

CERN-PH-TH/2007-009

Shadows and Twisted Variables

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Abstract

We explain how a new type of fields called shadows and the use of twisted variables allow for a better description of Yang–Mills supersymmetric theories.

(Based on lectures given in Cargèse, June 2006.)

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1 Introduction

Non-linear aspects and the non-existence of a supersymmetry-preserving regulator make the definition of supersymmetric theories a subtle task. We explain in these lectures notes that the introduction of new fields, called shadows, clarify the construction of Yang–Mills supersymmetric theories.

In the formalism that we develop, a supersymmetric theory is defined in terms of classical fields (gauge fields and matter fields), Faddeev–Popov ghosts and shadow fields. Gauge invariance is expressed by the BRST invariance, with a graded differential operator s . The shadows fields permit the replacement of the notion of the supersymmetry generators by that of a differential operator Q , consistent with s . The operator Q acts as an ordinary supersymmetry transformation on the gauge invariant functions of the physical fields. Moreover, there exist gauges for which Q annihilates both the classical action and the s -exact gauge-fixing action.

The advantage of having both operators s and Q acting on the extended set of fields is that two independent Slavnov–Taylor identities can be associated with supersymmetry and BRST invariances. Observables can be appropriately defined for understanding their gauge and supersymmetry covariance : they are the cohomology of the BRST symmetry. Anomalies and renormalization can be conventionally analyzed, considering insertions of arbitrary composite operators. This defines an unambiguous renormalization process of Yang–Mills supersymmetric theory, for any given choice of the regularization of divergences.

Shadows can be used to demonstrate non-renormalization theorems. Moreover, the proofs are greatly simplified by twisting the spinor fields in tensors. In fact, twisted variables permit one to determine off-shell closed sub-sectors of supersymmetry algebra that are relevant for the non-renormalization properties.

Both differential operators s and Q of supersymmetric theories satisfy extended curvature conditions, analogous to those of the topological BRST operator of topological quantum field theory. This similarity suggests that some of the relevant equations for the non-renormalization theorems have a geometrical meaning .

2 Introducing the shadow fields

To fix ideas, consider the $\mathcal{N} = 4$, $d = 4$ supersymmetric action in flat space. The physical fields of this gauge invariant theory with $SO(3,1)$ Lorentz symmetry are the

gauge field A_μ , the $SU(4)$ -Majorana spinor λ , and the six scalar fields ϕ^i in the vector representation of $SO(6) \sim SU(4)$. All fields are in the adjoint representation of a compact gauge group that we will suppose simple. The classical action is uniquely determined by supersymmetry, $Spin(3,1) \times SU(4)$ global symmetry and gauge invariance. It reads

$$S \equiv \int d^4x \text{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi^i D^\mu \phi_i + \frac{i}{2} (\bar{\lambda} \not{D} \lambda) - \frac{1}{2} (\bar{\lambda} [\phi, \lambda]) - \frac{1}{4} [\phi^i, \phi^j] [\phi_i, \phi_j] \right) \quad (1)$$

with $\phi \equiv \phi^i \tau_i$ and the supersymmetry transformations δ^{Susy}

$$\delta^{Susy} A_\mu = i(\bar{\epsilon} \gamma_\mu \lambda) \quad \delta^{Susy} \phi^i = -(\bar{\epsilon} \tau^i \lambda) \quad \delta^{Susy} \lambda = (\not{F} + i \not{D} \phi + \frac{1}{2} [\phi, \phi]) \epsilon \quad (2)$$

For the sake of convenience, we can chose the parameter ϵ as a commuting spinor. In this way, δ^{Susy^2} represents the commutator of two supersymmetry transformations, with

$$\delta^{Susy^2} \approx \delta^{\text{gauge}} (\bar{\epsilon} [\phi - iA] \epsilon) - i(\bar{\epsilon} \gamma^\mu \epsilon) \partial_\mu \quad (3)$$

Here \approx stands for the equality modulo equations of motion.

In view of the last equation, the quest of a quantum field theory with supersymmetry implies the following remarks.

The presence of equations of motion in the right-hand-side of (3) is a rather annoying technical difficulty. However, it can always be turned around in quantum field theory, by using the Batalin-Vilkowiski formalism. Moreover, as we will shortly see, even in the case where no auxiliary fields exist, it can be practically resolved in the proofs for the consistency of the quantum theory by using twisted variables.

The existence of the field dependent gauge transformation in the commutator of two supersymmetry transformations (3) is a deeper problems. It concretely implies that one cannot give sense to the notion of a δ^{Susy} -invariant gauge-fixing action. This fact explicitly shows up when one uses the Faddeev-Popov procedure. Suppose that one fixes the gauge, say in a Feynman-Landau gauge. This process is independent of supersymmetry and gives an action

$$S_{\text{gf}} = S + \int \text{Tr} \left(\frac{(\partial A)^2}{2\alpha} - \bar{\Omega} \partial D \Omega \right) \quad (4)$$

This lagrangian breaks gauge invariance in the desired way, but one cannot find a definition of δ^{Susy} acting on the scalar Faddeev-Popov ghosts Ω and $\bar{\Omega}$ that is compatible with the closure relation (3). This forbids one to define the Ward identities associated to supersymmetry with usual techniques. Therefore, one must improve the techniques currently used for ordinary global symmetries coupled to gauge invariance. Since there are

cases where an off-shell superfield formalism does not exist (in particular for the $\mathcal{N} = 4$ theory) and since no regulator exist that can maintain both supersymmetry and gauge invariance, such improvement must follow from new ideas.

One method for handling the problems caused by the gauge transformations in the closing relations for the supersymmetry transformations of classical fields is by introducing an additional anticommuting scalar field c valued in the Lie algebra of the gauge group. One can define in this way a differential operator Q out of δ^{Susy} , which is nilpotent modulo a translation [1]

$$Q^2 \approx -i(\bar{\epsilon}\gamma^\mu\epsilon)\partial_\mu \quad (5)$$

The way to do so is to define the action of Q on all the physical fields φ and c as follows

$$Q\varphi = \delta^{Susy}(\epsilon)\varphi - \delta^{\text{gauge}}(c)\varphi$$

with

$$Qc = (\bar{\epsilon}[\phi - iA]\epsilon) - c^2 \quad (6)$$

The field c will be called the shadow field, and its presence will allow one to solve at once all questions discussed above, with the conclusion that the notion of the operator δ^{Susy} must be replaced by that of the differential Q at the quantum level, in a way that is analogous to the enhancement of gauge invariance into BRST symmetry.

We see that the action of Q on the classical fields is linear in the global parameters ϵ and on the field c . Since, for the classical fields, Q is the sum of a supersymmetry transformation and a gauge transformation, δ^{Susy} invariance is the same as Q invariance for gauge invariant quantities.

The action of Q on c is quadratic both in c and ϵ , and $Q\epsilon = 0$. We have the existence of a grading equal to the shadow number, which is zero for the classical fields, and one for c and ϵ .

In practice, one must do computations with a BRST invariant gauge-fixed theory, where interacting Faddeev–Popov ghosts propagate. In fact, renormalization generally mixes gauge invariant operators with non gauge-invariant BRST-exact operators. Thus, observables must be defined through the cohomology of the BRST operator s for ordinary gauge symmetry. To control the covariance under supersymmetry of observables, the BRST Ward identity and the supersymmetry Ward identities must be disentangled. It follows that Q and s must be independent and consistent operators (i.e., Q and s must anticommute). Therefore the scalar field c cannot be identified with the Faddeev–Popov ghost Ω .

The idea of shadows [1] is thus to introduce new fields, in the form of BRST doublets, in order not to affect physical quantities, and to redefine the supersymmetry transformations of classical fields by addition of a compensating gauge transformations with a parameter equal to the shadow field c . Moreover, Eq. (5) must be satisfied for all fields.

The action of the BRST operator s on all physical fields is nothing but a gauge transformation of parameter Ω with

$$s\varphi = -\delta^{\text{gauge}}(\Omega)\varphi \quad s\Omega = -\Omega^2 \quad (7)$$

and since the shadow c must not affect the physical sector of the theory we introduce the commuting scalar μ such that (c, μ) builds a trivial BRST doublet

$$s c = \mu \quad s \mu = 0 \quad (8)$$

We want to impose Eq. (5) on all fields, as well as

$$s^2 = s Q + Q s = 0 \quad (9)$$

In fact, by a direct computation, we find that the algebra (5) and (9) is satisfied with

$$Q\Omega = -\mu - [c, \Omega] \quad Q\mu = -[(\bar{\epsilon}\phi\epsilon), \Omega] + i(\bar{\epsilon}\gamma^\mu\epsilon)D_\mu\Omega - [c, \mu] \quad (10)$$

We will shortly write a curvature equation that explains these transformation laws, and in particular the property

$$s c + Q\Omega + [c, \Omega] = 0 \quad (11)$$

In order to define the Ward identities associated to supersymmetry, we need a BRST-exact gauge-fixing that is Q -invariant. Such gauge-fixing will be said to be supersymmetric. To define it, we introduce the trivial quartet $\bar{\mu}$, \bar{c} , $\bar{\Omega}$, b , with

$$\begin{aligned} s\bar{\mu} &= \bar{c} & s\bar{c} &= 0 & s\bar{\Omega} &= b & sb &= 0 \\ Q\bar{\mu} &= \bar{\Omega} & Q\bar{c} &= -b & Q\bar{\Omega} &= -i(\bar{\epsilon}\gamma^\mu\epsilon)\partial_\mu\bar{\mu} & Qb &= i(\bar{\epsilon}\gamma^\mu\epsilon)\partial_\mu\bar{c} \end{aligned} \quad (12)$$

The quantum field theory has an internal bigrading, the ordinary ghost number and the new shadow number. The Q transformation of fields depend on the constant commuting supersymmetry parameter. The latter is understood as an ordinary gauge parameter for the quantum field theory, but observables will not depend on them, owing to BRST invariance.

3 Supersymmetric shadow dependent lagrangians

In order to control supersymmetry and renormalize the theory, we start from a renormalizable s and Q invariant gauge-fixed action, which determines the Feynman rules. A class of such actions is of the form:

$$S_{\text{gf}}[\varphi, \Omega, \bar{\Omega}, b, c, \bar{c}, \mu, \bar{\mu}] = S[\varphi] - s Q \int \text{Tr} \bar{\mu} \left(\partial A + \frac{\alpha}{2} b \right) \quad (13)$$

One has indeed

$$\begin{aligned} - s Q \int \text{Tr} \bar{\mu} \left(\partial A + \frac{\alpha}{2} b \right) &= - s \int \text{Tr} \left(\bar{\Omega} \left(\partial A + \frac{\alpha}{2} b \right) + \bar{\mu} Q \left(\partial A + \frac{\alpha}{2} b \right) \right) \\ &= \int \text{Tr} \left(-\frac{\alpha}{2} b^2 - b \partial A - \bar{\Omega} \partial D \Omega + \dots \right) \end{aligned} \quad (14)$$

Here, the dots stand for terms that imply a propagation of the pairs of shadows $\mu, \bar{\mu}$ and c, \bar{c} . They are given by an easy computation. They imply ϵ -dependent propagators and vertices. However, observables are defined by the cohomology of the BRST operator s , so that their expectation values are independent on the values of ϵ , since the later occur through an s -exact term.

In the absence of anomaly, one can enforce both Ward identities for the s and Q invariances. This means that one can concretely impose renormalization conditions which enforce these identities at any given finite order of perturbation theory, within the framework of any type of regularization for divergences.

The prize one has to pay for having shadows is that they generate a perturbative theory with more Feynman diagrams. If we consider physical composite operators that mix through renormalization with BRST-exact operators, the latter can depend on all possible fields that propagate, and we have in principle to consider a dependence on the whole set of fields in order to compute the supersymmetry-restoring non-invariant counterterms. For certain “simple” Green functions, which cannot mix with BRST-exact composite operators, there exist gauges in which some of the additional fields can be integrated out, in a way that justifies, a posteriori, the work of Stöckinger et al. for the $\mathcal{N} = 1$ theories [2]. By doing this elimination, one loses the algebraic meaning, but one may gain in computational simplicity.

The shadow dependent methodology is suitable for non-ambiguously computing the non-invariant counterterms that maintain supersymmetry, BRST invariance and the R-symmetry. It applies to the renormalization of all supersymmetric theories.

4 Renormalization

4.1 Ward identities for the theory

By introducing sources associated to the non-linear s , Q and sQ transformations of fields, we get the following ϵ -dependent action, which initiates a BRST-invariant supersymmetric perturbation theory¹

$$\begin{aligned} \Sigma \equiv & \frac{1}{g^2} S - \int d^4x \text{Tr} \left(b \partial^\mu A_\mu + \frac{\alpha}{2} b^2 - \bar{c} \partial^\mu (D_\mu c + i(\bar{\epsilon} \gamma_\mu \lambda)) - \frac{i\alpha}{2} (\bar{\epsilon} \gamma^\mu \epsilon) \bar{c} \partial_\mu \bar{c} \right. \\ & \left. + \bar{\Omega} \partial^\mu D_\mu \Omega - \bar{\mu} \partial^\mu (D_\mu \mu + [D_\mu \Omega, c] - i(\bar{\epsilon} \gamma_\mu [\Omega, \lambda])) \right) \\ & + \int d^4x \text{Tr} \left(A_\mu^{(s)} D^\mu \Omega + \bar{\lambda}^{(s)} [\Omega, \lambda] - \phi_i^{(s)} [\Omega, \phi^i] + A_\mu^{(Q)} Q A^\mu - \bar{\lambda}^{(Q)} Q \lambda + \phi_i^{(Q)} Q \phi^i \right. \\ & \left. + A_\mu^{(Qs)} s Q A^\mu - \bar{\lambda}^{(Qs)} s Q \lambda + \phi_i^{(Qs)} s Q \phi^i + \Omega^{(s)} \Omega^2 - \Omega^{(Q)} Q \Omega - \Omega^{(Qs)} s Q \Omega \right. \\ & \left. - c^{(Q)} Q c + \mu^{(Q)} Q \mu + \frac{g^2}{2} (\bar{\lambda}^{(Q)} - [\bar{\lambda}^{(Qs)}, \Omega]) M(\lambda^{(Q)} - [\lambda^{(Qs)}, \Omega]) \right) \quad (15) \end{aligned}$$

Because of the s and Q invariances, the action is invariant under the both Slavnov–Taylor identities defined in [1], which are associated respectively to gauge and supersymmetry invariance, $\mathcal{S}_{(s)}(\Sigma) = \mathcal{S}_{(Q)}(\Sigma) = 0$. For the sake of illustration, let us present the supersymmetry Slavnov–Taylor operator of the $\mathcal{N} = 4$ theory²

$$\begin{aligned} \mathcal{S}_{(Q)}(\mathcal{F}) \equiv & \int d^4x \text{Tr} \left(\frac{\delta^R \mathcal{F}}{\delta A^\mu} \frac{\delta^L \mathcal{F}}{\delta A_\mu^{(Q)}} + \frac{\delta^R \mathcal{F}}{\delta \lambda} \frac{\delta^L \mathcal{F}}{\delta \bar{\lambda}^{(Q)}} + \frac{\delta^R \mathcal{F}}{\delta \phi^i} \frac{\delta^L \mathcal{F}}{\delta \phi_i^{(Q)}} + \frac{\delta^R \mathcal{F}}{\delta c} \frac{\delta^L \mathcal{F}}{\delta c^{(Q)}} + \frac{\delta^R \mathcal{F}}{\delta \mu} \frac{\delta^L \mathcal{F}}{\delta \mu^{(Q)}} \right. \\ & + \frac{\delta^R \mathcal{F}}{\delta \Omega} \frac{\delta^L \mathcal{F}}{\delta \Omega^{(Q)}} - A_\mu^{(s)} \frac{\delta^L \mathcal{F}}{\delta A_\mu^{(Qs)}} + \bar{\lambda}^{(s)} \frac{\delta^L \mathcal{F}}{\delta \bar{\lambda}^{(Qs)}} - \phi_i^{(s)} \frac{\delta^L \mathcal{F}}{\delta \phi_i^{(Qs)}} + \Omega^{(s)} \frac{\delta^L \mathcal{F}}{\delta \Omega^{(Qs)}} - b \frac{\delta^L \mathcal{F}}{\delta \bar{c}} + \bar{\Omega} \frac{\delta^L \mathcal{F}}{\delta \bar{\mu}} \\ & - i(\bar{\epsilon} \gamma^\mu \epsilon) \left(-\partial_\mu A_\nu^{(Qs)} \frac{\delta^L \mathcal{F}}{\delta A_\nu^{(s)}} + \partial_\mu \bar{\lambda}^{(Qs)} \frac{\delta^L \mathcal{F}}{\delta \bar{\lambda}^{(s)}} - \partial_\mu \phi_i^{(Qs)} \frac{\delta^L \mathcal{F}}{\delta \phi_i^{(s)}} + \partial_\mu \Omega^{(Qs)} \frac{\delta^L \mathcal{F}}{\delta \Omega^{(s)}} - \partial_\mu \bar{c} \frac{\delta^L \mathcal{F}}{\delta b} + \partial_\mu \bar{\mu} \frac{\delta^L \mathcal{F}}{\delta \bar{\Omega}} \right. \\ & \left. + A_\nu^{(Q)} \partial_\mu A^\nu + \bar{\lambda}^{(Q)} \partial_\mu \lambda + \phi_i^{(Q)} \partial_\mu \phi^i + \Omega^{(Q)} \partial_\mu \Omega + c^{(Q)} \partial_\mu c + \mu^{(Q)} \partial_\mu \mu \right) \quad (16) \end{aligned}$$

If no anomaly occurs, the Slavnov–Taylor identities $\mathcal{S}_{(s)}(\Gamma) = \mathcal{S}_{(Q)}(\Gamma) = 0$ completely determines all ambiguities of the supersymmetric effective action Γ , order by order in perturbation theory.

¹ M is the 32×32 matrix $M \equiv \frac{1}{2}(\bar{\epsilon} \gamma^\mu \epsilon) \gamma_\mu + \frac{1}{2}(\bar{\epsilon} \tau_i \epsilon) \tau^i - \epsilon \bar{\epsilon}$. It occurs because Q^2 is a pure derivative only modulo equations of motion. The dimension of A_μ , λ , ϕ^i , Ω , $\bar{\Omega}$, b , μ , $\bar{\mu}$, c and \bar{c} are respectively 1, $\frac{3}{2}$, 1, 0, 2, 2, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$. Their ghost and shadow numbers are respectively (0, 0), (0, 0), (0, 0), (1, 0), (-1, 0), (0, 0), (1, 1), (-1, -1), (0, 1) and (0, -1).

² The linearized Slavnov–Taylor operator $\mathcal{S}_{(Q)}|_\Sigma$ [1] verifies $\mathcal{S}_{(Q)}|_\Sigma^2 = -i(\bar{\epsilon} \gamma^\mu \epsilon) \partial_\mu$, which solves in practice the fact that Q^2 is a pure derivative only modulo equations of motion.

4.2 Anomalies

In [1, 3], we showed the absence of anomaly for the $\mathcal{N} = 2, 4$ and the stability of the $\mathcal{N} = 1, 2, 4$ action Σ under renormalization. Thus, all Green functions of the complete theory involving shadows and ghosts can be renormalized, in any given regularization scheme, so that supersymmetry and gauge invariance are preserved at any given finite order.

Let us sketch the proof that no supersymmetry anomaly can exist for $\mathcal{N} = 2, 4$, and that for $\mathcal{N} = 1$ the only possible anomaly is the Adler–Bardeen anomaly.

An anomaly in a supersymmetry theory can only occur if a pair of local functionals \mathcal{A} and \mathcal{B} of the fields and sources can violate the pair of Ward identities for both s and Q invariances. For instance, when one renormalizes the theory at the one-loop level, the result of the computation can violate the Ward identities by local terms \mathcal{A} and \mathcal{B} , as follows

$$\mathcal{S}_{(s)|\Sigma}\Gamma^{1\text{ loop}} = \hbar \int \mathcal{A} \quad \mathcal{S}_{(Q)|\Sigma}\Gamma^{1\text{ loop}} = \hbar \int \mathcal{B} \quad (17)$$

If either \mathcal{A} and \mathcal{B} cannot be eliminated by adding local counterterms to $\Gamma^{1\text{ loop}}$, which means that they are not $\mathcal{S}_{(s)|\Sigma}$ and $\mathcal{S}_{(Q)|\Sigma}$ exact, one has an anomaly, and the theory cannot be renormalized while maintaining either supersymmetry or gauge invariance, or both. In [1, 3], we proved that the solution \mathcal{A} and \mathcal{B} of Eq. (17), modulo $\mathcal{S}_{(s)|\Sigma}$ and $\mathcal{S}_{(Q)|\Sigma}$ exact terms, can only depend on the fields, and thus, the consistency relation for s and Q implies:

$$s \int \mathcal{A} = 0 \quad Q \int \mathcal{A} + s \int \mathcal{B} = 0 \quad Q \int \mathcal{B} = 0 \quad (18)$$

In fact, the first equation implies that \mathcal{A} must be the consistent Adler-Bardeen anomaly, which descends formally from the Chern class $\text{Tr } FFF$. But then, the Q symmetry is so demanding that the second and third equations have no solution $\mathcal{B} \neq 0$ for $\mathcal{N} = 2, 4$. Thus there cannot be an anomaly for these cases. For $\mathcal{N} = 1$, the constraint is weaker, and the Adler-Bardeen anomaly admits a supersymmetric counterpart \mathcal{B} . However, the Adler–Bardeen theorem holds, and if the one-loop coefficient of the Adler–Bardeen anomaly cancels, it will cancel to all order.

Of course, these are well known facts. However, by having introduced the shadows, both Ward identities for supersymmetry and gauge invariance allow a safe verification of the status of gauge and supersymmetry anomalies by the standard consistency argument, valid to all order of perturbation theory.

4.3 Ward identities for the observables

Observables of a super-Yang–Mills theory are Green functions of local operators in the cohomology of the BRST linearized Slavnov–Taylor operator $\mathcal{S}_{(\iota)|\Sigma}$. From this definition, these Green functions are independent of the gauge parameters of the action, including ϵ . Classically, they are represented by gauge-invariant polynomials of the physical fields [1, 4]. We introduce classical sources u for all these operators. We must generalize the supersymmetry Slavnov–Taylor identity for the extended local action that depends on these sources. Since the supersymmetry algebra does not close off-shell, other sources v , coupled to unphysical $\mathcal{S}_{(\iota)|\Sigma}$ -exact operators, must also be introduced. We define the following field and source combinations φ^*

$$\begin{aligned} A_\mu^* &\equiv A_\mu^{(Q)} - \partial_\mu \bar{c} - [A_\mu^{(Q_s)} - \partial_\mu \bar{\mu}, \Omega] & c^* &\equiv c^{(Q)} - [\mu^{(Q)}, \Omega] \\ \phi_i^* &\equiv \phi_i^{(Q)} - [\phi_i^{(Q_s)}, \Omega] & \lambda^* &\equiv \lambda^{(Q)} - [\lambda^{(Q_s)}, \Omega] \end{aligned} \quad (19)$$

They verify $\mathcal{S}_{(\iota)|\Sigma} \varphi^* = -[\Omega, \varphi^*]$. The collection of local operators coupled to the v 's is made of all possible gauge-invariant (i.e. $\mathcal{S}_{(\iota)|\Sigma}$ -invariant) polynomials in the physical fields and the φ^* 's. These operators have ghost number zero, and their shadow number is negative, in contrast with the physical gauge-invariant operators, which have shadow number zero.

The relevant action is thus $\Sigma[u, v] \equiv \Sigma + \Upsilon[u, v]$, with

$$\begin{aligned} \Upsilon[u, v] &\equiv \int d^4x \left(u_{ij} \frac{1}{2} \text{Tr} \phi^i \phi^j + u_i^\alpha \text{Tr} \phi^i \lambda_\alpha + u_{ijk} \frac{1}{3} \text{Tr} \phi^i \phi^j \phi^k \right. \\ &\quad + \kappa u_{ij}^\mu \text{Tr} (i \phi^{[i} D_\mu \phi^{j]}) + \frac{1}{8} \bar{\lambda} \gamma_\mu \tau^{ij} \lambda) + \kappa u_i^{\mu\nu} \text{Tr} (F_{\mu\nu} \phi^i - \frac{1}{2} \bar{\lambda} \gamma_{\mu\nu} \tau^i \lambda) + \kappa u_\mu^5 \frac{1}{2} \text{Tr} \bar{\lambda} \gamma_5 \gamma^\mu \lambda \\ &\quad + c u_{ijk} \text{Tr} (\frac{1}{3} \phi^{[i} \phi^j \phi^{k]}) + \frac{1}{8} \bar{\lambda} \tau^{ijk} \lambda) + c u_{ij}^\mu \text{Tr} (i \phi^{[i} D_\mu \phi^{j]}) - \frac{1}{4} \bar{\lambda} \gamma_\mu \tau^{ij} \lambda) \\ &\quad + c u_i^{\mu\nu} \text{Tr} (F_{\mu\nu} \phi^i + \frac{1}{4} \bar{\lambda} \gamma_{\mu\nu} \tau^i \lambda) + u_{ij}^\alpha \text{Tr} \phi^i \phi^j \lambda_\alpha + i u_i^{\mu\alpha} \text{Tr} D_\mu \phi^i \lambda_\alpha + u^{\mu\nu\alpha} \text{Tr} F_{\mu\nu} \lambda_\alpha + \dots \\ &\quad + v_i^\alpha \text{Tr} \phi^i \lambda_\alpha^* + v^{\alpha\beta} \text{Tr} \lambda_\alpha \lambda_\beta^* + v_i^\mu \text{Tr} \phi^i A_\mu^* + v_{ij} \text{Tr} \phi^i \phi^{*j} + i v_i^{\mu\alpha} \text{Tr} D_\mu \phi^i \lambda_\alpha^* \\ &\quad \left. + v_i^\alpha \text{Tr} \lambda_\alpha \phi^{*i} + i v^{\mu\alpha\beta} \text{Tr} D_\mu \lambda_\alpha \lambda_\beta^* + i v_{ij}^\mu \text{Tr} D_\mu \phi^i \phi^{*j} + i^{-1} v_i^{\mu\alpha} \text{Tr} D_\mu \lambda_\alpha \phi^{*i} + \dots \right) \quad (20) \end{aligned}$$

Here, the \dots stand for all other analogous operators.

The Slavnov–Taylor operator $\mathcal{S}_{(Q)}$ can be generalized into a new one, $\mathcal{S}_{(Q)}^{\text{ext}}$, by addition of terms that are linear in the functional derivatives with respect to the sources u and v , in such a way that

$$\mathcal{S}_{(Q)}^{\text{ext}}(\Sigma[u, v]) = \mathcal{S}_{(Q)}(\Sigma) + \mathcal{S}_{(Q)|\Sigma}^{\text{ext}} \Upsilon + \int d^4x \text{Tr} \left(\frac{\delta^R \Upsilon}{\delta A^\mu} \frac{\delta^L \Upsilon}{\delta A_\mu^*} + \frac{\delta^R \Upsilon}{\delta \lambda} \frac{\delta^L \Upsilon}{\delta \lambda^*} + \frac{\delta^R \Upsilon}{\delta \phi^i} \frac{\delta^L \Upsilon}{\delta \phi_i^*} \right) = 0 \quad (21)$$

Indeed, if we were to compute $\mathcal{S}_{(Q)}(\Sigma[u, v])$ without taking into account the transformations of the sources u and v , the breaking of the Slavnov–Taylor identity would be a local functional linear in the set of gauge-invariant local polynomials in the physical fields, A_μ^* , c^* , ϕ_i^* and λ^* .

Eq. (21) defines the transformations $\mathcal{S}_{(Q)|\Sigma}^{\text{ext}}$ of the sources u and v . Simplest examples for the transformation laws of the u 's are for instance

$$\begin{aligned}\mathcal{S}_{(Q)|\Sigma}^{\text{ext}} u_{ij} &= -i[\gamma^\mu \tau_{\{i} \epsilon\}_{\alpha}} \partial_\mu u_j^\alpha + \partial_\mu \partial^\mu v_{\{ij\}} + 2u_{\{i|k} v_{j\}}^k + 2u_{\{i} v_{j\}}^\alpha - i\partial_\mu (u_{\{i|k} v_{j\}}^{\mu k} + u_{\{i} v_{j\}}^{\mu \alpha}) \\ \mathcal{S}_{(Q)|\Sigma}^{\text{ext}} u_i^\alpha &= [\bar{\epsilon} \tau^j]^\alpha (u_{ij} - i\partial_\mu ({}^K u_{ij}^\mu + {}^C u_{ij}^\mu)) - 2i[\bar{\epsilon} \gamma_\mu]^\alpha \partial_\nu ({}^K u_i^{\mu\nu} + {}^C u_i^{\mu\nu}) + i[\gamma^\mu]_\beta^\alpha \partial_\mu v_i^\beta \\ &\quad - u_{ij}{}^0 v^{j\alpha} - u_j^\alpha v_i^j + u_i^\beta v^\alpha{}_\beta + u^{\alpha\beta} v_{i\beta} + i\partial_\mu (u_{ij}{}^{-1} v^{j\mu\alpha} - u_i^\beta v_\beta^{\mu\alpha})\end{aligned}\quad (22)$$

These transformations are quite complicated in their most general expression. However, for many practical computations of non-supersymmetric local counterterms, we can consider them at $v = 0$. We define $Qu \equiv (\mathcal{S}_{(Q)|\Sigma}^{\text{ext}} u)_{|v=0}$. By using $\delta^{usy} \Upsilon[u] + \Upsilon[Qu] = 0$ we can in fact conveniently compute Qu . Notice that Q is not nilpotent on the sources, but we have the result that $\Upsilon[Q^2 u]$ is a linear functional of the equation of motion of the fermion λ .

It is a well-defined process to compute all observables, provided that a complete set of sources has been introduced. This lengthy process cannot be avoided because there exists no regulator that preserves both gauge invariance and supersymmetry. We must keep in mind that renormalization generally mixes physical observables with BRST-exact operators, and a careful analysis must be done [5].

5 Enforcement of supersymmetry

Once both Ward identities for the Green functions of fields and of observables have been established, it is a straightforward (but tedious) task to adjust the counterterms that are necessary to ensure supersymmetry and gauge symmetry at the quantum level. The possibility of that is warranted by the fact the theory is renormalizable by power counting, that no anomaly exist, and that the lagrangian is stable. The technical details are given in [6]. The question of not having a regulator that maintains supersymmetry is irrelevant. However, in practice, one wishes to preserve the symmetry of the bare action as much as it is possible, and thus, one uses dimensional reduction regularization, as in [7].

6 Twisted variables

Using twisted variables for the spines in four dimensions allows one to extract subalgebra of supersymmetry transformations that close without using equations of motion [3]. This property allows one to greatly simplify the proofs of finiteness in supersymmetric theories. Before coming to this point, let us sketch the way the twist works for the $\mathcal{N} = 4$ theory, by choosing the so-called first twist of this theory.

6.1 $\mathcal{N} = 4$ super-Yang–Mills theory in the twisted variables

The components of spinor and scalar fields λ^α and ϕ^i can be twisted, i.e., decomposed on irreducible representations of the following subgroup³

$$SU(2)_+ \times \text{diag}(SU(2)_- \times SU(2)_R) \times U(1) \subset SU(2)_+ \times SU(2)_- \times SL(2, \mathbb{H}) \quad (23)$$

We redefine $SU(2) \cong \text{diag}(SU(2)_- \times SU(2)_R)$. The $\mathcal{N} = 4$ multiplet is decomposed as follows

$$(A_\mu, \Psi_\mu, \eta, \chi^I, \Phi, \bar{\Phi}) \quad (L, h_I, \bar{\Psi}_\mu, \bar{\eta}, \bar{\chi}_I) \quad (24)$$

In this equation, the vector index μ is a “twisted world index”, which stands for the $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2)_+ \times SU(2)$. The index I is for the adjoint representation of the diagonal $SU(2)$. In fact, any given field X^I can be identified as a twisted antiselfdual 2-form $X_{\mu\nu^-}$,

$$X_{\mu\nu^-} \sim X_I \quad (25)$$

by using the flat hyperKähler structure $J_{\mu\nu}^I$.

All 16 components of the $SL(2, \mathbb{H})$ -Majorana spinors can therefore be mapped on the following multiplets of tensors.

$$\lambda \rightarrow (\Psi_\mu^{(1)}, \bar{\Psi}_\mu^{(-1)}, \chi_I^{(-1)}, \bar{\chi}_I^{(1)}, \eta^{(-1)}, \bar{\eta}^{(1)}) \quad (26)$$

The scalars ϕ^i in the fundamental representation of $SO(6)$ decompose as follows

$$\phi^i \rightarrow (\Phi^{(2)}, \bar{\Phi}^{(-2)}, L^{(0)}, h_I^{(0)}) \quad (27)$$

where the superscript states for the $U(1)$ representation. The 16 generators of the supersymmetry algebra and the corresponding parameter ϵ are respectively twisted into

$$\mathcal{Q}^{(1)}, \bar{\mathcal{Q}}^{(-1)}, \mathcal{Q}_\mu^{(1)}, \bar{\mathcal{Q}}_\mu^{(-1)}, \mathcal{Q}_I^{(1)}, \bar{\mathcal{Q}}_I^{(-1)} \quad (28)$$

³Usually, one means by twist a redefinition of the energy momentum tensor that we do not consider here.

and

$$\epsilon \rightarrow (\omega^{(1)}, \varpi^{(-1)}, \varepsilon^{(1)\mu}, \bar{\varepsilon}^{(-1)\mu}, v^{(1)I}, \bar{v}^{(-1)I}) \quad (29)$$

with

$$\delta^{susy} = \varpi \mathcal{Q} + \omega \bar{\mathcal{Q}} + \bar{\varepsilon}^\mu \mathcal{Q}_\mu + \varepsilon^\mu \bar{\mathcal{Q}}_\mu + \bar{v}^I \mathcal{Q}_I + v^I \bar{\mathcal{Q}}_I \quad (30)$$

The ten-dimensional super-Yang–Mills theory determines by dimensional reduction the untwisted $\mathcal{N} = 4$ super-Yang–Mills theory. Analogously, the twisted eight-dimensional $\mathcal{N} = 2$ theory determines the twisted formulation of the $\mathcal{N} = 4$ super-Yang–Mills theory in four dimensions by dimensional reduction [8, 3].

The twisted $\mathcal{N} = 2$, $d = 8$ symmetry contains a maximal supersymmetry subalgebra that closes without the equations of motion. It depends on nine twisted supersymmetry parameters, which are one scalar ϖ and one eight-dimensional vector ε^M .

By dimensional reduction (ϖ, ε^M) decomposes into $(\varpi, \varepsilon^\mu, \omega, v^I)$ and the off-shell representation of supersymmetry remains. The dimensionally reduced four-dimensional supersymmetry with 9 parameters is

$$\delta^{susy} = \varpi \mathcal{Q} + \omega \bar{\mathcal{Q}} + \varepsilon^\mu \bar{\mathcal{Q}}_\mu + v^I \bar{\mathcal{Q}}_I \quad (31)$$

It closes independently of equations of motions to

$$\delta^{susy^2} = \delta^{\text{gauge}}(\hat{\Phi}(\phi) + \varpi \varepsilon^\mu A_\mu) + \varpi \varepsilon^\mu \partial_\mu \quad (32)$$

with

$$\hat{\Phi}(\phi) \equiv \varpi^2 \Phi + \omega \varpi L + \varpi v^I h_I + (\omega^2 + \varepsilon^\mu \varepsilon_\mu + v^I v_I) \bar{\Phi} \quad (33)$$

Moreover, using the extended nilpotent differential $d + s + Q - \varpi i_\varepsilon$, the action of Q and s on all fields is simply given by the definition of the following extended curvature

$$\mathcal{F} \equiv (d + s + Q - \varpi i_\varepsilon)(A + \Omega + c) + (A + \Omega + c)^2 = F + \hat{\Psi}(\lambda) + \hat{\Phi}(\phi) \quad (34)$$

and the Bianchi relation that it satisfies

$$(d + s + Q - \varpi i_\varepsilon)\mathcal{F} + [A + \Omega + c, \mathcal{F}] = 0 \quad (35)$$

Here the linear function of the gluini $\hat{\Psi}(\lambda)$ is⁴

$$\hat{\Psi}(\lambda) \equiv \varpi \Psi + \omega \bar{\Psi} + v^I J_I(\bar{\Psi}) + g(\varepsilon)\eta + i_\varepsilon \chi \quad (36)$$

⁴Given a vector field V , one defines the 1-form $g(V) \equiv g_{\mu\nu} V^\mu dx^\nu$, and the vector $(J_I(V))^\mu \equiv J_I^\mu{}_\nu V^\nu$. $\hat{\Psi}(\lambda)_\mu$ can be written $\varpi \Psi_\mu + \omega \bar{\Psi}_\mu + v_{\mu\nu} \bar{\Psi}^\nu - \varepsilon_\mu \eta + \varepsilon^\nu \chi_{\mu\nu}$.

Eqs. (34) and (35) determine respectively the action of Q and s on A , c , Ω and on the fields on the right-hand-side of Eq. (34), by expansion in form degree.

Few degenerate component equations occur when solving Eqs. (34) and (35). They are solved by introducing the fields $\bar{\chi}_I$ and $\bar{\eta}$, the auxiliary fields H^I, T_μ and the shadow field μ . Notice that the auxiliary fields H_I and T_μ , carry a total of $7 = 3 + 4$ degrees of freedom. The latter compensate the deficit between the number of off-shell gauge-invariant degrees of freedom of fermions and bosons in the theory.

Eqs. (34) and (35) determine δ^{Susy} as

$$\begin{aligned}
\delta^{Susy} A &= \varpi \Psi + \omega \bar{\Psi} + g(\varepsilon)\eta + g(J_I \varepsilon)\chi^I + v_I J^I(\bar{\Psi}) \\
\delta^{Susy} \Psi &= -\varpi d_A \Phi - \omega(d_A L + T) + i_\varepsilon F + g(J_I \varepsilon)H^I + g(\varepsilon)[\Phi, \bar{\Phi}] - v_I(d_A h^I + J^I(T)) \\
\delta^{Susy} \Phi &= -\omega \bar{\eta} + i_\varepsilon \Psi - v_I \bar{\chi}^I \\
\delta^{Susy} \bar{\Phi} &= \varpi \eta \\
\delta^{Susy} \eta &= \varpi[\Phi, \bar{\Phi}] - \omega[\bar{\Phi}, L] + \mathcal{L}_\varepsilon \bar{\Phi} - v_I[\bar{\Phi}, h^I] \\
\delta^{Susy} \chi^I &= \varpi H^I + \omega[\bar{\Phi}, h^I] + \mathcal{L}_{J_I \varepsilon} \bar{\Phi} - v_I[\bar{\Phi}, L] + \varepsilon^I{}_{JK} v^J[\bar{\Phi}, h^K] \\
\delta^{Susy} H^I &= \varpi[\Phi, \chi^I] + \omega([L, \chi^I] - [\eta, h^I] - [\bar{\Phi}, \bar{\chi}^I]) - \mathcal{L}_{J_I \varepsilon} \eta - [\bar{\Phi}, i_{J_I \varepsilon} \Psi] + \mathcal{L}_\varepsilon \chi^I \\
&\quad + v_J[h^J, \chi^I] + v^I([\eta, L] + [\bar{\Phi}, \bar{\eta}]) - \varepsilon^I{}_{JK} v^J([\eta, h^K] + [\bar{\Phi}, \bar{\chi}^K]) \\
\delta^{Susy} L &= \varpi \bar{\eta} - \omega \eta + i_\varepsilon \bar{\Psi} - v_I \chi^I \\
\delta^{Susy} \bar{\eta} &= \varpi[\Phi, L] + \omega[\Phi, \bar{\Phi}] + \mathcal{L}_\varepsilon L + i_\varepsilon T + v_I(H^I + [h^I, L]) \\
\delta^{Susy} \bar{\Psi} &= \varpi T - \omega d_A \bar{\Phi} - g(\varepsilon)[\bar{\Phi}, L] + g(J_I \varepsilon)[\bar{\Phi}, h^I] + v_I J^I(d_A \bar{\Phi}) \\
\delta^{Susy} T &= \varpi[\Phi, \bar{\Psi}] + \omega(-d_A \eta - [\bar{\Phi}, \Psi] + [L, \bar{\Psi}]) - g(\varepsilon)([\eta, L] + [\bar{\Phi}, \bar{\eta}]) \\
&\quad + g(J_I \varepsilon)([\eta, h^I] + [\bar{\Phi}, \bar{\chi}^I]) + \mathcal{L}_\varepsilon \bar{\Psi} + v_I([h^I, \bar{\Psi}] + J^I(d_A \eta + [\bar{\Phi}, \bar{\Psi}])) \\
\delta^{Susy} h^I &= \varpi \bar{\chi}^I + \omega \chi^I - i_{J_I \varepsilon} \bar{\Psi} - v^I \eta - \varepsilon^I{}_{JK} v^J \chi^K \\
\delta^{Susy} \bar{\chi}^I &= \varpi[\Phi, h^I] + \omega([L, h^I] - H^I) + \mathcal{L}_\varepsilon h^I - i_{J_I \varepsilon} T + v^I[\Phi, \bar{\Phi}] + v_J[h^J, h^I] + \varepsilon^I{}_{JK} v^J H^K
\end{aligned} \tag{37}$$

One can verify that, for $T_\mu = H_I = 0$, the transformation laws of δ^{Susy} in Eq. (37) are the on-shell transformation laws of the twisted $\mathcal{N} = 4$ supersymmetry. It is quite remarkable that the supersymmetry transformations are the solution of the curvature equation (34) and its Bianchi identity (35). As we will shortly sketch, these equations play a key role in non-renormalization theorems.

6.2 Protected operators

Superconformal invariance implies that the so-called BPS local operators are protected from renormalization and their anomalous dimensions vanish to all orders in perturbation

theory [9]. In the $\mathcal{N} = 4$ theory, these operators play an important role for the *AdS/CFT* correspondence, since their non-renormalization properties allows to test the conjecture.

One wishes to prove that, without the assumption of the superconformal symmetry, $\mathcal{N} = 4$ supersymmetry implies that all 1/2 BPS primary operators, and thus all their descendants, have zero anomalous dimension. We will sketch the proof of this statement using only Ward identities associated to gauge and supersymmetry invariance. The 1/2 BPS primary operators are the gauge-invariant polynomials in the scalar fields of the theory in traceless symmetric representations of the $SO(5, 1)$ R-symmetry group.

In the gauge $\varepsilon^\mu = 0$ the operator Q is nilpotent.⁵ The linear function of the scalar fields $\hat{\Phi}(\phi)$ that characterizes the field dependent gauge transformations that appear in the commutators of two supersymmetries, depends in this case on five parameters,

$$\hat{\Phi}(\phi) = \varpi^2 \Phi + \varpi \omega L + \varpi v_I h^I + (\omega^2 + v_I v^I) \bar{\Phi} \quad (38)$$

The decomposition under the independent functions of the supersymmetric parameters of the invariant polynomial \mathcal{P} in $\hat{\Phi}(\phi)$ gives all the gauge invariant polynomials in the scalar fields that belongs to traceless symmetric representations of $SO(5, 1)$ [3]. Since Q is nilpotent with the restricted set of parameters, the shadow number 2 component of the curvature equation (34) is also a curvature equation

$$Qc + c^2 = \hat{\Phi}(\phi) \quad (39)$$

By comparison with the Baulieu–Singer curvature equation in TQFT’s, one interprets c as the component of the connexion of the space of gauge orbits along the fundamental vector field generating supersymmetry and $\hat{\Phi}(\phi)$ as the component of its curvature along the same fundamental vector field.⁶ The Chern–Simons formula then implies that any given invariant polynomial $\mathcal{P}(\hat{\Phi})$ can be written as a Q -exact term

$$\mathcal{P}(\hat{\Phi}(\phi)) = Q \Delta(c, \hat{\Phi}(\phi)) \quad (40)$$

⁵Remember that the supersymmetry parameters appearing in the differential Q can be understood in quantum field theory as gauge parameters of the Q -invariant gauge-fixing action.

⁶By this we mean the following. Given ω as the connection of the fiber bundle defined as the direct sum of the space of irreducible connexions and the space of matter fields of the theory, on which the group of pointed gauge transformations acts freely. Define Φ as the corresponding curvature $s\omega + \omega^2$. The supersymmetry transformations can be seen as generated by an anticommuting fundamental vector field v , such that $Q = L_v \equiv [I_v, s]$. With the reduced set of parameters, the vector field v commutes with itself. Then one has

$$L_v I_v \omega + (I_v \omega)^2 = \frac{1}{2} I_v^2 (s\omega + \omega^2) + \frac{1}{2} [L_v, I_v] \omega = \frac{1}{2} I_v^2 \Phi$$

where the Chern-Simons form Δ is given by

$$\Delta(c, \omega(\varphi)) \equiv \int_0^1 dt \mathcal{P}(c | t\omega(\varphi) + (t^2 - t)c^2) \quad (41)$$

Any given polynomial in the scalar fields belonging to a traceless symmetric representation of $SO(5, 1)$ has a canonical dimension which is strictly lower than that of all other operators in the same representation, made out of other fields. Thus, by power counting, the polynomials in the scalar fields can only mix between themselves under renormalization. Thus, if \mathcal{C} is the Callan–Simanzik operator, for any homogeneous polynomial \mathcal{P}_A of degree n in the traceless symmetric representation, renormalization can only produce anomalous dimensions that satisfy

$$\mathcal{C}[\mathcal{P}_A(\hat{\Phi}(\phi)) \cdot \Gamma] = \sum_B \gamma_A^B [\mathcal{P}_B(\hat{\Phi}(\phi)) \cdot \Gamma] \quad (42)$$

In this notation, given a local operator \mathcal{O} , $[\mathcal{O} \cdot \Gamma]$ means its insertion in the generating functional of one-particle irreducible Green functions Γ . Then, the Slavnov–Taylor identities imply

$$\mathcal{C}[\Delta_A(c, \hat{\Phi}(\phi)) \cdot \Gamma] = \sum_B \gamma_A^B [\Delta_B(c, \hat{\Phi}(\phi)) \cdot \Gamma] + \dots \quad (43)$$

where the dots stand for possible $\mathcal{S}_{(Q)|\Gamma}$ -invariant corrections. However, in the shadow-Landau gauge (i.e., the gauge (13) with $\alpha = 0$), $\Delta_A(c, \hat{\Phi}(\phi))$ cannot appear in the right-hand-side because such term would break the so-called ghost Ward identities [3]. One thus gets the result that $\gamma_A^B = 0$

$$\mathcal{C}[\mathcal{P}_A(\hat{\Phi}(\phi)) \cdot \Gamma] = 0 \quad (44)$$

Upon decomposition of this equation in function of the five independent supersymmetry parameters, one then gets the finiteness proof for each invariant polynomial $\mathcal{P}(\phi) \equiv \mathcal{P}(\phi^i, \phi^j, \phi^k, \dots)$ in the traceless symmetric representation of the R-symmetry group, namely

$$\mathcal{C}[\mathcal{P}(\phi) \cdot \Gamma] = 0 \quad (45)$$

Having proved that all 1/2 BPS primary operators have zero anomalous dimension, the Q -symmetry implies that all the operators generated from them, by applying $\mathcal{N} = 4$ super-Poincaré generators, have also vanishing anomalous dimensions. It follows that all the operators of the 1/2 BPS multiplets are protected operators.

It is worth considering as an example the simplest case of $\text{Tr } \hat{\Phi}(\phi)^2$. One has

$$\begin{aligned} Q\text{Tr} \left(\hat{\Phi}(\phi)c - \frac{1}{3}c^3 \right) &= \text{Tr } \hat{\Phi}(\phi)^2 & {}_s Q\text{Tr} \left(\hat{\Phi}(\phi)c - \frac{1}{3}c^3 \right) &= 0 \\ {}_s \text{Tr} \left(\hat{\Phi}(\phi)c - \frac{1}{3}c^3 \right) &= \text{Tr} \left(\mu(\hat{\Phi}(\phi) - c^2) - [\Omega, \hat{\Phi}(\phi)]c \right) \end{aligned} \quad (46)$$

These constraints imply that $\Delta_{\left[\frac{3}{2}\right]}^{(0,3)}$ is proportional to $\text{Tr} \left(\hat{\Phi}(\phi)c - \frac{1}{3}c^3 \right)$. Thus the three insertions that we have introduced can only be multiplicatively renormalized, with the same anomalous dimension. Moreover, the ghost Ward identities forbid the introduction of any invariant counterterm depending on the shadow field c , if it is not through a derivative term dc or particular combinations of c and the other fields that do not appear in the insertion $\text{Tr} \left(\hat{\Phi}(\phi)c - \frac{1}{3}c^3 \right)$. This gives the result that

$$\mathcal{C} [\text{Tr } \hat{\Phi}(\phi)^2 \cdot \Gamma] = 0 \quad (47)$$

Finally, by decomposition of the gauge-invariant operators upon independent combinations of the parameters, we obtain that all the 20 operators that constitute the traceless-symmetric tensor representation of rank two in $SO(5, 1)$ are protected operators

$$\begin{aligned} \text{Tr} (\Phi^2), \text{Tr} (\Phi L), \text{Tr} \left(\Phi \bar{\Phi} + \frac{1}{2}L^2 \right), \text{Tr} (\bar{\Phi} L), \text{Tr} (\bar{\Phi}^2), \\ \text{Tr} (\Phi h_I), \text{Tr} (L h_I), \text{Tr} (\bar{\Phi} h_I), \text{Tr} \left(\delta_{IJ} \Phi \bar{\Phi} + \frac{1}{2}h_I h_J \right) \end{aligned} \quad (48)$$

This constitutes the simplest application of Eq. (45), for $\mathcal{P}(\phi) \equiv \text{Tr} \left(\phi^i \phi_j - \frac{1}{6} \delta_j^i \phi_k \phi^k \right)$.

6.3 Cancellation of the β function form descent equations

To show that the coupling constant of the $\mathcal{N} = 4$ theory is not rescaled by renormalization, the key point is proving that the action $S = \int \mathcal{L}_4^0$ has vanishing anomalous dimension, in the sense that it cannot be renormalized by anything but a mixing with a BRST-exact counterterms. We will restrict here to the proof of this lemma, that is proving the Callan–Symanzik equation

$$\mathcal{C} \left[\int \mathcal{L}_4^0 \cdot \Gamma \right] = \mathcal{S}_{(\epsilon)}|_{\Gamma} [\hat{\Psi}^{(1)} \cdot \Gamma] \quad (49)$$

where $\hat{\Psi}^{(1)}$ is a functional of ghost number -1 and shadow number 0. (See [3] for a complete discussion.)

To prove (49), we will use the fact that descent equations imply that the lagrangian density is uniquely linked to a combination of protected operators (48), with coefficients that are fixed functions of the supersymmetric parameters.

As shown in [3], the reduced supersymmetry with the six generator \mathcal{Q} , $\bar{\mathcal{Q}}$ and $\bar{\mathcal{Q}}_\mu$ is sufficient to completely determine the classical action. For simplicity, we will thus restrict δ^{Susy} to these generators in this section ($v^I = 0$). Because \mathcal{L}_4^0 and $Ch_4^0 = Tr(FF)$ are supersymmetric invariant only modulo a boundary-term, the algebraic Poincaré lemma predicts series of cocycles, which are linked to \mathcal{L}_4^0 and Ch_4^0 by descent equations, as follows:

$$\begin{aligned}
\delta^{Susy} \mathcal{L}_4^0 + d\mathcal{L}_3^1 &= 0 & \delta^{Susy} Ch_4^0 + dCh_3^1 &= 0 \\
\delta^{Susy} \mathcal{L}_3^1 + d\mathcal{L}_2^2 &= \varpi i_\varepsilon \mathcal{L}_4^0 & \delta^{Susy} Ch_3^1 + dCh_2^2 &= \varpi i_\varepsilon Ch_4^0 \\
\delta^{Susy} \mathcal{L}_2^2 + d\mathcal{L}_1^3 &= \varpi i_\varepsilon \mathcal{L}_3^1 & \delta^{Susy} Ch_2^2 + dCh_1^3 &= \varpi i_\varepsilon Ch_3^1 \\
\delta^{Susy} \mathcal{L}_1^3 + d\mathcal{L}_0^4 &= \varpi i_\varepsilon \mathcal{L}_2^2 & \delta^{Susy} Ch_1^3 + dCh_0^4 &= \varpi i_\varepsilon Ch_2^2 \\
\delta^{Susy} \mathcal{L}_0^4 &= \varpi i_\varepsilon \mathcal{L}_1^3 & \delta^{Susy} Ch_0^4 &= \varpi i_\varepsilon Ch_1^3
\end{aligned} \tag{50}$$

Using the grading properties of the shadow number and the form degree, we conveniently define

$$\begin{aligned}
\mathcal{L} &\equiv \mathcal{L}_4^0 + \mathcal{L}_3^1 + \mathcal{L}_2^2 + \mathcal{L}_1^3 + \mathcal{L}_0^4 \\
Ch &\equiv Ch_4^0 + Ch_3^1 + Ch_2^2 + Ch_1^3 + Ch_0^4
\end{aligned} \tag{51}$$

The descent equations can then be written in a unified way

$$(d + \delta^{Susy} - \varpi i_\varepsilon) \mathcal{L} = 0 \quad (d + \delta^{Susy} - \varpi i_\varepsilon) Ch = 0 \tag{52}$$

Note that on gauge-invariant polynomials in the physical fields, δ^{Susy} can be identified to $s + Q$, in such way that the differential $(d + \delta^{Susy} - \varpi i_\varepsilon)$ is nilpotent on them. Since \mathcal{L}_4^0 and Ch_4^0 are the unique solutions of the first equation in (50), one obtains that \mathcal{L} and Ch are the only non-trivial solutions of the descent equations, that is, the only ones that cannot be written as $(d + \delta^{Susy} - \varpi i_\varepsilon) \Xi$ for a non trivial element of the s cohomology Ξ . The expression of the cocycles Ch_{4-s}^s can be simply obtained using the extended curvature (34) since the extended second Chern class

$$Ch = \frac{1}{2} \text{Tr} \left(F + \varpi \Psi + \omega \bar{\Psi} + g(\varepsilon) \eta + g(J_I \varepsilon) \chi^I + \varpi^2 \Phi + \varpi \omega L + (\omega^2 + |\varepsilon|^2) \bar{\Phi} \right)^2 \tag{53}$$

is $(d + \delta^{Susy} - \varpi i_\varepsilon)$ invariant by definition.

As for determining the explicit form of \mathcal{L}_{4-s}^s for $s \geq 1$, we found no other way than doing a brute force computation. In this way, one gets [3]

$$\mathcal{L}_0^4 = \frac{1}{2} \text{Tr} \left((\varpi^2 \Phi + \varpi \omega L + \omega^2 \bar{\Phi})^2 + \varpi^2 |\varepsilon|^2 \bar{\Phi}^2 \right) \quad (54)$$

The last cocycle \mathcal{L}_0^4 is a linear combination of the protected operators (48) and thus, its anomalous dimension is zero. This permits to prove that its ascendant \mathcal{L}_4^0 can only be renormalized by d -exact or $\mathcal{S}_{(\varepsilon)|\Sigma}$ -exact counterterms.

7 Conclusion

In the formalism that we have presented, the set of fields of a supersymmetric theory has been extended. With the introduction of shadow fields, one can express supersymmetry under the form of a nilpotent differential operator.

This clarifies many questions that arise when one builds the quantum field theory of a supersymmetric Yang–Mills theory, in particular for defining observables and study their renormalization. For instance, supersymmetric observables can be defined within the standard point of view of the cohomology of the BRST symmetry. In this framework, we have been able to define unambiguously the computation at all order in perturbation theory of all correlation functions, including insertions of gauge invariant local operators. The Slavnov–Taylor identities permit one to compute the non-invariant finite counterterms to maintain supersymmetry and gauge invariance of observables, independently of the choice of the regularization scheme.

By twisting the spinors, one can find subalgebra of supersymmetry with no equations of motions in the closure relations. This permits to simplify the proofs of various renormalization theorems for the $\mathcal{N} = 4$ super-Yang–Mills theory.

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