Something to Do With Schrödinger Spacetimes

by

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Abstract

We present a brief review of the AdS/CFT correspondence and the progress made toward building a realistic gravity/gauge theory duality for a non-relativisitc conformal field theory. In particular, we highlight many of the computational tools necessary for such a program before introducing one such model duality. The model presented exhibits the symmetry group of Schrödinger's equation along with conformal symmetry. A black hole can be placed in this spacetime to study a finite temperature duality. In the low-frequency, long-distance limit at finite temperature classical hydrodynamics can be used to determine the retarded Green's functions of the field theory, which can be computed from the gravity dual. This facilitates the calculation of several characteristic quantities including the shear viscosity and the shear diffusion constant giving results consistent with other hydrodynamic analyses of the system.

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1. Introduction

Perhaps the most remarkable development in string theory has been the formulation of the various gravity/gauge theory correspondences. Over the past decade, intense investigation has led to conjectured equivalences between field theories in flat spacetimes and string theories in higher dimensional curved spacetimes. The original and canonical example of such a duality is the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence. This particular duality is most commonly expressed between $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in Minkowski four-space and type IIB string theory in the ten dimensional spacetime $AdS_5 \times S^5$. This duality has gathered such intense interest because of a phenomenon known as strong/weak coupling duality. In the strong coupling (i.e. large t' Hooft coupling) limit of the field theory, the string theory becomes weakly coupled reducing to classical supergravity. Hence, physics in the non-perturbative regime of the field theory can be analytically determined by studying the classical low energy limit of the gravity theory.

Interest has grown in formulating such a gravity duality for more realistic gauge theories. A necessary step toward this goal is studying such at duality at finite temperature. In [1, 2], this was done for the AdS/CFT correspondence at finite temperature, in which they considered real-time correlators. While Euclidean correlators decay exponentially at finite distances, their Minkowski counterparts posses non-exponential tails that contain global information about the field theory. Although a complete analysis of all modes becomes analytically intractable, the behavior of the theory can be well understood in the hydrodynamic limit by using fluid mechanics allowing us to extract hydrodynamic information from the Minkowski correlators. This is essentially the statement that any interacting field theory in the low-frequency, long-wavelength limit should reduce to a theory describable by classical fluid mechanics. This places tight constraints on the behavior of the various field theoretic operators because they must be determined from just a few hydrodynamic quantities, and an analysis of the dual gravity theory provides a method for determining these.

This type of analysis has been extremely successful and extended to other systems at finite temperature such as [3, 4, 5, 6]. However, most of the strongly coupled relativistic field theories are not experimentally accessible. So, it can be asked what other types of gauge theory/gravity correspondences can be formulated with a particular interest toward finding a realistic field theory that can be studied in a laboratory. It turns out that there are several examples of strongly coupled conformal field theories in the non-relativistic regime, of which the most accesible is cold fermionic atoms at unitarity. In order to eventually realize a consistent gravity dual for such a system it is necessary to have a non-relativistic version of the AdS/CFT correspondence. This paper aims to move toward such a goal by studying a prototype model at finite temperature using a hydrodynamic analysis to determine the various transport coefficients describing the non-relativistic field theory.

This paper is structured as follows. In section 2, the original AdS/CFT correspondence will be presented along with a simple worked example to illustrate the procedure for determining the correlators in the field theory concluding with a prescription for computing real-time Minkowski correlators. This will also reveal several more subtle aspects of such computations. Subsequently, in section 3 the AdS/CFT duality will be studied at finite temperature after having introduced the AdS black hole. In section 4, the non-relativistic version of the correspondence will be formulated revolving around the construction of the necessary geometry. A brief overview of non-relativistic hydrodynamics in section 5 is presented before moving on to developing a systematic formalism for extracting correlators from the Schrödinger black hole in section 6.

2. An Introduction to Gravity/Gauge Duality

2.1 The Basics of AdS/CFT

In recent years, research in string theory has seen the development of an extremely fruitful program of deriving quantum field theories by taking limits of string/M theories. The AdS/CFT correspondence is historically the first and the canonical example of such a procedure. The original AdS/CFT correspondence between $\mathcal{N} = 4$ SYM and type IIB string theory was discovered via a rather roundabout route that involved studying black branes and D-branes in string theory. However, we will try to motivate the results through a much more direct approach by appealing to symmetries. We will subsequently develop the foundations of the AdS/CFT correspondence and indicate how it applies to similar dualities between different types of theories.

Symmetry narrows the search for determining how to relate a gauge theory to a theory of gravity by placing tight constraints on the types of geometries we can consider. Besides Poincare symmetry, which is common to all relativistic field theories, we would like to impose the additional constraint of conformal symmetry. Consequently, our symmetry algebra will have an extra generator corresponding to scale transformations on top of the usual Lorentz and translation generators. For a four dimensional quantum field theory, we would naively begin by searching for a string theory in four dimensions that respects the same symmetries as the desired field theory. It can be shown that there is no consistent and quantizable string theory in four flat dimensions, so in order to find our string theory an extra dimension must be included.

Hence, we broaden our scope by looking for a five dimensional geometry that has a four dimensional Poincare symmetry along with conformal invariance. Now, Poincare symmery restricts the general form of the metric to

$$ds^2 = f(r)^2(dx_i dx^i + dr^2), \tag{2.1}$$

where the radial coordinate r is left invariant under such a transformation and the coefficient in front of dr^2 can be set to 1 by rescaling. Imposing conformal symmetry requires that field theory be scale invariant i.e. $x^i \to \lambda x^i$ is a symmetry. Since the string theory has a natural scale set by the string tension, the only way the theory can respect this scale invariance is if this transformation is an isometry[1]. Hence, we additionally require that $r \to \lambda r$ and $f(r) = R/r$ to ensure invariance. This gives the AdS metric

$$ds^2 = \frac{R^2}{r^2}(dx_i dx^i + dr^2) \tag{2.2}$$

where R is called the AdS radius, which determines the constant negative curvature of the spacetime. Notice that in the limit $r \to 0$ the induced metric on the boundary is proportional to the metric[2] describing the field theory. Hence, it is often loosely said that AdS is dual to a field theory defined on the boundary.

Before proceeding, let's describe the two theories that have been claimed to be equivalent [7]. Firstly, $\mathcal{N} = 4$ SYM is the maximally supersymmetric gauge theory in four dimensions with gauge group $SU(N)$. Its field content contains a gauge field/gluon, four Weyl fermions/gluinos, and 6 real scalars all of which are defined in the adjoint representation of the color group. Although we will refrain from giving the Lagrangian of the theory, it is described by two parameters g_{YM}, the gauge coupling, and N, the number of colors. In addition to supersymmetry, the theory also respects a conformal invariance, which is preserved after quantization. In fact, the large number of symmetries of the theory yields a vanishing beta function for the coupling g_{YM}! As a result, $\mathcal{N} = 4$ SYM is often referred to as a finite field theory. Interest in such a theory is also obvious because of the clear similarities to QCD.

On the other side of the claimed duality is the ten dimensional type IIB string theory. This theory contains a few massless fields including a graviton, a dilaton, a one-form field strength, two three-form field strengths, and a self-dual five-form field strength along with an infinite number of massive string excitations. The string theory has two parameters, the string length l_s and the string coupling g_s. In the long

[1]Requiring that symmetry transformations of the field theory be isometries of the string theory metric will be essential idea later when we construct a non-relativistic version of the correspondence.

[2]Even though we started by imposing Poincare symmetry, we will later consider both Minkowski and Euclidean signatures beginning with the latter.

wavelength limit when all of the fields vary over distances much longer than the string length, the massive modes decouple from the theory yielding classical type IIB supergravity in ten dimensions. It can be shown that the AdS geometry we are looking for is contained in the metric

$$ds_{10}^2 = \frac{R^2}{r^2}(dx_i dx^i + dr^2) + R^2 d\Omega_5^2 \tag{2.3}$$

which is a solution to the supergravity equations of motion. The self-dual Ramond-Ramond five form F_5 contributes the stress-energy tensor necessary to support such a metric.

It has been claimed that the two theories are equivalent [8]. This has not been formally proven, but certain limits of the asserted duality have shown equivalence. For equivalence to hold, there should be a dictionary relating the content of one theory to the other. The two dimensionless coupling constants can be related by

$$g_{YM}^2 = 4\pi g_s, \tag{2.4}$$

$$g_{YM}^2 N = \frac{R^4}{l_s^4}. \tag{2.5}$$

Notice that in the large N limit with $g_{YM}^2 N \gg 1$, $l_s \ll R$ and $g_s \ll 1$. This was exactly the condition that was needed to decouple the massive string excitations, which gave a theory of classical supergravity. On the field theory side, the perturbative expansion is described by the 't Hooft coupling $\lambda = g_{YM}^2 N$. So, we see that the strong coupling regime of the field theory is dual to the weak coupling limit of the string theory, which allows us to use classical supergravity!

We can continue to develop this dictionary between the content of the dual theories. In particular, the gravity/gauge duality requires that for every bulk field Φ there is a corresponding gauge invariant operator $\hat{O}_\Phi$ in the boundary theory. The explicit statement of this correspondence relates the partition function of the bulk fields to the generating functional of the boundary field theory,

$$Z_{SG}[\phi_0] \equiv \int_{\Phi\sim\phi_0} D\Phi e^{-S_{SG}[\Phi]} = \langle \exp(-\int_{\partial AdS} \phi_0 \hat{O}_\Phi) \rangle_{QFT}, \tag{2.6}$$

where the expectation value on the right hand side is the path integral. To leading order in the saddle point approximation, this expression reduces to

$$S_{SG}[\phi_0(x)] = -\ln\langle \exp(-\int_{\partial AdS} \phi_0 \hat{O}_\Phi) \rangle_{QFT} \equiv -W_{QFT}[\phi_0(x)], \tag{2.7}$$

where the supergravity action is evaluated with an on-shell field that asymptotes to the desired boundary source. We also recognize $W_{QFT}[\phi_0(x)]$ as the generating function of connected correlators in the field theory. Thus, we see that correlation functions of the operator $\hat{O}_\Phi$ can be computed by functionally differentiating the supergravity action with respect to the source,

$$
\begin{aligned}
\langle \hat{O}(x) \rangle &= \frac{\delta S_{SG}}{\delta \phi_0(x)}|_{\phi_0=0}, \\
\langle \hat{O}(x_1)\hat{O}(x_2) \rangle &= -\frac{\delta^2 S_{SG}}{\delta \phi_0(x_1)\delta \phi_0(x_2)}|_{\phi_0=0}, \\
\langle \hat{O}(x_1)\ldots\hat{O}(x_n) \rangle &= (-1)^{n+1}\frac{\delta^n S_{SG}}{\delta \phi_0(x_1)\ldots\delta \phi_0(x_n)}|_{\phi_0=0}.
\end{aligned} \tag{2.8}
$$

Even though though this is a useful schematic relationship between the two theories, this statement is not well defined due to UV divergencies in the field theory. So, renormalization is needed to cancel the divergencies. We defer the reader to find formal development of the program of holographic renormalization in the literature [9]. However, we will in passing mention how renormalization ensures finite answers as need arises. The production of the necessary counter-terms will be illustrated by example.

2.2 A Massive Scalar Field

As a simple illustration of the mechanics of this duality, consider a scalar field ϕ in the Euclidean continuation of the AdS_5 spacetime[3] with a dimensionful mass m. Then the appropriate bulk action to consider for the scalar field is simply

$$S_\phi = \frac{\eta}{2}\int d^5x\sqrt{g}(\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2). \tag{2.9}$$

Requiring the variation of the action to be zero gives the equation of motion,

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,\partial^\mu\phi) - m^2\phi = 0. \tag{2.10}$$

The solution to this equation can be explicitly found by using translation invariance to Fourier transform the Euclidean coordinates. Use the transformation defined by

$$\phi(x,r) = \int \frac{d^4q}{(2\pi)^4}e^{iq_ix^i}f_q(r)\tilde{\phi}_0(q), \tag{2.11}$$

[3]We can neglect the S_5 portion of the metric. If we were to consider these extra dimensions, the only difference would be periodicity constraints in the spherical dimensions. Since the spherical dimensions are periodic, a Fourier series expansion could be performed giving an infinite set of effective massive modes in AdS_5. This is known as Kaluza-Klein dimensional reduction.

where $\tilde{\phi}_0(q)$ is the Fourier transform of the boundary source determined by

$$\tilde{\phi}_0(q) = \int d^4x\, e^{-iq_i x^i} \phi(x, 0). \tag{2.12}$$

We see that this requires $f_q(0) = 1$ for every Fourier mode. Then, in momentum space the equation of motion reduces to

$$-r^2 f_q'' + 3r f_q' + (q^2 r^2 + m^2 R^2) f_q = 0. \tag{2.13}$$

This equation has known solutions expressed in terms of modified Bessel functions of the second type,

$$f_q(r) = A r^2 K_\nu(qr) + B r^2 I_\nu(qr), \tag{2.14}$$

where $\nu = \sqrt{4 + m^2 R^2}$.

In order to compute the boundary operator correlators, boundary conditions need to be imposed. The correct solution is one that asymptotes to the boundary field and is finite in the interior. In the limit $r \to \infty$, $K_\nu \sim r^{-1/2} e^{-r}$ and $I_\nu \sim r^{-1/2} e^{r}$. The latter mode is clearly divergent, forcing the choice $B = 0$. To impose the remaining boundary condition, we need to examine the behavior of the solution near $r = 0$. Since $r^2 K_\nu(qr) \sim r^{2-\nu}$, the solution diverges at the boundary unless it is massless. Already, we can see that some regulator and a renormalization scheme will be necessary. So, we regulate the divergence by imposing the boundary condition at $r = \epsilon$. Then, we will take the limit $\epsilon \to 0$ at the end of the computation. The solution satisfying the new boundary conditions is

$$f_q(r) = \frac{r^2 K_\nu(qr)}{\epsilon^2 K_\nu(q\epsilon)}. \tag{2.15}$$

Without interactions the only non-trivial correlator is the two-point function. In order to compute the two-point function, the gravity/gauge prescription requires that we evaluate the action using the on-shell field with the aforementioned boundary conditions. A short calculation[4] shows that the action reduces to the boundary terms,

$$S = \int \frac{d^4q\, d^4q'}{(2\pi)^8} \tilde{\phi}_0(q) \tilde{\phi}_0(q') \mathcal{F}(r, q, q') \Big|_{r=\epsilon}^{r=\infty}, \tag{2.16}$$

where $\mathcal{F}(r, q, q')$ is the flux factor defined as

$$\mathcal{F}(r, q, q') = (2\pi)^4 \delta^4(q + q') \frac{\eta R^3}{2r^3} f_{q'}(r) f_q'(r). \tag{2.17}$$

[4]Evaluating the on-shell action consists of successively integrating by parts, using the equation of motion, and then applying Stokes' theorem.

We can now use the AdS/CFT correspondence (2.8) to get the momentum space correlator

$$\langle \hat{O}(q)\hat{O}(q')\rangle_\epsilon = -2\mathcal{F}(r,q,q')\Big|_{r=\epsilon}^{r=\infty}. \tag{2.18}$$

All that remains is a rather tedious evaluation of the flux factor at the boundary, since it is easy to see that $\mathcal{F}(\infty,q,q') = 0$. Note that the calculation depends on whether ν is an integer, because the Bessel function K_ν has a blocked exponent for an integer order.

After evaluating the flux factor at the boundary $r = 0$, we discover that there are in general divergent terms as $\epsilon \longrightarrow 0$. As shown in appendix A, these divergencies can be removed via renormalization by manufacturing a covariant boundary counter-term action. In fact, even finite contact terms are removed, terms that depend on squares of the momenta, because they don't contribute to the correlator at finite distances. Lastly, the boundary operator needs to be renormalized to give a non-zero result for $m \neq 0$. After renormalizing the theory, we find that two-point functions are

$$\langle \hat{O}(q)\hat{O}(q')\rangle = -(2\pi)^4\delta^4(q+q')\frac{\eta R^3}{2^{2\nu-1}}\frac{\Gamma(1-\nu)}{\Gamma(\nu)}q^{2\nu} \tag{2.19}$$

for non-integer ν and

$$\langle \hat{O}(q)\hat{O}(q')\rangle = -(2\pi)^4\delta^4(q+q')\frac{\eta R^3}{2^{2\nu-1}}\frac{(-1)^n}{(n-1)!^2}q^{2\nu}\ln q^2 \tag{2.20}$$

for integer ν.

2.3 Minkowski Space Correlators

As previously presented, the AdS/CFT correspondence allows us to compute correlators for boundary operators in the field theory with a Euclidean metric signature. Although it may be possible to determine Minkowski space correlators via Wick rotation (i.e. analytical continuation), a general prescription for evaluating these is necessary. The Minkowski space correlators can only be determined from their Euclidean counterparts via analytic continuation if the Euclidean space correlators are known for all frequencies. However, many computations require approximations making it necessary to directly compute the Minkowski space correlators. The hydrodynamic limit is such an approximation that necessitates a direct prescription.

To compute the Euclidean space correlators, we were simply able to functionally differentiate the on-shell supergravity action. For Minkowski correlators there is a complication, which is discussed in detail in [10]. Namely, no equivalence of the form of (2.7) can give complex Green's functions. This is partially due to the fact that there are a variety of Green's functions in the Minkowski case, so the Euclidean boundary conditions are not sufficient to determine a unique bulk field corresponding the boundary operator. But even after additional boundary conditions are imposed on the bulk solution, the imaginary components manage to cancel giving real Green's functions. However, Minkowski Green's functions are in general complex. This complication will be elucidated in a subsequent example.

Since there are complications for the Lorenztian signature theories, we will reduce our scope. In particular, we will simply present a prescription for two-point functions, which will be sufficient to later determine the various hydrodynamic quantities. Recall that the retarded Green's function for the boundary operator $\hat{O}_\Phi$ is defined by

$$G^R(q) = -i \int d^4x\, e^{ik_ix^i}\theta(t)\langle[\hat{O}_\Phi(x), \hat{O}_\Phi(0)]\rangle, \tag{2.21}$$

where $\theta(t)$ is the unit step function. All of the other Green's functions (advanced, Feynman, etc.) can be determined from the retarded Green's function. So, this is sufficient for understanding the Lorentzian signature theory. In analogy with (2.18), it was conjectured in [10] that the retarded Green's function should be given by

$$G^R(q) = -2\mathcal{F}(r, q, -q)\big|_{r=r_b}, \tag{2.22}$$

where the flux factor is found just as in the Euclidean case and r_b is the location of the boundary. The additional boundary condition for the retarded Green's function is that for timelike momenta the bulk solution asymptotes to an incoming wave at the horizion ($r = \infty$ for zero temperature). This makes sense from a physical viewpoint, because waves should only be able to propagate into a black hole. In every case where independent verification is possible, this prescription is found to be in agreement with the results from other methods.

3. Finite Temperature and the AdS Black Hole

3.1 Background Geometry

So far we have seen a gravity/gauge duality between $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ at zero temperature, and this follows because there is no natural scale for these theories. However, we should look for theories at finite temperature with the aim of finding more realistic models that could describe a system realizable in the laboratory. Originally, we presented the AdS geometry by arguing that the string theory background should respect certain symmetries if it were to be dual to a quantum field theory. Subsequently, we found such a geometry by studying the type IIB supergravity equations for motion.

In the same way, we can look for other solutions of type IIB supergravity equations of motion which asymptotically correspond to AdS [1]. In particular, a whole family of solutions are found related by a single parameter r_H, the black hole horizon. Such a solution is known as the AdS black hole or AdS black three-brane[5] at finite temperature and is given by the ten dimensional metric,

$$ds_{10}^2 = \frac{R^2}{r^2}(-f(r)dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{r^2 f(r)}dr^2 + R^2 d\Omega_5^2, \tag{3.1}$$

where $f(r) = 1 - r^4/r_H^4$. Notice than in the limit $r_H \to \infty$, this metric reduces to the Lorentzian signature AdS spacetime, and at the boundary we again recover the metric for the boundary field theory. Like all black holes, there is an intrinsic temperature and entropy. Here, the scale set by the horizon r_H determines the Hawking temperature $T = 1/\pi r_H$. Hence, the AdS solution is special in this family in that it corresponds to the zero temperature solution as claimed previously.

For the solutions with non-zero temperature, it is helpful to change variables by introducing a dimensionless radius $u = r^2/r_H^2$. Consequently, $f(u) = 1 - u^2$ and the metric assumes the form,

$$ds_5^2 = \frac{R^2}{r_H^2 u}(-f(u)dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{4u^2 f(u)}du^2. \tag{3.2}$$

Now the boundary corresponds to $u = 0$, and the horizon corresponds to $u = 1$ for all finite temperatures. As expected, this bulk gravity theory is dual to $\mathcal{N} = 4$ SYM with the same temperature $T = 1/\pi r_H$. Consequently, the same analysis developed previously effortlessly carries over to the finite temperature case.

3.2 A Hydrodynamic Illustration

Consider a massless scalar field ϕ defined on the AdS black hole background where we can neglect the spherical portion of the geometry as in AdS_5. The appropriate action to consider is

$$S_\phi = -\frac{\eta}{2}\int d^5x\sqrt{g}\,\partial_\mu\phi\,\partial^\mu\phi, \tag{3.3}$$

which is identical in form (modulo a sign) to (2.9) with $m = 0$ and the appropriate change in the volume element. In this geometry, the volume element for the black hole is $\sqrt{g} = R^5/2r_H^4 u^3$. Variation of the action then yields the equation of motion

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,\partial^\mu\phi) = 0. \tag{3.4}$$

[5] Unlike the Schwarzschild black hole, the AdS black hole has three flat directions at the horizon. Hence, it is a three-brane solution.

It is worth mentioning that this is Laplace' s equation for the AdS black hole geometry, so we can think of this as a scalar wave equation, which it is. Proceeding as we did in the zero temperature case, we will use the Fourier transform

$$\phi(x,u) = \int \frac{d^4q}{(2\pi)^4} e^{-i\omega t+iqz}\tilde{\phi}_q(u), \tag{3.5}$$

$$\tilde{\phi}_q(u) = \int d^4x\, e^{i\omega t-iqz}\phi(x,u), \tag{3.6}$$

where we have used the $O(3)$ rotational symmetry of the Euclidean dimensions to set the four-momentum to $(\omega, 0, 0, q)$. This then yields the equation

$$\tilde{\phi}_q'' - \frac{1+u^2}{uf}\tilde{\phi}_q' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2}\tilde{\phi}_q = 0, \tag{3.7}$$

where we have introduced dimensionless momenta defined by

$$\mathfrak{w} = \frac{r_H\omega}{2}, \qquad \mathfrak{q} = \frac{r_H q}{2}. \tag{3.8}$$

Now, (3.7) has no known solutions, so we can't proceed to calculate the retarded Green's function as we did before by evaluating the flux factor with the appropriate solution.

Even though the inclusion of finite temperature sufficiently complicates the equation of motion to make it analytically intractable, it also provides us with another mechanism to extract information out of the equation of motion that wasn't present in the zero temperature case. Namely, the temperature sets a scale for the system, so we can now perform a perturbative expansion in powers of the momenta $\mathfrak{w}$ and $\mathfrak{q}$. This is known as a hydrodynamic expansion.

Most of the mathematical details have been relegated to appendix B, but the basic idea is that we need to find a perturbative solution in powers of the momenta that satisfies the appropriate boundary conditions for the Minkowski space correlator. The case of the massless scalar is so simple that we could actually just guess the form of the solution, but it is worth illustrating a systematic approach that will be necessary later. After converting (3.7) to a first order system, we get the matrix equation

$$\begin{pmatrix} \tilde{\phi}_q \\ \tilde{\psi}_q \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{f} \\ \frac{\mathfrak{q}^2}{u} - \frac{\mathfrak{w}^2}{uf} & \frac{1}{u} \end{pmatrix} \begin{pmatrix} \tilde{\phi}_q \\ \tilde{\psi}_q \end{pmatrix}, \tag{3.9}$$

where $\tilde{\psi}_q = f\tilde{\phi}_q'$[6]. In vector notation, this equation is $\mathbf{y}' = A(u)\cdot\mathbf{y}$. Notice that the equation of motion has a regular singular point at $u = 1$. So, we can expand

[6]This choice was made in order to avoid an irregular singular point.

about the singularity imposing the correct boundary conditions at the horizion and still get complete solutions in the low momentum regime perturbatively. The two linearly independent solutions to (3.9) are of the form

$$\mathbf{y}(u) = (1-u)^{\nu}\mathbf{F}(u), \tag{3.10}$$

where ν is an eigenvalue of $(u-1)A(u)$ evaluated at the singularity $u = 1$, $\mathbf{F}(u)$ is holomorphic, and $\mathbf{F}(1)$ is the eigenvector associated with ν. Computation reveals that there are two possible eigenvalues to choose from $\nu = \pm i\mathfrak{w}/2$.

The prescription for fining Minkowski space correlators requires that we impose an additional boundary condition at the horizon beyond regularity. At the boundary, both solutions behave like

$$\mathbf{y}(u) \propto (1-u)^{\pm i\mathfrak{w}/2}, \tag{3.11}$$

where one solution is simply the conjugate of the other. Hence, both solutions are regular preventing us from choosing a unique solution, because any linear combination of the solutions satisfies the regularity condition. If we restore the time dependent phase from the Fourier transform we see that

$$e^{-i\omega t}(1-u)^{\pm\mathfrak{w}/2} = e^{-i\omega(t\mp v)}, \tag{3.12}$$

where $v = \ln(1-u)r_H/4$. The solution with $\nu = -i\mathfrak{w}/2$ consequently corresponds to an incoming wave at the horizon while its conjugate is an outgoing wave. Our prescription requires that we use the incoming wave solution, which is based on the physical intuition that nothing should come back from inside the horizion.

There is also a second problem in determining the the Minkowski space correlator. We see that one solution is the complex conjugate of the other and $\mathbf{y}^*(q,u) = \mathbf{y}(-q,u)$. If we simply functionally differentiate the equivalent of (2.16), we would find

$$G(q) = -\mathcal{F}(u,q)\Big|_{u=\epsilon}^{u=1} - \mathcal{F}(u,-q)\Big|_{u=\epsilon}^{u=1}. \tag{3.13}$$

However, this quantity is always real, but we expect the Green's function to be complex in general. It is unknown how to fix this by modifying the action principle. Instead, we simply use the ansatz given by (2.22).

Before we can use the prescription to calculate the retarded Green's function, we still need to calculate the solution with the incoming wave boundary condition ($\nu = -i\mathfrak{w}/2$). A double Taylor series expansion on $\mathbf{F}(u)$ gives to lowest order in the momenta

$$\mathbf{y}(u) = (1-u)^{-i\mathfrak{w}/2}\left[\mathbf{F}(u) + \mathfrak{q}\mathbf{F}_{\mathfrak{q},1}(u) + \mathfrak{w}\mathbf{F}_{1,\mathfrak{w}}(u) + \mathfrak{q}^2\mathbf{F}_{\mathfrak{q}^2,1}(u) + \mathcal{O}(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2, \mathfrak{q}^3)\right]. \tag{3.14}$$

Substituting this expansion into (3.9) and equating like powers in the momenta, we get a coupled system of differential equations for each combination of the momenta. Solving these equations recursively, we find that

$$\tilde{\phi}_q(u) = C(\mathfrak{q},\mathfrak{w})(1-u)^{-i\mathfrak{w}/2}\left[1 - \frac{i\mathfrak{w}}{2}\ln\frac{1+u}{2} - \mathfrak{q}^2\ln\frac{1+u}{2} + \mathcal{O}(\mathfrak{w}^2,\mathfrak{w}\mathfrak{q}^2,\mathfrak{q}^3)\right], \quad (3.15)$$

where $C(\mathfrak{q},\mathfrak{w})$ is an arbitrary constant that can depend on the momenta.

It remains to impose the boundary condition at the boundary $u=\epsilon$ by requiring the bulk field to asymptote to the boundary source that couples to the boundary operator in the field theory. Hence, require

$$\lim_{u\to\epsilon}\tilde{\phi}_q(u) = \tilde{\phi}^{\epsilon}_q. \quad (3.16)$$

Then to lowest order in the momenta, the full solution is

$$\tilde{\phi}_q(u) = \frac{2\tilde{\phi}^{\epsilon}_q(1-u)^{-i\mathfrak{w}/2}}{2+i\mathfrak{w}\ln 2 + 2\mathfrak{q}^2\ln 2}\left[1 - \frac{i\mathfrak{w}}{2}\ln\frac{1+u}{2} - \mathfrak{q}^2\ln\frac{1+u}{2} + \mathcal{O}(\mathfrak{w}^2,\mathfrak{w}\mathfrak{q}^2,\mathfrak{q}^3)\right]. \quad (3.17)$$

The flux factor is the same as in the Euclidean AdS case because the mass doesn't appear. Hence,

$$\begin{aligned}\mathcal{F}(\epsilon,q,q') &= -(2\pi)^4\delta^4(q+q')\frac{\eta}{2}\sqrt{g}g^{uu}\, f_{q'}(u)f'_q(u)\big|_{u=\epsilon}\\ &= -(2\pi)^4\delta^4(q+q')\frac{\eta R^3}{r_H^4}\frac{(\epsilon-1)\mathfrak{q}^2+i\epsilon\mathfrak{w}}{\epsilon}.\end{aligned} \quad (3.18)$$

The program of holographic renormalization is applicable for any asymptotically AdS space. We can then shamelessly drop the divergent term in the flux factor without systematically developing the counter-term action for our theory[7]. Finally, via (2.22) the retarded Green's function is then

$$G^R(q,q') = -(2\pi)^4\delta^4(q+q')\frac{2\eta R^3}{r_H^4}(i\mathfrak{w}+\mathfrak{q}^2). \quad (3.19)$$

Now that the procedure for determining the Minkowski correlators at finite temperature has been illustrated, we will move on to the original goal of studying a non-relativistic version of the AdS/CFT correspondence at finite temperature.

[7]In fact the counter-terms found for the AdS background at zero temperature should cancel the divergencies in the finite temperature case, because the renormalization should be independent of the background we are working in.

4. The Conformal Schrödinger Geometry

4.1 Zero Temperature and Chemical Potential

So far we have presented a well known duality between a conformal field theory and a higher dimensional black hole both parameterized by a scale that corresponds to temperature. Perhaps the most useful result of finding such a duality is the inverse relationship of the couplings in the dual theories. In the weak coupling limit of the gravitational theory, we can extract information in the strong coupling regime of the field theory. We would like to find a non-relativistic realization of such a duality, and so we begin by finding a geometry with the appropriate asymptotic symmetries just as we did in the relativistic case, which suggested that we use AdS.

In analogy with the relativistic case, we begin by specifying the symmetries of our sought after non-relativistic theory. In d spatial dimensions, the appropriate symmetry group is generated by the d-dimensional Schödinger algebra where we have additionally imposed conformal invariance. As the name suggests, the Schrödinger group is the symmetry group of Schrödinger's equation. When we were constructing a geometry that could support a duality between a relativistic conformal field theory and a string theory, we saw that it was necessary for the symmetry transformations to be isometries of the metric. So, we will do the equivalent thing here by looking for a metric that has the Schrödinger group as an isometry group. Such a construction can be found by recognizing that the Schrödinger group is a subgroup of the symmetry group of AdS, hence we can deform the AdS metric until it only exhibits the Schrödinger symmetry [11]. Alternatively, we can build it constructively (often known as guessing) by considering the symmetry generators acting on the coordinates [12]. Either way, the following metric is isometric under the desired symmetry group

$$ds_5^2 = \frac{R^2}{r^2}\left(-\frac{2\beta^2}{r^2}dt^2 + 2dtd\xi + d\vec{x}^2 + dr^2\right), \tag{4.1}$$

where $\vec{x} = (x, z)$ and β is a parameter with the dimensions of length. This geometry will be subsequently denoted as Sch_5[8].

There are a few surprises here, all of which are related. The first is the appearance of a new non-spatial coordinate ξ, which reduces the number of Euclidean spatial directions to two as opposed to the three in the AdS. This is a consequence of introducing a number operator into the symmetry algebra. There are now two operators which can be diagonalized in any representation of the algebra, the dimension and the

[8] Here, β is a quantity with the dimensions of length, which we could absorb into a redefinition of units. However, after discovering where this arises from in a few moments, we will see that keeping it serves as a useful check on later results by setting it to zero.

particle number. The ξ coordinate corresponds to the particle number, and so it seems likely that a d-dimensional non-relativistic field theory needs to be embedded in a $d+2$ dimensional spacetime. However, since ξ corresponds to the particle number, we need to compactify the ξ-dimension in order to have a gravity theory dual to non-relativistic field theory, because non-relativistic theories typically have discrete numbers of particles. This can be seen by supposing ξ has a period of L_ξ. Then, Fourier transforming a function $\phi(\xi)$ and imposing periodicity requires

$$e^{il\xi} = e^{il(\xi+L_\xi)} \qquad \Longrightarrow \qquad l = \frac{2\pi N}{L_\xi}, \tag{4.2}$$

where $N \in \mathbb{Z}$. Hence, there are a discrete number of particles.

Now that we have geometry with the right symmetries, we would like to claim that there is a gravity/gauge duality describing a non-relativistic conformal field theory on the boundary. Since, the AdS geometry can be found as a solution to the type IIB supergravity equations of motion, we might guess that $Sch_5 \times M$ is also a solution, where M is some compact manifold. A tedious computation shows that $Sch_5 \times S^5$ satisfies the equations, where the only difference in the background fields from AdS is the presence of a non-trivial two-form potential

$$B = A \wedge R\eta, \tag{4.3}$$

where A is a one-form with background value

$$A = \frac{\sqrt{2}R\beta}{r^2}dt \tag{4.4}$$

and η is a constant one-form to be described in more detail later. Since this spacetime is a solution of the type IIB supergravity equations just with different asymptotics than AdS, an similar equivalence between the bulk and boundary theories is expected and the formalism developed for AdS/CFT should carry over. However, before we proceed to study the boundary theory using the gravity dual, let's take a moment to move toward our second goal of finding such a duality between theories at finite temperature.

4.2 The Null Melvin Twist

The geometry $Sch_5 \times S^5$ is a solution of the type IIB equations of motion just like AdS, but now we would like to find the related solution that has a black hole but still has the same asymptotics. It turns out that there is a mechanism that can do this for us called the Null Melvin Twist, which was demonstrated in [13]. The Null Melvin Twist is a six step procedure that eats a solution of the type IIB equations of motion and spits out a new solution with different asymptotics. Since this procedure will preserve all of the curvature scalars, we can reliably use it in the supergravity approximation. So, we will use it to transform the familiar AdS black hole into the the Schrödinger black hole.

We will begin by outlining the melvinization procedure. We will need a solution to the type IIB equations of motion (both the field content and metric) that has a time coordinate t and two marked coordinates denoted y and φ, where φ is compact. To melvinize this solution:

Step I: Boost with parameter γ in the the y coordinate by applying the transformation

$$t' = t\cosh\gamma - y\sinh\gamma, \tag{4.5}$$

$$y' = -t\sinh\gamma + y\cosh\gamma. \tag{4.6}$$

This consequently mixes the dy and dt components of the metric and forms.

Step II: T-dualize along the y coordinate using the conventions:

$$g'_{yy} = \frac{1}{g_{yy}}, \qquad g'_{ay} = \frac{B_{ay}}{g_{yy}}, \qquad g'_{ab} = g_{ab} - \frac{g_{ay}g_{yb} + B_{ay}B_{yb}}{g_{yy}}, \tag{4.7}$$

$$\Phi' = \Phi - \frac{1}{2}\ln g_{yy}, \quad B'_{ay} = \frac{g_{ay}}{g_{yy}}, \qquad B'_{ab} = B_{ab} - \frac{g_{ay}B_{yb} + B_{ay}g_{yb}}{g_{yy}}. \tag{4.8}$$

Step III: Melvinize the marked compact coordinate via the transformation $\varphi' = \varphi + \alpha y$, where α is an undetermined parameter.

Step IV: T-dualize back along y.

Step V: Boost back along the y-coordinate using the parameter $-\gamma$.

Step VI: Finally, take the limit $\gamma \to \infty$ and $\alpha \to 0$ while holding $\beta' \equiv \alpha e^{\gamma}/2$ fixed.

Observe that melvinization not only changes the asymptotics, but it also mixes the metric, the dilaton, and the two-form potential B. The action of the melvinization on the other bosonic fields is unimportant for our purposes, but it is worth mentioning that this procedure leaves the five form unaffected.

We can now proceed to feed the AdS black hole solution to the Null Melvin twist. Previously, we had dropped the other fields present in the AdS background except for the Ramond-Ramond five form, because they are uniformly zero. However, we will discover that some of these components will become non-zero after melvinization. In fact, we know that the gauge field has to be non-zero, because its presence is necessary to support the zero temperature background. Recall that the black hole metric is

$$ds_{10}^2 = \frac{R^2}{r_H^2 u}(-f(u)d\tau^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2. \tag{4.9}$$

In order to proceed, we need to choose a coordinate to melvinize with on the five-sphere. A convenient choice of coordinates is given by the Hopf fibration,

$$ds_{S^5}^2 = ds_{\mathbb{CP}^2}^2 + (d\varphi + \mathcal{A})^2, \tag{4.10}$$

where $\mathbb{CP}^2$ is the complex projective space, φ is the local coordinate on the Hopf fiber, and $\mathcal{A}$ is the one form potential for the Kahler form on $\mathbb{CP}^2$ (consult the appendix of [13] for details). The one-form η identified earlier is the the fibration $d\varphi + \mathcal{A}$, which is needed to construct B. Melvinizing the AdS black hole is straight-forward but rather tedious. So, we will simply cite the result and defer the interested reader to [13] for a step by step treatement. In string frame, the result is

$$\begin{aligned} ds_{10}^2 &= \frac{R^2}{r_H^2 u^2 K(u)}\big[-f(u)(u+\delta^2)d\tau^2 - 2\delta^2 f(u)d\tau dy + (u - \delta^2 f(u))dy^2\big] + \dots \\ &\dots + \frac{R^2}{r_H^2 u}(dx^2 + dz^2) + \frac{R^2}{4u^2 f(u)}du^2 + R^2\left(ds_{\mathbb{CP}^2}^2 + \frac{1}{K(u)}(d\chi + \mathcal{A})^2\right), \end{aligned} \tag{4.11}$$

$$A^{Sch} = \frac{\delta R}{r_H u K(u)}[f(u)d\tau + dy], \tag{4.12}$$

where $\delta = \beta/r_H = R^2\beta'/r_H$[9] and $K(r) = 1 + \delta^2 u$. This can be made to look more like (4.1) by transforming to the light cone or null coordinates

$$t = \frac{y+\tau}{\sqrt{2}}, \qquad \xi = \frac{y-\tau}{\sqrt{2}}. \tag{4.13}$$

We have also identified the parameter β in (4.1) with the melvinization parameter β' and they are related by $\beta = R^2\beta'$. Besides mixing the two-form potential with the metric, the five-sphere is no longer a sphere like it was in the zero temperature case. Rather, it is "squashed" by the factor $1/K(u)$, which we will later see makes a number of things more complicated such as finding a consistent truncation. The temperature can also be computed for this black hole like it was in AdS. We find as similar result

$$T_H = \frac{\sqrt{2}}{\pi r_H}, \tag{4.14}$$

[9]This choice for δ is convenient because it makes δ dimensionless, since β has dimensions length and the melvinization parameter β' has dimensions of inverse length.

where the extract factor of $\sqrt{2}$ is related to the change to light cone coordinates. As a useful tool for checks on later results, we also note that the melvinization procedure can be undone by taking $\delta \to 0$, in which limit we recover the AdS black hole.

5. Non-Relativistic Hydrodynamics

Thus far we have presented a geometry, which solves the type IIB supergravity equations of motion and is expected to be dual to a non-relativistc conformal field theory defined on the boundary. In order to study the boundary field theory, we need to develop the constraints hydrodynamics places on non-relativistic fluids. We will stop to do that now, so we can ultimately proceed to extract information about the boundary theory from the gravity dual in the low momentum regime.

5.1 The Hydrodynamic Equations

In general, the non-equilibrium behavior of any many particle system is overwhelmingly complex. However, we can dramatically simplify the situation by considering systems in which the physical quantities vary slowly over space in time. Consequently, each portion of the system is almost in equilibrium at any point in time, and any variations can be completely described terms in of the local values of thermodynamic variables. Hence, we can apply classical fluid mechanics as an effective field theory, where we have integrated out the high energy degrees of freedom and only consider dynamics at low energy and large wavelengths.

Unlike other effective field theories, fluid mechanics is not formulated in terms of a Lagragian and an action principle. Instead, conservation equations are imposed because of the presence of dissipative effects in the system. In particular, for a one-component system we only need to consider the particle density $n(t,\vec{x})$, the momentum density $\vec{g}(t,\vec{x})$ and the energy density $\epsilon(t,\vec{x})$, which are constrained by the conservation laws

$$\partial_t n(t,\vec{x}) + \frac{1}{m}\vec{\nabla}\cdot\vec{g}(t,\vec{x}) = 0 \qquad \text{(number conservation)}, \tag{5.1}$$

$$\partial_t \vec{g}(t,\vec{x}) + \vec{\nabla}\cdot\tau(t,\vec{x}) = 0 \qquad \text{(momentum conservation)}, \tag{5.2}$$

$$\partial_t \epsilon(t,\vec{x}) + \vec{\nabla}\cdot\vec{j}^{\epsilon}(t,\vec{x}) = 0 \qquad \text{(energy conservation)}, \tag{5.3}$$

where τ is the stress tensor and $\vec{j}^{\epsilon}$ is the energy current density. Since we have assumed that all variations in time and space are slow, the system is effectively in thermal equilibrium and can be described by the local densities of conserved macroscopic quantities. These will be chosen to be the temperature, pressure, and average velocity. The average velocity can then be defined according to

$$\vec{v}(t,\vec{x}) = \frac{1}{\rho}\langle\vec{g}(t,\vec{x})\rangle, \tag{5.4}$$

with n being the equilibrium density of particles. Galilean invariance ensures that there is a conserved energy current for a system in complete equilibrium. However, when the system is only in local equilibrium there is an extra flow of energy according to temperature gradients allowing us to define

$$\langle\vec{j}^{\epsilon}(t,\vec{x})\rangle = (\epsilon + P)\vec{v}(t,\vec{x}) - \kappa\vec{\nabla}T(t,\vec{x}), \tag{5.5}$$

where ϵ and P are the equilibrium energy and pressure and κ is the thermal conductivity. Lastly, the stress tensor is given by

$$\langle\tau_{ij}(t,\vec{x})\rangle = \delta_{ij}P(t,\vec{x}) - \eta[\partial_j v_i(t,\vec{x}) + \partial_i v_j(t,\vec{x})] - \delta_{ij}\vec{\nabla}\cdot\vec{v}(t,\vec{x})\left(\zeta - \frac{2}{3}\eta\right), \tag{5.6}$$

where we have incorporated contributions due to the pressure $P(t,\vec{x})$, the shear viscosity η and the bulk viscosity ζ. It is also worth noting that the thermodynamic quantities such as the pressure and temperature are not independent of the other quantities. Since the system is in local equilibrium, the usual thermodynamic relations hold locally, which are

$$\partial_i T(t,\vec{x}) = \left.\frac{\partial T}{\partial n}\right|_{\epsilon}\partial_i\langle n(t,\vec{x})\rangle + \left.\frac{\partial T}{\partial \epsilon}\right|_{n}\partial_i\langle\epsilon(t,\vec{x})\rangle, \tag{5.7}$$

$$\partial_i P(t,\vec{x}) = \left.\frac{\partial P}{\partial n}\right|_{\epsilon}\partial_i\langle n(t,\vec{x})\rangle + \left.\frac{\partial P}{\partial \epsilon}\right|_{n}\partial_i\langle\epsilon(t,\vec{x})\rangle. \tag{5.8}$$

The conservation equations can be expressed in a more convenient form by combining them with the constitutive equations (5.4), (5.5), and (5.6). First, decompose the momentum density into longitudinal and transverse components

$$\vec{g}(t.\vec{x}) = \vec{g}_l(t,\vec{x}) + g_t(t,\vec{x}) \tag{5.9}$$

where

$$\vec{\nabla}\cdot\vec{g}_t(t,\vec{x}) = 0, \tag{5.10}$$

$$\vec{\nabla}\times\vec{g}_l(t,\vec{x}) = 0. \tag{5.11}$$

The transverse portion of the momentum conservation equation yields the diffusion equation

$$\partial_t\langle\vec{g}_t(t,\vec{x})\rangle = \frac{\eta}{\rho}\nabla^2\langle\vec{g}_t(t,\vec{x})\rangle \tag{5.12}$$

with the diffusion constant $D = \eta/\rho$. The divergence of the momentum conservation law (i.e. the longitudinal portion) combined with the number conservation law gives

$$\left[-m\partial^2 + \frac{1}{n}\left(\frac{4}{3}\eta + \zeta\right)\nabla^2\right]\langle n(t,\vec{x})\rangle + \nabla^2 P(t,\vec{x}) = 0. \tag{5.13}$$

Similarly, the momentum density can be eliminated from the energy conservation law to give

$$\partial_t\left[\langle\epsilon(t,\vec{x}) - \frac{\epsilon+P}{n}\langle n(t,\vec{x})\rangle\right] - \kappa\nabla^2 T(t,\vec{x}) = 0. \tag{5.14}$$

5.2 Expressions for the Transport Coefficients

Now that we have determined the non-relativistic hydrodynamic equations, we can see how these equations place constraints on the behavior of the local densities. Traditionally, the way to proceed is to study the linear response of the theory to small perturbations from equilibrium, which will determine the structure of the correlators. Define the response function

$$\tilde{\chi}''_{AB}(t-t',\vec{x}-\vec{x}') = \frac{1}{2}\langle[A(t,\vec{x}), B(t',\vec{x}')]\rangle. \tag{5.15}$$

It is then immediately apparent that this is related to the retarded Green's function we have been studying earlier. Using our convention for the Fourier transform,

$$\chi''_{AB}(\omega,q) = \int d^4x\, e^{-i\omega(t-t')+i\vec{q}\cdot(\vec{x}-\vec{x}')}\tilde{\chi}''_{AB}(t-t',\vec{x}-\vec{x}'). \tag{5.16}$$

Then, it is straight forward to show by substituting (5.15) into our definition for the retarded Green's function (2.21) that

$$\operatorname{Im} G^R_{AB}(\omega,q) = -\chi''_{AB}(\omega,q), \tag{5.17}$$

for real frequencies ω. Hence, any statement about the response function χ_{AB} is also a statement about the Green's function and vice versa. This will eventually enable us to determine the transport coefficients.

However, before continuing we can use symmetries and the conservation equations to learn about what the response functions should look like before we give explicit formulae for them. Of particular interest is the number density/number density response function χ''_{nn}. Time reversal and rotational invariance combined with the fact that this is a Hermitian operator requires that it be a real odd function of the frequency ω. The momentum density/momentum density response function $\chi''_{g_ig_j}$ is a tensor which can be decomposed into transverse and longitudinal portions

$$\chi''_{g_ig_j}(\omega,q) = \frac{q_iq_j}{q^2}\chi''_l(\omega,q) + \left(\delta_{ij} - \frac{q_iq_j}{q^2}\right)\chi''_t(\omega,q), \tag{5.18}$$

both of which χ_l'' and χ_t'' are real odd functions of the frequency. The number conservation law (5.1) can be used to establish the relation

$$\chi_{n,g_i}''(\omega, q) = \chi_{g_i,n}''(\omega, q) = \frac{q_i}{m\omega}\chi_l''(\omega, q). \tag{5.19}$$

Similarly, a double application of this law yields

$$\chi_{n,n}''(\omega, q) = \frac{q^2}{m^2\omega^2}\chi_l''(\omega, q). \tag{5.20}$$

This shows us that once we know $\chi_{n,n}''$ we can determine most of the remaining response functions in the longitudinal modes.

Now, that we have related the response functions to something familiar and have established relations among them we would like to use hydrodynamics to further constrain them. The procedure is to construct time-dependent perturbations that slowly shift the system from equilibrium at which point the perturbation turns off. The response of the system is completely described by the response functions defined above, hence the name. Working in momentum space, we can construct exactly what the low momentum behavior of the response function must be. This process is involved, so the reader is deferred to [14] for the details, and we will simply present the results that we need. Since the transverse momentum density satisfies its own diffusion equation, we find that

$$\chi_t''(\omega, q) = \frac{\eta\omega q^2}{\omega^2 + D^2 q^4}, \tag{5.21}$$

which has a diffusion pole at $\omega = iDq^2$. The transverse response function for the momentum density for our purposes will not be the most convenient quantity to compute. Instead, we can use the rotational invariance of the field theory to fix the momentum in the z-direction. Then, (5.18) gives us $\chi_t'' = \chi_{g_x g_x}''$. Combined with the momentum density conservation equation (5.2) and the definition of the response function, we find

$$\chi_t'' = \frac{q}{\omega}\chi_{\tau_{xz},\tau_{xz}}'', \tag{5.22}$$

and a second application gives

$$\chi_t'' = \frac{q^2}{\omega^2}\chi_{\tau_{xz},\tau_{xz}}''. \tag{5.23}$$

Now, (5.21) translates into the Kubo formula for the shear viscosity

$$\eta = -\lim_{\omega\to 0}\lim_{q\to 0}\frac{1}{\omega}\mathrm{Im}\, G_{\tau_{xz},\tau_{xz}}^R(\omega, q). \tag{5.24}$$

This is is a quantity we will be able to compute from the bulk theory. Similar, formulae can be expressed for the longitudinal modes to get the speed of sound, thermal conductivity, etc. [14].

6. Hydrodynamics and Holography

Having examined the hydrodynamic constraints placed upon the correlators, we can now determine the various transport coefficients by studying the gravity dual of the field theory. In particular, we can compute the retarded Green's functions in the low momentum regime, which subsequently allows us to simply read of the various transport coefficients by applying Kubo's formulae. The formalism we have developed thus far allows us to calculate Green's functions by considering first order fluctuations of the bulk fields around the equilibrium solution provided that the correspondence between bulk and boundary fields is known. We begin by sketching an argument that gives the relation between the bulk sources and the boundary operators. Once we know what field content determines the desired correlators, we present a few useful tools that we will be essential for solving the equations of motion in the hydrodynamic limit. Having established the gravity/gauge theory dictionary, we will digress into finding solutions in two different gauges in the AdS black hole, before finally using these solutions to determine correlators and transport coefficients in the Schrödinger spacetime.

6.1 Gauge/Gravity Dictionary

Now that we have a background metric and an understanding of what the conserved currents of the boundary theory are we would like to turn on sources coupled to these currents. These sources are the non-normalizable modes of the bulk field perturbations expanded about the background solutions. So, we need to match these non-normalizable modes to their dual boundary operators. This will be done by imposing the gauge invariance of the theory. We will simply sketch the procedure placing emphasis on the results, and the interested reader can consult [11] for more details. For simplicity, we will just consider the zero temperature case, because the generalization to finite temperature and chemical potential is trivial.

We need to consider perturbations of the bulk fields, which will correspond to background fields from which the boundary theory is constructed. Insight or experience tells us that the bulk field of primary interest is the metric, which should couple to some of boundary operators including the stress tensor. So, set

$$g_{\mu\nu}(x) = g^{(0)}_{\mu\nu}(x) + h_{\mu\nu}(x), \tag{6.1}$$

where $h_{\mu\nu}$ is taken to be a first order perturbation. At this point, we need to be able to identify what combinations of the $h_{\mu\nu}$ couple to the boundary operators. The gauge symmetry generated by diffeomorphisms can provide us with such a tool (section 6.3 has a few more details about gauge transformations for an unfamiliar reader). Firstly, it will be helpful to fix a gauge. The traditional choice of gauge is the transverse gauge

$h_{u\mu} = 0$, because it will greatly simplify the correspondence that will follow. The transverse gauge doesn't completely fix the gauge, because it leaves five residual gauge transformations. Since the bulk theory is gauge invariant and it is dual to the boundary theory, the boundary theory should also be invariant under the gauge transformations. Hence, we can require that the source terms in the boundary theory Lagrangian must be invariant under these residual gauge transformations. This restricts what bulk fields can couple to which boundary operators, and imposing the Schrödinger symmetry uniquely fixes it.

Start by postulating the following parameterization of the metric,

$$\frac{1}{R^2}ds^2 = -\frac{2e^{-2B_0}}{r^4}(\beta dt - B_i dx^i)^2 - \frac{2e^{-B_0}}{r^2}(\beta dt - B_i dx^i)\left(-\frac{1}{\beta}d\xi - A_0 dt - A_i dx^i\right) + \ldots$$
$$\ldots + \chi d\xi^2 + \frac{1}{r^2}\left(dx^i dx^j + H_{ij}dx^i dx^j + dr^2\right), \tag{6.2}$$

where for the moment we have set the AdS radius R equal to 1. This gives the background Sch_5 metric coupled to additional fields, which we will use to parameterize the metric fluctuations. Expanding the additional fields to first order and interpreting them as perturbations,

$$h_{\mu\nu} = \frac{R^2}{r^2}\begin{pmatrix} \frac{2\beta^2}{r^2}\left(A_0 r^2 + 2B_0\right) & -B_0 & \frac{\beta}{r^2}\left(A_1 r^2 + 2B_1\right) & \frac{\beta}{r^2}\left(A_2 r^2 + 2B_2\right) & 0 \\ -B_0 & r^2\chi & -\frac{1}{\beta}B_1 & -\frac{1}{\beta}B_2 & 0 \\ \frac{\beta}{r^2}\left(A_1 r^2 + 2B_1\right) & -\frac{1}{\beta}B_1 & H_{xx} & H_{zx} & 0 \\ \frac{\beta}{r^2}\left(A_2 r^2 + 2B_2\right) & -\frac{1}{\beta}B_2 & H_{zx} & H_{zz} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.3}$$

For the sake of computation, it will be useful to define dimensionless equivalents of these fields for finite temperature. In particular, we set

$$A_0 \longrightarrow \frac{A_0}{r_H^2}, \qquad A_i \longrightarrow \frac{A_i}{r_H}, \qquad B_i \longrightarrow r_H B_i. \tag{6.4}$$

Finally, in the (τ, y, u) coordinates we will use later the perturbations are then

$$h_{\mu\nu} = \frac{R^2}{r_H^2 u}\begin{pmatrix} A_0 + \frac{u+2\delta^2}{u}B_0 + \frac{u\chi}{2} & A_0 + \frac{2\delta^2}{u}B_0 - \frac{u}{2}\chi & \frac{\delta u A_1 + (u+2\delta^2)B_1}{\sqrt{2}\delta u} & \frac{\delta u A_2 + (u+2\delta^2)B_2}{\sqrt{2}\delta u} & 0 \\ A_0 + \frac{2\delta^2}{u}B_0 - \frac{u}{2}\chi & A_0 - \frac{u-2\delta^2}{u}B_0 + \frac{u\chi}{2} & \frac{\delta u A_1 - (u-2\delta^2)B_1}{\sqrt{2}\delta u} & \frac{\delta u A_2 - (u-2\delta^2)B_2}{\sqrt{2}\delta u} & 0 \\ \frac{\delta u A_1 + (u+2\delta^2)B_1}{\sqrt{2}\delta u} & \frac{\delta u A_1 - (u-2\delta^2)B_1}{\sqrt{2}\delta u} & H_{xx} & H_{zx} & 0 \\ \frac{\delta u A_2 + (u+2\delta^2)B_2}{\sqrt{2}\delta u} & \frac{\delta u A_2 - (u-2\delta^2)B_2}{\sqrt{2}\delta u} & H_{zx} & H_{zz} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.5}$$

All of the transformation rules for these perturbations under infinitesimal diffeomorphisms can be written out (though they are rather unenlightening), and a dual non-relativistic field theory can be constructed with the exact same symmetries. This identification then immediately furnishes an interpretation of these perturbations. The conclusion is that:

- H_{ij} couples to the stress tensor τ^{ij},
- A_μ couples to the mass current $(n, \vec{g})$,
- B_μ couples to the energy current $(\epsilon, \vec{j}^\epsilon)$.

The only mode that remains is χ, which has no immediate physical interpretation in the boundary theory. Notice that we have determined the bulk fields dual to all of the operators we considered in the hydrodynamic analysis. So, for our purposes the boundary operators coupled to the perturbations for other fields are unecessary, and we would like to set them to zero if possible.

Lastly, in order to compute the retarded Green's functions corresponding to these operators, we need to know the form of the on-shell action for the metric. In particular, we only need the quadratic terms, which are

$$S_{on-shell} = \frac{\pi^3 R^5}{8\kappa_{10}^2} \int d^4x \, \sqrt{g} g^{uu} \left(h^\mu{}_\nu \partial_u h^\nu{}_\mu + h^\mu{}_\mu \partial_u h^\nu{}_\nu \right) |_{u=\epsilon}^{u=1}, \tag{6.6}$$

where $\kappa_{10} = 2\pi^{5/2} R^4/N$ is the ten dimensional gravitational constant. At this point, we are ready to proceed to study these modes of the metric fluctuations.

6.2 Classification of the Modes

In the previous section it was argued that the only field which couples to the boundary operators of interest is the metric. So, we want to set as many perturbations to zero as possible while still keeping the metric perturbations unrestricted. The fermioninc content is trivially set to zero, and only a few bosonic fields will contribute. It is not hard to show that the only fields to consider are the metric, the dilaton, and the two-form potential.

Consider the perturbations

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + h_{\mu\nu}(x), \tag{6.7}$$
$$A_\mu(x) \to A_\mu(x) + \delta A_\mu(x), \tag{6.8}$$
$$\Phi(x) \to \Phi(x) + \delta\Phi(x). \tag{6.9}$$

As before, it is convenient to work with the momentum space equivalents of the perturbations related by the Fourier transform. For an arbitrary pertrubation $\delta h(x)$

$$\delta h(x,u) = \int \frac{d^4q}{(2\pi)^4} e^{i(-\epsilon\tau+q_y y+q_z z)} \delta\tilde{h}(q,u), \tag{6.10}$$

where we have again used the rotational symmetry to remove the x dependence in the Fourier kernel. There is an important distinction for the Schrödinger black hole that is not present in the AdS case. There is a second momentum that is not spatial, which is a consequence of the chemical potential. This will be important later, because we need to evaluate the Green's functions at equilibrium.

Let's stop for a moment and generalize. Instead suppose that we have a system with an $O(d)$ symmetry. This symmetry allows us to fix the momentum in just one direction, which is reflected in the kernel of the Fourier transform. However, there still remains an $O(d-1)$ residual symmetry, which consists of rotations about the marked axis. This residual symmetry must still be a good symmetry of the system, hence the equations of motion must respect it. Under the $O(d-1)$ group action, the fields can transform as scalars, vectors, or tensors, but the equations of motion are still invariant. Hence, modes that couple together in the equations of motion must transform in the same way under the residual $O(d-1)$ symmetry. This simplifies the analysis by allowing us to decompose the fluctuations into independent channels that can be studied separately.

For Sch_5, the residual symmetry is a trivial $O(1)$ symmetry, which is only a change of sign. However, this does ensure the decoupling of the perturbations into two channels[10]

$$\text{Scalars (Sound Channel): } h_{\tau\tau}, h_{y\tau}, h_{yy}, h_{xx}, h_{z\tau}, h_{zy}, h_{zz}, \delta A_\tau, \delta A_y, \delta A_z, \delta A_u, \delta\Phi \tag{6.11}$$

$$\text{Vectors (Shear Channel): } h_{x\tau}, h_{xy}, h_{zx}, \delta A_x. \tag{6.12}$$

It is also worth noting that in the $q_z \to 0$ limit, we regain the $O(2)$ symmetry and see even more decoupling. Here, h_{zx} decouples completely because it is the only component that transforms as a tensor. This will later simplify the computation of the shear viscosity, since we only need the zero momentum limit.

6.3 Residual Gauge Transformations

Seeing that the field perturbations separate into channels is helpful, because we can reduce the number of equations that need to be solved simultaneously. However, an

[10]The naming arises from considering the boundary operators to which these bulk fields couple. The "sound channel" has solutions which exhibit the sound pole in the boundary theory i.e. it couples to the longitudinal components of the momentum density. Conversely, the "shear channel" exhibits the diffusive behavior of the transverse components.

additional symmetry allows us to extract even more from the equations of motion without going through the laborious task of solving coupled second order equations. This symmetry is the gauge symmetry.

Since our bulk theory is simply classical supergravity which we have linearized about the background solutions, we have a gauge theory with diffeomorphisms as the gauge transformation. This is simply a statement that a change of coordinates shouldn't affect the physics of the theory. In general, an infinitesimal diffeomorphism acts on the fields according to

$$x^\mu \to x^\mu + \eta^\mu, \tag{6.13}$$

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) - \nabla_\mu \eta_\nu - \nabla_\nu \eta_\mu, \tag{6.14}$$

$$A_\mu(x) \to A_\mu(x) - \eta^\rho \nabla_\rho A_\mu(x) - A_\rho(x) \nabla_\mu \eta^\rho, \tag{6.15}$$

$$\Phi(x) \to \Phi(x) - \eta^\rho \nabla_\rho \Phi(x), \tag{6.16}$$

where we have recognized the patterns that each of the transformations is the Lie derivative of the field making it easy to generalize the transformation to higher component objects. It is also sufficient to consider only infinitesimal transformations, because global transformations can be constructed from them. Since we have given perturbations to all of our fields, we are free to let the background fields be invariant and only allow the perturbations to transform. Since the dictionary between bulk and boundary fields was given in the transverse gauge $h_{u\mu} = 0$, we will need to enforce this in the gauge transformation. This yields the constraint equations

$$\nabla_u \eta_\mu - \nabla_\mu \eta_u = \partial_\mu \eta_u + \partial_u \eta_\mu - 2\Gamma^\rho_{\mu u} \eta_\rho = 0. \tag{6.17}$$

To first order in the diffeomorphism, there are five solutions to this equation corresponding to five gauge transformations which preserve the transverse gauge. Because the action is invariant under the gauge transformation, the resulting equations of motion must be as well. Hence, linearity tells us that the gauge transformation acting on the trivial solution must also be a solution of the equations of motion. We are then instantly guaranteed to have five pure gauge solutions to the equations of motion which are exact to all orders in momenta.

We can go even farther than this and find gauge invariant parameterizations of the fluctuations [15]. By the same reasoning as before, the equation of motion that the gauge invariant variable satisfy must by definition be gauge invariant. Hence, we expect to see further decoupling because there are less gauge invariant parameterizations than there are fluctuations.

6.4 Finding the Bulk Solutions

The end goal is to compute retarded Green's functions of the boundary operators, and to do this we need to know the bulk solutions in the hydrodynamic limit. Naively, we could approach this problem just as we did before in computing the massless scalar coupled to the AdS black hole background by finding the equations of motion and directly solving them in the hydrodynamic limit. However, because AdS has a diagonal metric even the most complicated channel, the sound channel, is relatively easy to solve in the transverse gauge. This turns out not to be the case in the Schrödinger black hole[11].

There are two ways to proceed in finding these solutions at this point, and both involve melvinization. The insight is that the solutions in the Schrödinger spacetime can be determined from their AdS counterparts. They can either be directly melvinized, or we can use melvinization to find linear combinations of the metric perturbations that decouple many of the equations. In either approach, we need to know how the AdS perturbations are related to their Schrödinger counterparts. This can be found by melvinization.

Recall the melvinization procedure introduced earlier. We found that the non-trivial fields in the Schrödinger background are the metric, the dilaton, and the one-form potential, which in turn can be constructed from the same fields in the AdS background. So, we will want to melvinize the AdS perturbations of these fields to relate them to the perturbations in the Schrödinger spacetime. Now, it is not guaranteed that the transverse gauge in AdS melvinizes to the transverse gauge in the Schrödinger black hole. So, we will proceed without fixing a gauge until after we have melvinized. The AdS fields with fluctuations are

$$g_{\mu\nu}^{AdS} = g_{\mu\nu}^{(0)AdS} + h_{\mu\nu}^{AdS}, \tag{6.18}$$

$$A_{\mu}^{AdS} = \delta A_{\mu}^{AdS}, \tag{6.19}$$

$$\Phi^{AdS} = \delta\Phi^{AdS}. \tag{6.20}$$

When we originally melvinized the AdS black hole background, the process was not simple, but it could still be done by hand. Now that both the dilaton and gauge field are non-trivial and that the metric has off diagonal components, the melvinization procedure is significantly more complicated. We will simply cite the results that were found using the *Mathematica* package presented in the appendix.

[11] In more detail, the problem is that a large number of fields couple together in the equations of motion. After performing a hydrodynamic expansion, the recursive systems of equations in general have more than two coupled first order equations. There is no general solution for such a system. The AdS sound channel can be easily parameterized such that only coupled first order equations need to be solved

We will define the Schrödinger perturbations in the same way as their AdS relatives, with the only difference being the difference in the background values. The first metric components to consider are $h^{Sch}_{u\mu}$, which are related to the AdS perturbations by

$$h^{Sch}_{uu} = h^{AdS}_{uu}, \qquad h^{Sch}_{ux} = h^{AdS}_{ux}, \qquad h^{Sch}_{uz} = h^{AdS}_{uz}, \tag{6.21}$$

$$h^{Sch}_{uy} = \frac{u - f\delta^2}{uK} h^{AdS}_{uy} + \frac{\delta^2}{uK} h^{AdS}_{ut} - \frac{\delta R}{r_H u K} \delta A^{AdS}_{u}, \tag{6.22}$$

$$h^{Sch}_{u\tau} = \frac{u + \delta^2}{uK} h^{AdS}_{ut} - \frac{\delta^2 f}{uK} h^{AdS}_{uy} - \frac{\delta R f}{r_H u K} \delta A^{AdS}_{u}. \tag{6.23}$$

Observe that the melvinization mixes components of the gauge field with the metric. This is a the primary cause of the extra complication in determining the non-relativistic correlators. Now, we can use this result to relate the transverse gauge in the Schrödinger spacetime to a choice of gauge in AdS. We see that the gauge choice

$$h^{AdS}_{uu} = 0, \qquad h^{AdS}_{ux} = 0, \qquad h^{AdS}_{uz} = 0, \tag{6.24}$$

$$h^{AdS}_{ut} = \frac{\delta R f}{r_H u} \delta A^{AdS}_{u}, \quad h^{AdS}_{uy} = \frac{\delta R}{r_H u} \delta A^{AdS}_{u} \tag{6.25}$$

melvinizes to the transverse gauge $h^{Sch}_{u\mu} = 0$. This gauge choice will be referred to as the Schrödinger transverse gauge $h^{Sch}_{u\mu} = 0$, since the two are equivalent via melvinization. Now that we are familiar with the two gauges that are convenient to work in, we can write down the rest of the relations. However, since these formulae are rather complicated, we will defer the explicit presentation of these results to the appendices.

Now that we can relate the perturbations between the two asymptotically AdS spacetimes, we can proceed to find the solutions for the Schrödinger spacetime. As mentioned before, the Schrödinger fluctuations can be parameterized in terms of the AdS fluctuations, which should result in some decoupling in the equations of motion. This turns out to be true and it works beautifully for the shear channel, but it isn't straightforward to apply this parameterization to the sound channel. The problem is that ten dimensional Schrödinger geometry is not $Sch_5 \times S^5$. Instead, the five-sphere is "squashed" a little bit. Hence, the Kaluza-Klein reduction doesn't permit us to simply drop the pseudo-spherical geometry and proceed like we did in AdS. A consistent truncation does exist with the additional complication being the introduction of two new scalar fields [16]. These scalar field incorporate the effects of the "squashed" sphere on the Sch_5 portion of the geometry, and they complicate the sound channel equations enough such that they become nearly intractable to solve directly even after using the AdS parameterization. The easiest approach is then to solve for the AdS perturbations and then use the formulae derived by melvinization to determine the

Schrödinger solutions. This can all be done without considering the five dimensional consistent truncation.

We will solve the AdS perturbations in the same way we solved for the scalar field coupled to the black hole background. In fact we will do this twice, once for each gauge we want to consider. The reason for this is that the metric solutions for the transverse gauge $h^{AdS}_{u\mu} = 0$ are also solutions for the AdS gauge $h^{Sch}_{u\mu} = 0$ with the gauge field perturbation $\delta A^{AdS}_u = 0$. This reflects that fact that there are more nontrivial degrees of freedom in the Schrödinger black hole than in the AdS black hole.

6.4.1 The AdS Black Hole with $h^{AdS}_{u\mu} = 0$

We will begin by briefly applying several of the tools we have developed thus far by studying symmetries of the system and then solving the equations of motion. An interested reader can consult [1, 2] for more details about solving the AdS metric perturbations in the transverse gauge. Experience teaches us that the metric perturbations we want to consider have one raised and one lowered index. In fact, these are the modes that couple to the boundary stress-energy tensor. So, define

$$H_{\mu\nu} = h^{\mu}{}_{\nu}. \tag{6.26}$$

Since there are no off-diagonal elements in the metric, $H_{\mu\nu} \propto h_{\mu\nu}$ up to a radially dependent factor.

After fixing the momenta in the z-direction using the same Fourier transform as defined by (3.5), there is still an $O(2)$ residual symmetry in the xy plane. Hence, we see that there are three channels among which the perturbations are divided:

$$\text{Scalars (Sound Channel): } H_{zt}, H_{tt}, H_{zz}, (H_{xx}, H_{yy}, \text{or } H_{aa}) \tag{6.27}$$

$$\text{Vectors (Shear Channel): } H_{xt}, H_{zx}, \quad \text{and} \quad H_{yt}, H_{zy}, \tag{6.28}$$

$$\text{Tensors (Scalar Channel): } H_{xy}\,(H_{bb}). \tag{6.29}$$

It is worth noting that the tensor representation is reducible into a trace and traceless part. So, we can exchange $H_{aa} = H_{xx} + H_{yy}$ for H_{xx} and H_{yy} in the sound channel provided we add another scalar channel with $H_{bb} = H_{xx} - H_{yy}$. The full metric with the first order perturbations is then

$$g^{AdS}_{\mu\nu} = \frac{R^2}{r_H^2}\begin{pmatrix} -\frac{f}{u} + \frac{f}{u}H_{tt} & \frac{1}{u}H_{xt} & \frac{1}{u}H_{yt} & \frac{1}{u}H_{zt} & 0 \\ \frac{1}{u}H_{xt} & \frac{1}{u} + \frac{1}{u}H_{xx} & \frac{1}{u}H_{yx} & \frac{1}{u}H_{zx} & 0 \\ \frac{1}{u}H_{yt} & \frac{1}{u}H_{yx} & \frac{1}{u} + \frac{1}{u}H_{yy} & \frac{1}{u}H_{zy} & 0 \\ \frac{1}{u}H_{zt} & \frac{1}{u}H_{zx} & \frac{1}{u}H_{zy} & \frac{1}{u} + \frac{1}{u}H_{zz} & 0 \\ 0 & 0 & 0 & 0 & \frac{r_H^2}{4u^2(1-u^2)} \end{pmatrix}. \tag{6.30}$$

Variation of the action gives Einstein's equation

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{6.31}$$

where the cosmological constant is $\Lambda = -6/R^2$. These equations can be rearranged to yield Ricci's equation or the trace-reversed equation

$$R_{\mu\nu} = \frac{2\Lambda}{3} g_{\mu\nu}. \tag{6.32}$$

The traced reversed form of the equations are slightly more convenient to work with, but the only difference is in the linear combinations of the sound channel equations. The other channels yield equivalent expressions for corresponding components of the tensorial equation. Now, the scalar equations of motion resulting from this parameterization of the metric are

$$(E_{xy}) \qquad H''_{xy} - \frac{1+u^2}{uf} H'_{yx} + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2} H_{yx} = 0, \tag{6.33}$$

$$(E_{xx} - E_{yy}) \qquad H_{bb}'' - \frac{1+u^2}{uf} H_{bb}' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2} H_{bb} = 0, \tag{6.34}$$

where we have indicated the component of the Ricci equation that yielded each equation. The two shear channels are $(\alpha = x, y)$

$$(E_{u\alpha}) \qquad \frac{\mathfrak{w}}{f} H'_{\alpha t} + \mathfrak{q} H'_{z\alpha} = 0, \tag{6.35}$$

$$(E_{\alpha t}) \qquad H''_{\alpha t} - \frac{1}{u} H'_{\alpha t} - \frac{\mathfrak{q}}{uf}(\mathfrak{q} H_{\alpha t} + \mathfrak{w} H_{z\alpha}) = 0, \tag{6.36}$$

$$(E_{z\alpha}) \qquad H''_{z\alpha} - \frac{1+u^2}{uf} H'_{z\alpha} + \frac{\mathfrak{w}}{uf^2}(\mathfrak{q} H_{\alpha t} + \mathfrak{w} H_{z\alpha}) = 0. \tag{6.37}$$

Lastly, once we have made the further redefinition $H_{ii} = H_{aa} + H_{zz}$ the sound channel equations are

$$(E_{tt}) \quad H_{tt}'' - \frac{3(1+u^2)}{2uf}H_{tt}' + \frac{1+u^2}{2uf}H_{ii}' - \frac{2\mathfrak{q}\mathfrak{w}}{uf^2}H_{zt} - \frac{\mathfrak{q}^2}{uf}H_{tt} - \frac{\mathfrak{w}^2}{uf^2}H_{ii} = 0, \tag{6.38}$$

$$(E_{xx}+E_{yy}) \quad H_{aa}'' + \frac{1}{u}H_{tt}' - \frac{1}{u}H_{ii}' - \frac{1+u^2}{uf}H_{aa}' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2}H_{aa} = 0, \tag{6.39}$$

$$(E_{zt}) \quad H_{zt}'' - \frac{1}{u}H_{zt}' + \frac{\mathfrak{q}\mathfrak{w}}{uf}H_{aa} = 0, \tag{6.40}$$

$$\begin{aligned}(E_{zz}) \quad & H_{ii}'' - H_{aa}'' + \frac{1}{2u}H_{tt}' + \frac{3+u^2}{2uf}H_{ii}' + \frac{1+u^2}{uf}H_{aa}' + \dots \\ & \dots + \frac{2\mathfrak{q}\mathfrak{w}}{uf^2}H_{zt} + \frac{\mathfrak{q}^2}{uf}H_{tt} + \frac{\mathfrak{w}^2}{uf^2}H_{ii} - \frac{\mathfrak{w}^2 + \mathfrak{q}^2 f}{uf^2}H_{aa} = 0,\end{aligned} \tag{6.41}$$

$$(E_{ut}) \quad \mathfrak{q}H_{zt}' + \mathfrak{w}H_{ii}' + \frac{2\mathfrak{q}u}{f}H_{zt} + \frac{\mathfrak{w}u}{f}H_{ii} = 0, \tag{6.42}$$

$$(E_{uz}) \quad \mathfrak{w}H_{zt}' + \mathfrak{q}fH_{tt}' - \mathfrak{q}fH_{aa}' - \mathfrak{q}uH_{tt} = 0, \tag{6.43}$$

$$(E_{uu}) \quad H_{ii}'' - H_{tt}'' + \frac{3u}{f}H_{tt}' - \frac{u}{f}H_{ii}' = 0. \tag{6.44}$$

Notice that the scalar channel equations (6.33) and (6.34) are the same equation of motion that was derived for a massless scalar field coupled to the AdS background (3.7), hence the name scalar channel. We already know these solutions which are given by (3.17) with the appropriate exchange in field names.

In both the shear and sound channels, we notice that there is a redundancy in the equations, because there are fewer variables than there are equations. In particular, there are the so called gauge equations which are first order constraints imposed by our choice of gauge. In order to find the solutions, we need to select a complete set of equations, and then we can reduce them to a first order system which can be expanded in the hydrodynamic limit. To be completely rigorous, we should reduce the equations to a set from which we can derive all of the remaining equations. However, this is tedious in general, because the equations are not simply linear combinations of each other. Alternatively, we can select the same number of equations as there are degrees of freedom using trial and error until we find the correct physical behavior near the horizon as exhibited by the eigenvalues of the first order system. We will simply give the results from the later process.

For the shear channels, we need to select two equations since there are two degrees of freedom. We will choose to use (6.35) and (6.36), which means that the general solutions will have three degrees of freedom, which need to be fixed. The first order system constructed from these equations is

$$\begin{pmatrix} H_{\alpha t} \\ H_{z\alpha} \\ P_{\alpha t} \end{pmatrix}' = \begin{pmatrix} 0 & 0 & -\frac{\mathfrak{w}}{\mathfrak{q} f} \\ 0 & 0 & 1 \\ \frac{\mathfrak{q}\mathfrak{w}}{uf} & \frac{\mathfrak{q}^2}{uf} & \frac{1}{u} \end{pmatrix} \cdot \begin{pmatrix} H_{\alpha t} \\ H_{z\alpha} \\ P_{\alpha t} \end{pmatrix}. \tag{6.45}$$

We find three eigenvalues which are $\nu = 0, \pm i\mathfrak{w}/2$. The outgoing wave cån be discarded as unphysical. The $\nu = 0$ solution meets the incoming wave boundary condition because it corresponds to a pure gauge solution to the equations of motion, and the $\nu = -i\mathfrak{w}/2$ is the incoming wave that we expected to find. Thus, we have two degrees of freedom remaining, which can be fixed by requiring the bulk fields to asymptote to their dual boundary fields. In the hydrodynamic limit, the two solutions are then

$$H^I_{\alpha t} = \mathfrak{w}, \tag{6.46}$$

$$H^I_{z\alpha} = -\mathfrak{q}, \tag{6.47}$$

$$H^{inc}_{\alpha t} = (1-u)^{-i\mathfrak{w}/2}\left(\frac{i\mathfrak{q}f}{2} - \frac{\mathfrak{q}\mathfrak{w}}{4}(1-u)\left[(1+u)\ln\frac{1+u}{2} + u\right] + \dots \right.$$
$$\left. \dots + \frac{i\mathfrak{q}^3 f^2}{4} + \mathcal{O}(\mathfrak{q}^4, \mathfrak{q}^2\mathfrak{w}, \mathfrak{w}^2)\right), \tag{6.48}$$

$$H^{inc}_{z\alpha} = (1-u)^{-i\mathfrak{w}/2}\left(1 - \frac{i\mathfrak{w}}{2}\ln\frac{1+u}{2} + \mathcal{O}(\mathfrak{q}^4, \mathfrak{q}^2\mathfrak{w}, \mathfrak{w}^2)\right). \tag{6.49}$$

We might wonder what would have happened has we chosen to forgo using the first order gauge equation and selected the other second order dynamical equation. In that case we would end up with four degrees of freedom for a general solution, but we know that we still need to reduce this to two. Of course, discarding the outgoing wave solution eliminates one of the extra two degrees of freedom, but what happens to other one? The other degree of freedom has an exponent $\nu = 0$ and is actually a pure gauge solution that arises from a gauge transformation, which breaks the transverse gauge. In particular, the solution found corresponding to this exponent does not satisfy (6.35) and can be discarded. The solutions then are the same as we found using the gauge equation.

Now, we are prepared to move onto solving the sound channel equations of motion. There are four independent perturbations, which indicates that we need to select four of the seven available equations. Although there are several combinations of the listed equations that will work for this purpose, we will use (6.38), (6.40), (6.42), and (6.43). The resulting first order system is

$$\begin{pmatrix} H_{tt} \\ H_{aa} \\ H_{ii} \\ H_{zt} \\ P_{tt} \\ P_{zt} \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{f} & 0 \\ -\frac{u}{f} & 0 & 0 & 0 & \frac{1}{f} & \frac{\mathfrak{w}}{\mathfrak{q} f} \\ 0 & 0 & -\frac{u}{f} & -\frac{2\mathfrak{q} u}{\mathfrak{w} f} & 0 & -\frac{\mathfrak{q}}{\mathfrak{w}} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\mathfrak{q}^2}{u} & 0 & \frac{u(1+u^2)+2\mathfrak{w}^2}{2uf} & \frac{\mathfrak{q}\left(u^3+u+2\mathfrak{w}^2\right)}{\mathfrak{w} u f} & \frac{3-u^2}{2uf} & \frac{\mathfrak{q}\left(1+u^2\right)}{2\mathfrak{w} u} \\ 0 & -\frac{\mathfrak{q}\mathfrak{w}}{uf} & 0 & 0 & 0 & \frac{1}{u} \end{pmatrix} \cdot \begin{pmatrix} H_{tt} \\ H_{aa} \\ H_{ii} \\ H_{zt} \\ P_{tt} \\ P_{zt} \end{pmatrix} \tag{6.50}$$

The exponents of this system are $\nu = 0, 0, \pm 1/2$[12]$, \pm i\mathfrak{w}/2$. We need four solutions and the correct ones to choose have the exponents $\nu = -1/2, 0, 0, -i\mathfrak{w}/2$, which can be interpreted as three pure gauge solutions and an incoming wave. The complete solutions are

$$H^{I}_{ii} = -2\mathfrak{q}, \tag{6.51}$$
$$H^{I}_{zt} = \mathfrak{w}, \tag{6.52}$$

$$H^{II}_{tt} = -2\mathfrak{w}, \tag{6.53}$$
$$H^{II}_{zt} = \mathfrak{q} f, \tag{6.54}$$

$$H^{III}_{tt} = \frac{1 + u^2 + 2\mathfrak{w}^2 u}{\sqrt{f}}, \tag{6.55}$$
$$H^{III}_{aa} = -2\sqrt{f}, \tag{6.56}$$
$$H^{III}_{ii} = 2\mathfrak{q}^2 \sin^{-1} u - 3\sqrt{f}, \tag{6.57}$$
$$H^{II}_{zt} = -\mathfrak{q}\mathfrak{w}(\sin^{-1} u + u\sqrt{f}), \tag{6.58}$$

[12]It is at least worth mentioning in passing that the exponent $\nu = 1/2$ is blocked. Fortunately, its corresponding solution is not one we will need.

$$H_{tt}^{inc} = (1-u)^{-i\mathfrak{w}/2}\left[\frac{\mathfrak{q}^2}{3}(1-u) + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w})\right], \tag{6.59}$$

$$H_{aa}^{inc} = (1-u)^{-i\mathfrak{w}/2}\left[1 - \frac{i\mathfrak{w}}{2}\ln\frac{1+u}{2} + \frac{2\mathfrak{q}^2}{3}\left(1-u+\frac{1}{2}\ln\frac{1+u}{2}\right) + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w})\right], \tag{6.60}$$

$$H_{ii}^{inc} = (1-u)^{-i\mathfrak{w}/2}\left[1-u + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w})\right], \tag{6.61}$$

$$H_{zt}^{inc} = (1-u)^{-i\mathfrak{w}/2}\left[-\frac{i\mathfrak{q}f}{2} + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w})\right], \tag{6.62}$$

where we have only given the non-zero values for the pure gauge solutions.

6.4.2 The AdS Black Hole with $h_{u\mu}^{Sch} = 0$

Even though the metric perturbations which respect $h_{u\mu}^{AdS} = 0$ are not quite all the solutions we want, they are a subset of them. So, finding these solutions was a necessary task and a good warm up. There are several insights from solving the metric perturbations in the AdS transverse gauge that will help us find the solutions which melvinize to the Schrödinger transverse gauge. In particular, the same parameterization will be used and we will facilitate solving the equations by using the decoupling into independent channels.

For the $h_{u\mu}^{AdS} = 0$ gauge, we could simply discard the gauge field because its degrees of freedom are independent of the metric, but this is no longer the case in the transverse Schrödinger gauge. We now need to use following five dimensional action which results from the consistent truncation

$$S = \frac{\pi^3 R^5}{2\kappa_{10}^2}\int dx^5 \sqrt{g}\left(\mathcal{R} - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu\right), \tag{6.63}$$

where $\Lambda = -6/R^2$ and $m^2 = 8/R^2$. The gauge field here is the same one-form that appears in (4.3). This is simply the Proca action coupled to the AdS black hole, which is the result of the dimensional reduction on the five-sphere [16]. Note that we still need to impose the Lorentz gauge constraint

$$\nabla_\mu A^\mu = 0, \tag{6.64}$$

in addition to the equations of motion resulting from the variation of the action[13]. We will use the parameterization for the metric modes as specified by (6.26). However,

[13]The Lorentz gauge condition needs to be imposed, because the mass term in the Lagrangian is not gauge invariant. To find a gauge invariant action we need to consider the more general Stückleberg action, which can be made manifestly gauge invariant. The Proca action is then one possible gauge choice and the Lorentz gauge condition is inherited as a constraint.

we still need to choose a parameterization for the gauge field, and a little dimensional analysis will help. The bare gauge field perturbations (except for $[A_u] = L$) are dimensionless as required for the action (6.63) to be a dimensionless scalar. We have seen though that raised indices have less singular behavior near the boundary, so we will want to multiply each gauge field component by $R^2/r_H^2 u$. Introducing extra factors of u will make the resulting equations more complicated even though the solutions are more well behaved, so we will settle for the following compromise

$$g_{\mu\nu}^{AdS} = \frac{R^2}{r_H^2}\begin{pmatrix} -\frac{f}{u}+\frac{f}{u}H_{tt} & \frac{1}{u}H_{xt} & \frac{1}{u}H_{yt} & \frac{1}{u}H_{zt} & \frac{\delta R f}{u}\delta\mathcal{A}_u \\ \frac{1}{u}H_{xt} & \frac{1}{u}+\frac{1}{u}H_{xx} & \frac{1}{u}H_{yx} & \frac{1}{u}H_{zx} & 0 \\ \frac{1}{u}H_{yt} & \frac{1}{u}H_{yx} & \frac{1}{u}+\frac{1}{u}H_{yy} & \frac{1}{u}H_{zy} & \frac{\delta R}{u}\delta\mathcal{A}_u \\ \frac{1}{u}H_{zt} & \frac{1}{u}H_{zx} & \frac{1}{u}H_{zy} & \frac{1}{u}+\frac{1}{u}H_{zz} & 0 \\ \frac{\delta R f}{u}\delta\mathcal{A}_u & 0 & \frac{\delta R}{u}\delta\mathcal{A}_u & 0 & \frac{r_H^2}{4u^2(1-u^2)} \end{pmatrix}, \tag{6.65}$$

$$A_\mu^{AdS} = \frac{R^2}{r_H^2}\left(\delta\mathcal{A}_t,\, \delta\mathcal{A}_x,\, \delta\mathcal{A}_y,\, \delta\mathcal{A}_z,\, r_H\delta\mathcal{A}_u\right), \tag{6.66}$$

where we will agree to add the $1/u$ term later when needed. It turns out we will never need to worry about raising the index on the $\delta\mathcal{A}_u$ term, because we will shortly eliminate it.

Variation of the action yields two sets of equations of motion, one for the metric and one for the gauge field. The metric's equation of motion is Einstein's equation

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + T^A_{\mu\nu} + 3T^F_{\mu\nu}, \tag{6.67}$$

where $T^H_{\mu\nu}$ is the stress-energy tensor of the of the corresponding p-forms given by

$$T^H_{\mu\nu} = -\frac{2}{p(p+1)}\left(\frac{1}{4}g_{\mu\nu}H^2 - \frac{p}{2}H_{\mu\sigma\ldots}H_\nu{}^{\sigma\ldots}\right). \tag{6.68}$$

Like before, it is more convenient to work with Ricci's equations, so the stress-energy tensors need to be trace reversed. Conveniently, (6.68) immediately tells us that the stress-energy tensors will not contribute to the metric equations at first order in the perturbations, because the background value of the gauge field is zero. Hence, we can simply drop the stress-energy tensors and write down Ricci's equation as before

$$R_{\mu\nu} = \frac{2\Lambda}{3}g_{\mu\nu}. \tag{6.69}$$

Even though the tensorial structure of the equation of motion is the same, the individual components are not all the same due to the presence of δA_u.

Both of the scalar channels are unaffected by the change of gauge, and the solutions should be familiar at this point. We also find that the H_{xt}-H_{zx} shear channel is similarly unaffected. However, we find the first difference in the H_{yt}-H_{zy} channel, which is expected since the h_{uy} mode is no longer trivial. The equations of motion are

$$(E_{uy}) \qquad \frac{\mathfrak{w}}{f}H'_{yt} + \mathfrak{q}H'_{zy} + \frac{2i\Delta}{f}(\mathfrak{w}^2 - \mathfrak{q}^2 f)\delta\mathcal{A}_u = 0, \tag{6.70}$$

$$(E_{yt}) \qquad H''_{yt} - \frac{1}{u}H'_{yt} + 2i\Delta\mathfrak{w}\delta\mathcal{A}'_u - \frac{\mathfrak{q}}{uf}(\mathfrak{q}H_{\alpha t} + \mathfrak{w}H_{zy}) - \frac{2i\Delta\mathfrak{w}}{u}\delta\mathcal{A}_u = 0, \tag{6.71}$$

$$(E_{zy}) \quad H''_{zy} - \frac{1+u^2}{uf}H'_{zy} - 2i\Delta\mathfrak{q}\delta\mathcal{A}'_u + \frac{\mathfrak{w}}{uf^2}(\mathfrak{q}H_{yt} + \mathfrak{w}H_{z\alpha}) + \frac{2i\Delta\mathfrak{q}(1+u^2)}{uf}\delta\mathcal{A}_u = 0, \tag{6.72}$$

where we have rescaled $\Delta = \delta R/r_H$. Likewise, the sound channel is also affected,

$$\begin{aligned}(E_{tt}) \qquad & H''_{tt} - \frac{3(1+u^2)}{2uf}H'_{tt} + \frac{1+u^2}{2uf}H'_{ii} - 4i\Delta\mathfrak{w}\delta\mathcal{A}'_u - \frac{2\mathfrak{q}\mathfrak{w}}{uf^2}H_{zt} - \dots \\ & \dots - \frac{\mathfrak{q}^2}{uf}H_{tt} - \frac{\mathfrak{w}^2}{uf^2}H_{ii} + \frac{6i\Delta\mathfrak{w}(1+u^2)}{uf}\delta\mathcal{A}_u = 0,\end{aligned} \tag{6.73}$$

$$(E_{xx} + E_{yy}) \quad H''_{aa} + \frac{1}{u}H'_{tt} - \frac{1}{u}H'_{ii} - \frac{1+u^2}{uf}H'_{aa} + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2}H_{aa} - \frac{4i\Delta\mathfrak{w}}{u}\delta\mathcal{A}_u = 0, \tag{6.74}$$

$$(E_{zt}) \qquad H''_{zt} - \frac{1}{u}H'_{zt} + 2i\Delta\mathfrak{q}f\delta\mathcal{A}'_u + \frac{\mathfrak{q}\mathfrak{w}}{uf}H_{aa} - \frac{2i\Delta\mathfrak{q}}{u}(1+3u^2)\delta\mathcal{A}_u = 0, \tag{6.75}$$

$$\begin{aligned}(E_{zz}) \qquad & H''_{ii} - H''_{aa} + \frac{1}{2u}H'_{tt} + \frac{3+u^2}{2uf}H'_{ii} + \frac{1+u^2}{uf}H'_{aa} + \frac{2\mathfrak{q}\mathfrak{w}}{uf^2}H_{zt} + \dots \\ & \dots + \frac{\mathfrak{q}^2}{uf}H_{tt} + \frac{\mathfrak{w}^2}{uf^2}H_{ii} - \frac{\mathfrak{w}^2 + \mathfrak{q}^2 f}{uf^2}H_{aa} + \frac{2i\Delta\mathfrak{w}}{u}\delta\mathcal{A}_u = 0,\end{aligned} \tag{6.76}$$

$$(E_{ut}) \qquad \mathfrak{q}H'_{zt} + \mathfrak{w}H'_{ii} + \frac{2\mathfrak{q}u}{f}H_{zt} + \frac{\mathfrak{w}u}{f}H_{ii} + 2i\Delta\mathfrak{q}^2 f\delta\mathcal{A}_u = 0, \tag{6.77}$$

$$(E_{uz}) \qquad \mathfrak{w}H'_{zt} + \mathfrak{q}fH'_{tt} - \mathfrak{q}fH'_{aa} - \mathfrak{q}uH_{tt} - 2i\Delta\mathfrak{q}\mathfrak{w}f\delta\mathcal{A}_u = 0, \tag{6.78}$$

$$(E_{uu}) \qquad H''_{ii} - H''_{tt} + \frac{3u}{f}H'_{tt} - \frac{u}{f}H'_{ii} + 4i\Delta\mathfrak{w}\delta\mathcal{A}'_u - \frac{12i\Delta\mathfrak{w}u}{f}\delta\mathcal{A}_u = 0. \tag{6.79}$$

Observe that for $\delta\mathcal{A}_u = 0$, all of these equations are equivalent to their counterparts in the transverse gauge as we expected. Hence, all of our previous solutions are indeed solutions in this gauge as was claimed earlier, provided $\delta\mathcal{A}_u = 0$ is consistent with the gauge field equations of motion. This is obviously true because there is no background gauge field. Before we can proceed, we explicitly need to determine the gauge field equations of motion as well because of the coupling to $\delta\mathcal{A}_u$.

Varying the action with respect to the gauge field yields the Proca equation of for a massive vector particle coupled to a curved background,

$$\nabla_\mu F^{\mu\nu} - m^2 A^\nu = 0. \tag{6.80}$$

There are five equations of motion and one gauge constraint due to (6.64),

$$\delta\mathcal{A}_t'' + 2i\mathfrak{w}\delta\mathcal{A}_u' - \frac{\mathfrak{q}\mathfrak{w}}{uf}\delta\mathcal{A}_u - \frac{2+\mathfrak{q}^2 u}{u^2 f}\delta\mathcal{A}_t = 0, \tag{6.81}$$

$$\delta\mathcal{A}_\alpha'' - \frac{2u}{f}\delta\mathcal{A}_\alpha + \frac{\mathfrak{w}^2 u - (2+\mathfrak{q}^2 u)f}{u^2 f^2}\delta\mathcal{A}_\alpha = 0, \tag{6.82}$$

$$\delta\mathcal{A}_z'' - \frac{2u}{f}\delta\mathcal{A}_z' - 2i\mathfrak{q}\delta\mathcal{A}_u' + \frac{\mathfrak{q}\mathfrak{w}}{uf^2}\delta\mathcal{A}_t + \frac{4i\mathfrak{q}u}{f}\delta\mathcal{A}_u + \frac{\mathfrak{w}^2 u - 2f}{u^2 f^2}\delta\mathcal{A}_z = 0, \tag{6.83}$$

$$\mathfrak{w}\delta\mathcal{A}_t' + \mathfrak{q}f\delta\mathcal{A}_z' + \frac{2i}{u}(u\mathfrak{w}^2 - 2f - \mathfrak{q}^2 uf)\delta\mathcal{A}_u = 0, \tag{6.84}$$

$$f\delta\mathcal{A}_u' - \frac{1+u^2}{u}\delta\mathcal{A}_u + \frac{i\mathfrak{q}}{2u}\delta\mathcal{A}_z + \frac{i\mathfrak{w}}{2uf}\delta\mathcal{A}_t = 0, \tag{6.85}$$

where $\alpha = x, y$. A little bit of work shows that Lorentz gauge constraint (6.85) is actually automatically satisfied by the other five equations. Hence, we won't have to worry about imposing the Lorentz gauge. Consistent with symmetry considerations, we see that the modes in the xy-plane satisfy their own scalar equations, while the other modes couple together. We need the solutions for $\delta\mathcal{A}_u$ to determine the unknown metric perturbations, which are needed to get the complete melvinized solutions. However, we see that there can only be four propagating degrees of freedom for this five component gauge field, because there is no dynamical equation for $\delta\mathcal{A}_u$. This is consistent with what we should expect for a massive gauge field [17]. So, (6.84) determines $\delta\mathcal{A}_u$ in terms of the other gauge field components.

Let's stop for a moment before tackling the computation of the coupled shear and sound modes and compute the solutions for the transverse components of the gauge field, which we will need later anyway. Equation (6.82) can be expressed as the first order system

$$\begin{pmatrix} \delta\mathcal{A}_\alpha \\ P\delta\mathcal{A}_\alpha \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{f} \\ \frac{2}{u^2} - \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf} & 0 \end{pmatrix} \cdot \begin{pmatrix} \delta\mathcal{A}_\alpha \\ P\delta\mathcal{A}_\alpha \end{pmatrix}. \tag{6.86}$$

This system as two exponents $\nu = \pm i\mathfrak{w}/2$, and choosing the incoming wave condition corresponds to the solution

$$\delta\mathcal{A}_\alpha = (1-u)^{i\mathfrak{w}/2}\left[\frac{1}{u} + \frac{i\mathfrak{w}}{u}\left(1 - u + \frac{1}{2}\ln\frac{1+u}{2}\right) + \frac{\mathfrak{q}^2}{2u}(1-u) + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}^2\mathfrak{w}, \mathfrak{w}^2)\right]. \tag{6.87}$$

This solution also reveals a linear divergence in the gauge field modes with lower indices like we suspected earlier. By raising the index or equivalently multiplying by u, this component is immediately rendered finite. Even though we are primarily interested in the metric modes, this solution will be needed because the melvinization mixes the gauge field with the metric. Now all that remains is to solve the metric modes coupled to the gauge field.

Since the the gauge field equations of motion are self-contained, we could solve for $\delta\mathcal{A}_u$ and then use its solutions to source the metric equations. However, this breaks the formalism that has been developed so far and presented more completely in appendix B by introducing inhomogeneous terms. Instead, it turns out to be easier to solve the gauge field perturbations simultaneously with the metric. The source $\delta\mathcal{A}_u$ couples the sound and shear channels together, so the solutions are no longer completely independent. However, we can still treat the channels separately because $\delta\mathcal{A}_u$ is completely independent of the metric perturbations. Equation (6.84) will be used to eliminate $\delta\mathcal{A}_u$. Then, we will effectively solve for the gauge modes $\delta\mathcal{A}_t$ and $\delta\mathcal{A}_z$ twice, and then match the solutions between the shear and sound channels by identifying the equivalent gauge solutions.

Starting with the shear channel, equations (6.70) and (6.72) along with the gauge field equations (6.81) and (6.83) yield the first order system

$$\begin{pmatrix} H_{yt} \\ H_{zy} \\ \delta\mathcal{A}_t \\ \delta\mathcal{A}_z \\ P_{zy} \\ P_1 \\ P_4 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\mathfrak{q}}{\mathfrak{w}} & \frac{2\Delta}{\mathcal{K}} + \frac{\Delta}{f} & \frac{\Delta\mathfrak{q}u\left(\mathfrak{w}^2 - \mathfrak{q}^2 f\right)}{\mathfrak{w}\mathcal{K}} \\ 0 & 0 & 0 & 0 & \frac{1}{f} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{f} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{f} \\ -\frac{\mathfrak{q}\mathfrak{w}}{uf} & -\frac{\mathfrak{w}^2}{uf} & \frac{\Delta\mathfrak{q}\mathfrak{w}}{uf} & \frac{\Delta\mathfrak{q}^2}{u} & \frac{1}{u} & 0 & 0 \\ 0 & 0 & -\frac{\mathcal{K}}{u^2 f} & 0 & 0 & \frac{\mathfrak{w}^2 + 2u\left(2+\mathfrak{q}^2 u\right)}{\mathcal{K}} & \frac{\mathfrak{q}\mathfrak{w}\left(1+u^2\right)}{\mathcal{K}} \\ 0 & 0 & 0 & -\frac{\mathcal{K}}{u^2 f} & 0 & -\frac{\mathfrak{q}\mathfrak{w}}{\mathcal{K}} & -\frac{\mathfrak{q}^2 f}{\mathcal{K}} \end{pmatrix} \cdot \begin{pmatrix} H_{yt} \\ H_{zy} \\ \delta\mathcal{A}_t \\ \delta\mathcal{A}_z \\ P_{zy} \\ P_1 \\ P_4 \end{pmatrix} \tag{6.88}$$

where $\mathcal{K} = \mathfrak{w}^2 u - \left(2 + \mathfrak{q}^2 u\right) f$. Applying the procedure that should be familiar by now, the exponents of the system are $\nu = 0, \pm i\mathfrak{w}/2$, where there is a multiplicity of three for the complex values to give seven total. The four solutions we need are the three incoming waves and the gauge solution $\nu = 0$. The gauge solution and one incoming wave solution are unchanged in the new gauge as claimed. The only new solutions are the two new incoming waves, which couple to the metric modes. The new solutions are

$$H_{yt}^{inc2} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.89}$$

$$H_{zy}^{inc2} = (1-u)^{-i\mathfrak{w}/2} \left[-\frac{\Delta\mathfrak{q}^2}{2} \ln u + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}^2\mathfrak{w}, \mathfrak{w}^2) \right], \tag{6.90}$$

$$\delta\mathcal{A}_t^{inc2} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.91}$$

$$\delta\mathcal{A}_z^{inc2} = (1-u)^{-i\mathfrak{w}/2} \left[\frac{1}{u} + \frac{i\mathfrak{w}}{u} \left(f + \frac{1}{2} \ln \frac{1+u}{2} \right) + \mathcal{O}(\mathfrak{q}^2, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2) \right], \tag{6.92}$$

$$H_{yt}^{inc3} = (1-u)^{-i\mathfrak{w}/2} \left[\Delta + \frac{\Delta\mathfrak{q}^2}{4} \left(1 + u^2 + f \ln \frac{u-1}{u+1} \right) + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}) \right] \tag{6.93}$$

$$H_{zy}^{inc3} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.94}$$

$$\delta\mathcal{A}_t^{inc3} = (1-u)^{-i\mathfrak{w}/2} \left[1 + \frac{f}{2u} \ln \frac{u-1}{u+1} + \mathcal{O}(\mathfrak{q}^2, \mathfrak{q}\mathfrak{w}, \mathfrak{w}) \right], \tag{6.95}$$

$$\delta\mathcal{A}_z^{inc3} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.96}$$

where we have been particularly liberal about omitting higher order terms, which quickly become rather complicated expressions.

Proceeding with the sound channel in the exact same way, equations (6.73), (6.75), (6.77), (6.78),(6.81) and (6.83) were selected to construct the first order system. The explicit matrix is large and generally uninformative, so it will not be written down explicitly. The exponents of the system are exactly the same as in transverse gauge case except for the addition of two pairs of $\nu = \pm i\mathfrak{w}/2$. These solutions contain the two incoming wave modes that we need to complete our solutions. Just as we expected based on previous arguments, all of the transverse gauge solutions carry over. Hence, the only new information is contained in the solutions

$$H_{tt}^{inc2} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.97}$$

$$H_{aa}^{inc2} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.98}$$

$$H_{ii}^{inc2} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.99}$$

$$H_{zt}^{inc2} = (1-u)^{-i\mathfrak{w}/2} \left[\frac{\Delta\mathfrak{q}^2 f}{2} \ln u + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}^2\mathfrak{w}, \mathfrak{w}^2) \right], \tag{6.100}$$

$$\delta\mathcal{A}_t^{inc2} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.101}$$

$$\delta\mathcal{A}_z^{inc2} = (1-u)^{-i\mathfrak{w}/2} \left[\frac{1}{u} + \frac{i\mathfrak{w}}{u} \left(f + \frac{1}{2} \ln \frac{1+u}{2} \right) + \mathcal{O}(\mathfrak{q}^2, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2) \right], \tag{6.102}$$

$$H_{tt}^{inc3} = (1-u)^{-i\mathfrak{w}/2}\left[2\Delta\mathfrak{w} + \mathcal{O}(\mathfrak{q}^2, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2)\right], \tag{6.103}$$

$$H_{aa}^{inc3} = (1-u)^{-i\mathfrak{w}/2}\left[2i\Delta + \Delta\,\mathfrak{w}\left(1 + \ln\frac{1+u}{2}\right) + \mathcal{O}(\mathfrak{q}^2, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2)\right], \tag{6.104}$$

$$H_{ii}^{inc3} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^2, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.105}$$

$$H_{zt}^{inc3} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{6.106}$$

$$\delta\mathcal{A}_t^{inc3} = (1-u)^{-i\mathfrak{w}/2}\left[\mathfrak{w}\left(1 + \frac{f}{2u}\ln\frac{u-1}{u+1}\right) + \mathcal{O}(\mathfrak{q}^2, \mathfrak{q}\mathfrak{w}, \mathfrak{w})\right], \tag{6.107}$$

$$\delta\mathcal{A}_z^{inc3} = (1-u)^{-i\mathfrak{w}/2} \times \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}^2\mathfrak{w}, \mathfrak{w}^2). \tag{6.108}$$

We can now match the modes across the shear and sound channels. In particular, notice that (6.102) is identical to (6.92). Hence, all of the solutions labeled by H^{inc2} can be grouped together as one metric mode sourced by the gauge field. Similarly, note that (6.107) is a constant multiple of (6.95), where the constant is the dimensionless frequency. So, all of the H^{inc} solutions are coupled together, but the shear channel solutions need to be multiplied by $\mathfrak{w}$ in order to make the integration constant the same. Now that we have finished solving for the metric perturbations in the transverse Schrödinger gauge, we can return to computing the Schrödinger Green's functions.

6.5 Helicity Eigenstates and a Minor Complication

We know how to express the Schrödinger perturbations in terms of the AdS solutions, and we have the AdS solutions in the gauge $h_{u\mu}^{Sch} = 0$. So, we should be able to simply write out the solutions for the Schrödinger metric fluctuations and start computing the retarded Green's functions. Unfortunately, we're not quite there yet, because of one oversight. The relativistic boundary theory in AdS is a four dimensional theory, while the non-relativistic boundary theory in the Schrödinger spacetime is only three dimensional. This is reflected in the fact that AdS_5 has three Euclidean dimensions, while Sch_5 has only two. However, when we solved the solutions in AdS_5 we exploited the rotational invariance of the Euclidean dimensions to fix the momentum along the z-axis. We need to undo this in order to get the correct momentum structure after melvinization, because the y-coordinate needs to be treated differently than x and z.

We need to be a little more careful about how we treat the q_y momentum. The spatial momentum is $\vec{k} = (q_y, 0, q_z)$[14], where we have still used rotational symmetry to

[14]The coordinates are ordered (t, y, x, z, u) in order to match with the coordinates in Sch_5, with t set to τ after the melvinization. This is because the time coordinate in the Schrödinger spacetime is a null coordinate in the AdS parameterization we do the melvinization with.

set $q_x = 0$. Then, we can define an orthonormal momentum space basis

$$\hat{f}^t = (1, \vec{0}, 0), \quad \hat{f}^1 = (0, \hat{e}_1, 0), \quad \hat{f}^2 = (0, \hat{e}_2, 0), \quad \hat{f}^3 = (0, \hat{q}, 0), \quad \hat{f}^u = (0, \vec{0}, 1). \tag{6.109}$$

It will be convenient to choose $\hat{e}_2 = (0, 1, 0)$ and $\hat{e}_1 = \hat{e}_2 \times \hat{q} = \frac{1}{q}(-q_z, 0, q_y)$, where $q = |\vec{q}|$. Analogous to what was done earlier, we can decompose $h_{\mu\nu}^{AdS}$ into sums of helicity eigenstates according the to the $O(2)$ residual symmetry about the direction of the momentum. There are three helicity states $h = 0, 1, 2$ corresponding to scalar, vector, and tensor modes. These modes can be further subdivided into the tensor representations constructed from our vector momentum basis by taking the tensor product of any two basis vectors. For example, $h_{0,(tt)}^{AdS}$ is the zero helicity mode corresponding to the basis tensor constructed from $\hat{f}^t \otimes \hat{f}^t$. Thus,

$$h_{\mu\nu}^{AdS}(\omega, \vec{q}) = \sum_{h=0}^{2} \sum_{(ij)\in\mathcal{S}} h_{h,(ij)}^{AdS}(\omega, q) P_{\mu\nu}^{h,(ij)}(\vec{q}), \tag{6.110}$$

gives the decomposition of each metric mode into helicity eigenstates about momentum in the $\vec{k}$-direction, where $P_{\mu\nu}^{h,(ij)}(\vec{q})$ are the basis/projection tensors. The projection tensor indices run over the set $\mathcal{S} = \{(tt), (1t), (aa), (2t), (21), (bb), (3t), (31)(32), (33)\}$. The full set of projectors is

$$P_{\mu\nu}^{0,(tt)} = \hat{f}_\mu^t \hat{f}_\nu^t, \tag{6.111}$$

$$P_{\mu\nu}^{0,(3t)} = \hat{f}_\mu^t \hat{f}_\nu^3 + \hat{f}_\mu^3 \hat{f}_\nu^t, \tag{6.112}$$

$$P_{\mu\nu}^{0,(aa)} = \hat{f}_\mu^1 \hat{f}_\nu^1 + \hat{f}_\mu^2 \hat{f}_\nu^2, \tag{6.113}$$

$$P_{\mu\nu}^{0,(33)} = \hat{f}_\mu^3 \hat{f}_\nu^3, \tag{6.114}$$

$$P_{\mu\nu}^{1,(1t)} = \hat{f}_\mu^t \hat{f}_\nu^1 + \hat{f}_\mu^1 \hat{f}_\nu^t, \tag{6.115}$$

$$P_{\mu\nu}^{1,(2t)} = \hat{f}_\mu^t \hat{f}_\nu^2 + \hat{f}_\mu^2 \hat{f}_\nu^t, \tag{6.116}$$

$$P_{\mu\nu}^{1,(31)} = \hat{f}_\mu^1 \hat{f}_\nu^3 + \hat{f}_\mu^3 \hat{f}_\nu^1, \tag{6.117}$$

$$P_{\mu\nu}^{1,(32)} = \hat{f}_\mu^2 \hat{f}_\nu^3 + \hat{f}_\mu^3 \hat{f}_\nu^2, \tag{6.118}$$

$$P_{\mu\nu}^{2,(21)} = \hat{f}_\mu^1 \hat{f}_\nu^2 + \hat{f}_\mu^2 \hat{f}_\nu^1, \tag{6.119}$$

$$P_{\mu\nu}^{2,(bb)} = \hat{f}_\mu^1 \hat{f}_\nu^1 - \hat{f}_\mu^2 \hat{f}_\nu^2. \tag{6.120}$$

The solutions that we have found in the previous section are the $h_{h,(ij)}^{AdS}$. However, (6.110) along with the explicit expressions for all of the projection tensors can be used to express $h_{\mu\nu}^{AdS}$ as linear combinations of $h_{h,(ij)}^{AdS}$. The melvinization relations given in section 6.4 can then be used to express the Schrödinger perturbations in terms of the helicity eigenstate perturbations. Firstly, we find that

$$h_{tt}^{AdS} = h_{0,(tt)}^{AdS}, \quad h_{xt}^{AdS} = h_{1,(2t)}^{AdS}, \quad h_{xx}^{AdS} = h_{0,(aa)}^{AdS} - h_{2,(bb)}^{AdS}, \tag{6.121}$$

$$h_{yt}^{AdS} = \frac{1}{\sqrt{\mathfrak{q}_y^2 + \mathfrak{q}_z^2}}(\mathfrak{q}_y h_{0,(3t)}^{AdS} + \mathfrak{q}_z h_{1,(1t)}^{AdS}), \tag{6.122}$$

$$h_{yx}^{AdS} = \frac{1}{\sqrt{\mathfrak{q}_y^2 + \mathfrak{q}_z^2}}(\mathfrak{q}_y h_{1,(32)}^{AdS} + \mathfrak{q}_z h_{2,(21)}^{AdS}), \tag{6.123}$$

$$h_{yy}^{AdS} = \frac{1}{\mathfrak{q}_y^2 + \mathfrak{q}_z^2}(\mathfrak{q}_y^2 h_{0,(33)}^{AdS} + 2\mathfrak{q}_y\mathfrak{q}_z h_{1,(31)}^{AdS} + \mathfrak{q}_z^2 h_{0,(aa)}^{AdS} + \mathfrak{q}_z^2 h_{2,(bb)}^{AdS}), \tag{6.124}$$

$$h_{zt}^{AdS} = \frac{1}{\sqrt{\mathfrak{q}_y^2 + \mathfrak{q}_z^2}}(\mathfrak{q}_z h_{0,(3t)}^{AdS} - \mathfrak{q}_y h_{1,(1t)}^{AdS}), \tag{6.125}$$

$$h_{zx}^{AdS} = \frac{1}{\sqrt{\mathfrak{q}_y^2 + \mathfrak{q}_z^2}}(\mathfrak{q}_z h_{1,(32)}^{AdS} - \mathfrak{q}_y h_{2,(21)}^{AdS}), \tag{6.126}$$

$$h_{zy}^{AdS} = \frac{1}{\mathfrak{q}_y^2 + \mathfrak{q}_z^2}\left[\mathfrak{q}_y\mathfrak{q}_z(h_{0,(33)}^{AdS} - h_{0,(aa)}^{AdS} - h_{2,(bb)}^{AdS}) + (\mathfrak{q}_z^2 - \mathfrak{q}_y^2)h_{1,(31)}^{AdS}\right], \tag{6.127}$$

$$h_{zz}^{AdS} = \frac{1}{\mathfrak{q}_y^2 + \mathfrak{q}_z^2}(\mathfrak{q}_y^2 h_{0,(aa)}^{AdS} + \mathfrak{q}_y^2 h_{2,(bb)}^{AdS} - 2\mathfrak{q}_y\mathfrak{q}_z h_{1,(31)}^{AdS} + \mathfrak{q}_z^2 h_{0,(33)}^{AdS}). \tag{6.128}$$

Observe that for $\mathfrak{q}_y = 0$, all of the expressions for $h_{\mu\nu}^{AdS}$ reduce exactly to the decomposition that was used previously in solving for the AdS perturbations. This can also be done for the gauge field, which has the much simpler representation

$$A_\mu^{AdS}(\omega, \vec{q}) = A_{h,(i)}^{AdS}(\omega, q)\hat{f}_\mu^i(\vec{q}). \tag{6.129}$$

Hence,

$$A_t^{AdS} = A_{0,(t)}^{AdS}, \quad A_x^{AdS} = A_{0,(2)}^{AdS}, \quad A_u^{AdS} = A_{0,(u)}^{AdS}, \tag{6.130}$$

$$A_y^{AdS} = \frac{1}{\sqrt{\mathfrak{q}_y^2 + \mathfrak{q}_z^2}}\left(\mathfrak{q}_z A_{1,(1)}^{AdS} + \mathfrak{q}_y A_{1,(3)}^{AdS}\right), \tag{6.131}$$

$$A_z^{AdS} = \frac{1}{\sqrt{\mathfrak{q}_y^2 + \mathfrak{q}_z^2}}\left(\mathfrak{q}_z A_{1,(3)}^{AdS} - \mathfrak{q}_y A_{1,(1)}^{AdS}\right). \tag{6.132}$$

It is now straightforward to find the expressions for the Schrödinger perturbations given in terms of the $h_{h,(ij)}^{AdS}$.

6.6 Shear Physics

At this point, we have finally developed all of the formalism we will need to begin computing retarded Green's functions in order to extract the transport coefficients.

The simplest transport coefficient to calculate is the shear viscosity, but we will use our formalism to extract a bit more. Although the following procedure is akin to hunting a gerbil with an elephant gun, this process will illustrate the procedure necessary for computing the more elusive sound channel quantities. For a much simpler derivation of the shear viscosity the interested reader can consult section five of [12], in which the $O(2)$ symmetry in the $\vec{q}=0$ limit is immediately capitalized upon. Our approach will serve as a good check on that result.

It was identified earlier (6.12) that there are four modes that compose the shear channel, of which only three have a known physical interest. There are the three metric perturbations A_1, B_1, H_{zx} which couple to $g_x, j_x^\epsilon, \tau_{zx}$ in the boundary theory respectively. The fourth perturbation is the gauge field component δA_x^{Sch}. But, since it doesn't couple to any of the hydrodynamic operators in the boundary theory, we will almost immediately try to set it to zero. Recall from the hydrodynamic analysis of a non-relativistic fluid, we were able to relate the shear viscosity to the retarded Green's function $G^R_{\tau_{xz},\tau_{xz}}$ via (5.24). So, we will begin by computing this retarded Green's function.

For convenience, we will define dimensionless quantities that simplify the resulting expressions, which is done by effectively raising an index on the various fields modulo some scaling. In particular,

$$\delta\mathcal{A}_x^{Sch} = \frac{r_H u}{R}\delta A_3^{Sch}, \qquad H^{AdS}_{h,(ij)} = \frac{r_H^2 u}{R^2} h^{AdS}_{h,(ij)}. \tag{6.133}$$

Technically, for the $h^{AdS}_{h,(tt)}$ mode we need to include a factor of f, but it appears nowhere in the shear channel. So, it doesn't matter. Consulting the results of the melvinization listed in appendix C.2, we can relate them to the parameterization given by (6.5). Finally, after substituting in the results of the previous section, we find

$$A_1^{Sch} = \frac{1}{\sqrt{2K}}\left[\mathfrak{q}H^{AdS}_{1,(2t)} + (1+2\delta^2 u)\left(\mathfrak{q}_y H^{AdS}_{1,(32)} + \mathfrak{q}_z H^{AdS}_{2,(21)}\right) + \delta\mathfrak{q}(u^2-2K)\delta\mathcal{A}^{AdS}_{0,(2)}\right] \tag{6.134}$$

$$B_1^{Sch} = \frac{1}{\sqrt{2K}}\left(\mathfrak{q}H^{AdS}_{1,(2t)} - \mathfrak{q}_y H^{AdS}_{1,(32)} - \mathfrak{q}_z H^{AdS}_{2,(21)} + \delta\mathfrak{q}u^2\delta\mathcal{A}^{AdS}_{0,(2)}\right) \tag{6.135}$$

$$H_{zx}^{Sch} = \mathfrak{q}_z H^{AdS}_{1,(32)} - \mathfrak{q}_y H^{AdS}_{2,(21)}, \tag{6.136}$$

$$\delta\mathcal{A}_x^{Sch} = \frac{1}{uK}\left[-\delta\mathfrak{q}H^{AdS}_{1,(2t)} + \delta\left(\mathfrak{q}_y H^{AdS}_{1,(32)} + \mathfrak{q}_z H^{AdS}_{2,(21)}\right) + u\mathfrak{q}\delta\mathcal{A}^{AdS}_{0,(2)}\right], \tag{6.137}$$

where linearity of the solutions was used to remove an overall factor of $1/\mathfrak{q}$. Here we have related four Schrödinger perturbations to four AdS perturbations[15], and so the

[15]Note at this point it is important to distinguish the time τ in the AdS black hole from the time

number of degrees of freedom agree. The four solutions will be classified according to the corresponding $H^{AdS}_{h,(ij)}$ mode that generates them. As a result, we can already infer that there will be three propagating wave modes and one gauge solution.

Explicitly, the four sets of solutions are then

$$A_1^{2,(21)} = \frac{\mathfrak{q}_z(1+2\delta^2 u)(1-u)^{-i\mathfrak{e}/2}}{\sqrt{2}K}\left(1-\frac{i\mathfrak{e}}{2}\ln\frac{1+u}{2}-\mathfrak{q}^2\ln\frac{1+u}{2}\right), \tag{6.138}$$

$$B_1^{2,(21)} = -\frac{\mathfrak{q}_z(1-u)^{-i\mathfrak{e}/2}}{\sqrt{2}K}\left(1-\frac{i\mathfrak{e}}{2}\ln\frac{1+u}{2}-\mathfrak{q}^2\ln\frac{1+u}{2}\right), \tag{6.139}$$

$$H_{zx}^{2,(21)} = -\mathfrak{q}_y(1-u)^{-i\mathfrak{e}/2}\left(1-\frac{i\mathfrak{e}}{2}\ln\frac{1+u}{2}-\mathfrak{q}^2\ln\frac{1+u}{2}\right), \tag{6.140}$$

$$\delta\mathcal{A}_x^{2,(21)} = \frac{\delta\mathfrak{q}_z(1-u)^{-i\mathfrak{e}/2}}{K}\left(1-\frac{i\mathfrak{e}}{2}\ln\frac{1+u}{2}-\mathfrak{q}^2\ln\frac{1+u}{2}\right), \tag{6.141}$$

$$A_1^I = \frac{1}{\sqrt{2}K}\left[\mathfrak{e}-(1+2u\delta^2)\mathfrak{q}_y\right], \tag{6.142}$$

$$B_1^I = \frac{1}{\sqrt{2}K}\left(\mathfrak{e}+\mathfrak{q}_y\right), \tag{6.143}$$

$$H_{zx}^I = -\mathfrak{q}_z, \tag{6.144}$$

$$\delta\mathcal{A}_x^I = \frac{\delta}{K}\left(\mathfrak{e}+\mathfrak{q}_y\right), \tag{6.145}$$

$$A_1^{inc} = \frac{1}{\sqrt{2}K}\left[\mathfrak{q}_y(1+2\delta^2 u)+\frac{i\mathfrak{q}^2 f}{2}-\frac{i\mathfrak{q}_y\mathfrak{e}}{2}(1+2\delta^2 u)\ln\frac{1+u}{2}\right], \tag{6.146}$$

$$B_1^{inc} = -\frac{1}{\sqrt{2}K}\left[\mathfrak{q}_y-\frac{i\mathfrak{q}^2 f}{2}-\frac{i\mathfrak{q}_y\mathfrak{e}}{2}\ln\frac{1+u}{2}\right], \tag{6.147}$$

$$H_{zx}^{inc} = \mathfrak{q}_z(1-u)^{-i\mathfrak{e}}\left(1-\frac{\mathfrak{e}}{2}\ln\frac{1+u}{2}\right), \tag{6.148}$$

$$\delta\mathcal{A}_x^{inc} = \frac{\delta(1-u)^{-i\mathfrak{e}/2}}{K}\left(\mathfrak{q}_y-\frac{i\mathfrak{q}^2 f}{2}-\frac{i\mathfrak{q}_y\mathfrak{e}}{2}\ln\frac{1+u}{2}\right), \tag{6.149}$$

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t in the Schrödinger black hole, which is a light cone coordinate. Even though we used $\mathfrak{w}$ for the momentum conjugate to τ in previous sections, we will now reserve $\mathfrak{w}$ exclusively for the Schrödinger time coordinate t. Instead, ϵ and $\mathfrak{e}$ will be the dimensionful and dimensionless momenta conjugate to τ respectively.

$$A_1^{\delta A_x} = \frac{\Delta\mathfrak{q}(u^2-2K)(1-u)^{-i\mathfrak{e}/2}}{\sqrt{2}K}\left[\frac{1}{u}+\frac{i\mathfrak{e}}{u}\left(1-u+\frac{1}{2}\ln\frac{1+u}{2}\right)+\frac{\mathfrak{q}^2}{2}(1-u)\right] \tag{6.150}$$

$$B_1^{\delta A_x} = \frac{\Delta\mathfrak{q}u(1-u)^{-i\mathfrak{e}/2}}{\sqrt{2}K}\left[1+i\mathfrak{e}\left(1-u+\frac{1}{2}\ln\frac{1+u}{2}\right)+\frac{\mathfrak{q}^2u}{2}(1-u)\right] \tag{6.151}$$

$$H_{zx}^{\delta A_x} = 0, \tag{6.152}$$

$$\delta\mathcal{A}_x^{\delta A_x} = \frac{R(1-u)^{-i\mathfrak{e}}}{r_H K}\left[1+i\mathfrak{e}\left(1-u+\frac{1}{2}\ln\frac{1+u}{2}\right)+\frac{\mathfrak{q}^2}{2}(1-u)\right], \tag{6.153}$$

up to an arbitrary overall normalization due to linearity of the solutions. We have also suppressed higher order terms.

A couple of things should be noted at this point. As claimed, we explicitly see the decoupling of the H_{zx} mode when the spatial momentum $\mathfrak{q}_z$ is set to zero, and we could capitalize upon this simplification to easily compute the shear viscosity. Secondly, the coordinates we have been using are not the light cone coordinates (t,ξ), which have the physical meaning in the Schrödinger spacetime. So, the $\mathfrak{q}_y$ momenta is not a spatial momentum and $\mathfrak{e}$ is not a physical frequency. Recall the transformation that related the AdS coordinates to the light cone coordinates (4.13). The corresponding relations for the dimensionless momenta are

$$\mathfrak{q}_y = \frac{\mathfrak{w}+\mathfrak{l}}{\sqrt{2}}, \qquad \mathfrak{e} = \frac{\mathfrak{w}-\mathfrak{l}}{\sqrt{2}}. \tag{6.154}$$

It doesn't matter at what stage we make this substitution as long as we keep track of the powers of the momenta when discarding higher order terms. All that remains is to impose the correct boundary conditions on our melvinized solutions.

The general solutions meeting the incoming wave boundary conditions[16] are linear combinations of the four linearly independent vector solutions just listed. For example, we have

$$A_1 = C_{2,(12)}A_1^{2,(21)} + C_I A_1^I + C_{inc}A_1^{inc} + C_{\delta A_x}A_1^{\delta A_x}. \tag{6.155}$$

[16]It is not quite so obvious that the AdS solutions that satisfy the incoming wave boundary condition melvinize directly to the Schrödinger solutions that also satisfy the incoming wave condition. In our choice of coordinates this confusion doesn't arise because $\mathfrak{w} \propto \mathfrak{e}$; however, we could have made a difference choice of light cone coordinates that would change the sign between the two. The resolution is that the AdS coordinates are the correct local description of the black hole, hence determine the incoming wave condition. The Schrödinger coordinates are related to asymptotic difference between AdS and the Schrödinger spacetime, and even though they give a different local descriptions at the horizon they should not be used.

The prescription for computing real-time correlators requires that these solutions asymptote to the boundary sources. Explicitly,

$$\lim_{u\to\epsilon} A_1(u) = A_1^\epsilon, \qquad \lim_{u\to\epsilon} B_1(u) = B_1^\epsilon, \qquad \lim_{u\to\epsilon} H_{zx}(u) = H_{zx}^\epsilon. \tag{6.156}$$

However, we see that (6.150) contributes a linear divergence to A_1 near the boundary. Since a finite solution is needed, this requires that we set $C_{\delta A_x} = 0$. As we will soon see, this divergence is slightly problematic, because we are left with three degrees of freedom but four boundary condition left to fix including setting the gauge field component to zero. However, there is a small surprise. The boundary values of the four fields are not linearly independent. In particular,

$$\lim_{u\to\epsilon} \delta A_x(u) = -\sqrt{2}\delta \lim_{u\to\epsilon} B_1(u). \tag{6.157}$$

In order to set the gauge field to zero at the boundary, we will also have to set $B_1^\epsilon = 0$ as well. This is not terrible, but it prevents us from computing the retarded Green's functions related to the energy density.

Using the metric perturbation on-shell action (6.6) and the five dimensional Einstein frame metric, the flux factor reduces to

$$\mathcal{F} = \frac{N^2}{8\pi^2 r_H^4}\left[\frac{A_1^{\epsilon^2}}{2}\left(\frac{\mathfrak{q}^2 + i\sqrt{2}\mathfrak{e}}{\mathfrak{q}^2 - i\sqrt{2}\mathfrak{e}}\right) + \frac{A_1^\epsilon H_{zx}^\epsilon \mathfrak{q}_z \mathfrak{e}}{\mathfrak{q}^2 - i\sqrt{2}\mathfrak{e}} + H_{zx}^{\epsilon^2}\left(\frac{5\delta^4}{9} + \frac{\mathfrak{e}^2}{\mathfrak{q}^2 - i\sqrt{2}\mathfrak{w}}\right)\right]. \tag{6.158}$$

Dropping the lone δ-dependent contact term, because we want finite distance correlators, the retarded Green's functions to lowest order are then

$$G^R_{g_x,g_x}(\omega,q) = -\frac{\pi^2 L_\xi N^2 T_H^4}{32}\left(\frac{q_z^2 + 2\pi i T_H \omega}{q_z^2 - 2\pi i T_H\omega}\right), \tag{6.159}$$

$$G^R_{g_x,\tau_{zx}}(\omega,q) = -\frac{\pi^2 L_\xi N^2 T_H^4 q_z \omega}{16(q_z^2 - 2\pi i T_H \omega)}, \tag{6.160}$$

$$G^R_{\tau_{zx},\tau_{zx}}(\omega,q) = -\frac{\pi^2 L_\xi N^2 T_H^4 \omega^2}{16(q_z^2 - 2\pi i T_H \omega)}. \tag{6.161}$$

where we have returned to dimensionful momenta and the light cone coordinates. We have also suppressed the volume factor which becomes $(2\pi)^3\delta^3(0)$ after the Fourier transform, because it is convention dependent and neatly cancels when we extract the shear viscosity. Since, we need to compute the Green's function at zero particle number, we shouldn't divide out by the volume factor associated with ξ. Now, using the Kubo's formula derived earlier, the shear viscosity is

$$\eta = -\lim_{\omega\to 0}\lim_{q\to 0}\frac{1}{\omega}\mathrm{Im}\, G^R_{\tau_{zx},\tau_{zx}} = \frac{\pi L_\xi N^2 T_H^3}{32}. \tag{6.162}$$

The shear diffusion pole is also present in the correlators. This gives a diffusion constant,

$$D = \frac{1}{2\pi T_H}. \tag{6.163}$$

Lastly, all three of the retarded Green's functions perfectly satisfy the relations (5.22) and (5.23) derived solely from hydrodynamical considerations.

These results provide a non-trivial confirmation of the calculation of the shear viscosity in [12], and produce a diffusion constant consistent with their calculations for the other hydrodynamic quantities. From our results, the equilibrium density is

$$\rho = \frac{\eta}{D} = \frac{\pi^2 L_\xi N^2 T_H^4}{16}, \tag{6.164}$$

which is consistent with their result from calculating one-point functions up to a choice of units. Also, it is unavoidable to mention that the entropy density for a black hole can be calculated from the Bekenstein-Hawking formula, which gives

$$s = \frac{\pi^2 N^2 L_\xi T_H^3}{8}. \tag{6.165}$$

This reproduces the well known result $\eta/s = 1/4\pi$.

7. Conclusion

Our Schrödinger black hole has passed several non-trivial tests demonstrating that it is indeed dual to a non-relativistic field theory defined on the boundary, and this is but one illustration of many such examples that can be studied by using such a gravity/gauge theory duality. In particular, all of the transverse/shear operators computed beautifully satisfy the constraints placed on them by a purely hydrodynamic analysis of a non-relativistic fluid. Other thermodynamic constraints placed upon the relationships between the density, the entropy, and the shear viscosity held. However, several more transport coefficients still need to be computed.

In this work, we have provided the essential tools for tackling the problem of computing correlators in the sound channel. Through melvinization, all of the metric perturbations can be evaluated; however, there are still several residual difficulties that need to be addressed. As was seen in the shear channel, there was a linear divergence in the melvinized gauge field that caused A_0 to diverge. In removing this divergence, the remaining fields were no longer linearly independent near the boundary limiting the number of correlators we could compute. This problem becomes pathological in the sound channel as the number of divergences dramatically increases. A necessary future step and possible resolution to these difficulties is to melvinize two of the spherical modes of the AdS metric, which appear in the consistent truncation. These components have non-zero values in the Schrödinger spacetime and mix with the sound channel in the five dimensional Einstein frame. However, we leave further investigation of the sound channel to future work.

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A. A Detailed Derivation of the Scalar Correlator

The action for the massive scalar field was given by (2.9), which yielded the momentum space solution

$$f_q(r) = \frac{r^2 K_\nu(qr)}{\epsilon^2 K_\nu(q\epsilon)} \tilde{\phi}_0(q), \tag{A.1}$$

once the appropriate boundary conditions were imposed. The modified Bessel function of the second kind K_ν has two series expansions depending on whether or not ν is an integer, which is a consequence of a blocked exponent in the defining differential equation. For non-integer ν, the series expansion is

$$K_\nu(z) = z^{-\nu} \sum_{i=0}^{\infty} a_{2i} z^{2i} + z^{\nu} \sum_{i=0}^{\infty} b_{2i} z^{2i}, \tag{A.2}$$

where the coefficients are given by

$$a_{2i} = \frac{\Gamma(1-\nu)\Gamma(\nu)}{2^{-\nu+2i+1} i! \Gamma(1-\nu+i)}, \tag{A.3}$$

$$b_{2i} = -\frac{\Gamma(1-\nu)\Gamma(\nu)}{2^{\nu+2i+1} i! \Gamma(1+\nu+i)}. \tag{A.4}$$

In order to evaluate the flux factor, the following results will be useful. For $\epsilon \ll 1$,

$$\frac{1}{K_\nu(q\epsilon)} \approx \frac{1}{(q\epsilon)^{-\nu}\left[a_0 + (q\epsilon)^{2\nu} b_0\right]} \approx \frac{(q\epsilon)^\nu}{a_0}\left[1 - (q\epsilon)^{2\nu}\frac{b_0}{a_0}\right]. \tag{A.5}$$

At first sight, this expansion seems unnecessary, because the second term should go to zero much faster than the first. However, if we did that, we would have taken the $\epsilon \to 0$ limit prematurely. The subleading contribution here will combine later with another term in the numerator to contribute to the flux factor. Also,

$$r\partial_r[(qr)^2 K_\nu(qr)]\big|_{r=\epsilon} = (q\epsilon)^{-\nu+2} \sum_{i=0}^{\infty} (2-\nu+2i)\, a_{2i}(q\epsilon)^{2i} + \dots$$

$$\dots + (q\epsilon)^{\nu+2} \sum_{i=0}^{\infty} (2+\nu+2i)\, b_{2i}(q\epsilon)^{2i}. \tag{A.6}$$

Thus,

$$\begin{aligned}
r^{-3}\partial_r \left(\frac{r^2 K_\nu(qr)}{\epsilon^2 K_\nu(q\epsilon)}\right)\Big|_{r=\epsilon} &= \epsilon^{-4}\frac{(q\epsilon)^{\nu-2}}{a_0}\left[1-(q\epsilon)^{2\nu}\frac{b_0}{a_0}\right] \times \\
\times &\left[(q\epsilon)^{-\nu+2}\sum_{i=0}^{\infty}(2-\nu+2i)\, a_{2i}(q\epsilon)^{2i} + (q\epsilon)^{\nu+2}\sum_{i=0}^{\infty}(2+\nu+2i)\, b_{2i}(q\epsilon)^{2i}\right] \\
&= \frac{\epsilon^{-4}}{a_0}\left[\sum_{i=0}^{\infty}(2-\nu+2i)\, a_{2i}(q\epsilon)^{2i} - (2-\nu)b_0(q\epsilon)^{2\nu} + (2+\nu)b_0(q\epsilon)^{2\nu}\right] \\
&= \epsilon^{-4}\left[\sum_{i=0}^{[\nu]}(2-\nu+2i)\,\frac{a_{2i}}{a_0}(q\epsilon)^{2i} + 2\nu\frac{b_0}{a_0}(q\epsilon)^{2\nu}\right].
\end{aligned} \tag{A.7}$$

Only one of these terms is finite in the $\epsilon \to 0$ limit. So, the rest need to be removed by the program of holographic renormalization [9]. This is not a problem, because all of these terms depend on square powers of the momenta. After the inverse Fourier transform we get contact terms (delta functions), but we are interested in the correlator at finite distances. Observe,

$$\int d^4q\, e^{-iq_i x^i} q^{2n}\tilde{\phi}_0(q)\tilde{\phi}_0(-q) = (-1)^n \int d^4x d^4x' \delta^{(2n)}(x-x')\phi_0(x)\phi_0(x'), \tag{A.8}$$

where $\delta^{(2n)}(x)$ is the $2n$-th distributional derivative of the delta function. Hence, this can only contribute to two-point functions with both arguments evaluated at the same point.

These contact terms, which are often divergent, can simply be removed by subtracting them from the on-shell action. The counter-term action for the non-integer case is then

$$S_{ct} = \frac{\eta}{2R}\int \frac{d^4q d^4q'}{(2\pi)^4}\,\delta^4(q+q')\,\sqrt{\gamma}\left[\sum_{i=0}^{[\nu]}\frac{(2-\nu+2i)\Gamma(1-\nu)}{2^{2i}i!\Gamma(1-\nu+i)}(q\epsilon)^{2i}\right]\tilde{\phi}_0(q)\tilde{\phi}_0(q'), \tag{A.9}$$

where γ is the induced metric on the boundary giving $\sqrt{\gamma} = R^4/\epsilon^4$. This counter-term action is clearly covariant and guarantees that the Euclidean correlators are finite. After subtracting the divergencies via renormalization, the flux factor is finite and gives the correlator

$$\langle \hat{O}(q)\hat{O}(q')\rangle_\epsilon = -(2\pi)^4\delta^4(q+q')\frac{\eta R^3}{2^{2\nu-1}}\frac{\Gamma(1-\nu)}{\Gamma(\nu)}q^{2\nu}\epsilon^{2\nu-4}. \tag{A.10}$$

In the limit $\epsilon \to 0$, this correlator goes to zero. We see that we also need to renormalize the boundary operator to get a finite result. This is also equivalent to renormalizing the source. The renormalized source is then

$$\phi_0^{ren} = \epsilon^{2-\nu}\phi_0, \tag{A.11}$$

which gives a finite correlator.

In the case of integer $\nu = n$, the situation is almost identical with the only significant difference being the presence of a logarithm in the series expansion. We find that

$$K_n(z) = z^{-n}\sum_{i=0}^{\infty} a_{2i}^* z^{2i} + z^n \ln\frac{z}{2}\sum_{i=0}^{\infty} b_{2i}^* z^{2i}, \tag{A.12}$$

with the coefficients

$$a_{2i}^* = \begin{cases} \frac{(-1)^i}{2^{-n+2i+1}}\frac{(n-i-1)!}{i!} & i \le n-1 \\ \frac{(-1)^n}{2^{n+2i+1}}\frac{\psi(i+1)+\psi(n+i+1)}{i!(n+i)!} & i \ge n \end{cases} \tag{A.13}$$

$$b_{2i}^* = \frac{(-1)^{n+1}}{2^{n+2i}}\frac{1}{i!\Gamma(n+i+1)}, \tag{A.14}$$

where ψ is the digamma function. A computation identical to the integer order case gives

$$\begin{aligned}
r^{-3}\partial_r\left(\frac{r^2K_n(qr)}{\epsilon^2K_n(q\epsilon)}\right)\Big|_{r=\epsilon} &= \epsilon^{-4}\frac{(q\epsilon)^{n-2}}{a_0^*}\Bigg\{(q\epsilon)^{-n+2}\sum_{i=0}^{\infty}(2-n+2i)\,a_{2i}^*(q\epsilon)^{2i} + \ldots \\
&\quad \ldots + (q\epsilon)^{n+2}\sum_{i=0}^{\infty}\left[(2+n+2i)\ln\frac{q\epsilon}{2}+1\right]b_{2i}^*(q\epsilon)^{2i}\Bigg\}\left[1-(q\epsilon)^{2n}\frac{b_0^*}{a_0^*}\ln\frac{q\epsilon}{2}\right] \\
&= \frac{\epsilon^{-4}}{a_0^*}\Bigg\{\sum_{i=0}^{n}(2-n+2i)\,a_{2i}^*(q\epsilon)^{2i} + (q\epsilon)^{2n}\left[(n+2)\ln\frac{q\epsilon}{2}+1\right]b_0^* + \ldots \\
&\quad \ldots - (q\epsilon)^{2n}\ln\frac{q\epsilon}{2}b_0^*\,(2-n)\Bigg\} \\
&= \epsilon^{-4}\left\{\sum_{i=0}^{n}(2-n+2i)\,\frac{a_{2i}^*}{a_0^*}(q\epsilon)^{2i} + (q\epsilon)^{2n}\left(2n\ln\frac{q\epsilon}{2}+1\right)\frac{b_0^*}{a_0^*}\right\}.
\end{aligned}$$

Again, we use counter-terms in the action to subtract off all of the contact terms. The appropriate counter-term action is then

$$S_{ct} = \frac{\eta}{2R} \int \frac{d^4q d^4q'}{(2\pi)^4} \delta^4(q+q') \sqrt{\gamma} \left[\sum_{i=0}^{n} \frac{(2-\nu+2i)\Gamma(1-\nu)}{2^{2i} i! \Gamma(1-\nu+i)} (q\epsilon)^{2i} \right.$$
$$\left. \ldots + (q\epsilon)^{2n} \left(2n \ln \frac{\epsilon}{2} + 1 \right) \right] \tilde{\phi}_0(q) \tilde{\phi}_0(q'). \tag{A.15}$$

Using the same source renormalization scheme as before, the two-point function is then

$$\langle \hat{O}(q) \hat{O}(q') \rangle = -(2\pi)^4 \delta^4(q+q') \frac{\eta R^3}{2^{2\nu-1}} \frac{(-1)^n}{(n-1)!^2} q^{2\nu} \ln q^2. \tag{A.16}$$

B. Mathematics of the Hydrodynamic Expansion

While it is possible to simply guess the form of the solution for simple examples such as the massless scalar, we would like to develop a systematic procedure for the hydrodynamic expansion of the solutions. In general, the equations of motion for any combination of the field content are at most coupled systems of second order linear equations with typically no more than two singular points. The approach for getting solutions in the hydrodynamic limit consists of two steps. The first is reducing the second order system to a first order system with a regular singular point, and the second is expanding solutions in powers of the momenta about the regular singular point. These steps we be discussed in reverse order to help illustrate the importance of the ensuring there is only a regular singular point in the first order system.

B.1 Expansion About a Regular Singular Point

Consider a linear system of differential equations given by the equation

$$(x - x_0) Y' = A(x) Y, \tag{B.1}$$

where $A(x)$ is holomorphic at $x = x_0$. Then, $x = x_0$ is said to be a *regular singular point* or a *singularity of the first kind.* A unique solution to such a system always exists, and can be computed by a generalization of the method of Frobenius [18]. Consequently, the solution can inherit all of the subtleties related to repeated and blocked exponents. However, in practice blocked exponents rarely occur, so we won't consider them further.

Suppose that no two eigenvalues of $A(x_0)$ differ by a positive integer (i.e. blocked), then the equation (B.1) has the general solution

$$Y = P(x)(x - x_0)^{A(x_0)}, \tag{B.2}$$

where $P(x_0) = I$ and $P(x)$ is holomorphic at $x = x_0$. The power series expansion of $P(x)$ has the same radius of convergence as does $A(x)$ and its coefficients can be computed via algebraic operations. It is worth noting that there is a further simplification if there is a complete basis of eigenvectors for $A(x_0)$[17]. Then, $A(x_0)$ can be diagonalized to give $A(x_0) = S\Lambda S^{-1}$, where S is the eigenvector matrix and Λ is diagonal with eigenvalue entries. The solution given by (B.2) simplifies to

$$\begin{aligned} Y &= P(x)(x - x_0)^{S\Lambda S^{-1}} \\ &= P(x)S(x - x_0)^{\Lambda} S^{-1}. \end{aligned} \tag{B.3}$$

It is now easy to show that

$$Z = Q(x)(x - x_0)^{\Lambda} \tag{B.4}$$

is also a solution to (B.1), where $Q(x_0) = S$ and has the same radius of convergence as $A(x)$. The biggest advantage to this simplification is that $(x - x_0)^{\Lambda}$ is now diagonal, hence each column of Z is a linearly independent solution of (B.1).

This gives the solution as a power series expansion about the regular singular point, but it is not the hydrodynamic expansion we are looking for. This can easily be seen, because we need to impose boundary conditions at different coordinate points. Hence, we need exact solutions and not a series expansion about the singular point. If there is natural scale set by some parameter of the system, then this problem is solved by taking a hydrodynamic expansion in powers of the momenta.

Consider a single vector solution given by $\mathbf{z} = \mathbf{F}(x)(x - x_0)^{\nu}$, where ν is an eigenvalue of $A(x_0)$ and $\mathbf{q}(x_0)$ is the corresponding eigenvector. Instead of expanding the solution as a power series in x, we can expand as a double series in the momenta

$$\mathbf{F}(x) = \mathbf{F}(x_0) + \mathbf{F}_{1,1}(x) + \mathbf{F}_{\mathfrak{q},1}(x)\mathfrak{q} + \mathbf{F}_{1,\mathfrak{w}}(x)\mathfrak{w} + \mathbf{F}_{\mathfrak{q}^2,1}(x)\mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3, \mathfrak{q}\mathfrak{w}, \mathfrak{w}^2), \tag{B.5}$$

where $\mathbf{F}_{\mathfrak{q}^m,\mathfrak{w}^n}(x_0) = 0$. The differential equation for the single vector solutions reduces to

$$(x - x_0)\mathbf{F}'(x) = A(x)\mathbf{F}(x) - \nu\mathbf{F}(x), \tag{B.6}$$

Equating powers in (B.6) after substituting in the hydrodynamic expansion given by (B.5), recursive sets of differential equations are obtained. We also have one boundary condition to impose at every order to fix the integration constant. Ideally we can determine $\mathbf{q}(x)$ order by order in the momenta to get solutions that are exact in the coordinate x in the radius of convergence.

[17] Again, this happens almost always in the sorts of systems studied via gravity/gauge dualities. However, if there are degenerate eigenvectors, we could put $A(x_0)$ in Jordan canonical form instead. This means that some solutions will get coupled together spoiling some later simplifications.

B.2 Reducing the Second Order System

Now that we know how to find the general solution to a first order system with a regular singular point, we need to convert the initial second order system to a first order system. Given an arbitrary second order linear system of equations of functions y_i, one can reduce it to a first order system using the substitution $y_i' = p_i$, which gives a first order system at the expense of doubling the number of equations. Consequently, every linear system is equivalent to a first order linear system. However, we have to be careful that the resulting first order system has only a regular singular point, because the theory of irregular singular points is significantly more complicated, especially for matrix solutions.

It is often the case that the coefficients of the second order equations have second order poles. Let $y_i'' = F_i(y_j', y_k)$ be such an equation, where F_i is linear and second order poles can only occur in the coefficients of the second argument[18]. Then, the linear substitution $y_i' = p_i$ results in an irregular singular point because

$$p_i' = y_i'' = F_i(p_j, y_k),$$

and $F_i(y_j', y_k)$ has a second order pole. This can be corrected by instead making the substitution $y_i' = p_i/f_i(x)$, where we will soon determine $f_i(x)$. So,

$$p_i' = f_i(x)F_i\left(\frac{p_j}{f_j(x)}, y_k\right) + f_i'(x)y_i. \tag{B.7}$$

This indicates that if $y_i'' = F_i(y_j', y_k)$ has a second order pole, (B.7) has only a first order pole if $f_i(x)$ is linear in $x - x_0$. It is common to let $f_i(x)$ be the emblackening factor of the metric, which in the case of the AdS and Schrödinger black holes is $f_i(x) = x_0 - x^2$.

C. Conventional Confusion

C.1 β and δ and Null Coordinates, Oh My!

Since it seems that every author has a different convention for studying both the AdS and Schrödinger black holes and without fail they all like to use the same variable names for related but very different quantities, it will be helpful consolidate the various

[18] This restriction on the form of $F_i(y_j', y_k)$ is completely general for our purposes, because the fact that we are deriving equations of motion from an action principle is a strong constraint. The goal is to expand about the horizion, and in the metrics considered singularities can only occur via the emblackening factor in the denominator. Only the time and radial coordinates of any field have these functions. So, only second order poles can occur on the zero order terms, with one power coming from lowering the index on $\mathfrak{w}$ and the other from lowering the ∂^u index.

conventions and relate them to our convention. We will start with the AdS black hole, since it can be recovered by taking the appropriate limit of the Schrödinger black hole. Then, we will list several common coordinate choices and parameterizations for the Schrödinger black hole.

Recall our coordinate description of the AdS black hole,

$$ds^2_{AdS} = \frac{R^2}{r^2}(-f(r)d\tau^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{r^2 f(r)}dr^2 + R^2 d\Omega_5^2, \tag{C.1}$$

$$f(r) = 1 - \frac{r^4}{r_H^4}, \tag{C.2}$$

where we have made the choice of denoting the time coordinate by τ since we want to make sure that the AdS time coordinate is distinct from the Schrödinger time coordinate after melvinization. In this coordinate description, the horizon is at $r = r_H$ and the boundary is at $r = 0$. In [1, 2], an alternative parameterization is used which inverts the radial coordinate but preserves the fact that the radial coordinate is a length by multiplying by the appropriate factor of the AdS radius. In these coordinates,

$$ds^2_{AdS} = \frac{r'^2}{R^2}(-f(r')d\tau^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{r'^2 f(r')}dr'^2 + R^2 d\Omega_5^2, \tag{C.3}$$

$$r' = \frac{R^2}{r}, \qquad r_0 = \frac{R^2}{r_H}, \qquad f(r') = 1 - \frac{r_0^4}{r'^4}. \tag{C.4}$$

These coordinates have placed the horizon at $r = r_0$ and the boundary at spatial infinity. Fortunately, the conventions for choosing a dimensionless radius are the same. In particular,

$$u = \frac{r^2}{r_H^2} = \frac{r_0^2}{r'^2} = \frac{R^4}{r_H^2 r'^2}. \tag{C.5}$$

For completeness, this gives

$$ds^2_{AdS} = \frac{R^2}{r_H^2 u}(-f(u)dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2 \tag{C.6}$$

$$= \frac{r_0^2}{R^2 u}(-f(u)dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2. \tag{C.7}$$

The diversity in parameterizations for the AdS black hole is minimal. With the additional of an extra dimensionful quantity, there are many choices for coordinates in the Schrödinger black hole. We will start by considering a representative sample of the options for coordinates at zero temperature, and the finite temperature metrics are all related to these. Our choice at zero temperature was

$$ds^2_{Sch} = \frac{R^2}{r^2}\left(-\frac{2\beta^2}{r^2}dt^2 + 2dtd\xi + dx^2 + dz^2 + dr^2\right), \tag{C.8}$$

from which we can recover the AdS metric by setting $\beta = 0$ and identifying

$$y = \frac{t+\xi}{\sqrt{2}}, \qquad \tau = \frac{t-\xi}{\sqrt{2}}. \tag{C.9}$$

Alternatively, a redefinition of t and ξ can remove the factor of the 2β giving

$$ds^2_{Sch} = \frac{R^2}{r^2}\left(-\frac{dt'^2}{r^2} + 2dt'd\xi' + dx^2 + dz^2 + dr^2\right), \tag{C.10}$$

$$t' = \sqrt{2}\beta t, \qquad \xi' = \frac{1}{\sqrt{2}\beta}\xi, \tag{C.11}$$

which is used in the most recent verion of [13]. These are related to the AdS coordinates by setting

$$t' = \beta(y+\tau), \qquad \xi' = \frac{1}{2\beta}(y-\tau). \tag{C.12}$$

The last metric worth considering is from [19], which is

$$ds^2_{Sch} = \frac{r'^2}{R^2}(-r'^2du^2 - 2dudv + dx^2 + dz^2) + \frac{R^2}{r'^2}dr'^2, \tag{C.13}$$

$$u = \frac{t'}{R^2} = \frac{\sqrt{2}\beta t}{R^2}, \qquad v = -R^2\xi' = -\frac{R^2}{\sqrt{2}\beta}\xi, \qquad r' = \frac{R^2}{r} \tag{C.14}$$

To recover the AdS coordinates from the null coordinates use

$$u = \frac{\beta}{R^2}(\tau + y), \qquad v = \frac{R^2}{2\beta}(\tau - y). \tag{C.15}$$

C.2 Melvinized AdS Black Hole Perturbations

In section 6.4, it was explained how the Schrödinger field perturbations could be expressed in terms of the AdS perturbations via an application of the null melvin twist. For completeness, the explicit results of the calculation are given. For additional clarity, all of the perturbations are the bare perturbations in string frame without any extra dimensional factors introduced. For the ungauged theory, we find that:

$$h^{Sch}_{xx} = h^{AdS}_{xx}, \qquad h^{Sch}_{zx} = h^{AdS}_{zx} \qquad h^{Sch}_{zz} = h^{AdS}_{zz}, \tag{C.16}$$

$$h^{Sch}_{uu} = h^{AdS}_{uu}, \qquad h^{Sch}_{ux} = h^{AdS}_{ux}, \qquad h^{Sch}_{uz} = h^{AdS}_{uz}, \tag{C.17}$$

$$\begin{aligned} h^{Sch}_{\tau\tau} = \frac{1}{u^2K^2}\Big[&(u+\delta^2)^2 h^{AdS}_{tt} - \delta f\left(-\delta^3 f h^{AdS}_{yy} + 2\delta(u+\delta^2)h^{AdS}_{yt} + \dots\right. \\ &\left.\left. \dots + \frac{2R(u+\delta^2)}{r_H}\delta A^{AdS}_{\tau} - \frac{2\delta^2 Rf}{r_H}\delta A^{AdS}_{y}\right)\right], \end{aligned} \tag{C.18}$$

$$h_{y\tau}^{Sch} = \frac{1}{u^2K^2}\left\{(u^2K - 2\delta^4 f)h_{yt}^{AdS} + \delta\left[\delta(u+\delta^2)h_{tt}^{AdS} - \delta f(u-\delta^2 f)h_{yy}^{AdS}\right] - \dots\right.$$
$$\left.\dots - \frac{\delta R}{r_H}\left[u - (2-u^2)\delta^2\right]\left(\delta A_\tau^{AdS} - f\delta A_y^{AdS}\right)\right\}, \quad \text{(C.19)}$$
$$h_{yy}^{Sch} = \frac{1}{u^2K^2}\left[(u-\delta^2 f)^2 h_{yy}^{AdS} + \delta\left(\delta^3 f h_{tt}^{AdS} + 2\delta(u-\delta^2 f)h_{yt}^{AdS} - \dots\right.\right.$$
$$\left.\left.\dots - \frac{2\delta^2 R}{r_H}\delta A_\tau^{AdS} - \frac{2R(u-\delta^2 f)}{r_H}\delta A_y^{AdS}\right)\right], \quad \text{(C.20)}$$
$$h_{x\tau}^{Sch} = \frac{1}{uK}\left[(u+\delta^2)h_{xt}^{AdS} - \delta^2 f h_{yx}^{AdS} - \frac{\delta R f}{r_H}\delta A_x\right], \quad \text{(C.21)}$$
$$h_{xy}^{Sch} = \frac{1}{uK}\left[(u-\delta^2 f)h_{yx}^{AdS} + \delta^2 h_{xt}^{AdS} - \frac{\delta R}{r_H}\delta A_x\right], \quad \text{(C.22)}$$
$$h_{z\tau}^{Sch} = \frac{1}{uK}\left[(u+\delta^2)h_{zt}^{AdS} - \delta^2 f h_{zy}^{AdS} - \frac{\delta R f}{r_H}\delta A_z\right], \quad \text{(C.23)}$$
$$h_{zy}^{Sch} = \frac{1}{uK}\left[(u-\delta^2 f)h_{zy}^{AdS} + \delta^2 h_{zt}^{AdS} - \frac{\delta R}{r_H}\delta A_z\right], \quad \text{(C.24)}$$
$$h_{u\tau}^{Sch} = \frac{u+\delta^2}{uK}h_{ut}^{AdS} - \frac{\delta^2 f}{uK}h_{uy}^{AdS} - \frac{\delta R f}{r_H uK}\delta A_u^{AdS}, \quad \text{(C.25)}$$
$$h_{uy}^{Sch} = \frac{u - f\delta^2}{uK}h_{uy}^{AdS} + \frac{\delta^2}{uK}h_{ut}^{AdS} - \frac{\delta R}{r_H uK}\delta A_u^{AdS}, \quad \text{(C.26)}$$
$$\delta A_\tau^{Sch} = \frac{r_H}{RuK^2}\left[-\delta^3 f h_{yy}^{AdS} - \delta(u+\delta^2)h_{tt}^{AdS} + \delta(u+(2-u^2)\delta^2)h_{yt}^{AdS} + \dots\right.$$
$$\left.\dots + R\frac{u-(2-u^2)\delta^2}{r_H}\delta A_\tau - \frac{2\delta^2 R f}{r_H}\delta A_y^{AdS}\right], \quad \text{(C.27)}$$
$$\delta A_y^{Sch} = \frac{r_H}{uRK^2}\left[-\delta(u-(2-u^2)\delta^2)h_{yt}^{AdS} + \delta(u-\delta^2 f)h_{yy}^{AdS} - \delta^3 h_{tt}^{AdS} + \dots\right.$$
$$\left.\dots + \frac{2\delta^2 R}{r_H}\delta A_\tau^{AdS} + R\frac{u-(2-u^2)\delta^2}{r_H}\delta A_y\right], \quad \text{(C.28)}$$
$$\delta A_x^{Sch} = \frac{r_H\delta(h_{yx}^{AdS} - h_{xt}^{AdS}) + R\delta A_x^{AdS}}{RK}, \quad \text{(C.29)}$$
$$\delta A_z^{Sch} = \frac{r_H\delta(h_{zy}^{AdS} - h_{zt}^{AdS}) + R\delta A_z^{AdS}}{RK}, \quad \text{(C.30)}$$
$$\delta A_u^{Sch} = \frac{r_H\delta(h_{uy}^{AdS} - h_{ut}^{AdS}) + R\delta A_u^{AdS}}{RK}, \quad \text{(C.31)}$$
$$\delta\Phi^{Sch} = -\frac{r_H^2\delta}{2R^2K}\left[\delta(h_{tt}^{AdS} - 2h_{yt}^{AdS} + h_{yy}^{AdS}) - \frac{2R}{r_H}(\delta A_\tau^{AdS} - \delta A_y^{AdS})\right] \quad \text{(C.32)}$$

where $f = 1 - u^2$ and $K = 1 + \delta^2 u$.

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