Braneworld cosmological singularities

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The purpose of this brief report is to present some results of our on-going project on the asymptotic behaviour of braneworld-type solutions on approach to their possible finite 'time' singularities. Cosmological singularities in such frameworks have served as means to attack the cosmological constant problem (see[1] and references therein). The main mathematical tool of our analysis is the method of asymptotic splittings introduced in Ref.[2]

Below we study a model consisting of a 3−brane configuration embedded in a five dimensional bulk space with a scalar field being minimally coupled to the bulk and conformally coupled to the fields restricted on the brane. The total action is taken to be $S_{total} = S_{bulk} + S_{brane}$, where

$$S_{bulk} = \int d^4x dY \sqrt{g_5}\left(\frac{R}{2\kappa_5^2} - \frac{\beta}{2}(\nabla\phi)^2\right), \; S_{brane} = -\int d^4x\sqrt{g_4}f(\phi), \text{ at } Y = Y_*,$$

with Y denoting the fifth bulk dimension, $\kappa_5^2 = M_*^{-3}$, M_* being the five dimensional Planck mass and $f(\phi)$ is the tension of the brane depending on the scalar field ϕ. We assume a bulk metric of the form $ds^2 = a^2(Y)d\tilde{s}^2 + dY^2$, where $d\tilde{s}^2$ is the four dimensional flat, de Sitter or anti-de Sitter metric. Then varying the above action we obtain the field equations:

$$\frac{a'^2}{a^2} = \frac{\beta\kappa_5^2\phi'^2}{12} + \frac{kH^2}{a^2} \tag{1}$$

$$\frac{a''}{a} = -\frac{\beta\kappa_5^2\phi'^2}{4}, \quad \phi'' + 4\frac{a'}{a}\phi' = 0, \tag{2}$$

where $k = 0, 1$ or -1, and H^{-1} is the de Sitter curvature radius. Assuming further that the unknowns are invariant under a $Y \to -Y$ symmetry and solving the field equations on the brane we may express the solution in the form

$$a'(Y_*) = -\frac{\kappa_5^2}{6}f(\phi(Y_*))a(Y_*), \quad \phi'(Y_*) = \frac{f'(\phi(Y_*))}{2\beta}. \tag{3}$$

We now apply the method of asymptotic splittings to look for the possible asymptotic behaviours of the general solution. Setting $x = a$, $y = a'$, $z = \phi'$, where the differentiation is considered with respect to $\Upsilon = Y - Y_s$ (Y_s being the position of the singularity), the field equations (2), become the following system of ordinary differential equations:

$$x' = y, \; y' = -\beta A z^2 x, \; z' = -4yz/x, \tag{4}$$

*On leave from CPHT (UMR CNRS 7644) Ecole Polytechnique, 91128 Palaiseau Cedex, France where $A = \kappa_5^2/4$. Hence, we have the vector field $\mathbf{f} = (y, -\beta A z^2 x, -4yz/x)^\intercal$. Equation (1) does not include any terms containing derivatives with respect to Υ; it is the constraint equation of the above system. In terms of the new variables, the constraint has the form

$$y^2/x^2 = A\beta/3z^2 + kH^2/x^2. \tag{5}$$

Substituting the forms $(x, y, z) = (\alpha\Upsilon^p, \gamma\Upsilon^q, \delta\Upsilon^r)$, with $(p, q, r) \in \mathbb{Q}^3$ and $(\alpha, \gamma, \delta) \in \mathbb{C}^3 - \{\mathbf{0}\}$, in the dynamical system (4), we seek to determine the possible *dominant balances* in the neighborhood of the singularity, that is pairs of the form $\mathcal{B} = \{\mathbf{a}, \mathbf{p}\}$, where $\mathbf{a} = (\alpha, \gamma, \delta)$ and $\mathbf{p} = (p, q, r)$. For our system we find:

$$\mathcal{B}_1 = \{(\alpha, \alpha/4, \sqrt{3}/4\sqrt{A\beta}), (1/4, -3/4, -1)\} \tag{6}$$

$$\mathcal{B}_2 = \{(\alpha, \alpha, 0), (1, 0, -1)\} \tag{7}$$

$$\mathcal{B}_3 = \{(\alpha, 0, 0), (0, -1, -1)\}. \tag{8}$$

Since (4) is a weight-homogeneous system, the scale invariant solutions given by the above balances are exact solutions of the system. The balance $\mathcal{B}_1$ satisfies the constraint equation (5) only for $k = 0$, corresponding thus to a general solution for a flat brane, whereas $\mathcal{B}_2$ corresponds to a particular solution for a curved brane since it satisfies Eq. (5) for $k \neq 0$ and $\alpha^2 = kH^2$. Finally the balance $\mathcal{B}_3$ represents a static universe conformal to Minkowski space and will not be analyzed further.

Next we calculate the Kowalevskaya exponents, i.e., the eigenvalues of the matrix given by $\mathcal{K} = D\mathbf{f}(\mathbf{a}) - \text{diag}(\mathbf{p})$; for $\mathcal{B}_1$ we find that $\text{spec}(\mathcal{K}) = \{-1, 0, 3/2\}$, whereas for $\mathcal{B}_2$, $\text{spec}(\mathcal{K}) = \{-1, 0, -3\}$. These exponents correspond to the indices of the series coefficients where arbitrary constants first appear. The -1 exponent signals the arbitrary position of the singularity, Y_s. Since we have two non-negative integer eigenvalues the solution we are constructing will be a general solution (full number of arbitrary constants).

Let us now focus on each of the two possible balances separately and build series expansions in the neighborhood of the singularity. For the first balance, we substitute in the system (4) the series expansions $\mathbf{x} = \Upsilon^{\mathbf{p}}(\mathbf{a} + \Sigma_{j=1}^{\infty}\mathbf{c}_j\Upsilon^{j/s})$, where $\mathbf{x} = (x, y, z)$, $\mathbf{c}_j = (c_{j1}, c_{j2}, c_{j3})$, s is the least common multiple of the denominators of positive eigenvalues (here $s = 2$), and we arrive at the asymptotic solution

$$x = \alpha\Upsilon^{1/4} + \frac{4}{7}c_{32}\Upsilon^{7/4} + \cdots, \quad y = x', \quad z = \frac{\sqrt{3}}{4\sqrt{A}}\Upsilon^{-1} - \frac{4\sqrt{3}}{7\alpha\sqrt{A\beta}}c_{32}\Upsilon^{1/2} + \cdots. \tag{9}$$

The last step is to check if, for each j satisfying $j/s = \rho$ with ρ a positive eigenvalue corresponding to an eigenvector $\mathbf{v}$ of the $\mathcal{K}$ matrix, the compatibility conditions hold, i.e. $\mathbf{v}^\top \cdot \mathbf{P}_j = 0$, where $\mathbf{P}_j$ are polynomials in $\mathbf{c}_i, \ldots, \mathbf{c}_{j-1}$ given by $\mathcal{K}\mathbf{c}_j - (j/s)\mathbf{c}_j = \mathbf{P}_j$. Here the corresponding relation $j/2 = 3/2$ is valid only for $j = 3$ and the compatibility condition indeed holds. We therefore conclude that near the singularity at finite distance Y_s from the brane, the asymptotic forms of the variables are $a \to 0$, $a' \to \infty$, $\phi' \to \infty$. This is exactly the asymptotic behaviour of the solution found previously by Arkani-Hammed *et al* in Ref.[1]

However, the previous behaviour is not the only possible one. The second balance has two distinct negative Kowalevskaya exponents and we therefore expect to find an infinite expansion of a *particular* solution around the presumed singularity at Y_s. Expanding the variables in series with descending powers of Υ, in order to meet the two arbitrary constants occurring $j = -1$ and $j = -3$, and substituting back in the system (4) we find the forms

$$x = \alpha\Upsilon + c_{-1\,1} + \cdots, \quad y = \alpha + \cdots, \quad z = c_{-3\,3}\Upsilon^{-4} + \cdots \tag{10}$$

Therefore as $\Upsilon \to 0$, or equivalently as $S = 1/\Upsilon \to \infty$, we have that $a \to \infty$, $a' \to \infty$ and $\phi' \to \infty$.

We thus conclude that there exist two possible outcomes for these braneworld models, the dynamical behaviours of which strongly depend on the spatial geometry of the brane. For a flat brane the model experiences a finite distance singularity through which all the vacuum energy decays, whereas for a de Sitter or anti-de Sitter brane the singularity is now located at an infinite distance. We can choose the coupling such that to allow only for that behaviour met in flat solutions and, in fact, the particular form of the coupling used by Arkani-Hammed *et al* in[1] is the only choice to make this possible. This easily follows by using equations (3) and solving the Friedmann equation (1) on the brane for kH^2, i.e.

$$kH^2 = \frac{a^2(Y_*)\kappa_5^2}{4}\left(\frac{\kappa_5^2}{9}f^2(\phi) - \frac{f'^2(\phi)}{4\beta^2}\right).$$

Clearly then k is identically zero if and only if $f'(\phi)/f(\phi) = (2\beta/3)\kappa_5$, or equivalently, if and only if $f(\phi) \propto e^{(2\beta/3)\kappa_5\phi}$ (Arkani-Hammed *et al* in[1] have $\beta = 3$). By working with other couplings we can allow for non-flat, maximally symmetric solutions to exist and avoid in this way having the singularity at a finite distance away from the position of the brane.

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References

1. N. Arkani-Hammed, S. Dimopoulos, N. Kaloper, R. Sundrum, Phys. Lett. **B480** (2000) 193-199, arXiv:hep-th/0001197v2; S. Kachru, M. Schulz, E. Silverstein, Phys. Rev. **D62** (2000) 085003, arXiv:hep-th/0002121.
2. S. Cotsakis, J. D. Barrow, *The Dominant Balance at Cosmological Singularities*, arXiv:gr-qc/0608137; to appear in the Proceedings of the Greek Relativity Meeting NEB12, June 29-July 2, 2006, Nauplia, Greece.