XVIII. CIRCUIT THEORY

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A. RC BRIDGE OSCILLATORS

The use of RC bridge networks for tuning purposes is widespread. In particular, oscillators and feedback amplifiers in the low-frequency range use RC tuning more than LC tuning because of the simplicity and wide range afforded. The research reported here is a study of RC bridge oscillators operating as linear harmonic oscillators.

Consider the block diagram of Fig. XVIII-1 where the RC bridge is taken as a Wien bridge, or any of its possible modifications arising from the interchange of the branches, reversal of the polarity, and so on. The amplifier is assumed to have a gain K, n high cutoff frequencies, and n low cutoff frequencies (or the equivalent). Furthermore, two limiting cases of input-output impedances are considered: one has an infinite input impedance and a zero output impedance; the other has a zero input impedance and an infinite output impedance.

The first system that we shall consider is shown in Fig. XVIII-2, where R_3 or R_4, or both, are temperature-sensitive (thermistor or varistor), and the amplifier has a voltage gain K with the equivalent frequency response of n RC coupled stages. The voltage transfer function of the bridge is

$$\alpha(s) = \frac{E_{in}}{E_{out}} = \left[\frac{1}{(1 + C_2/C_1 + R_1/R_2) + R_1C_2s + 1/R_2C_1s} - \frac{R_4}{R_3 + R_4}\right] \tag{1}$$

The amplifier voltage gain is

$$K(j\omega) = \frac{K_o}{\prod_{i=1}^{n}\left(1 + j\frac{\omega}{\omega_{hi}}\right)\left(1 - j\frac{\omega_{\ell i}}{\omega}\right)} \tag{2}$$

The loop gain $G(j\omega)$ for the oscillator system is then

$$G(j\omega) = \alpha(j\omega)\, K(j\omega) \tag{3}$$

Several cases arise: positive or negative K_o, positive or negative $\alpha(j0)$, increasing or decreasing value of $R_4/(R_3 + R_4)$ as E_{out} increases, different values of n, and identical or different cutoff frequencies for each stage.

For K_o positive, $\alpha(j0)$ positive, and identical cutoff frequencies, the Nyquist plots for $\omega = 0$ to $\omega = \infty$ are shown in Fig. XVIII-3a, b, c, and d for n = 0, n = 1, n = 2, and n = 3. For n = 0, the Nyquist plot is a circle; and if $R_4/(R_3 + R_4)$ increases with an increase in E_{out}, the circle will shift to the left until it sits on 1 + j0. The stable

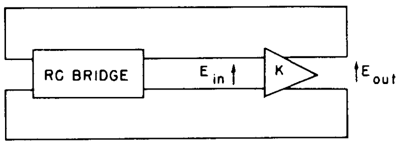

Fig. XVIII-1. Block diagram of RC bridge oscillator.

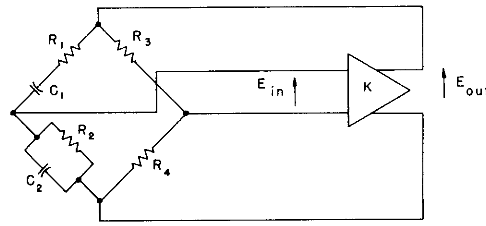

Fig. XVIII-2. Wien bridge oscillator.

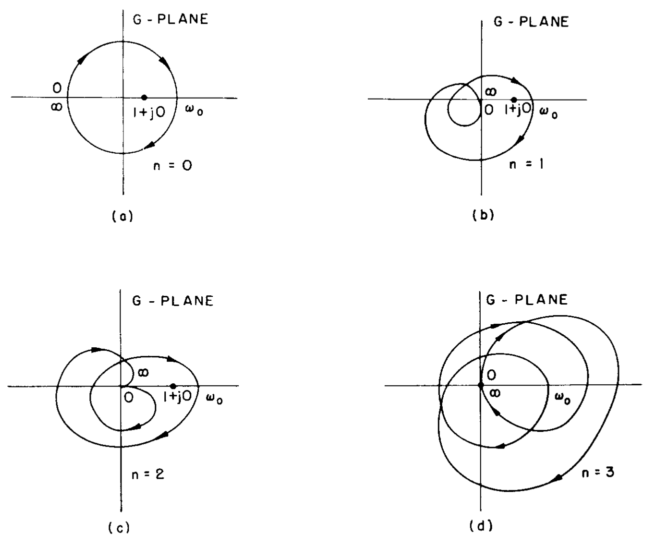

Fig. XVIII-3. Nyquist plot of $G(j\omega)$ for positive K_o and positive $\alpha(j0)$.

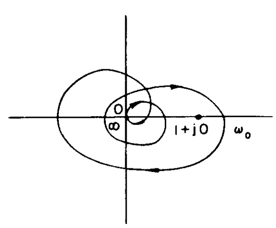

Fig. XVIII-4. Nyquist plot of $G(j\omega)$ for positive K_o, positive $\alpha(j0)$, and widely different cutoff frequencies.

frequency of oscillation is $\omega = (R_1R_2C_1C_2)^{-1/2} = \omega_o$. If $R_4/(R_3 + R_4)$ decreases as E_{out} increases, the circle always encloses 1 + j0. For n = 1 and n = 2, stable oscillation at $\omega = \omega_o$ is obtained if $R_4/(R_3 + R_4)$ increases with an increase in E_{out}. Otherwise, the systems are unstable and eventually lead to nonsinusoidal oscillations. For n = 3, or more, and for ordinary values of gain (anticipating that the system might work when the bridge is almost balanced), the system is unstable no matter what type of temperature dependence R_3 or R_4 might have (1). For very low values of K_o it might be possible to produce stable oscillation, but the frequency of oscillation is a very strong function of the cutoff frequencies rather than $R_1R_2C_1C_2$. The operation is essentially that of a phase-shift oscillator.

If the cutoff frequencies are not identical, it is possible to produce stable oscillation at $\omega = \omega_o$ for n = 3, or more. However, the requirements are more stringent: (n - 1) or (n - 2) of the amplifier stages must not have noticeable phase shift until the attenuation through one or two stages is of the order of the reciprocal of the amplifier gain K_o. Of course it is understood that the cutoff frequencies of the other two interstage networks are far from the operating range of the oscillator, so that their phase-shift contribution is negligible in the operating range. A typical Nyquist plot might look like Fig. XVIII-4. Thus, for a system using a 3-stage amplifier, if the desired frequency of operation is around 1000 cps, one or two of the stages might have a high cutoff frequency of 100 kc; and if the nominal gain K_o is around 1000, then the third high cutoff frequency should be of the order of 50 mc! If the gain K_o is made higher, the frequency spread is even larger. As to the lower frequency cutoff, one or two of the stages might have a cutoff of 10 cps, and the third stage must have a cutoff of approximately 0.02 cps! The use of more than two stages imposes an unusually large requirement on the bandwidth of the additional stages. And making all the stages wideband (cutoff frequencies almost identical) is not any better: the resultant system will not work as a bridge oscillator!

For positive K_o and negative $\alpha(j0)$, the Nyquist plots are shown in Fig. XVIII-5a, b, c, and d. For n = 0, n = 1, and n = 2, oscillation is impossible. For n = 3 or more, it is possible that the Nyquist plot will enclose 1 + j0 at least once and oscillation will start to grow. If $R_4/(R_3 + R_4)$ increases with an increase in E_{out}, 1 + j0 will always be enclosed; the system is then unstable. If $R_4/(R_3 + R_4)$ decreases with an increase in E_{out}, then the intersections on the positive real axis move to the left, while the intersection on the negative real axis corresponding to $\omega = \omega_o$ moves to the right. The process continues until 1 + j0 is on the Nyquist plot without being enclosed. This happens when the gain is unusually low or when the cutoff frequencies are not made identical in such a manner that 1 + j0 is almost sitting on the Nyquist plot to start with. The frequency of stable oscillation is very much a function of the cutoff frequencies. If the gain is appreciable or if the stages are identical, $\alpha(j0)$ might reverse its sign during the

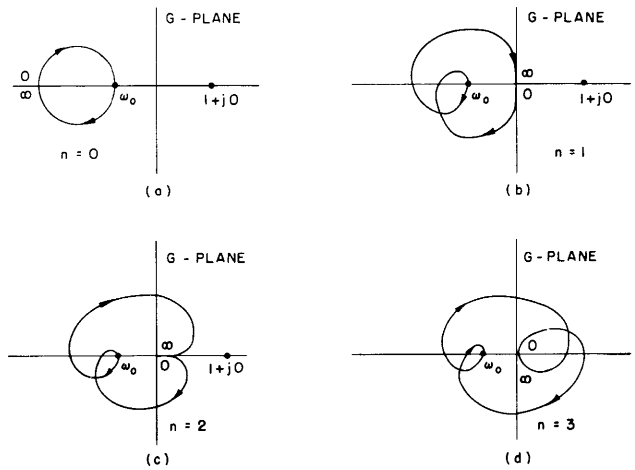

Fig. XVIII-5. Nyquist plot of $G(j\omega)$ for K_o positive, and $a(j0)$ negative.

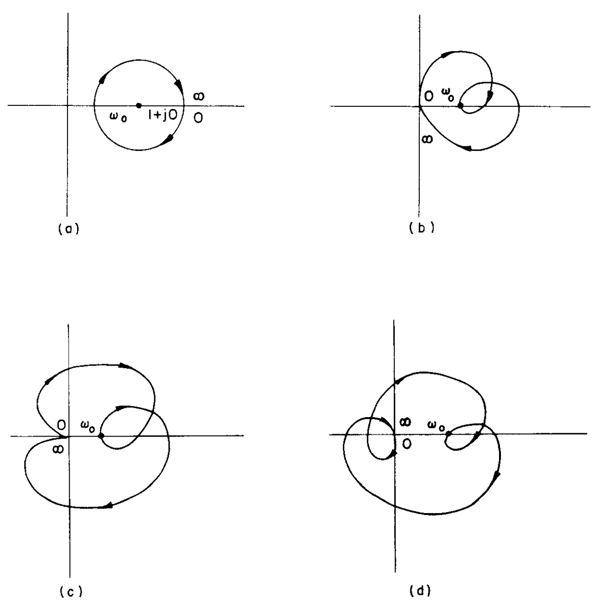

Fig. XVIII-6. Nyquist plot of $G(j\omega)$ for K_o negative and $a(j0)$ negative.

process so that the intersection corresponding to ω_o is on the positive real axis. This intersection will eventually move to the right of $1 + j0$, and it will always remain inside the Nyquist plot; the system is then unstable.

For negative K_o and negative $a(j0)$ the Nyquist plots are shown in Fig. XVIII-6a, b, c, and d. For $n = 0$, if $1 + j0$ is enclosed to start with, and if $R_4/(R_3 + R_4)$ increases as E_{out} increases, the circle will move to the right until $1 + j0$ just sits on it. The stable frequency of oscillation is $\omega = (R_1R_2C_1C_2)^{-1/2}$. If $R_4/(R_3 + R_4)$ decreases with an increase in E_{out}, the circle moves to the left, reversing the sign of $\alpha(j0)$ if necessary until $1 + j0$ sits on it. The stable frequency of oscillation is either zero or infinity.

For n = 1 or more, and if $1 + j0$ is enclosed to start with, oscillations will grow. If $R_4/(R_3 + R_4)$ increases as E_{out} increases, $1 + j0$ will always remain enclosed. Making $R_4/(R_3 + R_4)$ decrease as E_{out} increases will produce a stable system, but the frequency of oscillation will be strongly dependent on the cutoff frequencies, whether or not they are identical.

When K_o is negative and $\alpha(j0)$ is positive there will be a similar result: the system will be stable only if $R_4/(R_3 + R_4)$ decreases as E_{out} increases, whether or not the interstage networks are identical.

Modification of the Wien bridge by all possible interchanges of the branches was found to be unsatisfactory in the sense that the stable frequency of oscillation (if there is any) is either a strong function of the cutoff frequencies, amplifier phase shift, or all of the elements in the bridge, including the thermal elements.

If the amplifier is assumed to have zero input impedance and infinite output impedance, the Wien bridge input and output terminals could be reversed, and all of the results previously derived will apply to this dual termination case (1, 2).

The stability of the frequency with respect to variations in amplifier phase shift and amplifier gain, and the stability of the amplitude with respect to variations in amplifier gain were all found to be directly proportional to the magnitude of amplifier gain. Thus, it might be desirable to use more than two amplifier stages even at the expense of reduced frequency range. For a detailed derivation of the expressions for the stability of frequency and stability of amplitude, see reference 1.

J. B. Cruz, Jr.

References

1. J. B. Cruz, Jr., S. M. Thesis, Department of Electrical Engineering, M.I.T., 1956.
2. Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., Jan. 15, 1956, p. 116.

B. DESIGN OF A CATHODE-RAY TUBE FOR HIGH-SPEED ANALOG MULTIPLICATION

1. Introduction

The advent of the electronic analog computer established the need for a multiplier with electronic inputs. There are many ways of accomplishing electronic multiplication. In the present design (1), the potentiometer is used for multiplication, its resistance being a resistive strip that is used as a target in a cathode-ray tube, the variable tap

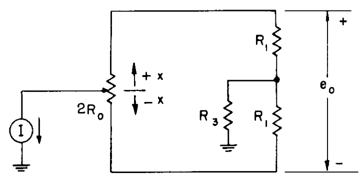

Fig. XVIII-7. The current divider, illustrating the principle of multiplication.

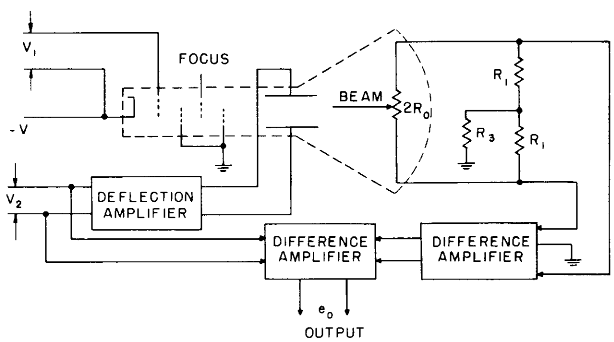

Fig. XVIII-8. Block diagram of the multiplier.

of the potentiometer corresponding to the electron-beam in the tube. The target is made by evaporating a resistive material (chromium was used) on a glass disc. The tube parameters used for multiplication are beam current and beam deflection. The output of the multiplier is the voltage across the resistive strip.

2. Discussion

The operation of the multiplier can be best understood by considering its current divider, shown in Fig. XVIII-7. The current I flows out of the movable tap of the potentiometer $2R_o$. The displacement per cent x of the movable tap is measured from the electric center of the potentiometer resistance $2R_o$; it increases to 100 per cent at either end of the resistance.

The output voltage e_o is given (2) by the equation

$$e_o = -\left(\frac{R_1 R_o}{R_1 + R_o}\right) I\, x$$

It is clear that the output voltage e_o is proportional to the product of the current I and the displacement per cent x. Since the grid voltage on the cathode-ray tube is proportional to the beam current, it is used as one input to the multiplier. The voltage on the deflection plates is used as the other input. Figure XVIII-8 is a block diagram of the multiplier.

In the current divider the current I can flow in both directions, but in the cathode-ray tube the current can flow only from the target to the electron gun. In order to accommodate negative voltages at the grid input, it is necessary to bias the grid to get some steady-state value of beam current. The positive and negative input voltages correspond to an increase and decrease in beam current from the steady-state value. The output voltage caused by the steady-state beam current will vary as the deflection voltage. Therefore, if some proportion of the deflection voltage is subtracted from the output voltage, we have an output voltage e_o that is proportional to the product of the incremental grid voltage V_1 and the deflection voltage V_2. The output is positive when the inputs are both positive or both negative. The output is negative when the inputs have different signs. The details of the design and construction of this multiplier are covered in reference 1. The use of feedback for linearization is also discussed there.

3. Conclusions and Recommendations

The multiplier, without the use of feedback, was accurate to within 0.04 per cent of the maximum output. The addition of feedback might improve the accuracy to within 0.004 per cent if the increased feedback did not cause oscillation. However, no tests were made with the feedback.

The frequency response was limited by the external amplifiers. The difference amplifiers used had a high-frequency response of 10 kc; at higher frequencies the phase shift caused inaccurate subtraction. Careful design and construction of these amplifiers might give a frequency response from zero to 1 mc.

Further study of this multiplier must be made in order to perfect it. Theoretically, it can operate over a wide frequency range with very good accuracy.

J. F. Mueller

References

1. J. F. Mueller, S. B. Thesis submitted to the Department of Electrical Engineering, M.I.T., May 1956.
2. Those interested in the derivation of this equation may consult the thesis.

C. SOME LIMITATIONS OF AMPLIFYING DEVICES

A study was undertaken to determine how the characteristics of an amplifying device limit the performance that can be achieved from circuits constructed from these devices and passive circuit elements. The first step was the investigation of the gain-bandwidth limitations of an amplifying device operating with signals small enough for linear analysis.

1. Representation of Characteristics of Amplifying Devices

In the present discussion, it is assumed that all amplifying devices can be described by linear differential equations in such a manner that terminal voltages and currents can be related by a hybrid matrix. The term "amplifying device" refers to any multi-terminal circuit element which, when used in conjunction with dc power supplies and passive circuit elements, is capable of effecting power gain from a source to a load. Both real and imaginary parts of the hybrid parameters will, in general, be frequency-dependent; it is assumed that these variations are known for the band of frequencies in which the amplifying device is useful.

Other workers made detailed studies of the limitations of a device that is described by a single-frequency hybrid matrix; they determined quantities such as maximum available power gain, input and output impedance, and unilateral power gain. The present study attempts to determine how the frequency-dependent terms limit the ability of a device to provide high gain and rapid response simultaneously.

If we consider only relatively low frequencies, a hybrid parameter can usually be expanded in a Taylor series of the form

$$h = h^o + h^1 s + h^2 s^2 \ . \ . \ . \tag{1}$$

where the zero-order term, h^o, is the dc resistance, conductance, or dimensionless ratio; and the higher-order terms determine the frequency-dependent behavior. If we restrict the frequency range sufficiently, we can usually neglect the second-order terms, but, even though the first-order terms may be small they appear to be the determining factors in limiting high-gain, wideband operation. In this study, therefore, it is assumed that all hybrid parameters can be approximated by the h^o and h^1 terms alone. In terms of this approximation, we have

$$h = h^o + h^1 s \ ; \qquad h^o = h\Big|_{s=0} \ ; \qquad h^1 = \frac{dh}{ds}\Big|_{s=0} \tag{2}$$

2. Energy and Power in an Amplifying Device

If we define the independent voltages and currents by the matrix $[X]$ and the dependent variables by $[Y]$, we can write

$$[Y] = [H][X] = [H^o][X] + [H^1]s[X] \tag{3}$$

where $[H^o]$ is the dc hybrid matrix, and $[H^1]$ is the matrix describing the first-order frequency dependence of $[H]$. Recognizing that $[X^t][Y]$ is the instantaneous rate of energy dissipation, we see that the energy consumed in a time interval T is given by

$$[W] = \int_0^T [X^t][H^o][X]dt + \int_{X(t=0)}^{X(t=T)} [X^t][H^1][dX] \tag{4}$$

The first term on the right in Eq. 4 is interpreted as the energy dissipated by the device, since it is irrecoverable and depends on the resistive parameters and the length of time during which the voltages are applied. The second term is interpreted as the stored energy, since it depends on the voltage and current variables at the start and end of the time interval but not on the length of time the signals are applied. If $[H^1]$ is symmetric, Eq. 4 can be written

$$\text{Energy supplied} = \text{Energy dissipated} + \text{Change in energy stored} \tag{5}$$

$$\text{Energy dissipated} = \int \sum h_{ij}^{o} x_i x_j \, dt \; ; \quad \text{Energy stored} = \sum \tfrac{1}{2} h_{ij}^{1} x_i x_j \tag{6}$$

If $[H^1]$ is not symmetric, the problem is more difficult because the change in energy stored depends on the way in which the voltage and current variables are changed. Equation 6 is still correct, however, if we interpret energy stored as the average exchange of energy involved in going from one state to another and back again by an identical process. Thus, if we supply one unit of energy to charge a network and remove three units of energy when discharging it, the stored energy according to Eq. 6 would be two units. This interpretation of stored energy simplifies to $[I][L][I]/2$ and $[V][C][V]/2$ for simple capacitive and inductive networks but has a more general interpretation for nonreciprocal networks.

If the device is to be capable of effecting power gain, it is apparent that we must be able to adjust the terminal voltages and currents so as to dissipate a negative power; in other words, the amplifying device must supply a net incremental power to the external circuit. To change the power supplied we must change the energy stored, which, in turn, requires time if the power levels are always finite. The inability to change power levels instantaneously is, in essense, the limitation on speed of response of an amplifier.

If we compute the ratio, R, of power dissipated to energy stored, we see that

$$R = 2\,\frac{[X^t][H^o][X]}{[X^t][H^1][X]} \tag{7}$$

where the numerator and denominator of R are assumed to be expanded in quadratic form. If we now attempt to maximize R by adjusting the ratios of the terminal voltages and currents, it is easily shown that M, defined as the maximum possible ratio of power output to twice the energy stored, is given by the largest root of Eq. 9.

$$M = \frac{1}{2}(-R)_{max} = \text{Maximum power output per unit energy stored} \tag{8}$$

$$\left[H^{o} + H^{ot}\right] + M\left[H^{1} + H^{1t}\right] = 0 \tag{9}$$

3. Dependence of Transient Response on Energy Storage

The parameter M, defined as in Eq. 8, is important in limiting the power gain that can be achieved by an amplifier with a given transient response. If we attempt to build a power amplifier utilizing a particular amplifying device and arbitrary passive coupling it is of interest to determine how the device characteristics limit the realizable gain. The definition of M and the knowledge that passive circuits can only dissipate and store positive energy make it apparent that the amplifier cannot have a larger M than the devices used to build the amplifier have. This idea is illustrated by a simple example.

If we attempt to build a voltage amplifier with input admittance $g_1 + sC_1$, output admittance $g_2 + sC_2$, and forward transconductance g_m, the $\left[H^{o}\right]$ and $\left[H^{1}\right]$ matrices will be

$$\left[H^{o}\right] = \begin{bmatrix} g_1 & 0 \\ g_m & g_2 \end{bmatrix} \quad ; \qquad \left[H^{1}\right] = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \tag{10}$$

and M will be given by

$$4(g_1 + MC_1)(g_2 + MC_2) = g_m^2 \tag{11}$$

If we insist that the M of the amplifier cannot exceed a value determined by the characteristics of the devices used to build the amplifier, it is readily seen that the maximum g_m is given by

$$g_m = 2\left[(g_1 + MC_1)(g_2 + MC_2)\right]^{1/2} \tag{12}$$

If these amplifiers are connected in cascade so that the input admittance for one stage is the load admittance on the previous stage, we see that

$$K_v = \frac{g_m}{g_1 + g_2} = \text{Steady-state voltage gain per stage} \tag{13}$$

$$W = \frac{g_1 + g_2}{C_1 + C_2} = \text{Half-power bandwidth per stage} \tag{14}$$

Furthermore, we have the limiting relations

$$(K_v - 1)W = M, \text{ if } g_1 - g_2 = M(C_2 - C_1) \tag{15}$$

$$(K_v - 1)W < M, \text{ if } g_1 - g_2 \neq M(C_2 - C_1) \tag{16}$$

so that M is a kind of maximum available gain bandwidth product. It should be noted, however, that M represents a device limitation that ideal circuit design cannot hope to circumvent. The fact that M is the gain minus one multiplied by the bandwidth instead of the usual gain-bandwidth product can be attributed to the fact that an ideal amplifier does not waste power in the coupling circuits but rather adds the input voltage to the output so that the actual gain contributed by the device is only $K_v - 1$.

If the amplifier is to be constructed from vacuum tubes described by

$$\left[H^o\right] = \begin{bmatrix} 0 & 0 \\ g_m & g_p \end{bmatrix}; \qquad \left[H^1\right] = \begin{bmatrix} C_g & -C_m \\ -C_m & C_p \end{bmatrix}; \qquad \frac{g_m}{g_p} = \mu \tag{17}$$

$$M = \frac{g_m/2}{(C_m + C_g/\mu) + \left[(C_m + C_g/\mu)^2 + C_gC_p - C_m^2\right]^{1/2}} \approx \frac{g_m/2}{C_m + \left[C_gC_p\right]^{1/2}} \quad ; \; \mu >> 1 \tag{18}$$

we know that the best circuit design cannot produce more voltage gain per stage than is given by Eq. 15 with M as given by Eq. 18. In principle, it is immaterial whether we use triodes or pentodes, since the only important parameter is M. If there is grid-to-plate capacity, this can be neutralized with an external feedback network so that a triode and pentode with the same M will have the same input and output capacitance and zero effective grid-to-plate capacitance when properly neutralized. The only advantage of using a pentode in a wideband amplifier is the inherent neutralization that makes possible simplified circuit design.

To achieve the optimum response given by Eq. 15, we must couple from one stage to the next with a transformer-turns ratio of $(C_p/C_g)^{1/2}$ and neutralize with a feedback capacitor of value $C_m(C_p/C_g)^{1/2}$. Under these conditions, we find that $(K_v - 1)W = M$ but unfortunately we are obliged to settle for a K_v of $\mu(C_p/C_g)^{1/2}$. The only convenient way to reduce the gain and increase the bandwidth is to use a load resistor in the interstage network. Ideally we would like to vary the μ of the tube without changing the M but this appears to be difficult; nevertheless, it would seem worthwhile to investigate the problem of building high M tubes with a low μ so that we need not resort to distributed amplifiers for wideband operation. The distributed amplifier is inherently inefficient because the gains are additive instead of multiplicative and we shall necessarily fall short of optimum performance. The major difficulty is that we do not have anything approaching an ideal gyrator for use in coupling circuits and we cannot add the input power to the output without destroying the unilateralness of the amplifier.

The significance of the parameter M was investigated in some detail for a number of devices and circuit configurations. A more detailed report is being prepared. An important feature of the analysis described above is that the frequency variable s can be replaced by the variable $s_1 = s - s_o$, where s_o is the center frequency of a tuned amplifier. If we now determine $\left[H^o\right]$ and $\left[H^1\right]$ at the frequency s_o and apply the complex-power theorem, it is possible to determine M at the frequency s_o. In general, an amplifying device will have an M-versus-frequency plot that is positive for only a limited band of frequencies. If $M = M_o$ at a frequency $s_o = j\omega_o$, M is a measure of the gain-bandwidth limitations of a narrow-band amplifier tuned to ω_o; if M_o is negative it is not possible to build an amplifier with power gain. If we construct a single-time-constant, narrow-band amplifier with a center frequency ω_o, we can increase the gain with positive feedback. But in the limit, as the feedback is increased the gain approaches infinity and the bandwidth approaches zero in such a way that the product is always less than M_o.

In this discussion the energy stored in an amplifying device is related to the problem of building wideband, high-gain amplifiers. The parameter M, defined as one-half the maximum power that can be delivered by a device per unit energy stored in the device, is found to be a convenient measure of the gain-bandwidth capabilities of the device. M expresses an inherent device limitation and, therefore, can be used to compare the capabilities of various devices or to determine the effectiveness of a circuit in realizing optimum performance. The parameter M can be determined as a function of frequency and can be used to determine items such as voltage gain-bandwidth products, allowed frequencies of oscillation, and optimum methods of neutralizing internal feedback.

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