XVI. CIRCUIT THEORY

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A. SOME LIMITATIONS OF LINEAR AMPLIFIERS

If a network is to be constructed by imbedding a linear device in a network of ideal transformers, there are certain allowed natural frequencies for the resulting network. They are readily determined by studying the properties of a linear amplifier that supplies power to a complex load. A method of determining the allowed frequencies for a linear amplifier will be discussed.

1. The Powerless Concept

If a linear device, described by a complex admittance matrix $[Y]$, is imbedded in an ideal transformer network and connected to a load, conservation of power enables us to write

$$[V]^\dagger [Y][V] + P = 0 \tag{1}$$

where V is the complex column matrix of voltages at the terminals of the device, P is the total power delivered to the load, and $[V]^\dagger$ is the conjugate transpose of $[V]$.

Before continuing the discussion it is well to interpret the meaning of power for signals with exponentially increasing or decreasing amplitudes. If a voltage v, given by $v = 2^{1/2} |V| e^{\sigma t} \cos(\omega t + \beta)$, is applied to a one-ohm resistor, the instantaneous power will be given by $p = 2|V|^2 e^{2\sigma t} \cos^2(\omega t + \beta)$. For convenience of notation, we speak of the complex rms voltage amplitude as $V = |V| e^{j\beta}$ and the complex average power amplitude as $P = |P| e^{j\theta}$. Admittedly, the envelopes of the voltage and power will be increasing or decreasing exponentially, but we can consider that V and P determine the amplitudes at $t = 0$. Alternatively, we can neglect σ and assume that the sensitivity of the voltmeter and wattmeter are exponential functions of time, so that a quasi-steady state condition may be assumed. In reality, the entire analysis is purely mathematical and depends upon analytic continuation for its justification. We cannot, in general, measure waveforms that die away faster than the natural frequencies of the system. With these thoughts in mind, we continue the analysis as though we were dealing with pure sinusoids, except that the admittance of circuit components will be assumed to be a function of a mystical parameter σ.

With reference to Eq. 1, we can make the following statement:

> If it is not possible to find a $[V]$ with the property that $[V]^\dagger [Y(s_1)][V] + P = 0$, then it is not possible for the device described by $Y(s)$ to deliver a power P at a complex frequency s_1. $\tag{2}$

Expression 2 establishes a necessary condition that must be satisfied if the

amplifier is to be realizable. We must now show that this is also a sufficient condition. The proof of sufficiency is best established by a direct synthesis procedure.

2. Synthesis of a Linear Amplifier

Assume that a V that satisfies Eq. 1 has been found, and consider the question of how we might construct a circuit to supply the power P to a physical load. Since we are interested in synthesizing an amplifier for an arbitrary complex frequency, we can no longer use L's and C's as phase-shifting elements; we must use an artifice that is equivalent to an ideal phase-shifting transformer. It appears that this can best be done by interpreting a complex voltage Ve^{st} as the sum of two real voltages which, for convenience, will be called the even and odd components.

$$v_e = |V|\, e^{\sigma t}\cos(\omega t + \beta) \qquad\qquad v_o = |V|\, e^{\sigma t}\sin(\omega t + \beta) \qquad\qquad (3)$$

If we interpret a single complex voltage as the sum of two real voltages which are 90° out of phase, we can synthesize all circuits in a symmetric form in such a manner that all voltages appear as if they were on a two-phase transmission line. In terms of this two-phase representation we can visualize the general amplifier, as shown in Fig. XVI-1. We could, of course, use more than two phases but the redundancy is not necessary; in fact, in many instances, the synthesis can be accomplished without resort to a multi-phase system. For general synthesis, however, we must consider a complex voltage as a mathematical representation of two real voltages so that a network of real transformers can effect both amplitude and phase transformation.

With reference to Fig. XVI-1, let us choose

$$[V_1] = [N_r][V_e] + [N_i][V_o] \qquad\qquad [I_1] = [Y][V_1]$$
$$[V_2] = -[N_i][V_e] + [N_r][V_o] \qquad\qquad [I_2] = [Y][V_2] \qquad (4)$$

where $[V_1]$ and $[V_2]$ are the voltage matrices for devices 1 and 2 in Fig. XVI-1, and $[N_r]$ and $[N_i]$ are arbitrary real transformations. It is then readily shown that for $V_e = V'$,
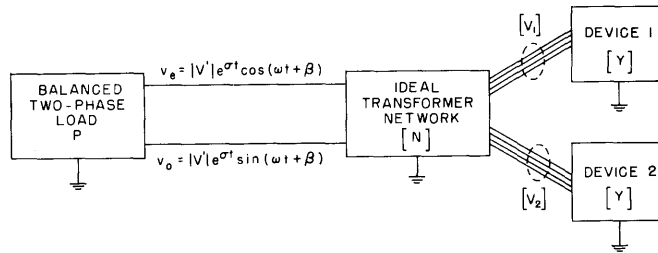

Fig. XVI-1. Synthesis of linear amplifier in two-phase form.

$V_o = jV'$ (i.e., V_e and V_o are two voltages 90° out of phase), the power dissipated in the devices is given by

$$P_1 = V_1^\dagger I_1 = P_2 = V_2^\dagger I_2 = |V'|^2 [N]^\dagger [Y][N]$$

$$N = [N_r] + j[N_i]$$

(5)

Thus, by choosing $V'[N] = V$, we have ensured that the total power supplied by each device is $P = -[V]^\dagger [Y][V]$, and thus Eq. 1 is satisfied. Since the load is completely arbitrary, we can think of it as a connection of ideal transformers plus a pair of load devices described by a matrix $[Y_L]$, and interconnected in a manner analogous to that described in connection with the amplifier synthesis. (In reality, there is only an arbitrary distinction between which is called the amplifier and which the load.)

In light of our analysis, we can make the following statement.

> If it is possible to find a V with the property that
> $[V][Y][V] + P = 0$, then it is possible to build an amplifier
> to supply a power P to an arbitrary load. The con-
> struction of this amplifier will, in general, require the
> use of two devices plus ideal transformers, and the out-
> put voltage will appear as if it were on a two-phase
> transmission line.

(6)

We then see that the existence of a V that satisfies Eq. 1 is a necessary and suf-ficient condition for the realization of an amplifier if we allow all networks to be con-structed in symmetric two-phase form. There is no assurance that other natural frequencies will not be present and, in general, an amplifier synthesized by the afore-mentioned technique may have an infinity of natural frequencies. The merit of this synthesis procedure is that it allows us to establish necessary and sufficient bounds on the natural frequencies of a system.

3. Complex Power and Its Relation to the Gyrator

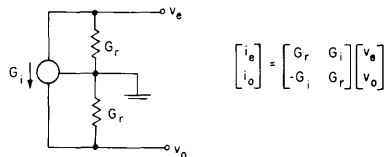The canonic form of a balanced two-phase load, which is indicated in Fig. XVI-1, can be thought of as a combination of two equal conductances and an ideal gyrator, as shown in Fig. XVI-2. For two-phase excitation with $V_e = V$, $V_o = jV$, we see that $P = 2|V|^2 G_r + j2|V|^2 G_i$. Thus the real power $2|V|^2 G_r$ is dissipated in the two conductances, while the "imaginary" power is "dissipated" in the gyrator. The term "dissipated" may appear strange,

$$\begin{bmatrix} i_e \\ i_o \end{bmatrix} = \begin{bmatrix} G_r & G_i \\ -G_i & G_r \end{bmatrix} \begin{bmatrix} v_e \\ v_o \end{bmatrix}$$

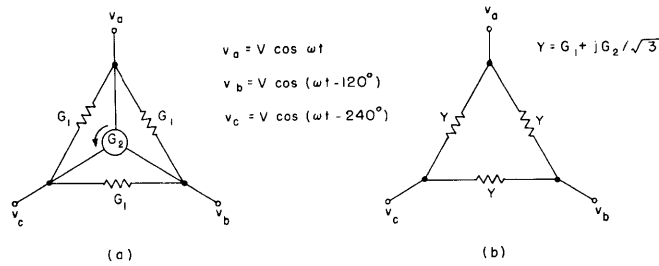Fig. XVI-2. Canonic form of a bal-anced two-phase load.

Fig. XVI-3. Complex gyrator representation for a three-phase system.
(a) Gyrator circuit. (b) Equivalent complex representation.

since, by turning the network upside down, the sign of the imaginary power is reversed and the gyrator becomes a generator of power. The important concept seems to be not so much what happens to the imaginary power but rather that for equilibrium to exist both the real and the imaginary power must be zero for the system as a whole. Note that the gyrator is associated entirely with the imaginary power, while the conductances are associated entirely with the real power.

In terms of the canonic form, we see that for a balanced multiphase transmission system any balanced load can be realized by resistors and gyrators (at a single frequency) without recourse to reactive elements. A simple example of a three-phase delta-connected load is shown in Fig. XVI-3a. Note that this load, containing only resistors and gyrators, can be analyzed exactly by use of the equivalent "reactive" circuit of Fig. XVI-3b. Thus for a multiphase system "reactive" power need not be associated with energy-storage elements, and hence the term "imaginary" power has been used to avoid the implication of reactance. In terms of a picture such as Fig. XVI-2, we see that imaginary power can be interpreted as "transmitted," or "interchanged," power, since gyrators do not store energy but merely interchange the power between the two phases of the excitation voltage. Thus a device such as a "synchronous capacitor," which is commonly used for power-factor correction, is more nearly a gyrator; ideally the machine would neither store nor dissipate energy.

4. Allowed Poles and Residues

The previous analysis enables us to determine some important bounds on network behavior. As an example, suppose that we wish to determine the allowed poles and residues of a driving-point impedance. In the vicinity of the pole the admittance is approximately $Y = (s - s_1)/A$, where s_1 is the location of the pole, and A is the residue. For $s = s_1$, $V^{\dagger} YV = 0$, and we see that this is the special case of an oscillator supplying zero power to an open circuit. If we deviate slightly from $s = s_1$, the conditions for oscillations become

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$$V^\dagger YV + P = 0 \quad \text{or} \quad (s - s_1) + PA/|V|^2 = 0 \tag{7}$$

In the immediate vicinity of s_1 we can write

$$s - s_1 = \delta e^{j\alpha} \qquad A = |A| e^{j\beta} \qquad P = |P| e^{j} \tag{8}$$

and the oscillator synthesis question becomes

$$\delta e^{j\alpha} + \left| \frac{PA}{V^2} \right| e^{j(\beta+\theta)} \overset{?}{=} 0 \tag{9}$$

Equation 9 means: For all values of δ, α, and β will only the "allowed" values of θ satisfy Eq. 9? If the driving-point impedance is constructed by imbedding a particular device in an arbitrary transformer network, we know that there is only a certain allowed range of values of θ, and therefore values of θ that satisfy Eq. 9 must fall within this range. As δ becomes vanishingly small, the allowed θ will generally become independent of δ and, since $|V|$ may be any real number, Eq. 9 can be written:

$$\alpha \overset{?}{=} \beta + \theta + \pi \tag{10}$$

The allowed values of θ must then include $\alpha - \beta \pm \pi$ if the driving-point impedance is to be realizable.

As an example, consider the problem of constructing an LC circuit with a pole of impedance at $s = j\omega_1$, $\omega_1 \neq 0$. It can be shown that the allowed θ for an LC circuit can be determined by

$$\frac{\sin \theta}{\omega} - \frac{\cos \theta}{\sigma} \leqslant 0 \tag{11}$$

As δ becomes vanishingly small, this equation can only be satisfied by

$$\sigma > 0, \ \sigma \to 0 \quad \Longrightarrow \quad -\frac{\pi}{2} < \theta + \pi < \frac{\pi}{2}$$
$$\sigma < 0, \ \sigma \to 0 \quad \Longrightarrow \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \tag{12}$$

Or, in terms of $\alpha = \tan^{-1}[(\omega - \omega_1)/\sigma]$, for $\delta = |s - s_1| \to 0$, we have

$$-\frac{\pi}{2} < \alpha < \frac{\pi}{2} \quad \Longrightarrow \quad -\frac{\pi}{2} < \theta + \pi < \frac{\pi}{2}$$
$$-\frac{\pi}{2} < \alpha + \pi < \frac{\pi}{2} \quad \Longrightarrow \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \tag{13}$$

This limitation on α immediately indicates that $\beta = 0$ is the only possible phase angle that satisfies Eq. 10, and hence it is the only possible angle for the residue of the pole at ω_1. This restriction of positive real residues is, of course, readily determined from conventional methods of analysis. If we had attempted to place a pole at a

frequency $s = \sigma_1 + j\omega_1$, $\sigma_1 \neq 0$, we would have found it impossible to satisfy Eq. 10 and we would reach the correct conclusion that all the poles must be on the $j\omega$-axis and all the residues must be positive real.

Thus we see that the limitations on poles and residues of driving-point impedances can be determined as a special case of the general limitations of linear amplifiers. Some of the other implications and consequences of the linear amplifier limitations will be discussed in a later report.

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