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Attractor Horizon Geometries of Extremal Black Holes

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Abstract

We report on recent advances in the study of critical points of the "black hole effective potential" V_{BH} (usually named *attractors*) of $\mathcal{N}=2$, $d=4$ supergravity coupled to n_V Abelian vector multiplets, in an asymptotically flat extremal black hole background described by $2n_V+2$ dyonic charges and (complex) scalar fields which are coordinates of an n_V-dimensional Special Kähler manifold.

Contents

1 Introduction

After some seminal papers [1]-[5] of some years ago, extremal black hole (BH) attractors have been recently widely investigated [6]- [34]. Such a *renaissance* is mainly due to the (re)discovery of new classes of solutions to the attractor equations corresponding to non-BPS (Bogomol'ny-Prasad-Sommerfeld) horizon geometries.

An horizon extremal BH attractor geometry is in general supported by particular configurations of the $1 \times (2n_V + 2)$ symplectic vector of the BH field-strength fluxes, *i.e.* of the BH magnetic and electric charges:

$$\widetilde{\Gamma} \equiv \left(p^\Lambda, q_\Lambda\right), \qquad p^\Lambda \equiv \frac{1}{4\pi}\int_{S^2_\infty} \mathcal{F}^\Lambda, \quad q_\Lambda \equiv \frac{1}{4\pi}\int_{S^2_\infty} \mathcal{G}_\Lambda, \quad \Lambda = 0, 1, ..., n_V, \tag{1.1}$$

where, in the case of $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravity theories (MESGTs), n_V denotes the number of Abelian vector supermultiplets coupled to the supergravity one (containing the Maxwell vector A^0, usually named *graviphoton*). Here $\mathcal{F}^\Lambda = dA^\Lambda$ and $\mathcal{G}_\Lambda$ is the "dual" field-strength two-form [35, 36].

In the present brief review we will consider only *non-degenerate* ($\frac{1}{2}$-BPS as well as non-BPS) geometries, *i.e.* geometries yielding a finite, non-vanishing horizon area, corresponding to the so-called *"large"* BHs. Due to the well-known Attractor Mechanism [1]-[5], such BH horizon geometries are actually *critical*, because their Bekenstein-Hawking entropy [37] can be obtained by extremizing a properly defined, positive-definite "effective BH potential" function $V_{BH}\left(\phi, \widetilde{\Gamma}\right)$, where "$\phi$" denotes the set of real scalars relevant for the Attractor Mechanism.

In $\mathcal{N} = 2$, $d = 4$ MESGTs, non-degenerate attractor horizon geometries correspond to BH solitonic states belonging to $\frac{1}{2}$-BPS "short massive multiplets" or to "long massive multiplets" violating the BPS bound [39], respectively with[1]

$$
\begin{aligned}
&\tfrac{1}{2}\text{-}BPS\text{: } 0 < |Z|_H = M_{ADM,H};\\
&non\text{-}BPS \begin{cases} Z \neq 0\text{: } 0 < |Z|_H < M_{ADM,H};\\ Z = 0\text{: } 0 = |Z|_H < M_{ADM,H}, \end{cases}
\end{aligned}
\tag{1.2}
$$

where Z denotes the $\mathcal{N}=2$, $d=4$ *central charge function*, and the Arnowitt-Deser-Misner (ADM) mass [40] at the BH horizon is obtained by extremizing $V_{BH}\left(\phi,\widetilde{\Gamma}\right)$ with respect to its dependence on the moduli:

$$
M_{ADM,H}\left(\widetilde{\Gamma}\right)=\sqrt{\left.V_{BH}\left(\phi,\widetilde{\Gamma}\right)\right|_{\partial_\phi V_{BH}=0}}. \tag{1.3}
$$

The charge-dependent BH entropy S_{BH} is given by the Bekenstein-Hawking entropy-area formula [37, 5]

$$
S_{BH}\left(\widetilde{\Gamma}\right)=\pi M^2_{ADM,H}\left(\widetilde{\Gamma}\right)=\frac{A_H\left(\widetilde{\Gamma}\right)}{4}=\pi\left.V_{BH}\left(\phi,\widetilde{\Gamma}\right)\right|_{\partial_\phi V_{BH}=0}=\pi V_{BH}\left(\phi_H\left(\widetilde{\Gamma}\right),\widetilde{\Gamma}\right), \tag{1.4}
$$

where A_H is the event horizon area. The charge-dependent horizon configuration $\phi_H\left(\widetilde{\Gamma}\right)$ of the real scalars is obtained by extremizing $V_{BH}\left(\phi,\widetilde{\Gamma}\right)$, *i.e.* by solving the criticality conditions

$$
\partial_\phi V_{BH}\left(\phi,\widetilde{\Gamma}\right)=0. \tag{1.5}
$$

Strictly speaking, $\phi_H\left(\widetilde{\Gamma}\right)$is an *attractor* if the critical $(2n_V+2)\times(2n_V+2)$ real symmetric Hessian matrix

$$
\left.\frac{\partial^2 V_{BH}\left(\phi,\widetilde{\Gamma}\right)}{\partial\phi\partial\phi}\right|_{\phi=\phi_H\left(\widetilde{\Gamma}\right)} \tag{1.6}
$$

is a strictly positive-definite matrix[2].

Although non-supersymmetric (non-BPS) BH attractors arise also in $\mathcal{N}>2$, $d=4$ and $d=5$ supergravities [38, 21] (see [41] for a recent review), the richest casistics pertains to $\mathcal{N}=2$, $d=4$ MESGTs, where the manifold parameterized by the scalars is endowed with a special Kähler (SK) metric structure.

The plan of the paper is as follows.

[1]Here and in what follows, the subscript "H" will denote values at the BH event horizon.

[2]It is worth pointing out that the opposite is in general not true, *i.e.* there can be attractor points corresponding to critical Hessian matrices with some "flat" directions (*i.e.* vanishing eigenvalues). In general, in such a case one has to look at higher-order covariant derivatives of V_{BH} evaluated at the considered point, and study their sign. Dependingly on the configurations of the supporting BH charges, one can obtains stable or unstable critical points. Examples in literature of investigations beyond the Hessian level can be found in [10, 26, 27].

In Sect. 2 we sketchily recall the fundamentals of the local SK geometry. Thence, in Sect. 3 we introduce the effective BH potential for a generic $\mathcal{N}=2$, $d=4$ MESGT, and consider its $\frac{1}{2}$-BPS critical points [1]- [5], which turn out to be always stable, and thus attractors in strict sense. Sects. 4 and 5 are devoted to the discussion of the non-BPS, $Z\neq 0$ case, with an explicit application to the one-modulus case $n_V=1$; in particular, in 4 non-BPS, $Z\neq 0$ critical points of V_{BH} and the related eigenvalue problem are presented, whereas Sect. 5 deals with the issue of stability of such a class of points. Finally, Sect. 6 contains some summarizing observations and general remarks, as well as an outlook of possible future further developments along the considered research directions.

2 Special Kähler Geometry

In the present Section we briefly recall the fundamentals of the SK geometry underlying the scalar manifold $\mathcal{M}_{n_V}$ of $\mathcal{N}=2$, $d=4$ MESGT, n_V being the number of Abelian vector supermultiplets coupled to the supergravity multiplet ($dim_{\mathbb{C}}\mathcal{M}_{n_V}=n_V$).

It is convenient to switch from the Riemannian $2n_V$-dim. parameterization of $\mathcal{M}_{n_V}$ given by the local real coordinates $\{\phi^a\}_{a=1,\dots,2n_V}$ to the Kähler n_V-dim. holomorphic/antiholomorphic parameterization given by the local complex coordinates $\left\{z^i,\overline{z}^{\overline{i}}\right\}_{i,\overline{i}=1,\dots,n_V}$. This corresponds to the following *unitary Cayley transformation*:

$$z^k\equiv\frac{\varphi^{2k-1}+i\varphi^{2k}}{\sqrt{2}},\quad k=1,\dots,n_V. \tag{2.1}$$

The metric structure of $\mathcal{M}_{n_V}$ is given by the covariant (special) Kähler metric tensor $g_{i\overline{j}}(z,\overline{z})=\partial_i\overline{\partial}_{\overline{j}}K(z,\overline{z})$, $K(z,\overline{z})$ being the real Kähler potential.

Usually, the $n_V\times n_V$ Hermitian matrix $g_{i\overline{j}}$ is assumed to be non-degenerate (*i.e.* invertible, with non-vanishing determinant and rank n_V) and with strict positive Euclidean signature (*i.e.* with all strictly positive eigenvalues) *globally* in $\mathcal{M}_{n_V}$. We will so assume, even though we will be concerned mainly with the properties of $g_{i\overline{j}}$ at those peculiar points of $\mathcal{M}_{n_V}$ which are critical points of V_{BH}.

It is worth here remarking that various possibilities arise when going beyond the assumption of *global strict regular* $g_{i\overline{j}}$, namely:

- (*locally*) *not strictly regular* $g_{i\overline{j}}$, *i.e.* a (*locally*) non-invertible metric tensor, with some strictly positive and some vanishing eigenvalues (rank $<n_V$);

- (*locally*) *non-regular non-degenerate* $g_{i\overline{j}}$, *i.e.* a (*locally*) invertible metric tensor with *pseudo-Euclidean signature*, namely with some strictly positive and some strictly negative eigenvalues (rank $=n_V$);

- (*locally*) *non-regular degenerate* $g_{i\overline{j}}$, *i.e.* a (*locally*) non-invertible metric tensor with some strictly positive, some strictly negative, and some vanishing eigenvalues (rank $<n_V$).

The *local* violation of strict regularity of $g_{i\bar{j}}$ would produce some kind of "phase transition" in the SKG endowing $\mathcal{M}_{n_V}$, corresponding to a breakdown of the 1-dim. effective Lagrangian picture (see [5], [42], and also [18] and [41]) of $d=4$ (extremal) BHs obtained by integrating all massive states of the theory out, unless new massless states appear [5].

The previously mentioned $\mathcal{N}=2$, $d=4$ *central charge function* is defined as

$$\begin{aligned} Z\left(z,\overline{z};q,p\right) &\equiv \widetilde{\Gamma}\Omega V\left(z,\overline{z}\right)=q_{\Lambda}L^{\Lambda}\left(z,\overline{z}\right)-p^{\Lambda}M_{\Lambda}\left(z,\overline{z}\right)=e^{\frac{1}{2}K(z,\overline{z})}\widetilde{\Gamma}\Omega\Pi\left(z\right)= \\ &=e^{\frac{1}{2}K(z,\overline{z})}\left[q_{\Lambda}X^{\Lambda}\left(z\right)-p^{\Lambda}F_{\Lambda}\left(z\right)\right]\equiv e^{\frac{1}{2}K(z,\overline{z})}W\left(z;q,p\right), \end{aligned} \tag{2.2}$$

where Ω is the $(2n_V+2)$-dim. square symplectic metric (subscripts denote dimensions of square sub-blocks)

$$\Omega\equiv\left(\begin{array}{cc} 0_{n_V+1} & -\mathbb{I}_{n_V+1} \\ & \\ \mathbb{I}_{n_V+1} & 0_{n_V+1} \end{array}\right), \tag{2.3}$$

and $V\left(z,\overline{z}\right)$ and $\Pi\left(z\right)$ respectively stand for the $(2n_V+2)\times 1$ covariantly holomorphic (Kähler weights $(1,-1)$) and holomorphic (Kähler weights $(2,0)$) period vectors in symplectic basis:

$$\begin{gathered} \overline{D}_{\overline{i}}V\left(z,\overline{z}\right)=\left(\overline{\partial}_{\overline{i}}-\tfrac{1}{2}\overline{\partial}_{\overline{i}}K\right)V\left(z,\overline{z}\right)=0,\quad D_{i}V\left(z,\overline{z}\right)=\left(\partial_{i}+\tfrac{1}{2}\partial_{i}K\right)V\left(z,\overline{z}\right) \\ \Updownarrow \\ V\left(z,\overline{z}\right)=e^{\frac{1}{2}K(z,\overline{z})}\Pi\left(z\right),\quad \overline{D}_{\overline{i}}\Pi\left(z\right)=\overline{\partial}_{\overline{i}}\Pi\left(z\right)=0,\quad D_{i}\Pi\left(z\right)=\left(\partial_{i}+\partial_{i}K\right)\Pi\left(z\right), \\ \Pi\left(z\right)\equiv\left(\begin{array}{c} X^{\Lambda}\left(z\right) \\ \\ F_{\Lambda}\left(X\left(z\right)\right) \end{array}\right)=exp\left(-\tfrac{1}{2}K\left(z,\overline{z}\right)\right)\left(\begin{array}{c} L^{\Lambda}\left(z,\overline{z}\right) \\ \\ M_{\Lambda}\left(z,\overline{z}\right) \end{array}\right), \end{gathered} \tag{2.4}$$

with $X^{\Lambda}\left(z\right)$ and $F_{\Lambda}\left(X\left(z\right)\right)$ being the holomorphic sections of the $U(1)$ line (Hodge) bundle over $\mathcal{M}_{n_V}$. $W\left(z;q,p\right)$ is the so-called *holomorphic* $\mathcal{N}=2$ *central charge function*, also named $\mathcal{N}=2$ *superpotential*. Up to some particular choices of local symplectic coordinates in $\mathcal{M}_{n_V}$, the covariant symplectic holomorphic sections $F_{\Lambda}\left(X\left(z\right)\right)$ may be seen as derivatives of an *holomorphic prepotential* function F (with Kähler weights $(4,0)$):

$$F_{\Lambda}\left(X\left(z\right)\right)=\frac{\partial F\left(X\left(z\right)\right)}{\partial X^{\Lambda}}. \tag{2.5}$$

In $\mathcal{N}=2$, $d=4$ MESGT the holomorphic function F is constrained to be homogeneous of degree 2 in the contravariant symplectic holomorphic sections $X^{\Lambda}\left(z\right)$, *i.e.* (see [36] and Refs. therein)

$$2F\left(X\left(z\right)\right)=X^{\Lambda}\left(z\right)F_{\Lambda}\left(X\left(z\right)\right). \tag{2.6}$$

The normalization of the holomorphic period vector $\Pi\left(z\right)$ is such that

$$K\left(z,\overline{z}\right)=-ln\left[i\left\langle\Pi\left(z\right),\overline{\Pi}\left(\overline{z}\right)\right\rangle\right]\equiv -ln\left[i\Pi^{T}\left(z\right)\Omega\overline{\Pi}\left(\overline{z}\right)\right]=-ln\left\{i\left[\overline{X}^{\Lambda}\left(\overline{z}\right)F_{\Lambda}\left(z\right)-X^{\Lambda}\left(z\right)\overline{F}_{\Lambda}\left(\overline{z}\right)\right]\right\}, \tag{2.7}$$

where $\langle \cdot, \cdot \rangle$ stands for the symplectic scalar product defined by Ω. Note that under a Kähler transformation $K(z,\overline{z}) \longrightarrow K(z,\overline{z}) + f(z) + \overline{f}(\overline{z})$ ($f(z)$ being a generic holomorphic function), the holomorphic period vector transforms as $\Pi(z) \longrightarrow \Pi(z)\, e^{-f(z)}$, and therefore $X^{\Lambda}(z) \longrightarrow X^{\Lambda}(z)\, e^{-f(z)}$. This means that, at least locally, the contravariant holomorphic symplectic sections $X^{\Lambda}(z)$ can be regarded as a set of homogeneous coordinates on $\mathcal{M}_{n_V}$, provided that the Jacobian complex $n_V \times n_V$ holomorphic matrix

$$e_i^a(z) \equiv \frac{\partial}{\partial z^i}\left(\frac{X^a(z)}{X^0(z)}\right), \quad a = 1, ..., n_V \tag{2.8}$$

is invertible. If this is the case, then one can introduce the local projective symplectic coordinates

$$t^a(z) \equiv \frac{X^a(z)}{X^0(z)}, \tag{2.9}$$

and the SKG of $\mathcal{M}_{n_V}$ turns out to be based on the holomorphic prepotential $\mathcal{F}(t) \equiv (X^0)^{-2} F(X)$. By using the t-coordinates, Eq. (2.7) can be rewritten as follows ($\mathcal{F}_a(t) = \partial_a \mathcal{F}(t)$, $\overline{t}^a = \overline{t^a}$, $\overline{\mathcal{F}}_a(\overline{t}) = \overline{\mathcal{F}_a(t)}$):

$$K(t,\overline{t}) = -ln\left\{ i\left|X^0(z(t))\right|^2 \left[2\left(\mathcal{F}(t) - \overline{\mathcal{F}}(\overline{t})\right) - \left(t^a - \overline{t}^a\right)\left(\mathcal{F}_a(t) + \overline{\mathcal{F}}_a(\overline{t})\right)\right]\right\}. \tag{2.10}$$

By performing a Kähler gauge-fixing with $f(z) = ln(X^0(z))$, yielding that $X^0(z) \longrightarrow 1$, one thus gets

$$K(t,\overline{t})\big|_{X^0(z)\longrightarrow 1} = -ln\left\{ i\left[2\left(\mathcal{F}(t) - \overline{\mathcal{F}}(\overline{t})\right) - \left(t^a - \overline{t}^a\right)\left(\mathcal{F}_a(t) + \overline{\mathcal{F}}_a(\overline{t})\right)\right]\right\}. \tag{2.11}$$

In particular, one can choose the so-called *special coordinates*, i.e. the system of local projective t-coordinates such that

$$e_i^a(z) = \delta_i^a \Leftrightarrow t^a(z) = z^i \left(+c^i,\ c^i \in \mathbb{C}\right). \tag{2.12}$$

Thus, Eq. (2.11) acquires the form

$$K(t,\overline{t})\big|_{X^0(z)\longrightarrow 1, e_i^a(z)=\delta_i^a} = -ln\left\{ i\left[2\left(\mathcal{F}(z) - \overline{\mathcal{F}}(\overline{z})\right) - \left(z^j - \overline{z}^{\overline{j}}\right)\left(\mathcal{F}_j(z) + \overline{\mathcal{F}}_{\overline{j}}(\overline{z})\right)\right]\right\}. \tag{2.13}$$

Moreover, it should be recalled that Z has Kähler weights $(p,\overline{p}) = (1,-1)$, and therefore its Kähler-covariant derivatives read

$$D_i Z = \left(\partial_i + \frac{1}{2}\partial_i K\right) Z, \quad \overline{D}_{\overline{i}} Z = \left(\overline{\partial}_{\overline{i}} - \frac{1}{2}\overline{\partial}_{\overline{i}} K\right) Z. \tag{2.14}$$

The fundamental differential relations of SK geometry are[3] (see *e.g.* [36]):

$$\left\{\begin{array}{l} D_i Z = Z_i; \\ \\ D_i Z_j = iC_{ijk} g^{k\overline{k}} \overline{D}_{\overline{k}} \overline{Z} = iC_{ijk} g^{k\overline{k}} \overline{Z}_{\overline{k}}; \\ \\ D_i \overline{D}_{\overline{j}} \overline{Z} = D_i \overline{Z}_{\overline{j}} = g_{i\overline{j}} \overline{Z}; \\ \\ D_i \overline{Z} = 0, \end{array}\right. \tag{2.15}$$

where the first relation is nothing but the definition of the so-called *matter charges* Z_i, and the fourth relation expresses the Kähler-covariant holomorphicity of Z. C_{ijk} is the rank-3, completely symmetric, covariantly holomorphic tensor of SK geometry (with Kähler weights $(2,-2)$) (see *e.g.*[4] [36, 45, 46]):

$$\left\{\begin{array}{l} C_{ijk} = \langle D_i D_j V, D_k V \rangle = e^K \left(\partial_i \mathcal{N}_{\Lambda\Sigma}\right) D_j X^\Lambda D_k X^\Sigma = \\ \\ = e^K \left(\partial_i X^\Lambda\right) \left(\partial_j X^\Sigma\right) \left(\partial_k X^\Xi\right) \partial_\Xi \partial_\Sigma F_\Lambda (X) \equiv e^K W_{ijk}, \quad \overline{\partial}_{\overline{l}} W_{ijk} = 0; \\ \\ C_{ijk} = D_i D_j D_k \mathcal{S}, \quad \mathcal{S} \equiv -iL^\Lambda L^\Sigma Im \left(F_{\Lambda\Sigma}\right), \quad F_{\Lambda\Sigma} \equiv \frac{\partial F_\Lambda}{\partial X^\Sigma}, F_{\Lambda\Sigma} \equiv F_{(\Lambda\Sigma)} \ ; \\ \\ C_{ijk} = -ig_{i\overline{l}} \overline{f}^{\overline{l}}_\Lambda D_j D_k L^\Lambda, \quad \overline{f}^{\overline{l}}_\Lambda \left(\overline{DL}^\Lambda_{\overline{s}}\right) \equiv \delta^{\overline{l}}_{\overline{s}}; \end{array}\right. \tag{2.16}$$

$$\overline{D}_{\overline{i}} C_{jkl} = 0 \ (\textit{covariant holomorphicity});$$

$$R_{i\overline{j}k\overline{l}} = -g_{i\overline{j}} g_{k\overline{l}} - g_{i\overline{l}} g_{k\overline{j}} + C_{ikp} \overline{C}_{\overline{j}\overline{l}\overline{p}} g^{p\overline{p}} \text{ (usually named } \textit{SKG constraints});$$

$$D_{[i} C_{j]kl} = 0,$$

where the last property is a consequence, through the SKG constraints and the covariant holomorphicity of C_{ijk}, of the Bianchi identities for the Riemann tensor $R_{i\overline{j}k\overline{l}}$, and square brackets denote antisymmetrization with respect to enclosed indices. It is worth recalling that in a generic Kähler geometry $R_{i\overline{j}k\overline{l}}$ reads

$$\begin{array}{l} R_{i\overline{j}k\overline{l}} = g^{m\overline{n}} \left(\overline{\partial}_{\overline{l}} \overline{\partial}_{\overline{j}} \partial_m K\right) \partial_i \overline{\partial}_{\overline{n}} \partial_k K - \overline{\partial}_{\overline{l}} \partial_i \overline{\partial}_{\overline{j}} \partial_k K = g_{k\overline{n}} \partial_i \overline{\Gamma}_{\overline{l}\overline{j}}{}^{\overline{n}} = g_{n\overline{l}} \overline{\partial}_{\overline{j}} \Gamma_{ki}{}^n, \\ \\ \overline{R_{i\overline{j}k\overline{l}}} = R_{j\overline{i}l\overline{k}} \quad (\textit{reality}), \\ \\ \Gamma_{ij}{}^l = -g^{l\overline{l}} \partial_i g_{j\overline{l}} = -g^{l\overline{l}} \partial_i \overline{\partial}_{\overline{l}} \partial_j K = \Gamma_{(ij)}^{\ \ l}, \end{array} \tag{2.17}$$

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[3] Actually, there are different (equivalent) defining approaches to SK geometry. For subtleties and further elucidation concerning such an issue, see *e.g.* [43] and [44].

[4] Notice that the third of Eqs. (2.16) correctly defines the Riemann tensor $R_{i\overline{j}k\overline{l}}$, and it is actual the opposite of the one which may be found in a large part of existing literature. Such a formulation of the so-called *SKG constraints* is well defined, because, as we will mention at the end of Sect. 5, it yields negative values of the constant scalar curvature of ($n_V = $ 1-dim.) homogeneous symmetric compact SK manifolds.

where ${\Gamma_{ij}}^{l}$ stand for the Christoffel symbols of the second kind of the Kähler metric $g_{i\overline{j}}$.

In the first of Eqs. (2.16), a fundamental entity, the so-called kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(z,\overline{z})$ of $\mathcal{N}=2$, $d=4$ MESGT, has been introduced. It is an $(n_V+1)\times(n_V+1)$ complex symmetric, moduli-dependent, Kähler gauge-invariant matrix defined by the following fundamental *Ansätze* of SKG, solving the *SKG constraints* (given by the third of Eqs. (2.16)):

$$M_\Lambda=\mathcal{N}_{\Lambda\Sigma}L^\Sigma,\quad D_iM_\Lambda=\overline{\mathcal{N}}_{\Lambda\Sigma}D_iL^\Sigma. \tag{2.18}$$

By introducing the $(n_V+1)\times(n_V+1)$ complex matrices ($I=1,...,n_V+1$)

$$f_I^\Lambda(z,\overline{z})\equiv\left(\overline{D}_{\overline{i}}\overline{L}^\Lambda(z,\overline{z}),L^\Lambda(z,\overline{z})\right),\quad h_{I\Lambda}(z,\overline{z})\equiv\left(\overline{D}_{\overline{i}}\overline{M}_\Lambda(z,\overline{z}),M_\Lambda(z,\overline{z})\right), \tag{2.19}$$

the *Ansätze* (2.18) univoquely determine $\mathcal{N}_{\Lambda\Sigma}(z,\overline{z})$ as

$$\mathcal{N}_{\Lambda\Sigma}(z,\overline{z})=h_{I\Lambda}(z,\overline{z})\circ\left(f^{-1}\right)_\Sigma^I(z,\overline{z}), \tag{2.20}$$

where $\circ$ denotes the usual matrix product, and $(f^{-1})_\Sigma^I f_I^\Lambda=\delta_\Sigma^\Lambda$, $(f^{-1})_\Lambda^I f_J^\Lambda=\delta_J^I$.

The covariantly holomorphic $(2n_V+2)\times 1$ period vector $V(z,\overline{z})$ is *symplectically orthogonal* to all its Kähler-covariant derivatives:

$$\left\{\begin{array}{l}\langle V(z,\overline{z}),D_iV(z,\overline{z})\rangle=0;\\[2mm] \left\langle V(z,\overline{z}),\overline{D}_{\overline{i}}V(z,\overline{z})\right\rangle=0;\\[2mm] \left\langle V(z,\overline{z}),D_i\overline{V}(z,\overline{z})\right\rangle=0;\\[2mm] \left\langle V(z,\overline{z}),\overline{D}_{\overline{i}}\overline{V}(z,\overline{z})\right\rangle=0.\end{array}\right. \tag{2.21}$$

Morover, it holds that

$$\begin{array}{l}g_{i\overline{j}}(z,\overline{z})=-i\left\langle D_iV(z,\overline{z}),\overline{D}_{\overline{j}}\overline{V}(z,\overline{z})\right\rangle=\\[2mm] =-2Im\left(\mathcal{N}_{\Lambda\Sigma}(z,\overline{z})\right)D_iL^\Lambda(z,\overline{z})\overline{D}_{\overline{i}}\overline{L}^\Sigma(z,\overline{z})=2Im\left(F_{\Lambda\Sigma}(z)\right)D_iL^\Lambda(z,\overline{z})\overline{D}_{\overline{i}}\overline{L}^\Sigma(z,\overline{z});\end{array} \tag{2.22}$$

$$\left\langle V(z,\overline{z}),D_i\overline{D}_{\overline{j}}V(z,\overline{z})\right\rangle=iC_{ijk}g^{k\overline{k}}\left\langle V(z,\overline{z}),\overline{D}_{\overline{k}}\overline{V}(z,\overline{z})\right\rangle=0. \tag{2.23}$$

The fundamental $(2n_V+2)\times 1$ vector identity defining the geometric structure of SK manifolds read as follows [47, 9, 14, 17, 18, 26]:

$$\widetilde{\Gamma}^T-i\Omega\mathcal{M}(\mathcal{N})\widetilde{\Gamma}^T=-2iZ\overline{V}-2ig^{j\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_jV. \tag{2.24}$$

The $(2n_V+2)\times(2n_V+2)$ real symmetric matrix $\mathcal{M}(\mathcal{N})$ is defined as [36, 3, 4]

$$\begin{array}{rcl}\mathcal{M}(\mathcal{N}) & = & \mathcal{M}\left(Re(\mathcal{N}),Im(\mathcal{N})\right)\equiv \qquad (2.25)\\ & \equiv & \left(\begin{array}{cc}Im(\mathcal{N})+Re(\mathcal{N})(Im(\mathcal{N}))^{-1}Re(\mathcal{N}) & -Re(\mathcal{N})(Im(\mathcal{N}))^{-1}\\ -(Im(\mathcal{N}))^{-1}Re(\mathcal{N}) & (Im(\mathcal{N}))^{-1}\end{array}\right),\end{array}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is a complex symmetric matrix playing a key role in $\mathcal{N}=2$, $d=4$ MESGT (see *e.g.* the report [36]). It is worth reminding that $\mathcal{M}(\mathcal{N})$ is symplectic with respect to the metric Ω defined in Eq. (2.3), *i.e.* it satisfies $\left(\left(\mathcal{M}(\mathcal{N})\right)^T=\mathcal{M}(\mathcal{N})\right)$

$$\mathcal{M}(\mathcal{N})\,\Omega\mathcal{M}(\mathcal{N})=\Omega. \tag{2.26}$$

By using Eqs. (2.7), (2.21), (2.22) and (2.23), the identity (2.24) implies the following relations:

$$\left\{\begin{array}{l}\left\langle V,\widetilde{\Gamma}^T-i\Omega\mathcal{M}(\mathcal{N})\,\widetilde{\Gamma}^T\right\rangle=-2Z;\\[2ex]\left\langle \overline{V},\widetilde{\Gamma}^T-i\Omega\mathcal{M}(\mathcal{N})\,\widetilde{\Gamma}^T\right\rangle=0;\\[2ex]\left\langle D_iV,\widetilde{\Gamma}^T-i\Omega\mathcal{M}(\mathcal{N})\,\widetilde{\Gamma}^T\right\rangle=0;\\[2ex]\left\langle \overline{D}_{\overline{i}}\overline{V},\widetilde{\Gamma}^T-i\Omega\mathcal{M}(\mathcal{N})\,\widetilde{\Gamma}^T\right\rangle=-2\overline{D}_{\overline{i}}\overline{Z}.\end{array}\right. \tag{2.27}$$

There are only $2n_V$ independent real relations out of the $4n_V+4$ real ones yielded by the $2n_V+2$ complex identities (2.24). Indeed, by taking the real and imaginary part of the SKG vector identity (2.24) one respectively obtains

$$\widetilde{\Gamma}^T=-2Re\left[iZ\overline{V}+iG^{j\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_jV\right]=-2Im\left[\overline{Z}V+G^{j\overline{j}}\left(D_jZ\right)\left(\overline{D}_{\overline{j}}\overline{V}\right)\right]; \tag{2.28}$$

$$\Omega\mathcal{M}(\mathcal{N})\,\widetilde{\Gamma}^T=2Im\left[iZ\overline{V}+iG^{j\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_jV\right]=2Re\left[\overline{Z}V+G^{j\overline{j}}\left(D_jZ\right)\left(\overline{D}_{\overline{j}}\overline{V}\right)\right]. \tag{2.29}$$

Consequently, the imaginary and real parts of the SKG vector identity (2.24) are *linearly dependent* one from the other, being related by the $(2n_V+2)\times(2n_V+2)$ real matrix

$$\Omega\mathcal{M}(\mathcal{N})=\left(\begin{array}{cc}\left(Im(\mathcal{N})\right)^{-1}Re(\mathcal{N}) & -\left(Im(\mathcal{N})\right)^{-1}\\ Im(\mathcal{N})+Re(\mathcal{N})\left(Im(\mathcal{N})\right)^{-1}Re(\mathcal{N}) & -Re(\mathcal{N})\left(Im(\mathcal{N})\right)^{-1}\end{array}\right). \tag{2.30}$$

Put another way, Eqs. (2.28) and (2.29) yield

$$Re\left[Z\overline{V}+G^{j\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_jV\right]=\Omega\mathcal{M}(\mathcal{N})\,Im\left[Z\overline{V}+G^{j\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_jV\right], \tag{2.31}$$

expressing the fact that the real and imaginary parts of the quantity $Z\overline{V}+G^{j\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_jV$ are simply related through a *symplectic rotation* given by the matrix $\Omega\mathcal{M}(\mathcal{N})$, whose simplecticity directly follows from the symplectic nature of $\mathcal{M}(\mathcal{N})$. Eq. (2.31) reduces the number of independent real relations implied by the identity (2.24) from $4n_V+4$ to $2n_V+2$.

Moreover, it should be stressed that vector identity (2.24) entails 2 *redundant* degrees of freedom, encoded in the homogeneity (of degree 1) of (2.24) under complex rescalings of $\widetilde{\Gamma}$. Indeed, by using the definition (2.2), it is easy to check that the right-hand side of (2.24) gets rescaled by an overall factor λ under the following transformation on $\widetilde{\Gamma}$:

$$\widetilde{\Gamma}\longrightarrow\lambda\widetilde{\Gamma},\quad\lambda\in\mathbb{C}. \tag{2.32}$$

Thus, as announced, only $2n_V$ real independent relations are actually yielded by the vector identity (2.24).

This is clearly consistent with the fact that the $2n_V+2$ complex identities (2.24) express nothing but a *change of basis* of the BH charge configurations, between the Kähler-invariant $1\times(2n_V+2)$ symplectic (magnetic/electric) basis vector $\widetilde{\Gamma}$ defined by Eq. (1.1) and the complex, moduli-dependent $1\times(n_V+1)$ *matter charges* vector (with Kähler weights $(1,-1)$)

$$\mathcal{Z}(z,\overline{z})\equiv\left(Z(z,\overline{z}),Z_i(z,\overline{z})\right)_{i=1,\dots,n_V}. \tag{2.33}$$

It should be recalled that the BH charges are conserved due to the overall $(U(1))^{n_V+1}$ gauge-invariance of the system under consideration, and $\widetilde{\Gamma}$ and $\mathcal{Z}(z,\overline{z})$ are two *equivalent* basis for them. Their very equivalence relations are given by the SKG identities (2.24) themselves. By its very definition (1.1), $\widetilde{\Gamma}$ is *moduli-independent* (at least in a stationary, spherically symmetric and asymptotically flat extremal BH background, as it is the case being treated here), whereas Z is *moduli-dependent*, since it refers to the eigenstates of the $\mathcal{N}=2$, $d=4$ supergravity multiplet and of the n_V Maxwell vector supermultiplets.

3 Supersymmetric Attractors

The "effective BH potential" of $\mathcal{N}=2$, $d=4$ MESGT has the following expression [3, 4, 36] :

$$V_{BH}(z,\overline{z};q,p)=|Z|^2(z,\overline{z};q,p)+g^{j\overline{j}}(z,\overline{z})D_jZ(z,\overline{z};q,p)\overline{D}_{\overline{j}}\overline{Z}(z,\overline{z};q,p). \tag{3.1}$$

An elegant way to obtain V_{BH} is given by left-multiplying the SKG vector identity (2.24) by the $1\times(2n_V+2)$ complex moduli-dependent vector $-\frac{1}{2}\widetilde{\Gamma}\mathcal{M}(\mathcal{N})$; due to the symplecticity of the matrix $\mathcal{M}(\mathcal{N})$, one obtains [3, 4, 36]

$$V_{BH}(z,\overline{z};q,p)=-\frac{1}{2}\widetilde{\Gamma}\mathcal{M}(\mathcal{N})\widetilde{\Gamma}^T. \tag{3.2}$$

Thus, V_{BH} is identified with the first (of two), lowest-order (-quadratic- in charges), positive-definite real invariant I_1 of SK geometry (see *e.g.* [26, 36]). It is worth noticing that the result (3.2) can also be derived from the SK geometry identities (2.24) by using the relation (see [21], where a generalization for $\mathcal{N}>2$-extended supergravities is also given)

$$\frac{1}{2}(\mathcal{M}(\mathcal{N})+i\Omega)\mathcal{V}=i\Omega\mathcal{V}\Leftrightarrow\mathcal{M}(\mathcal{N})\mathcal{V}=i\Omega\mathcal{V}, \tag{3.3}$$

where $\mathcal{V}$ is a $(2n_V+2)\times(n_V+1)$ matrix defined as:

$$\mathcal{V}\equiv\left(V,\overline{D}_{\overline{1}}\overline{V},\dots,\overline{D}_{\overline{n_V}}\overline{V}\right). \tag{3.4}$$

By differentiating Eq. (3.1) with respect to the scalars, it is easy to check that the general criticality conditions (1.5) acquire the peculiar form [5]

$$D_i V_{BH} = \partial_i V_{BH} = 0 \Leftrightarrow 2\overline{Z} D_i Z + g^{j\overline{j}} \left(D_i D_j Z \right) \overline{D}_{\overline{j}} \overline{Z} = 0; \tag{3.5}$$

this is what one should rigorously call the $\mathcal{N} = 2$, $d = 4$ MESGT attractor Eqs. (AEs). By means of the features of SKG given by Eqs. (2.15), the $\mathcal{N} = 2$ AEs (3.5) can be re-expressed as follows [5]:

$$2\overline{Z} Z_i + i C_{ijk} g^{j\overline{j}} g^{k\overline{k}} \overline{Z}_{\overline{j}} \overline{Z}_{\overline{k}} = 0. \tag{3.6}$$

It is evident that the tensor C_{ijk} is crucial in relating the $\mathcal{N} = 2$ central charge function Z (*graviphoton charge*) and the n_V *matter charges* Z_i (coming from the n_V Abelian vector supermultiplets) at the critical points of V_{BH} in the SK scalar manifold $\mathcal{M}_{n_V}$.

The static, spherically symmetric, asymptotically flat BHs are known to be described by an effective $d = 1$ Lagrangian ([5], [42], and also [18] and [41]), with an effective scalar potential and effective fermionic "mass terms" terms controlled by the vector $\widetilde{\Gamma}$ of the field-strength fluxes (defined by Eq. (1.1)). The *"apparent" gravitino mass* is given by Z, whereas the *gaugino mass matrix* Λ_{ij} reads (see the second Ref. of [46])

$$\Lambda_{ij} = C_{ijk} g^{k\overline{k}} \overline{Z}_{\overline{k}}. \tag{3.7}$$

The *supersymmetry breaking order parameters*, related to the mixed gravitino-gaugino couplings, are nothing but the *matter charge(function)s* $D_i Z = Z_i$ (see the first of Eqs. (2.15)).

As evident from the AEs (3.5) and (3.6), the conditions

$$(Z \neq 0,)\ D_i Z = 0 \quad \forall i = 1, ..., n_V \tag{3.8}$$

determine a (*non-degenerate*) critical point of V_{BH}, namely a $\frac{1}{2}$-BPS critical point, which preserve 4 supersymmetry degrees of freedom out of the 8 pertaining to the $\mathcal{N} = 2$, $d = 4$ Poincarè superalgebra related to the asymptotical Minkowski background. The horizon ADM squared mass at $\frac{1}{2}$-BPS critical points of V_{BH} saturates the BPS bound, reading [1]-[5]:

$$M^2_{ADM,H,\frac{1}{2}-BPS} = V_{BH}|_{\frac{1}{2}-BPS} = |Z|^2_{\frac{1}{2}-BPS} + \left[g^{i\overline{i}} \left(D_i Z \right) \left(\overline{D}_{\overline{i}} \overline{Z} \right) \right]_{\frac{1}{2}-BPS} = |Z|^2_{\frac{1}{2}-BPS} > 0. \tag{3.9}$$

In general, $\frac{1}{2}$-BPS critical points are (at least local) minima of V_{BH} in $\mathcal{M}_{n_V}$, and therefore they are stable; thus, they are *attractors* in strict sense. Indeed, the $2n_V \times 2n_V$ (covariant) Hessian matrix (in $(z, \overline{z})$-coordinates) of V_{BH} evaluated at such points is strictly positive-definite [5] :

$$\begin{aligned} &\left(D_i D_j V_{BH} \right)_{\frac{1}{2}-BPS} = \left(\partial_i \partial_j V_{BH} \right)_{\frac{1}{2}-BPS} = 0, \\ &\left(D_i \overline{D}_{\overline{j}} V_{BH} \right)_{\frac{1}{2}-BPS} = \left(\partial_i \overline{\partial}_{\overline{j}} V_{BH} \right)_{\frac{1}{2}-BPS} = 2 \left(g_{i\overline{j}} V_{BH} \right)_{\frac{1}{2}-BPS} = 2\ g_{i\overline{j}}\big|_{\frac{1}{2}-BPS} |Z|^2_{\frac{1}{2}-BPS} > 0, \end{aligned} \tag{3.10}$$

where here and below the notation "> 0" ("< 0") is understood as strict positive-(negative-)definiteness. Eqs. (3.10) yield that the Hermiticity and (strict) positive-definiteness of the (covariant) Hessian matrix (in $(z, \overline{z})$-coordinates) of V_{BH} at the $\frac{1}{2}$-BPS critical points are due to the Hermiticity and - assumed - (strict) positive-definiteness (actually holding globally) of the metric $g_{i\overline{j}}$ of $\mathcal{M}_{n_V}$.

Considering the $\mathcal{N} = 2$, $d = 4$ MESGT Lagrangian in a static, spherically symmetric, asymptotically flat BH background, and denoting by ψ and λ^i respectively the gravitino and gaugino fields, it is easy to see that such a Lagrangian contains terms of the form (see the second and third Refs. of [46])

$$
\begin{aligned}
&Z\psi\psi; \\
&C_{ijk} g^{k\overline{k}} \left(\overline{D}_{\overline{k}}\overline{Z}\right) \lambda^i \lambda^j; \\
&(D_i Z)\, \lambda^i \psi.
\end{aligned}
\tag{3.11}
$$

Thus, the ($\frac{1}{2}$)-BPS conditions (3.8) implies the gaugino mass term and the $\lambda\psi$ term to vanish at the $\frac{1}{2}$-BPS critical points of V_{BH} in $\mathcal{M}_{n_V}$. It is interesting to remark that the gravitino "apparent mass" term $Z\psi\psi$ is in general non-vanishing, also when evaluated at the considered $\frac{1}{2}$-BPS attractors; this is ultimately a consequence of the fact that the extremal BH horizon geometry at the $\frac{1}{2}$-BPS (as well as at the non-BPS) attractors is Bertotti-Robinson $AdS_2 \times S^2$ [48, 49, 50].

4 Non-BPS Critical Points of V_{BH} with $Z \neq 0$

It is here worth recalling once again that what we call extremal BH *attractor* in (asymptotically flat) $\mathcal{N} = 2$, $d = 4$ MESGT is, strictly speaking, a configuration of the scalar fluctuations which is a(n at least local) minimum for the "effective BH potential" V_{BH} (as also pointed out in [7]), seen as a positive-definite, real function in the SK scalar manifold $\mathcal{M}_{n_V}$. Put another way, an extremal BH attractor (horizon) scalar configuration satisfies the AEs (3.5) or (3.6), and it is furthermore constrained by the condition of positive-definiteness of the Hessian matrix of V_{BH}, shorthand denoted as

$$
(\partial_i \partial_j V_{BH})_{\partial V_{BH}=0} > 0. \tag{4.1}
$$

Obviously, the $\frac{1}{2}$-BPS conditions (3.8) are not the most general ones satisfying the AEs (3.5) or (3.6). For instance, one might consider critical points of V_{BH} (thus satisfying the AEs (3.5) or (3.6)) characterized by

$$
\begin{cases}
D_i Z \neq 0, \text{ for } \textit{(at least one) } i, \\
Z \neq 0.
\end{cases}
\tag{4.2}
$$

Such critical points are *non-supersymmetric* ones (*i.e.* they do *not* preserve any of the 8 supersymmetry degrees of freedom of the asymptotical Minkowski background), and they correspond to an extremal, non-BPS BH background. They are commonly named *non-BPS* $Z \neq 0$ *critical points of* V_{BH}. We will devote the present Sect. (and, after a general treatment, also next Sect. 5) to present their main features.

The horizon ADM squared mass corresponding to non-BPS $Z \neq 0$ critical points of V_{BH} does *not* saturate the BPS bound ([9], [14], [16]):

$$\begin{aligned} & M^2_{ADM,H,non-BPS,Z\neq 0} = V_{BH}|_{non-BPS,Z\neq 0} = \\ & = |Z|^2_{non-BPS,Z\neq 0} + \left[g^{i\bar{i}} \left(D_i Z \right) \left(\overline{D}_{\bar{i}} \overline{Z} \right) \right]_{non-BPS,Z\neq 0} > |Z|^2_{non-BPS,Z\neq 0} . \end{aligned} \tag{4.3}$$

As implied by AEs (3.6), if at non-BPS $Z \neq 0$ critical points it holds that $D_i Z \neq 0$ for at least one index i and $Z \neq 0$, then

$$\left(C_{ijk} \right)_{non-BPS,Z\neq 0} \neq 0, \quad \text{for some } (i,j,k) \in \{1, ..., n_V\}^3 , \tag{4.4}$$

i.e. the SKG rank-3 symmetric tensor will for sure have some non-vanishing components in order for criticality conditions (3.6) to be satisfied at non-BPS $Z \neq 0$ critical points.

Moreover, the general criticality conditions (3.5) for V_{BH} can be recognized to be the general Ward identities relating the gravitino mass Z, the gaugino masses $D_i D_j Z$ and the supersymmetry-breaking order parameters $D_i Z$ in a generic spontaneously broken supergravity theory [51]. Indeed, away from $\frac{1}{2}$-BPS critical points (*i.e.* for $D_i Z \neq 0$ for some i), the AEs (3.5) can be re-expressed as follows:

$$\left(\mathbf{M}_{ij} h^j \right)_{\partial V_{BH}=0} = 0, \tag{4.5}$$

with

$$\mathbf{M}_{ij} \equiv D_i D_j Z + 2 \frac{\overline{Z}}{\left[g^{k\bar{k}} \left(D_k Z \right) \left(\overline{D}_{\bar{k}} \overline{Z} \right) \right]} \left(D_i Z \right) \left(D_j Z \right), \ (\text{Kähler weights } (1,-1)), \tag{4.6}$$

and

$$h^j \equiv g^{j\bar{j}} \overline{D}_{\bar{j}} \overline{Z}, \ (\text{Kähler weights } (-1,1)). \tag{4.7}$$

For a non-vanishing contravariant vector h^j (*i.e.* away from $\frac{1}{2}$-BPS critical points, as pointed out above), Eq. (4.5) admits a solution iff the $n_V \times n_V$ complex symmetric matrix $\mathbf{M}_{ij}$ has vanishing determinant (implying that it has at most $n_V - 1$ non-vanishing eigenvalues) at the considered (non-BPS) critical points of V_{BH} (however, notice that $\mathbf{M}_{ij}$ is symmetric but not necessarily Hermitian, thus in general its eigenvalues are not necessarily real).

$n_V = 1$ SKG represents a noteworthy case, in which major simplifications occur.

Indeed, in the one-modulus case the condition of vanishing determinant trivially reads ($z^1 \equiv z$)

$$\mathbf{M}_{11} = 0, \tag{4.8}$$

and (away from $\frac{1}{2}$-BPS critical points, *i.e.* for $D_z Z \neq 0$) it is equivalent to the criticality condition $\partial_z V_{BH} = 0$. $n_V = 1$ AEs (3.6) consist of the unique complex Eq.

$$\partial_z V_{BH} = 0 \Leftrightarrow 2\overline{Z} D_z Z + iCg^{-2} \left(\overline{D}_{\overline{z}} \overline{Z} \right)^2 = 0, \tag{4.9}$$

where we defined $C_{111} \equiv C(z, \overline{z}) \in \mathbb{C}$ and $g_{1\overline{1}} \equiv g(z, \overline{z}) \in \mathbb{R}_0^+$. From the treatment given above, it necessarily holds that

$$\begin{cases} C_{non-BPS,Z\neq 0} \neq 0, \\ \\ |D_z Z|^2_{non-BPS,Z\neq 0} = 4 \left[g^4 \frac{|Z|^2}{|C|^2} \right]_{non-BPS,Z\neq 0} > 0. \end{cases} \tag{4.10}$$

Consequently, the horizon ADM squared mass at non-BPS $Z \neq 0$ critical points of V_{BH} in $n_V = 1$ SKG reads

$$\begin{gathered} M^2_{ADM,H,non-BPS,Z\neq 0} = V_{BH}|_{non-BPS,Z\neq 0} = |Z|^2_{non-BPS,Z\neq 0} + g^{-1} |D_z Z|^2_{non-BPS,Z\neq 0} = \\ = |Z|^2_{non-BPS,Z\neq 0} \left[1 + 4 \left(\frac{g^3}{|C|^2} \right)_{non-BPS,Z\neq 0} \right] > |Z|^2_{non-BPS,Z\neq 0} . \end{gathered} \tag{4.11}$$

Eq. (4.11) yields an interesting feature of non-BPS $Z \neq 0$ critical points of V_{BH} in $n_V = 1$ SKG: the entropy $S_{BH,non-BPS,Z\neq 0} = \pi \left. V_{BH} \right|_{non-BPS,Z\neq 0}$ is multiplicatively (and increasingly) "renormalized" (with respect to its formal expression in the $\frac{1}{2}$-BPS case - see Eq. (3.9) -) as follows:

$$S_{BH,non-BPS,Z\neq 0} = \pi\gamma \, |Z|^2_{non-BPS,Z\neq 0} , \tag{4.12}$$

with

$$\gamma - 1 \equiv 4 \left(\frac{g^3}{|C|^2} \right)_{non-BPS,Z\neq 0} > 0. \tag{4.13}$$

Now, let us introduce the so-called non-BPS $Z \neq 0$ *scalar "supersymmetry breaking order parameter"* as

$$\mathcal{O}_{non-BPS,Z\neq 0} \equiv \frac{\left(g^{i\overline{j}} D_i Z \overline{D}_{\overline{j}} \overline{Z} \right)_{non-BPS,Z\neq 0}}{|Z|^2_{non-BPS,Z\neq 0}} > 0, \tag{4.14}$$

the strict positivity bound directly coming from the assumed (global) strict positive definiteness of the metric $g_{i\overline{j}}$ of $\mathcal{M}_{n_V}$. The actual independence of $\mathcal{O}_{non-BPS,Z\neq 0}$ on $|Z|^2_{non-BPS,Z\neq 0}$ determines the multiplicative (and increasing) "renormalization" of $S_{BH,non-BPS,Z\neq 0}$ to occur. Nevertheless, the definition (4.14) clearly holds $\forall n \in \mathbb{N}$, also when no multiplicative "renormalization" takes place.

It is immediate to conclude that $\gamma - 1$ can be identified with $\mathcal{O}_{non-BPS,Z\neq 0}$ in the $n_V = 1$ case:

$$\gamma - 1 = \mathcal{O}_{non-BPS,Z\neq 0,n_V=1} \equiv \frac{g^{-1} |D_z Z|^2_{non-BPS,Z\neq 0}}{|Z|^2_{non-BPS,Z\neq 0}} > 0. \tag{4.15}$$

Apriori, Eqs. (4.13)-(4.15) do depend on the particular non-BPS $Z \neq 0$ critical point of V_{BH} being considered, *i.e.* they are dependent on the particular set of BH charges at hand, chosen among the BH charge configurations supporting non-BPS $Z \neq 0$ critical points of V_{BH}. Put another way, one would *apriori* conclude that $\mathcal{O}_{non-BPS,Z\neq 0}$ changes its value depending on which configuration of BH charges is chosen among the ones supposrting non-BPS $Z \neq 0$ critical points of V_{BH} in $\mathcal{M}_{n_V}$ ($dim_{\mathbb{C}}\mathcal{M}_{n_V} = n_V$).

This is not the case for homogeneous symmetric and non-symmetric SKGs, as respectively computed in [24] and [34]. For such SKGs $\gamma = 4$ regardless of the peculiar non-BPS $Z \neq 0$ critical point of V_{BH} being considered. As claimed in [10], $\gamma = 4$ seemingly holds true for every non-BPS $Z \neq 0$ critical point of V_{BH} in generic n_V-dim. cubic (not necessarily symmetric, nor homogeneous) SKG.

The strict positivity of $\mathcal{O}_{non-BPS,Z\neq 0}$ (and the subsequent increasing nature of the multiplicative "renormalization" of $S_{BH,non-BPS,Z\neq 0}$ with respect to the formal expression of $S_{BH,\frac{1}{2}-BPS}$, when it actually occurs) yields that (at least *formally*, and in the considered framework) the $\frac{1}{2}$-BPS and non-BPS $Z \neq 0$ species of critical points of V_{BH} are *"discretely disjoint"* one from the other.

5 Stability of non-BPS Critical Points of V_{BH}

In order to decide whether a critical point of V_{BH} is an attractor in strict sense, one has to consider the following condition:

$$H_{\mathbb{R}}^{V_{BH}} \equiv H_{ab}^{V_{BH}} \equiv D_a D_b V_{BH} > 0 \quad \text{at} \quad D_c V_{BH} = \frac{\partial V_{BH}}{\partial \phi^c} = 0 \quad \forall c = 1, ..., 2n_V, \tag{5.1}$$

i.e. the condition of (strict) positive-definiteness of the real $2n_V \times 2n_V$ Hessian matrix $H_{\mathbb{R}}^{V_{BH}} \equiv H_{ab}^{V_{BH}}$ of V_{BH} (which is nothing but the squared mass matrix of the moduli) at the critical points of V_{BH}, expressed in the real parameterization through the ϕ-coordinates. Since V_{BH} is positive-definite, a stable critical point (namely, an attractor in strict sense) is necessarily a(n at least local) minimum, and therefore it fulfills the condition (5.1).

In general, $H_{\mathbb{R}}^{V_{BH}}$ may be block-decomposed in $n_V \times n_V$ real matrices:

$$H_{\mathbb{R}}^{V_{BH}} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{B} \end{pmatrix}, \tag{5.2}$$

with $\mathcal{A}$ and $\mathcal{B}$ being $n_V \times n_V$ real symmetric matrices:

$$\mathcal{A}^T = \mathcal{A},\ \mathcal{B}^T = \mathcal{B} \Leftrightarrow \left(H_{\mathbb{R}}^{V_{BH}}\right)^T = H_{\mathbb{R}}^{V_{BH}}. \tag{5.3}$$

In the local complex $(z, \overline{z})$-parameterization, the $2n_V \times 2n_V$ Hessian matrix of V_{BH} reads

$$H_{\mathbb{C}}^{V_{BH}} \equiv H_{\widehat{ij}}^{V_{BH}} \equiv \begin{pmatrix} D_i D_j V_{BH} & D_i \overline{D}_{\overline{j}} V_{BH} \\ D_j \overline{D}_{\overline{i}} V_{BH} & \overline{D}_{\overline{i}} \overline{D}_{\overline{j}} V_{BH} \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{ij} & \mathcal{N}_{i\overline{j}} \\ \overline{\mathcal{N}_{i\overline{j}}} & \overline{\mathcal{M}_{ij}} \end{pmatrix}, \tag{5.4}$$

where the hatted indices $\hat{\imath}$ and $\hat{\jmath}$ may be holomorphic or antiholomorphic. $H_{\mathbb{C}}^{V_{BH}}$ is the matrix actually computable in the SKG formalism presented in Sect. 2 (see below, Eqs. (5.5) and (5.6)). Let us here recall that the invertible unitary Cayley transformation (2.1) expresses the change between the Riemannian $2n_V$-dim. ϕ-parameterization of $\mathcal{M}_{n_V}$ and the Kähler n_V-dim. holomorphic/antiholomorphic $(z,\overline{z})$-parameterization of $\mathcal{M}_{n_V}$, used in previous Sects..

As pointed out above, for SKGs having a globally strict positive-definite metric tensor $g_{i\overline{j}}$ the condition (5.1) is automatically satisfied at the $\frac{1}{2}$-BPS critical points of V_{BH} (defined by Eq. (3.8)). On the other hand, non-BPS $Z \neq 0$ critical points of V_{BH} does not automatically fulfill the condition (5.1), and a more detailed analysis [24, 18] is needed.

Using the properties of SKG, one obtains:

$$\begin{aligned} &\mathcal{M}_{ij} \equiv D_i D_j V_{BH} = D_j D_i V_{BH} = \\ &= 4i\overline{Z}C_{ijk}g^{k\overline{k}}\left(\overline{D}_{\overline{k}}\overline{Z}\right) + i\left(D_j C_{ikl}\right)g^{k\overline{k}}g^{l\overline{l}}\left(\overline{D}_{\overline{k}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right); \end{aligned} \tag{5.5}$$

$$\begin{aligned} &\mathcal{N}_{i\overline{j}} \equiv D_i\overline{D}_{\overline{j}}V_{BH} = \overline{D}_{\overline{j}}D_i V_{BH} = \\ &= 2\left[g_{i\overline{j}}\left|Z\right|^2 + \left(D_i Z\right)\left(\overline{D}_{\overline{j}}\overline{Z}\right) + g^{l\overline{n}}C_{ikl}\overline{C}_{\overline{j}\overline{m}\overline{n}}g^{k\overline{k}}g^{m\overline{m}}\left(\overline{D}_{\overline{k}}\overline{Z}\right)\left(D_m Z\right)\right]. \end{aligned} \tag{5.6}$$

Here we limit ourselves to point out that further noteworthy elaborations of $\mathcal{M}_{ij}$ and $\mathcal{N}_{i\overline{j}}$ can be performed in homogeneous symmetric SK manifolds, where $D_j C_{ikl} = 0$ globally [24], and that the Kähler-invariant $(2,2)$-tensor $g^{l\overline{n}}C_{ikl}\overline{C}_{\overline{j}\overline{m}\overline{n}}$ can be rewritten in terms of the Riemann-Christoffel tensor $R_{i\overline{j}k\overline{m}}$ by using the so-called "SKG constraints" (see the third of Eqs. (2.16)) [18]. Moreover, the differential Bianchi identities for $R_{i\overline{j}k\overline{m}}$ imply $\mathcal{M}_{ij}$ to be symmetric (see comment below Eqs. (2.16)).

Thus, one gets the following global properties:

$$\mathcal{M}^T = \mathcal{M}, \quad \mathcal{N}^\dagger = \mathcal{N} \Leftrightarrow \left(H_{\mathbb{C}}^{V_{BH}}\right)^T = H_{\mathbb{C}}^{V_{BH}}, \tag{5.7}$$

implying that

$$\left(H_{\mathbb{C}}^{V_{BH}}\right)^\dagger = H_{\mathbb{C}}^{V_{BH}} \Leftrightarrow \mathcal{M}^\dagger = \mathcal{M}, \quad \mathcal{N}^T = \mathcal{N} \Leftrightarrow \overline{\mathcal{M}} = \mathcal{M}, \quad \overline{\mathcal{N}} = \mathcal{N}. \tag{5.8}$$

It should be stressed clearly that the symmetry but non-Hermiticity of $H_{\mathbb{C}}^{V_{BH}}$ actually does not matter, because what one is ineterested in are the eigenvalues of the real form $H_{\mathbb{R}}^{V_{BH}}$, which is real and symmetric, and therefore admitting $2n_V$ *real* eigenvalues.

In order to relate $H_{\mathbb{R}}^{V_{BH}}$ expressed by Eq. (5.2) with $H_{\mathbb{C}}^{V_{BH}}$ given by Eq. (5.4), we exploit the invertible unitary Cayley transformation (2.1), obtaining the following relations between the $n_V \times n_V$ sub-blocks of $H_{\mathbb{R}}^{V_{BH}}$ and $H_{\mathbb{C}}^{V_{BH}}$:

$$\left\{ \begin{array}{l} \mathcal{M} = \frac{1}{2}\left(\mathcal{A} - \mathcal{B}\right) + \frac{i}{2}\left(\mathcal{C} + \mathcal{C}^T\right); \\ \\ \mathcal{N} = \frac{1}{2}\left(\mathcal{A} + \mathcal{B}\right) + \frac{i}{2}\left(\mathcal{C}^T - \mathcal{C}\right), \end{array} \right. \tag{5.9}$$

or its inverse

$$\begin{cases} \mathcal{A} = Re\mathcal{M} + Re\mathcal{N}; \\ \\ \mathcal{B} = Re\mathcal{N} - Re\mathcal{M}; \\ \\ \mathcal{C} = Im\mathcal{M} - Im\mathcal{N}. \end{cases} \tag{5.10}$$

The matrix action of the invertible Cayley unitary transformation (2.1) may be encoded in a matrix $\mathcal{U} \in U(2n_V)$ ($\Leftrightarrow \mathcal{U}^{-1} = \mathcal{U}^{\dagger}$):

$$H_{\mathbb{R}}^{V_{BH}} = \mathcal{U}^{-1} H_{\mathbb{C}}^{V_{BH}} \left(\mathcal{U}^{T}\right)^{-1}, \tag{5.11}$$

or equivalently

$$H_{\mathbb{C}}^{V_{BH}} = \mathcal{U} H_{\mathbb{R}}^{V_{BH}} \mathcal{U}^{T}. \tag{5.12}$$

The structure of the Hessian matrix gets simplified at the critical points of V_{BH}, because the covariant derivatives may be substituted by the flat ones; the critical Hessian matrices in complex holomorphic/antiholomorphic and real local parameterizations respectively read

$$H_{\mathbb{C}}^{V_{BH}}\Big|_{\partial V_{BH}=0} \equiv \left(\begin{array}{cc} \partial_i \partial_j V_{BH} & \partial_i \overline{\partial}_{\overline{j}} V_{BH} \\ \\ \partial_j \overline{\partial}_{\overline{i}} V_{BH} & \overline{\partial}_{\overline{i}} \overline{\partial}_{\overline{j}} V_{BH} \end{array} \right)_{\partial V_{BH}=0} = \left(\begin{array}{cc} \mathcal{M} & \mathcal{N} \\ \\ \overline{\mathcal{N}} & \overline{\mathcal{M}} \end{array} \right)_{\partial V_{BH}=0}; \tag{5.13}$$

$$H_{\mathbb{R}}^{V_{BH}}\Big|_{\partial V_{BH}=0} = \left.\frac{\partial^2 V_{BH}}{\partial \phi^a \partial \phi^b}\right|_{\partial V_{BH}=0} = \left(\begin{array}{cc} \mathcal{A} & \mathcal{C} \\ \\ \mathcal{C}^T & \mathcal{B} \end{array} \right)_{\partial V_{BH}=0}. \tag{5.14}$$

Thus, the study of the condition (5.1) finally amounts to the study of the *eigenvalue problem* of the real symmetric $2n_V \times 2n_V$ critical Hessian matrix $H_{\mathbb{R}}^{V_{BH}}\big|_{\partial V_{BH}=0}$ given by Eq. (5.14), which is the Cayley-transformed (through Eq. (5.11)) of the complex (symmetric, but not necessarily Hermitian) $2n_V \times 2n_V$ critical Hessian $H_{\mathbb{C}}^{V_{BH}}\big|_{\partial V_{BH}=0}$ given by Eq. (5.13).

Once again, the situation strongly simplifies in $n_V = 1$ SKG.

Indeed, for $n_V = 1$ the moduli-dependent matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{M}$ and $\mathcal{N}$ introduced above are simply scalar functions. In particular, $\mathcal{N}$ is real, since $\mathcal{C}$ trivially satisfies $\mathcal{C} = \mathcal{C}^T$. The stability condition (5.1) can thus be written as

$$H_{\mathbb{R}}^{V_{BH}} \equiv D_a D_b V_{BH} > 0, \ (a, b = 1, 2) \quad \text{at} \quad D_c V_{BH} = \frac{\partial V_{BH}}{\partial \phi^c} = 0 \quad \forall c = 1, 2. \tag{5.15}$$

It may be easily shown that such a stability condition for critical points of V_{BH} in $n_V = 1$ SKG can be equivalently reformulated as the strict bound

$$\mathcal{N}|_{\partial V_{BH}=0} > |\mathcal{M}|_{\partial V_{BH}=0}, \tag{5.16}$$

where

$$\mathcal{N} \equiv D_z\overline{D}_{\overline{z}}V_{BH} = \overline{D}_{\overline{z}}D_zV_{BH} = 2\left[g\left|Z\right|^2 + \left|D_zZ\right|^2 + \left|C\right|^2 g^{-3}\left|D_zZ\right|^2\right]; \quad (5.17)$$

$$\mathcal{M} \equiv D_zD_zV_{BH} = 4i\overline{Z}Cg^{-1}\left(\overline{D}_{\overline{z}}\overline{Z}\right) + i\left(D_zC\right)g^{-2}\left(\overline{D}_{\overline{z}}\overline{Z}\right)^2. \quad (5.18)$$

As it has to be from the treatment given in Sect. 3, $\frac{1}{2}$-BPS critical points of V_{BH} (determined in the $n_V = 1$ case by the unique differential condition $D_zZ = 0$) automatically satisfies the strict bound (5.16).

Let us now consider the non-BPS, $Z \neq 0$ critical points of V_{BH} introduced in Sect. 4. By evaluating the functions $\mathcal{N}$ and $\mathcal{M}$ at such a class of points and using the second of relations (4.10), one gets[17]

$$\mathcal{N}|_{non-BPS,Z\neq 0} = 2\left[\left|D_zZ\right|^2\left(1 + \frac{5}{4}\left|C\right|^2 g^{-3}\right)\right]_{non-BPS,Z\neq 0}; \quad (5.19)$$

$$\left|\mathcal{M}\right|^2_{non-BPS,Z\neq 0} = 4\left\{\left|D_zZ\right|^4\left[\left|C\right|^4 g^{-6} + \frac{1}{4}g^{-4}\left|D_zC\right|^2 + 2g^{-3}Re\left[C\left(\overline{D}_{\overline{z}}\overline{C}\right)\left(\overline{D}_{\overline{z}}ln\overline{Z}\right)\right]\right]\right\}_{non-BPS,Z\neq 0}. \quad (5.20)$$

By substituting Eqs. (5.19) and (5.20) into the strict inequality Eq. (5.16), one finally obtains the stability condition for non-BPS, $Z \neq 0$ critical points of V_{BH} in $n_V = 1$ SKG:

$$\begin{gathered}\mathcal{N}|_{non-BPS,Z\neq 0} > \left|\mathcal{M}\right|_{non-BPS,Z\neq 0}; \quad (5.21)\\ \Updownarrow \\ 1 + \frac{5}{4}\left(\left|C\right|^2 g^{-3}\right)_{non-BPS,Z\neq 0} > \\ > \sqrt{\left[\left|C\right|^4 g^{-6} + \frac{1}{4}g^{-4}\left|D_zC\right|^2 + 2g^{-3}Re\left[C\left(\overline{D}_{\overline{z}}\overline{C}\right)\left(\overline{D}_{\overline{z}}ln\overline{Z}\right)\right]\right]_{non-BPS,Z\neq 0}}.\end{gathered}$$

It is immediate to notice that Eq. (5) is satisfied for sure when the function C is globally covariantly constant, *i.e.* when $D_zC = 0$ globally [52, 53]. Because of the fact that $\forall n_V \in \mathbb{N}$ quadratic (homogeneous symmetric) SKGs does not admit non-BPS, $Z \neq 0$ critical points of V_{BH} [24], the $n_V = 1$ homogeneous symmetric SKG automatically satisfying the condition (5) corresponds to the SK manifold $\frac{SU(1,1)}{U(1)}$, endowed with a cubic holomorphic prepotential which (in a suitable system of local special symplectic coordinates) reads

$$F(t) = \lambda t^3,\ \lambda \in \mathbb{C}, \quad (5.22)$$

and constrained by the condition $Im\left(t\right) < 0$ (usually in the literature $\lambda = \frac{1}{3}$, but such a choice does not yield any loss of generality).

Such an $n_V = 1$ SKG may be obtained by putting $n = -2$ in the so-called cubic reducible rank-3 infinite sequence of homogeneous symmetric SK manifolds $\frac{SU(1,1)}{U(1)} \otimes$

$\frac{SO(2,2+n)}{SO(2)\otimes SO(2+n)}$ (where $n_V = n + rank = n + 3$). It should be noticed that $\frac{SU(1,1)}{U(1)}$ endowed with $F(t) = \lambda t^3$ actually is the "smallest" element of the infinite family $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2)\otimes SO(2+n)}$, which indeed does *not* admit the $n_V = 0$ case, *i.e.* the pure $\mathcal{N} = 2$, $d = 4$ supergravity theory, as a limit case[5] (pure supergravity would indeed be reached by putting $n = 3$, but such a case is not admitted).

Moreover, the manifold $\frac{SU(1,1)}{U(1)}$ endowed with $F(t) = \lambda t^3$ corresponds to nothing but a peculiar *triality-symmetry-destroying degeneration* of the noteworthy $n_V = 3$ *stu* SKG, based on the manifold $\left(\frac{SU(1,1)}{U(1)}\right)^3$ and endowed with a cubic holomorphic prepotential which (in a suitable system of manifestly triality-invariant[6] local special symplectic coordinates) reads $F(s,t,u) = stu$ [55] (see also [56] and [26]). Indeed, $F(t) = \lambda t^3$ can be obtained from $F(s,t,u) = stu$ *e.g.* by identifying $s = t = u$, and by a further suitable rescaling, *e.g.* by rescaling every modulus by $\lambda^{-\frac{1}{3}}$ (in the choice $\lambda = \frac{1}{3}$, rescaling by $\sqrt[3]{3}$).

It should be also pointed out that the $n_V = 1$-dim. (in $\mathbb{C}$) SK manifold $\frac{SU(1,1)}{U(1)}$ can also be obtained as the $n = 0$ element of the quadratic irreducible rank-1 infinite sequence $\frac{SU(1,1+n)}{U(1)\otimes SU(1+n)}$, but in such a case it would be endowed with a quadratic holomorphic prepotential function reading - in a suitable system of local special symplectic coordinates - $F(t) = \frac{i}{4}\left(t^2 - 1\right)$ (see [24] and Refs. therein). Such differences at the level of prepotential determine actual different geometrical properties. For instance, by working in a suitable system of local special symplectic coordinates and using the first and third of Eqs. (2.16), one obtains the following values for the scalar curvature R:

$$
\begin{aligned}
&\frac{SU(1,1)}{U(1)}, F(t) = \lambda t^3 : R \equiv g^{-2} R_{1\overline{1}1\overline{1}} = -\frac{2}{3};\\
&\frac{SU(1,1)}{U(1)}, F(t) = \frac{i}{4}\left(t^2 - 1\right) : R \equiv g^{-2} R_{1\overline{1}1\overline{1}} = -2,
\end{aligned}
\tag{5.23}
$$

where $R_{1\overline{1}1\overline{1}}$ denotes the unique component of the Riemann tensor in $n_V = 1$ (S)KG, and the global values $C = 0$ for the quadratic case and[7] $|C|^2 g^{-3} = \frac{4}{3}$ for the cubic case were respectively used.

Clearly, the cubic homogeneous symmetric $n_V = 1$ SKG based on $\frac{SU(1,1)}{U(1)}$ is not the only one admitting non-BPS, $Z \neq 0$ critical points satisfying the stability condition (5).

[5]The only homogeneous symmetric SKG admitting a consistent (and obtained by vanishing some moduli) $n_V = 0$ limit (reached for $n = -1$) is the quadratic one of the irreducible rank-1 infinite sequence $\frac{SU(1,1+n)}{U(1)\otimes SU(1+n)}$ (see [24] and Refs. therein). The homogeneous non-symmetric SKGs (see *e.g.* [34] and Refs. therein), because of they all are cubic, do not admit a consistent (and obtained by vanishing some moduli) $n_V = 0$ limit.

[6]The noteworthy *triality symmetry* of the *stu* $n_V = 3$ SKG has been recently related to quantum information theory [57]-[62].

[7]The global value $|C|^2 g^{-3} = \frac{4}{3}$ for homogeneous symmetric cubic $n_V = 1$ SKGs actually is nothing but the $n_V = 1$ case of the general global relation holding in a generic homogeneous symmetric cubic n_V-dimensional SKG [53, 63]:

$$
C_{p(kl}C_{ij)n} g^{n\overline{n}} g^{p\overline{p}} \overline{C}_{\overline{npm}} = C_{(p|(kl}C_{ij)|n)} g^{n\overline{n}} g^{p\overline{p}} \overline{C}_{\overline{npm}} = \frac{4}{3} g_{(l|\overline{m}} C_{|ijk)}.
$$

In the general case $(D_z C)_{non-BPS,Z\neq 0}$ is the fundamental geometrical quantity playing a key role in determining the stability of non-BPS, $Z \neq 0$ critical points of V_{BH} in $n_V = 1$ SKG.

6 Further Results, Some Developments and Outlook

The present report dealt with some recent advances in the study of extremal BH attractors in $\mathcal{N} = 2$, $d = 4$ MESGT.

We discussed the AEs for a generic number n_V of moduli in a static, spherically symmetric, asymptotically flat extremal BH background. Such Eqs. are nothing but the criticality conditions for a real, positive-definite "effective BH potential" function V_{BH} defined on the SK vector supermultiplets' scalar manifold $\mathcal{M}_{n_V}$.

V_{BH} is one of the two invariants of the SK geometry of $\mathcal{M}_{n_V}$ which are quadratic (and thus lowest-order) in the BH charges, defined as the electric and magnetic fluxes of the field-strength two-forms of the $n_V + 1$ Maxwell vector fields of the $\mathcal{N} = 2$, $d = 4$ MESGT being considered (n_V is the number of Abelian vector multiplets, and also the graviphoton from the supergravity multiplet has to be taken into account).

Due to staticity and spherical symmetry, the (bosonic sector of the) considered $\mathcal{N} = 2$, $d = 4$ MESGT can be described by an effective 1-dimensional Lagrangian in the radial (evolution) variable. Peculiar features of a spontaneously broken supergravity theory arise in such a Lagrangian effective formalism, in which the condition of existence of non-BPS critical points of V_{BH} (with non-vanishing central charge Z) is given by the vanishing of the determinant of a (fermionic) gaugino mass matrix.

Concerning the stability of the critical points of V_{BH}, because of V_{BH} is positive-definite, they necessarily must be (at least local) minima in order to correspond to *attractor* horizon scalar configurations in strict sense. In general, the stability is controlled by the SKG of $\mathcal{M}_{n_V}$: in addition to the rank-3, completely symmetric, covariantly holomorphic tensor C_{ijk}, also its covariant derivatives $D_i C_{jkl}$ (related, through the so-called *SK geometry contraints*, to the covariant derivatives $D_m R_{i\bar{j}k\bar{l}}$ of the Riemann-Christoffel tensor) turn out to be crucial. This can easily be seen by considering the explicit expression of $H_{\mathbb{C}}^{V_{BH}}$, the $2n_V \times 2n_V$ covariant Hessian matrix in the complex holomorphic/anti-holomorphic parameterization of $\mathcal{M}_{n_V}$. In order to decide whether a critical point of V_{BH} actually gives rise to an attractor in strict sense, one has actually to study the eigenvalue problem for $H_{\mathbb{R}}^{V_{BH}}$, real form of $H_{\mathbb{C}}^{V_{BH}}$, properly evaluated at the considered critical point.

The so-called $\frac{1}{2}$-BPS critical points of V_{BH} (treated in Sect. 3) correspond to horizon scalar configurations which preserve half of the supersymmetry degrees of freedom of the asymptotical Minkowski background (namely, 4 out of 8). They are always stable, thus corresponding to *attractors* in strict sense. Other two species of critical points of V_{BH} exist in $\mathcal{N} = 2$, $d = 4$ MESGT, *i.e.* the non-BPS $Z \neq 0$ (treated in Sects. 4 and 5) and non-BPS $Z = 0$ ones. In general, both such classes of critical points are not necessarily stable; the condition(s) for their stability can be formulated in purely geometrical terms, by using the properties of the SKG of $\mathcal{M}_{n_V}$.

As it happens for the study of SKG, also the eigenvalue problem of $H_{\mathbb{R}}^{V_{BH}}$ strongly simplifies in the case $n_V = 1$, *i.e.* in the case in which only 1 Maxwell vector multiplet is coupled to the supergravity multiplet. Consequently, only 2 Abelian vector fields are present in such a case: the graviphoton one and the one coming from the unique Abelian supermultiplet. The stability condition for non-BPS, $Z \neq 0$ critical points of V_{BH} in a generic $n_V = 1$ SKG can be shown to be equivalent to a strict inequality, involving the fundamental geometrical entities of the SKG of $\mathcal{M}_{n_V=1}$ (this actually happens also for the non-BPS, $Z = 0$ case [30], not treated in the present report).

Recently, in [24] the general solutions to the AEs were obtained and classified by group-theoretical methods for the peculiar class of $\mathcal{N} = 2$, $d = 4$ MESGTs having an *homogeneous symmetric* SK scalar manifold, *i.e.* for those $\mathcal{N} = 2$, $d = 4$ MESGTs in which $\mathcal{M}_{n_V}$, beside being SK, is a coset $\frac{G}{H}$ with a globally covariantly constant Riemann-Christoffel tensor $R_{i\bar{j}k\bar{l}}$: $D_m R_{i\bar{j}k\bar{l}} = 0$. Such a conditions can be transported on C_{ijk} by means of the so-called *SK geometry contraints*, obtaining: $D_l C_{ijk} = 0$.

The considered $\mathcal{N} = 2$, $d = 4$ MESGTs are usually named *homogeneous symmetric* MESGTs, and they have been classified in literature [52, 53, 54].

With the exception of the ones based on[8] $\frac{SU(1,1+n)}{U(1)\otimes SU(1+n)}$, *all* homogeneous symmetric SKGs are endowed with cubic holomorphic prepotentials. In all rank-3 homogeneous symmetric cubic SK manifolds $\frac{G}{H=H_0\otimes U(1)}$ (being the vector supermultiplets' scalar manifolds of $\mathcal{N} = 2$, $d = 4$ MESGTs defined by Jordan algebras of degree 3), the solutions to AEs have been shown to exist in three distinct classes, one $\frac{1}{2}$-BPS and the other two non-BPS, one of which corresponds to vanishing central charge $Z = 0$. It is here worth remarking that the non-BPS $Z = 0$ class of solutions to AEs has no analogue in $d = 5$, where a similar classification has been recently given [23].

Furthermore, the three classes of critical points of V_{BH} in $\mathcal{N} = 2$, $d = 4$ homogeneous symmetric cubic MESGTs have been put in *one-to-one* correspondence with the non-degenerate charge orbits of the actions of the U-duality groups G on the corresponding BH charge configuration spaces. In other words, the three species of solutions to AEs in $\mathcal{N} = 2$, $d = 4$ homogeneous symmetric cubic MESGTs are supported by configurations of the BH charges lying along the non-degenerate typologies of charge orbits of the U-duality group G in the real (electric-magnetic field strengths) representation space R_V. The results obtained in [24] are summarized in Table 1.

In all the $\mathcal{N} = 2$, $d = 4$ homogeneous symmetric MESGTs based on rank-3 SK cubic

[8]The quadratic irreducible rank-1 infinite sequence $\frac{SU(1,1+n)}{U(1)\otimes SU(1+n)}$ has $C_{ijk} = 0$ globally. As shown in App. I of [24], such a family has only two classes of non-degenerate solutions to the AEs: one $\frac{1}{2}$-BPS and one non-BPS with $Z = 0$.

	$\frac{1}{2}$-BPS orbits $\mathcal{O}_{\frac{1}{2}-BPS} = \frac{G}{H_0}$	non-BPS, $Z \neq 0$ orbits $\mathcal{O}_{non-BPS,Z\neq 0} = \frac{G}{\widehat{H}}$	non-BPS, $Z = 0$ orbits $\mathcal{O}_{non-BPS,Z=0} = \frac{G}{\widetilde{H}}$
Quadratic Sequence	$\frac{SU(1,n+1)}{SU(n+1)}$	$-$	$\frac{SU(1,n+1)}{SU(1,n)}$
Cubic Sequence	$\frac{SU(1,1)\otimes SO(2,2+n)}{SO(2)\otimes SO(2+n)}$	$\frac{SU(1,1)\otimes SO(2,2+n)}{SO(1,1)\otimes SO(1,1+n)}$	$\frac{SU(1,1)\otimes SO(2,2+n)}{SO(2)\otimes SO(2,n)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\otimes SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

Table 1: **Non-degenerate orbits of $\mathcal{N} = 2$, $d = 4$ homogeneous symmetric MESGTs**

manifolds, the classical BH entropy is given by the Bekenstein-Hawking entropy-area formula [37]

$$S_{BH} = \frac{A_H}{4} = \pi \left. V_{BH} \right|_{\partial V_{BH}=0} = \pi \sqrt{|I_4|}, \tag{6.1}$$

where I_4 is the (lowest order, quartic[9] in the BH charges) moduli-independent G-invariant built out of the (considered non-degenerate charge orbit in the) representation R_V. $\frac{1}{2}$-BPS and non-BPS $Z = 0$ classes have $I_4 > 0$, while the non-BPS $Z \neq 0$ class is characterized by $I_4 < 0$.

The critical mass spectra split in different ways, depending on the considered class of non-degenerate charge orbits. In general, both at non-BPS $Z \neq 0$ and at non-BPS $Z = 0$, the critical Hessian matrix (1.6) usually exhibit zero modes (*i.e.* "flat" directions), whose actual attractor nature seemingly further depends on additional conditions on the charge vector $\widetilde{\Gamma}$, other than the ones given by the extremality conditions (1.5) (see *e.g.* [10]).

[9] For the quadratic irreducible rank-1 infinite sequence $\frac{SU(1,1+n)}{U(1)\otimes SU(1+n)}$ the lowest-order G-invariant is instead quadratic in the BH charges; it is positive for $\frac{1}{2}$-BPS orbits and negative for the non-BPS ($Z = 0$) ones (see App. I of [24]).

An interesting direction for further investigations concerns the study of extremal BH attractors in more general, *non-cubic* SK geometries. A noteworthy example is given by the SKGs of the scalar manifolds of those $\mathcal{N} = 2$, $d = 4$ MESGTs obtained as effective, low-energy theories of $d = 10$ Type IIB superstrings compactified on Calabi-Yau threefolds (CY_3s), away from the limit of large volume of CY_3s.

Recently, [30] studied the extremal BH attractors in $n_V = 1$ SKGs obtained by compactifications (away from the limit of large volume of the internal manifold) on a peculiar class of CY_3s, given by the so-called (mirror) *Fermat* CY_3s. Such threefolds are classified by the *Fermat parameter* $k = 5, 6, 8, 10$, and they were firstly found in [64]. The fourth order linear Picard-Fuchs (PF) ordinary differential Eqs. determining the holomorphic fundamental period 4×1 vector for such a class of 1-modulus CY_3s were found some time ago for $k = 5$ in [65, 66] (see also [67]), and for $k = 6, 8, 10$ in [68].

More specifically, [30] dealt with the so-called *Landau-Ginzburg* (LG) extremal BH attractors, *i.e.* the solutions to the AEs near the origin $z = 0$ (named *LG point*) of the moduli space $\mathcal{M}_{n_V=1}$ ($dim_{\mathbb{C}}\mathcal{M}_{n_V=1} = 1$), and the BH charge configurations supporting $z = 0$ to be a critical point of V_{BH} were explicitly determined, as well.

An intriguing development in such a framework would amount to extending to the *Fermat* CY_3-compactifications (away from the limit of large volume of the threefold) the conjecture formulated in Sect. 5 of [26]. The conjecture was formulated in the framework of (the large volume limit of CY_3-compactifications leading to) the previously mentioned *triality-symmetric* cubic *stu* model [55, 56, 26], and it argues that the instability of the considered non-BPS ($Z \neq 0$) critical points of V_{BH} might be only apparent, since such attractors might correspond to multi-centre stable attractor solutions (see also [69] and Refs. therein), whose stable nature should be "resolved" only at sufficiently small distances. The extension of such a tempting conjecture to non-BPS extremal BH LG attractors in *Fermat* CY_3-compactifications would be interesting; in particular, the extension to the non-BPS $Z = 0$ case might lead to predict the existence (at least in the considered peculiar $n_V = 1$ framework) of *non-BPS lines of marginal stability* [70, 71] with $Z = 0$.

Moreover, it should be here recalled that the PF Eqs. of *Fermat* CY_3s ([65]-[68], see also[30]) exhibit other two species of *regular singular* points, namely the k-th roots of unity ($z^k = 1$, the so-called *conifold points*) and the *point at infinity* $z \longrightarrow \infty$ in the moduli space, corresponding to the so-called *large complex structure modulus limit.* Thus, it would be interesting to solve the AEs in proximity of such regular singular points, *i.e.* it would be worth investigating *extremal BH conifold attractors* and *extremal BH large complex structure attractors* in the moduli space of 1-modulus (*Fermat*) CY_3s. Such an investigation would be of interest, also in view of recent studies of extremal BH attractors in peculiar examples of $n_V = 2$-moduli CY_3-compactifications [27].

Despite the considerable number of papers written on the Attractor Mechanism in the extremal BHs of the supersymmetric theories of gravitation along the last years, still much remains to be discovered along the way leading to a deep understanding of the inner dynamics of (eventually extended) space-time singularities in supergravities, and hopefully in their fundamental high-energy counterparts, such as $d = 10$ superstrings and $d = 11$ M-theory.

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