambda)\, d\lambda \right| \leq \sqrt{2}\, K_f \, \|\Omega\| \, W \lambda_o \quad (22)$$

so that the bound on the error is independent of pulse shape.

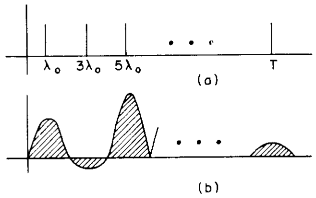

Fig. VIII-4. Singular response and corresponding smoothed response.

Since unidirectional pulses are precisely what Cerrillo calls "windows," Eq. 22 shows that it is possible to associate a singular response with every window function, with a very clear estimate of the error incurred by this procedure. Furthermore, since a singular response is just a numerical operator, the procedure shows how to use the results of numerical analysis to obtain the Cerrillo kernel appropriate to a given operation directly.

More generally, since the pulse $p(\lambda)$ may be chosen rectangular, Eq. 22 shows the relationship between singular responses and simple-function, or staircase, responses. Since it is known that the set of simple functions is uniformly dense in the set of time-limited continuous functions, Eq. 22 also shows that it is possible to associate a singular response with any continuous (time-limited) response, and that this can be done with arbitrarily small error at the output, for all members of a specified ensemble of inputs.

Several other applications of these results, plus a study of the geometry of the space of the Ω's will be given in a future report.

R. E. Wernikoff

C. PROPERTIES OF SECOND-ORDER CORRELATION FUNCTIONS

The present study has been completed. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science and the degree of Electrical Engineer, Department of Electrical Engineering, M.I.T., June 1957, and will also be presented as Technical Report 330.

J. Y. Hayase

D. TIME-DOMAIN SYNTHESIS BY AREA APPROXIMATION

1. Introduction

a. Motivation for study of area approximation

In the formulation of time-domain synthesis problems, the singular impulse response (an impulse response consisting of a finite set of impulses of finite area distributed over a finite interval) is becoming increasingly important. For example, Wernikoff (1) and Cerrillo (2) have suggested techniques for obtaining networks to perform specified linear operations on a given class of inputs. These networks are specified by singular impulse responses. Also Ba Hli (3) has suggested a method for calculating an appropriate singular impulse response when the design data are given in the form of two time functions, one representing the output from the desired network when the other is applied to the input. A limitation of the use of these techniques is that, presently, a singular impulse response can be realized only by use of a tapped delay line. A feature of synthesis by area approximation is that it permits the approximation of singular impulse response characteristics by lumped, linear networks. A further advantage is that a simple expression is available for the time-domain error.

b. Basis of synthesis by area approximation

A singular impulse response associates the area under the response curve with a finite set of points. A first-order approximation by a smooth curve is achieved if the integrals of the two curves, i.e., the step responses, are forced to coincide at a set of points, although not necessarily at the set of points on which the singular response is defined. This approximation technique is referred to as "area approximation." Unfortunately, the shape of the smooth response does enter into the error calculations, but, for the class of approximating functions considered in Section VIII-D3, a reasonable extimate of the error is obtained by assuming straight-line connections between coincident points.

The synthesis of networks with smooth (nonsingular) impulse responses can also be accomplished by area approximations, although a shift in the emphasis of the approximation step is required. Classically, the success of the approximation is judged by how closely the output of the approximating network resembles the output of the desired network for impulse excitation. No control is exercised over the outputs for other inputs, and there is certainly no guarantee that small imperfections in the approximative impulse response will not be exaggerated during convolution. If this situation is recognized, and the criterion for success of the approximation step is shifted to reproduction of outputs stimulated by inputs chosen from a specified class of possible inputs, synthesis by area approximation becomes applicable. Ease of error calculation is slightly less for smooth responses than for singular responses.

c. Error criterion

Wernikoff (1) has provided a simple proof that the output of a network with a singular response converges uniformly to the output of a network with smooth response as the number of coincident points is increased. This proof is directly applicable to synthesis by area approximation, and allows a bound to be placed on the approximation error. Therefore, a singular network can be made indistinguishable from a smooth network, for a given class of inputs, since all physical detectors possess finite resolution.

2. Error Analysis of Synthesis by Area Approximation (3)

a. Derivation of error expression

Our object is to determine an expression for the maximum difference in output between two networks that are excited by a common input, when the step responses of the two networks agree at a finite set of points. The networks are assumed to have impulse responses that are of finite duration, and the input is assumed to possess all orders of derivatives.

Let the points of coincidence between the step responses be at times $T_0', T_1', \ldots, T_{m+1}'$; and let T_i be the midpoint of the interval (T_i', T_{i+1}'). The length of the interval (T_i', T_{i+1}') is d_i. The output of a linear network is related to the input and to the impulse response by the convolution integral, which can be written as

$$g(t) = \sum_{m=0}^{M} \int_{-d_m/2}^{d_m/2} h_m(y)\, f(t - T_m - y)\, dy, \qquad t > T \tag{1}$$

where $g(t)$ is the output; $h(t)$ is the impulse response, and $h_m(t) = h(t + T_m)$; $f(t)$ is the input, and $T = T_{m+1}'$ is the duration of the impulse response.

If we expand the input in a Taylor series, and let the area under the impulse response in the interval (T_i', T_{i+1}') be a_i, the convolution integral becomes

$$g(t) = \sum_{m=0}^{M} \left[a_m f(t - T_m) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} f_m^{(n)} \int_{-d_m/2}^{d_m/2} y^n h_m(y)\, dy \right], \qquad t > T \tag{2}$$

where $f_m^{(n)}$ is the n^{th} derivative of the input evaluated at $x = t - T_m$.

The error is now defined as the difference in output between the desired network and the approximative network, and, by use of Eq. 2 and an integration-by-parts, it can be expressed as

$$e(t) = \sum_{m=0}^{M} \left[\sum_{n=1}^{\infty} 2(2n-1) \frac{f_m^{(2n-1)}}{(2n-1)!} \int_{0}^{d_m/2} y^{2n-2} D_{me}(y)\, dy \right.$$

$$\left. - \sum_{n=1}^{\infty} 2(2n) \frac{f_m^{(2n)}}{(2n)!} \int_{0}^{d_m/2} y^{2n-1} D_{mo}(y)\, dy \right], \qquad t > T \tag{3}$$

where if $A(t)$ and $A^*(t)$ are the step responses of the desired and of the approximative networks, then $D_e(t)$ and $D_o(t)$ are the even and odd parts of the difference $A(t) - A^*(t)$. The subscript m refers, as before, to a translation in time, $y = t + T_m$.

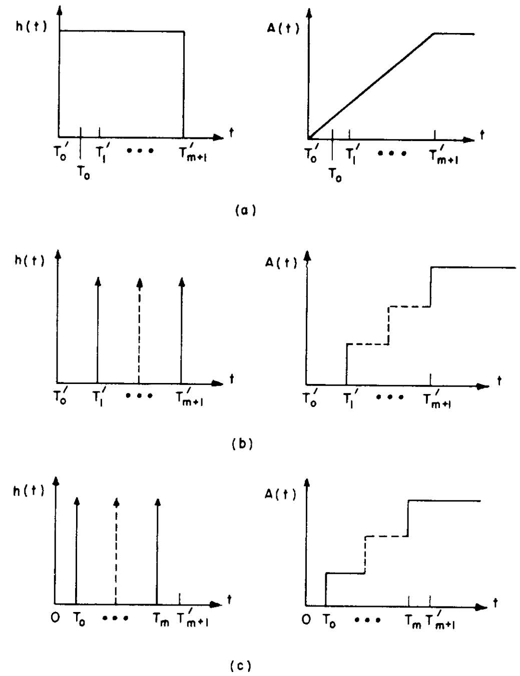

Fig. VIII-5. Singular responses of three networks equivalent under an approximation: (a) network A; (b) network B; (c) network C.

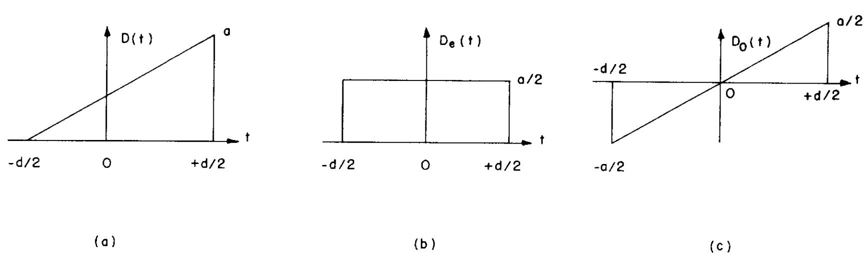

Fig. VIII-6. Difference functions for approximation between networks A and B of Fig. VIII-5.

For the input $f(t) = \epsilon^{j\omega t}$, the summations on n can be expressed in closed form, and Eq. 3 becomes

$$e(t) = \sum_{m=0}^{M} \epsilon^{j\omega(t-T_m)} \left[\int_0^{d_m/2} 2\omega \sin \omega y \, D_{mo}(y) \, dy + j \int_0^{d_m/2} 2\omega \cos \omega y \, D_{me}(y) \, dy \right], \qquad t > T \tag{4}$$

The dependence of the error on the shape of the step response enters through the difference functions, $D_{me}(t)$ and $D_{mo}(t)$.

b. Applications of the error expression

Figure VIII-5 illustrates the impulse and step responses of three networks that are equivalent under area approximation. Networks B and C both have singular responses. The necessary difference functions for an error calculation between the outputs of networks A and B and networks A and C are given in Figs. VIII-6 and VIII-7. The use of Eq. 4 in the calculations is particularly straightforward, and the resulting errors are plotted in Fig. VIII-9. For convenience, the errors are normalized to the output of the singular networks.

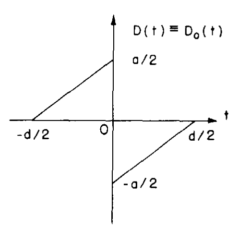

Fig. VIII-7. Difference function for approximation between networks A and C of Fig. VIII-5.

A second application of the error expression, Eq. 4, is in the calculation of the error caused by sine- and cosine-like differences between the step responses. The difference functions are illustrated in Fig. VIII-8, and the associated error is plotted in Fig. VIII-10. Notice that the error becomes very large as the input frequency approaches the perturbation frequency. This illustrates the point made in Section VIII-D1b, since large errors may result if even a small periodicity is ignored in the approximation.

Equation 4 can also be applied to inputs with more complicated spectra than a single frequency by the use of Fourier transform theory (2). An interesting result is that the relative error in the outputs, for any input chosen from the class of inputs bandlimited to a frequency W, is bounded by the error obtained for a sinusoidal input of frequency W. For this calculation, the error is normalized with respect to the peak amplitude of the output that results from that number of the input class whose output is a maximum (when all members of the class have the same energy).

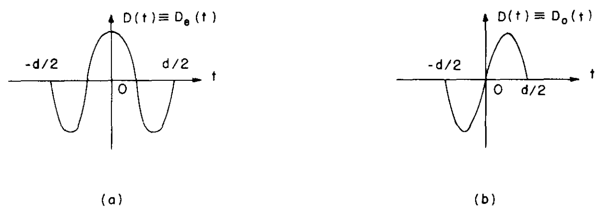

Fig. VIII-8. Cosine-like and sine-like contributions to difference function.

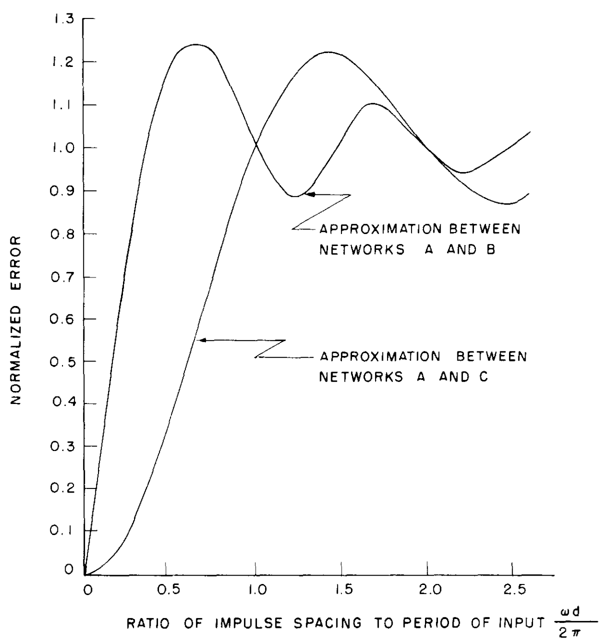

Fig. VIII-9. Normalized error versus ratio of impulse spacing to period of input sinusoid for approximations between networks of Fig. VIII-5.

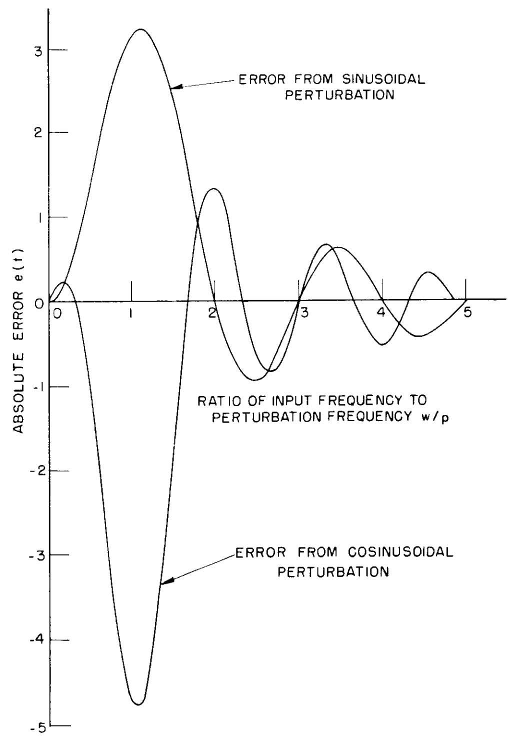

Fig. VIII-10. Absolute error resulting from sinusoidal and cosinusoidal perturbations on difference function versus ratio of input frequency to perturbation frequency.

3. Solution of the Approximation Problem

To complete the approximation step of the synthesis procedure, it is necessary to specify a function with a rational Laplace transform that can be passed through the desired coincident points of the step response. The difficulty is enhanced by the restriction that the function must maintain its last assigned value as t approaches infinity so that the duration of the approximative response will equal that of the desired impulse response.

An approximative function that fulfills these requirements is

$$A^*(t) = K + \sum_{m=0}^{M} b_m t^m \epsilon^{-t/d} + R^*(t)\,\epsilon^{-t/d} \tag{5}$$

The first two terms are passed through the coincident points by an appropriate choice of the coefficients. The third term is used to cancel the response after time T. The time constant in the exponential terms is chosen as d seconds, since this value causes the m^{th} term in the series to attain a maximum at time $t = m \times d$. Thus, if the coincident points are equally spaced d seconds apart, the maxima fall on the coincident

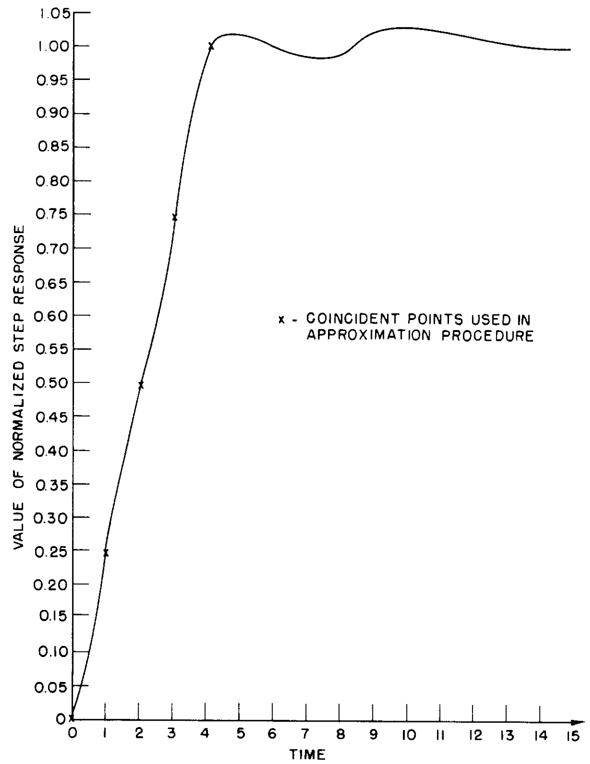

Fig. VIII-11. Approximative function for networks of Fig. VIII-5, obtained with five coincident points.

points, and the coefficients b_m are kept within the same order of magnitude.

The polynomial $R^*(t)$ is chosen with zeros at each of the coincident points, and with a multiple-order zero at a time T', when the second term is essentially zero. By properly choosing the order of the multiple zero, the third term can be made to cancel the overshoot of the second term. A systematic procedure for calculating the various coefficients is given in reference 4. An approximative function for the networks of Fig. VIII-5, utilizing five coincident points, is illustrated in Fig. VIII-11. A particularly simple realization of the step response of Eq. 5 is given by Wernikoff (5).

4. Limitation on Synthesis by Area Approximation

Although synthesis by area approximation does allow the realization of networks characterized by a singular impulse response, the number of coincident points required to achieve, for example, a 1 per cent relative error, may be of the order of 20 or 30. Therefore, the standard passive realization techniques are not practical, and the stability problems in an active realization may be overly severe.

This work, which was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science, Department of Electrical Engineering, M.I.T., June 1957, will be published as Technical Report 331.

I. M. Jacobs

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3. F. Ba Hli, Network synthesis by impulse response for specified input and output in the time domain, Sc.D. Thesis, Department of Electrical Engineering, M.I.T., Sept. 1953.
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E. AN ANALOG PROBABILITY DENSITY ANALYZER

Experimental probability density distributions of both periodic and noise functions were discussed in the Quarterly Progress Report of April 15, 1957, page 81. Additional experimental work, which is related to the analysis of speech, is described in the present report.

The analysis of speech requires longer integration times than those that were used for the periodic and noise functions. The speech probability density function, shown in Fig. VIII-12, was obtained with a 17-sec integrator time constant. The actual value of the RC integrator product was 8.5 sec, but the effective value of the time constant was doubled by playing back the recorded speech sample with the recording speed doubled. A one-minute sample of speech was continuously repeated to obtain the probability

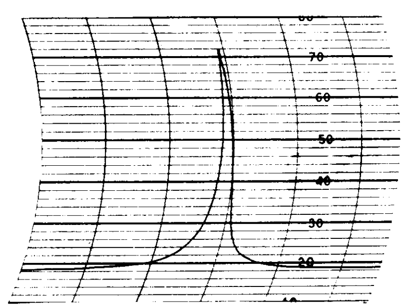

Fig. VIII-12. Speech probability density function.

density function. Since the length of the sample is much longer than the integration time, the probability density function should approximate the stationary characteristics of the speech source.

This report concludes a series of reports entitled "An Analog Probability Density Analyzer." A complete description of the development and construction of the analog probability density analyzer and experimental results will be published as Technical Report 326. This work was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science, Department of Electrical Engineering, M.I.T., June 1957.

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