Prof. E. A. Guillemin
Dr. M. V. Cerrillo
Dr. P. M. Lewis II
E. F. Bolinder
J. C. Pinson

A. STUDY OF THE MOTIONS OF TWO POLES IN THE COMPLEX FREQUENCY PLANE BY A GRAPHICAL METHOD

In the Quarterly Progress Report of July 15, 1956, page 82, an interesting analogy between the fixed points in the complex impedance plane (Z-plane) and two saddlepoints in the complex frequency plane (s-plane, $s = \sigma + j\omega$) was pointed out. This analogy can be extended to enclose the motions of two poles in the s-plane.

Let us study a simple example. The input impedance of a simple parallel resonance circuit, which has an inductance L with a series resistance r, and a capacitance C with a conductance G, is

$$Z(s) = \frac{1}{C} \frac{s + r/L}{s^2 + (r/L + G/C)\,s + (1+rG)/LC} \tag{1}$$

This equation can be written

$$Z(s) = \frac{1}{C} \frac{1}{s + \dfrac{sG/C + (1+rG)/LC}{s + r/L}} \tag{2}$$

The positions of the poles, s_{p1} and s_{p2}, are obtained by making the denominator equal to zero:

$$s_p = \frac{-s_p G/C - (1+rG)/LC}{s_p + r/L} \tag{3}$$

Equation 3 is analogous to the equation used in obtaining the fixed points Z_{f1} and Z_{f2} in the Z-plane:

$$Z_f = \frac{aZ_f + b}{cZ_f + d}\,; \qquad ad - bc = 1 \tag{4}$$

To obtain an exact analogy, the coefficients in Eq. 3 have to obey the condition $ad - bc = 1$. Equation 3 then transforms into

$$s_p = \frac{-s_p G(L/C)^{1/2} - (1+rG)/(LC)^{1/2}}{s_p (LC)^{1/2} + r(C/L)^{1/2}} \tag{5}$$

In the Z-plane the positions of the fixed points are easily obtained from the positions of the isometric circles in the nonloxodromic case, when a + d is real (Quarterly Progress Report, April 15, 1956, p. 123). Analogous conditions yield two circles with

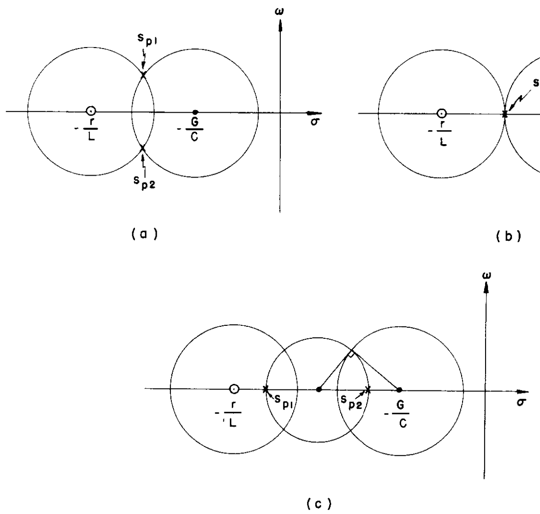

Fig. XIII-1. Graphical construction of pole positions.

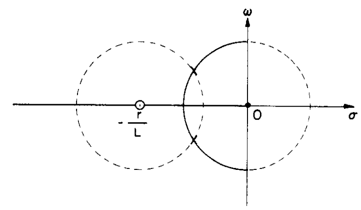

Fig. XIII-2. Example of pole trajectories.

centers at $-d/c = -r/L$ and $a/c = -G/C$, both having the radius $1/|c| = 1/(LC)^{1/2} = \omega_o$, that immediately specify the pole positions. See Fig. XIII-1. If the two circles intersect (Fig. XIII-1a), two complex conjugate poles are obtained, corresponding to the oscillating case. If the two circles are tangent (Fig. XIII-1b), two coalescing real poles are obtained, corresponding to the cutoff case. If, finally, the two circles are external (Fig. XIII-1c), two real poles are obtained as the crossover points of a circle that is orthogonal to the two circles, corresponding to the below-cutoff case. In every case, a zero is situated in the center of one of the circles, at $\sigma_o = -r/L$.

In various textbooks one usually finds that the pole trajectories shown in Fig. XIII-2 are obtained from an rLC circuit by the variation of the resistance r. This figure is immediately explained by the graphical method that has been described. With $G = 0$, one circle is fixed with its center at the origin, and the other is moved along the negative σ-axis as r is varied. The trajectories are therefore the real axis and the fixed circle.

E. F. Bolinder