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A note on C -Parity Conservation and the Validity of Orientifold Planar Equivalence

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Abstract

We analyze the possibility of a spontaneous breaking of C -invariance in gauge theories with fermions in vector-like — but otherwise generic — representations of the gauge group. QCD, supersymmetric Yang–Mills theory, and orientifold field theories, all belong to this class. We argue that charge conjugation is not spontaneously broken as long as Lorentz invariance is maintained. Uniqueness of the vacuum state in pure Yang–Mills theory (without fermions) and convergence of the expansion in fermion loops are key ingredients. The fact that C -invariance is conserved has an interesting application to our proof of planar equivalence between supersymmetric Yang–Mills theory and orientifold field theory on R^4 , since it allows the use of charge conjugation to connect the large- N limit of Wilson loops in different representations.

1 Introduction

There are very few tools that enable us to explore QCD in the nonperturbative regime. Recently, building on an earlier idea due to Strassler [1], we suggested a new tool for analyzing nonperturbative QCD [2, 3]. We argued that one-flavor QCD can be approximated, within a $1/N$ error, by $\mathcal{N} = 1$ super-Yang–Mills theory [3] (for a review see [4]). The relation between QCD and super-Yang–Mills was established by observing that $SU(N)$ gauge theory with the Dirac two-index antisymmetric fermion (to be referred to as the *orientifold field theory*) is nonperturbatively equivalent to super-Yang–Mills in a well defined bosonic subsector at $N \rightarrow \infty$. Planar equivalence led to several strong predictions concerning QCD. Among them are the value of the quark condensate [5] and the degeneracy of the σ and the η' mesons [3] in one-flavor QCD. These predictions were supported by recent lattice simulations [6, 7].

In Ref. [8] we gave a formal proof of planar equivalence (see Ref. [9] for a lattice strong-coupling version of the proof). Our proof assumes, implicitly, charge conjugation invariance. The proof does *not* hold on compact spaces, such as $R^3 \times S^1$, as was first demonstrated in [10].

Recently it was pointed out [11] that a necessary and *sufficient* condition for orientifold planar equivalence to hold is the absence of spontaneous breaking of charge conjugation symmetry. To further explore this observation the authors of [11] considered the orientifold field theory on $R^3 \times S^1$ (with a small radius of S^1 , so that the one-loop analysis can be trusted) and demonstrated that C -parity *is* spontaneously broken in this case, the order parameter being the Polyakov line in the compactified direction.¹ It was concluded that planar equivalence does not hold on $R^3 \times S^1$, at least for sufficiently small radii.²

The above result was advertised (see e.g. the title of [11]) as raising doubts concerning the validity of planar equivalence on R^4 . Shortly after, the phase

¹In a revised version of the paper [11], it was argued that not only C -parity is broken, but CPT as well.

²In fact, a more careful reading of [11] implies that C parity conservation and planar equivalence are invalid only if periodic boundary conditions are imposed on the fermions. With antiperiodic boundary conditions, both C parity and planar equivalence do hold, in contrast to claims otherwise [12]. The sensitivity to the boundary conditions is a clear-cut indication that this is a finite-size effect. The effects found in [11] are just Casimir-like effects that vanish as the theory decompactifies, i.e. as $R \rightarrow \infty$.

structure of the orientifold field theory on $S^3 \times S^1$ was analyzed [13]. It was shown that, if the S^3 radius is sufficiently small so that perturbation theory can be trusted, the theory undergoes a phase transition as the S^1 radius increases. At large radius charge conjugation is restored. Although the analysis of [13] certainly cannot be trusted in the domain of large S^3 radii, it could still be considered as pointing in the opposite direction, i.e. that planar equivalence holds on R^4 . Another indication that planar equivalence holds on R^4 is provided by a recent lattice simulation [14], showing that QCD on a circle undergoes a phase transition from a C -parity violating phase to a C -parity preserving phase above a critical radius.

The purpose of this paper is to argue that C -parity does not break spontaneously in any vector-like gauge theory on R^4 . Although we will not be able to give a rigorous mathematical proof of the type known for spatial parity [15], we will present several convincing physical arguments that seem impossible to overcome.

In addition, we clarify certain aspects of our proof [8] and point out which particular aspects of the set-up of Ref. [11] are to blame for the failure of C -parity and planar equivalence on $R^3 \times S^1$.

2 C -parity in pure Yang–Mills and vector-like gauge theories

In this section we will first argue that C -parity is not spontaneously broken in Yang–Mills theories on R^4 . We will then argue that, if C is not broken in pure Yang–Mills theory, it cannot be spontaneously broken if we add fermions in vector-like (real) representations.

The impossibility of spontaneous breaking of P -parity at $\theta = 0$ was proven long ago [15]. This proof is in essence nondynamical and is based only on certain general features of Yang–Mills theories. Unlike spatial parity, the issue of spontaneous breaking of C invariance depends on the dynamics. This is the reason why we cannot prove our assertion at the same level of rigor as that of Ref. [15]. Instead, we rely on a number of independent physical arguments which exploit known features of the gauge dynamics.

Consider first pure Yang–Mills theory on the cylinder $R^3 \times S^1$ with a large radius, and the suspected order parameter for the spontaneous C -parity

breaking, the Polyakov line in the compact direction (let us call it t),

$$P = \text{Tr} \exp \left(i \int A_0 dt \right). \quad (1)$$

If C is spontaneously broken, the vacuum expectation value (VEV) of the Polyakov line will acquire an imaginary part, corresponding to two degenerate vacua with $\text{Im}\langle P \rangle = \pm K$ with K a non-vanishing constant.

However, such a nonvanishing VEV would contradict color confinement in pure Yang–Mills theory. Indeed, the order parameter (1) is simultaneously an order parameter for the group’s center. It must vanish in the confinement phase: and it does at low temperatures, a well-established fact. There is a critical temperature below which color confinement is recovered, and, correspondingly, $\langle P \rangle = 0$.

There is a subtle point in this argument. One can say that, as the radius of the cylinder grows and eventually crosses the critical value beyond which $\langle P \rangle = 0$, the Polyakov line no longer represents an appropriate order parameter for the spontaneous breaking of C invariance of the theory.

Therefore, let us look at this problem from a more general perspective. The Yang–Mills Lagrangian is C invariant. Therefore, for the spontaneous breaking to take place in pure Yang–Mills theory its vacuum structure must be nontrivial. There should exist two (more generally, an even number of) degenerate vacua with opposite C parities.

A certain amount of knowledge has been accumulated regarding the vacuum structure of pure Yang–Mills theory. From Witten’s work [16] we know that the vacuum of pure Yang–Mills theory is nondegenerate at $\theta = 0$. The vacuum energy is given by the partition function

$$\begin{aligned} Z &= \int DA_\mu \exp \left\{ - \int_{R^4} d^4x \left[-\frac{1}{2g^2} \text{Tr} F_{\mu\nu}^2 + i \frac{\theta}{16\pi^2} \text{Tr} F \tilde{F} \right] \right\} \\ &\equiv \exp(-V E(\theta)). \end{aligned} \quad (2)$$

The vacuum at $\theta = 0$ is the absolute minimum of energy with respect to other θ vacua since it is a sum of positive contributions (in the Euclidean formulation). What is of relevance for us is that it is also nondegenerate. The latter property can be seen explicitly at large N by considering the string dual to pure Yang–Mills [16].

The uniqueness (nondegeneracy) of the pure Yang–Mills vacuum can be argued from a different side, starting from a slightly broken supersymmetric gluodynamics (i.e. $\mathcal{N} = 1$ supersymmetric pure Yang–Mills theory). This theory has N degenerate vacua due to the spontaneous breaking of the Z_{2N} chiral symmetry down to Z_2 . The order parameter is the gluino condensate which was exactly calculated in [17].

When a small mass $m/\Lambda \ll 1$ is given to the gluino field, the vacuum degeneracy is lifted resulting in a theory with a nondegenerate vacuum. Pure Yang–Mills theory is recovered by taking the limit $m/\Lambda \rightarrow \infty$. If the vacua of pure Yang–Mills theory were degenerate, in order to end up with degenerate vacua starting from a nondegenerate one, one would have a phase transition to occur at an intermediate value of m . This is certainly impossible if the expansion in fermion loops (see below) is convergent.

Finally, let us note that the uniqueness of the Yang–Mills vacuum is supported by lattice simulations.

Let us now include fermions. In vector-like theories C invariance is preserved at the Lagrangian level. As in the previous case of pure Yang–Mills, spontaneous breaking requires degenerate vacua. Taking the uniqueness of the vacuum state of pure Yang–Mills for granted (at $\theta = 0$), what must one require in order to have a vacuum degeneracy after the inclusion of dynamical fermions?

Since fermions appear in the action bilinearly, it is possible to integrate them out in the general form. The resulting theory is defined by the partition function

$$\mathcal{Z} = \int DA_\mu \exp \{ -S_{\text{YM}} + \ln \det (\not{D} + m) \} , \quad (3)$$

where m is a mass parameter ($m \neq 0$, which can be viewed as an infrared regulator), and the expression (3) is given in Euclidean space. The exponent is nonlocal, due to the fermion determinant (which is taken in the appropriate representation of $SU(N)_{\text{color}}$) but this is irrelevant for our purposes.

The expansion in fermion loops is the expansion of the exponent in $\ln \det (\not{D} + m)$. The leading term in this expansion corresponds to pure Yang–Mills. If the expansion in fermion loops is convergent, the unique nondegenerate vacuum of pure Yang–Mills is inherited by the theory with quarks. It is certainly the case for sufficiently large m . A sudden switch to a divergent regime possibly accompanied by occurrence of a vacuum degeneracy at a small value of m , not supported by any symmetry, is highly unlikely. Vacuum

uniqueness implies no spontaneous breaking of C . Note, incidentally, that planar equivalence also holds at finite m : it becomes an equivalence between a non-supersymmetric theory and one with softly broken supersymmetry [4].

3 Implications for planar equivalence

According to [11], the absence of spontaneous breaking of C -parity is a necessary and sufficient condition for the validity of orientifold planar equivalence. The necessity of this condition is absolutely obvious. The very possibility of a spontaneous breaking of C -parity is related to the possible existence of two degenerate vacua of the theory transforming into each other under the action of C parity. In pure Yang–Mills theory the vacuum is unique.³

Since Dirac quarks do not alter the conclusion – charge conjugation cannot be broken on R^4 – we can assert that planar equivalence holds when the theory is formulated on R^4 . In this section we wish to elucidate this issue with regards to the “refined” proof of Ref. [8].

Let us recall the main points of [8]. It was shown there that the partition functions of $\mathcal{N} = 1$ super-Yang–Mills and the orientifold field theory, after integration over the fermions, coincide at large N . The corresponding derivation is based on the expansion of the partition functions in powers of $\ln \det (i \not{D} - m)$ and then expressing generic terms of the expansion in terms of Wilson loops⁴. Then we compare the results, term by term,

$$\langle W_{\text{SYM}} W_{\text{SYM}} \dots W_{\text{SYM}} \rangle_{\text{conn.}} = \langle W_{\text{QCD-OR}} W_{\text{QCD-OR}} \dots W_{\text{QCD-OR}} \rangle_{\text{conn.}}, \quad (4)$$

at large N . The subscripts above are self-evident.

For example, for a single Wilson loop, Eq. (4) reduces to

$$\langle W_{\text{SYM}} \rangle = \langle W_{\text{QCD-OR}} \rangle. \quad (5)$$

The reason why Eq. (5) holds is that, at large N ,

$$\langle W_{\text{SYM}} \rangle = \langle \text{Tr } U \text{ Tr } U^\dagger \rangle = \langle \text{Tr } U \rangle \langle \text{Tr } U^\dagger \rangle, \quad (6)$$

$$\langle W_{\text{QCD-OR}} \rangle = \langle \text{Tr } U \text{ Tr } U \rangle = \langle \text{Tr } U \rangle \langle \text{Tr } U \rangle, \quad (7)$$

³For simplicity in this section we set $\theta = 0$.

⁴It is essential to add a small quark mass, see comments at the end of this section.

where U is the Wilson loop in the fundamental representation. Under the assumption that C -parity is not broken,

$$\langle \text{Tr } U \rangle = \langle \text{Tr } U^\dagger \rangle,$$

and Eq. (5) is obviously satisfied. The equality of all connected correlation functions similarly follows [8] under the assumption that C -parity is not broken.

A closer look at our proof reveals that in fact we needed to make only the very mild assumption of no C -parity breaking in pure Yang–Mills theory (on R^4). This is due to the fact that all the above Wilson loop correlation functions are calculated in the Yang–Mills vacuum, and not in the vacuum of the theory with quarks. How is the presence of quarks felt here?

Since we expand in fermion loops, starting from pure Yang–Mills theory, if this expansion is convergent, the number of vacua cannot change compared to that in pure Yang–Mills theory. The latter has a unique vacuum, and so does the theory with quarks defined by this expansion. In order to ensure the convergence of the fermion loop expansion on R^4 we introduced the infrared cut-off – a mass term for the fermions. We believe that on R^4 it is sufficient, but we have to assume that the limit $m \rightarrow 0$ is nonsingular. Practically, it is impossible to think otherwise, since there are no massless particles in the theories under consideration at $m = 0$. If the vacuum is unique, certainly there is no place for the spontaneous breaking of C -parity.

There is another subtle issue which requires a mild assumption in our proof [8]. We assume that the sum of the correlation functions commutes with factorization. Lattice simulations can also shed light on this point.

4 Discussion and conclusions

The result of Ref. [11], and of the previous investigation [18], can be interpreted as follows. The expansion in fermion loops does not exist in the orientifold field theory on $R^3 \times S^1$ with periodic boundary conditions, provided the S^1 radius is small enough. Inclusion of fermions drastically changes the vacuum structure compared to that of pure Yang–Mills theory. Technically, the spontaneous breaking of C -parity in the theory on $R^3 \times S^1$, demonstrated in [11], is associated with an order parameter that breaks P invariance. Indeed, an expectation value of the Wilson line in the compactified direction

is generated. This is equivalent to the expectation value of a component of a vector. Needless to say, this is an artifact of working with a Lorentz-non-invariant theory. There are no lessons to be drawn from this analysis for Lorentz-invariant theories.

Our work was motivated by the potential importance of the claimed non-perturbative planar (large N) equivalence between supersymmetric gluodynamics and the orientifold field theory (one-flavor QCD at $N = 3$). Such an equivalence, once firmly established, will provide a unique analytic tool in studying both theories nonperturbatively. Although numerical methods are — or will become — available for light dynamical fermions, the importance of making reliable analytic predictions can hardly be overestimated.

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