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Unitarity cuts and reduction to master integrals in d dimensions for one-loop amplitudes

Charalampos Anastasiou
CERN, 1211 Geneva, 23, Switzerland
E-mail: babis@cern.ch

Ruth Britto
Institute for Theoretical Physics, University of Amsterdam
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
E-mail: rbritto@science.uva.nl

Bo Feng
Center of Mathematical Science, Zhejiang University, Hangzhou, China, and
Blackett Laboratory and The Institute for Mathematical Sciences, Imperial College
London, SW7 2AZ, U.K.
E-mail: b.feng@imperial.ac.uk

Zoltan Kunszt
Institute of Theoretical Physics, ETH Zurich
8093 Zurich, Switzerland
E-mail: kunszt@itp.phys.ethz.ch

Pierpaolo Mastrolia
Institute for Theoretical Physics, University of Zurich,
8057 Zurich, Switzerland
E-mail: mastrol@physik.unizh.ch

Abstract: We present an alternative reduction to master integrals for one-loop amplitudes using a unitarity cut method in arbitrary dimensions. We carry out the reduction in two steps. The first step is a pure four-dimensional cut-integration of tree amplitudes with a mass parameter, and the second step is applying dimensional shift identities to master integrals. This reduction is performed at the integrand level, so that coefficients can be read out algebraically.

Keywords: NLO Computations, QCD.

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Contents

1. Introduction

At the Large Hadron Collider (LHC), complex final states will be observed frequently and therefore the predictions of the Standard Model for the production rate of such events will have to be evaluated precisely. This requires the calculation of next-to-leading order QCD corrections to the cross-section of a large number of production processes, such as multi-jets,

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vector bosons and jets, top quarks and jets, etc. We need multileg amplitudes at one loop accuracy. In the traditional method, one generates the amplitudes according to Feynman diagrams, expresses the tensor loop integral in terms of scalar form factors and reduces the required scalar integrals to master integrals using recurrence identities. In the case of five or more particle processes the use of this straightforward method becomes cumbersome. The number of Feynman diagrams and terms generated have factorial growth. Furthermore, the presence of Gram determinants in the denominators of the reduction coefficients requires suitably optimized numerical techniques.

In the last few years remarkable progress has been achieved in replacing the traditional method with more efficient approaches which build on the properties of Yang-Mills theories more explicitly. In particular, very efficient recursive algorithms are available for the calculation of tree amplitudes and facilitate the calculation of one loop amplitudes.[1] Many of these new ideas have been stimulated by the suggestion of Witten to transform QCD amplitudes to twistor space [3].

The new techniques could also be extended to the calculation of loop amplitudes. One can apply the unitarity cut method [4], which uses tree amplitudes as input and so avoids the generation of Feynman diagrams. Recently, the four-dimensional unitarity cut method has been further developed as an efficient systematic tool to calculate QCD amplitudes [5, 6], building on techniques inspired by twistor space geometry [3, 7 – 13]. The phase-space integration and the reduction to master integrals are carried out explicitly in terms of spinors. Many, mostly supersymmetric, amplitudes can be reconstructed fully with this technique [5, 6, 14]. However, the mapping to master integrals is in general incomplete, since it misses rational contributions that arise from multiplying $1/\epsilon$ poles of master integrals with $\mathcal{O}(\epsilon)$ coefficients. This gap has been filled recently by methods that target these rational contributions specifically, either by developing [15 – 19] recursion relations for amplitudes [20, 21], or by using specialized diagrammatic reductions [22 – 25]. As a result, for example, short analytic formulas are now available for all the one-loop six gluon QCD helicity amplitudes.

We can, however, reconstruct the full amplitude with the unitarity cut method, provided the cut integrals are treated in $d = 4 - 2\epsilon$ dimensions [26]. A complete method for one-loop calculations was developed in the pioneering work of Bern *et al.* [27 – 29], and it was recently re-examined in [30]. The idea is that the (-2ϵ)-dimensional component of momentum can be considered as constant, and orthogonal to the 4-dimensional components. From this point of view, a massless d-dimensional scalar can be traded for a massive 4-dimensional scalar. Then, unitarity cuts can be applied to constrain the coefficients of master integrals. However, the calculation of unitarity cuts is generally difficult because of the reduction of d-dimensional tensor integrals. Eventually, in the papers mentioned above one resorts to traditional reduction methods to complete the computation.

In [31] we have reported results on an efficient implementation of the d-dimensional unitary cut method. One-loop amplitudes can be reduced to master integrals for arbitrary values of the dimension parameter. An important result is that we can read out

[1] For reviews of this progress, see for example [1, 2].

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the coefficients of the master integrals without fully carrying out the d-dimensional phase space integrals. The problem is reduced to four dimensional integration, where we can now capitalize on the recent advances in computation. The four dimensional tensor integrals can be calculated using spinor integration for light-like momenta [12, 13, 5, 6, 32]. Recursion relations for amplitudes with massive scalars have been developed [33] specifically for generating the tree-level input for d-dimensional unitarity cuts.

In this paper we outline the d-dimensional unitarity cut method in detail and give simple illustrative applications using spinorial integration for tensor reduction. Here we work entirely in terms of standard double cuts; some work with generalized unitarity cuts in d dimensions has appeared in [30, 34, 32]. A different method of constructing the master integral coefficients of one-loop amplitudes, from the values of the loop momentum that correspond to unitarity cuts, has been presented in [35].

In section 2 we discuss the parametrization of d-dimensional cut integrals. We identify the 4-dimensional integral within the d-dimensional integral, leaving the final (-2ϵ)-dimensional integral for last. The 4-dimensional integral can be performed by a method of choice; here we proceed in terms of spinorial variables. We show the cut bubble as a prototype of any cut integral and then set up the integral for a general amplitude.

In section 3, we derive the integral representations of the cut master integrals, namely scalar bubbles, triangles, boxes, and pentagons. The physical arguments of [6, 14, 32] state that all possible integrands are related to these basis integrals simply by polynomial factors in the (-2ϵ)-dimensional mass parameter. We relate these general integrands to the basis integrands by dimensional shift identities, which here take the form of recursion and reduction relations. (Recursion refers to the degree of the polynomial.) We derive these identities and explain their application.

In section 4, we work through the examples of the five-gluon all-plus amplitude and the four-gluon amplitudes and confirm that our results agree with [27, 29, 30]. The reduction is done using spinorial integration [6, 14]. These spinorial integrals are evaluated using Schouten identities, Feynman parameter integrals and the holomorphic anomaly formula [8]. Since in the d-dimensional unitarity method the integrand of spinorial integrals depends on an additional mass parameter, the size of the expressions is larger and the recognition of the scalar master integrals is more involved than in the four-dimensional case.

Appendix A discusses the kinematic region and domains of integration for a unitarity cut. Appendix B gives further details of the various master integrals. Appendix C contains helpful identities and formalisms for spinor integration, in particular with regard to quadratic denominator factors.

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2. The d-dimensional unitarity method

The n-point scalar function is defined by[2]

$$I_n = \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p-K_1)^2(p-K_1-K_2)^2\dots(p-\sum_{j=1}^{n-1}K_j)^2}. \tag{2.1}$$

We operate in the four-dimensional helicity (FDH) scheme, in which all external momenta are in four dimensions. In this formula, therefore, the loop momentum p is $(4-2\epsilon)$-dimensional, while all the K_i are 4-dimensional. We can decompose $p = \widetilde{\ell} + \vec{\mu}$ where $\widetilde{\ell}$ is 4-dimensional and $\vec{\mu}$ is (-2ϵ)-dimensional. Then the integration measure becomes

$$\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} = \int \frac{d^4\widetilde{\ell}}{(2\pi)^4} \int \frac{d^{-2\epsilon}\ell_\epsilon}{(2\pi)^{-2\epsilon}} = \int \frac{d^4\widetilde{\ell}}{(2\pi)^4} \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int d\mu^2\ (\mu^2)^{-1-\epsilon},$$

and the scalar function is

$$I_n = \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int d\mu^2 (\mu^2)^{-1-\epsilon} \int \frac{d^4\widetilde{\ell}}{(2\pi)^4} \frac{1}{(\widetilde{\ell}^2-\mu^2)((\widetilde{\ell}-K_1)^2-\mu^2)\dots((\widetilde{\ell}-\sum_{j=1}^{n-1}K_j)^2-\mu^2)}. \tag{2.2}$$

We will use spinor integration when we cut the 4D momentum $\widetilde{\ell}$, so we choose to decompose it into a linear combination of a light-like momentum variable and a fixed vector K:

$$\widetilde{\ell} = \ell + zK, \qquad \ell^2 = 0, \qquad \Longrightarrow \int d^4\widetilde{\ell} = \int dz\ d^4\ell\ \delta^+(\ell^2)(2\ell\cdot K). \tag{2.3}$$

Eventually it will be convenient to choose K to be the momentum through the unitarity cut. This is one of the most important ideas that enables our whole program to work. Further we define

$$u = \frac{4\mu^2}{K^2}. \tag{2.4}$$

As will be clear from discussions in appendix A, in our cut calculation we have $u \in [0,1]$. Therefore

$$\frac{(4\pi)^\epsilon}{(2\pi)^4\Gamma(-\epsilon)} \int d\mu^2\ (\mu^2)^{-1-\epsilon} \to \frac{(4\pi)^\epsilon}{(2\pi)^4\Gamma(-\epsilon)} \left(\frac{K^2}{4}\right)^{-\epsilon} \int_0^1 du\ u^{-1-\epsilon}. \tag{2.5}$$

Since $\frac{(4\pi)^\epsilon}{(2\pi)^4\Gamma(-\epsilon)} \left(\frac{K^2}{4}\right)^{-\epsilon}$ is an universal factor appearing on both sides of every cut calculation, we will neglect it throughout the rest of the paper.

Finally we arrive at the equation

$$I_n = \int_0^1 du\ u^{-1-\epsilon} \int dz\ d^4\ell\ \delta^+(\ell^2)(2\ell\cdot K) \frac{1}{(\widetilde{\ell}^2-\mu^2)((\widetilde{\ell}-K_1)^2-\mu^2)\dots((\widetilde{\ell}-\sum_{j=1}^{n-1}K_j)^2-\mu^2)}, \tag{2.6}$$

where μ^2 is related to u through (2.4).

At this point we are ready to carry out the 4-dimensional cut integration. We do this in the language of spinors.

[2]Our convention is to omit the prefactor $i(-1)^{n+1}(4\pi)^{D/2}$ that is common elsewhere in the literature.

2.1 The cut-integration of bubble functions

The cut of a scalar bubble is the simplest kind of unitarity cut, so it is instructive as well as useful to go through this case in detail. Then we will be able to set up the framework for any other cut of master integrals or amplitudes.

The expression of the (double) cut of the bubble function is given by (2.6)

$$C[I_2(K)] = \int_0^1 du\ u^{-1-\epsilon} \int dz\ d^4\ell\ \delta^+(\ell^2)(2\ell\cdot K)\delta(\widetilde{\ell}^2-\mu^2)\delta((\widetilde{\ell}-K)^2-\mu^2), \quad (2.7)$$

where $\widetilde{\ell} = \ell + zK$ with $\ell^2 = 0$, and μ^2 is related to u by (2.4). We make use of the delta functions to rewrite the integral as follows.

$$\begin{aligned} C[I_2(K)] &= \int_0^1 du\ u^{-1-\epsilon} \int dz\ d^4\ell\ \delta^+(\ell^2)(2\ell\cdot K)\delta(\widetilde{\ell}^2-\mu^2)\delta(K^2-2K\cdot\widetilde{\ell}) \\ &= \int_0^1 du\ u^{-1-\epsilon} \int dz\ d^4\ell\ \delta^+(\ell^2)(2\ell\cdot K)\delta(z^2K^2+2zK\cdot\ell-\mu^2)\delta((1-2z)K^2-2K\cdot\ell) \\ &= \int_0^1 du\ u^{-1-\epsilon} \int dz(1-2z)K^2\delta(z(1-z)K^2-\mu^2) \int d^4\ell\ \delta^+(\ell^2)\delta((1-2z)K^2-2K\cdot\ell). \end{aligned}$$

Here we have brought the integral into a form where one of the delta-functions, $\delta(z(1-z)K^2-\mu^2)$, does not depend on ℓ. Now we continue by transforming the integral to spinor coordinates [7]:

$$\ell = t\lambda\widetilde{\lambda}, \quad (2.8)$$

so that the measure transforms as

$$\int d^4\ell\delta^{(+)}(\ell^2)\ (\bullet) = \int_0^\infty dt\ t \int \langle\lambda\ d\lambda\rangle\ [\tilde{\lambda}\ d\tilde{\lambda}](\bullet). \quad (2.9)$$

Here t ranges over the positive real line, and $\lambda, \widetilde{\lambda}$ are homogeneous spinors, also written respectively as $|\ell\rangle, |\ell]$ in many expressions involving spinor products. The first step in spinor integration is to integrate over the variable t. This is never true integration, because all we need is to solve the delta function of the *second* cut propagator. Thus we find

$$\begin{aligned} C[I_2(K)] = \int_0^1 du\ u^{-1-\epsilon} &\int dz(1-2z)K^2\delta(z(1-z)K^2-\mu^2) \\ &\int \langle\ell\ d\ell\rangle\ [\ell\ d\ell] \int dt\ t\ \delta((1-2z)K^2-2K\cdot\ell). \end{aligned}$$

The spinor and t-integrations are similar to the four-dimensional case [6, 14]; the only new feature is the factor of $(1-2z)$. After this integration, we get[3]

$$C[I_2(K)] = \int_0^1 du\ u^{-1-\epsilon} \int dz(1-2z)K^2\delta(z(1-z)K^2-u\frac{K^2}{4})(1-2z).$$

[3]In this paper we take residue instead of the negative of residue when we do phase space integration. This is just a matter of convention because when we calculate both sides of the cut equation, the sign cancels out.

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Using the formula

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \tag{2.10}$$

where the x_i's are the roots of $g(x)$, we can finish the z-integration to get

$$C[I_2(K)] = \int_0^1 du\ u^{-1-\epsilon}\sqrt{1-u}, \tag{2.11}$$

where we have used the fact (see appendix A) that only one root is allowed, specifically, $z = (1 - \sqrt{1-u})/2$. Equation (2.11) is simple enough that we can finish the u integration directly to find[4]

$$C[I_2(K)] = \frac{\sqrt{\pi}\Gamma(-\epsilon)}{2\Gamma(\frac{3}{2} - \epsilon)}, \quad \mathrm{Re}(\epsilon) < 0. \tag{2.12}$$

We can check the result (2.12) on the well known scalar bubble function given by

$$I_2^{\mathrm{old}}(K^2) = \frac{r_\Gamma}{\epsilon(1-2\epsilon)}\left(-K^2\right)^{-\epsilon}, \quad r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}. \tag{2.13}$$

To take the imaginary part we need to use[5]

$$\mathrm{Im}(-K^2)^{-\epsilon} = 2i\sin(\pi\epsilon)|K^2|^{-\epsilon}, \tag{2.14}$$

thus

$$C[I_2^{\mathrm{old}}(K^2)] = \frac{2i\sin(\pi\epsilon)\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon(1-2\epsilon)\Gamma(1-2\epsilon)}(K^2)^{-\epsilon}. \tag{2.15}$$

When we try to compare with our new result (2.12), we must multiply (2.12) by the following two factors: (1) $\frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)}\left(\frac{K^2}{4}\right)^{-\epsilon}$ from the discussion below (2.5); (2) $i(4\pi)^{2-\epsilon}$ from our non-standard definition of the scalar function in (2.1). Considering these two facts, one can check that $\frac{C[I_2^{\mathrm{our}}(K^2)]}{C[I_2^{\mathrm{old}}(K^2)]} = 8\pi$, which is just a matter of a different normalization when we take $\int d^4\ell\ \delta^+(\ell^2)$.

2.2 Cut-integral of an amplitude

Now we discuss how to apply the integration technique to the cut of an amplitude. The general expression will be

$$\begin{aligned} C = &\int_0^1 du\ u^{-1-\epsilon}\int dz\ (1-2z)K^2\delta(z(1-z)K^2 - \mu^2) \\ &\int d^4\ell\ \delta^+(\ell^2)\delta((1-2z)K^2 - 2K\cdot\ell)A_L(\widetilde{\ell}_1, \widetilde{\ell}_2)A_R(\widetilde{\ell}_1, \widetilde{\ell}_2), \end{aligned} \tag{2.16}$$

[4]The following analytic expression is right only when $\mathrm{Re}(\epsilon) < 0$. This is the condition for us to use integration by parts to derive all recursion and reduction formulas.

[5]To compute the cut with momentum K, we work in the kinematic region where only $K^2 > 0$ and all other momentum invariants are negative.

where A_L, A_R are the tree-level amplitudes on either side of the cut. In this formula, K is the cut-momentum and $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ are the (massive) cut 4D-momenta satisfying

$$\widetilde{\ell}_2 = K - \widetilde{\ell}_1, \quad \widetilde{\ell}_1^2 = \widetilde{\ell}_2^2 = \mu^2, \quad \widetilde{\ell}_1 = \ell + zK. \tag{2.17}$$

Now we explain the meaning of the expression (2.16). The second line is simply a 4D cut-integration that depends on the parameter z. The techniques developed in [6, 14, 32] can be applied directly. Then this result can be put into the first line, and the z-integration can be performed trivially by using the delta-function. We arrive at the final expression

$$C = \int_0^1 du\ u^{-1-\epsilon} \int d^4\ell\ \delta^+(\ell^2)\delta((1-2z)K^2 - 2K\cdot\ell) A_L(\widetilde{\ell}_1, \widetilde{\ell}_2) A_R(\widetilde{\ell}_1, \widetilde{\ell}_2), \tag{2.18}$$

where

$$\mu^2 = \frac{K^2}{4}u, \qquad z = \frac{1-\sqrt{1-u}}{2}. \tag{2.19}$$

This is our setup for all calculations in this paper.

3. Identifying integrands: cuts of master integrals

In this section we study the cuts of the master integrals. Our aim is to relate these with cuts of the amplitude, at the integrand level, so that we can read off the coefficients.

The general integrand arising from the cut of an amplitude looks like a series of terms that are related to cuts of master integrals by factors that are polynomial in u. Therefore, we define classes of integrals related to the cuts of master integrals by additional powers of u. Through integration by parts, the powers of u can be stripped away. The result is a set of "recursion and reduction identities" that relate any integrand to cuts of master integrals. With these identities it is possible to read off the coefficients without any actual integration.

3.1 Cut bubbles

Here we consider the whole class of integrands that will be related to bubbles by a recursion formula. For $n \geq 0$, we define the following new function.

$$\mathrm{Bub}^{(n)} \equiv \int_0^1 du\ u^{-1-\epsilon} u^n \sqrt{1-u}. \tag{3.1}$$

The physical cut of the bubble master integral is $C[I_2(K)] = \mathrm{Bub}^{(0)}$. The function $\mathrm{Bub}^{(n)}$ represents a term that may arise from a general cut amplitude. It is simple enough to evaluate this integral directly, but what we want is to relate it to the master integral. We carry this out in rather general terms, to illustrate the idea for the more complicated master integrals. Let us see how to find a recursion relation in n and eventually write $\mathrm{Bub}^{(n)}$ in

terms of $\text{Bub}^{(0)}$. For $n \geq 1$, we integrate by parts to get

$$\begin{aligned}\text{Bub}^{(n)} &= -\frac{2}{3}(1-u)^{3/2}u^{-1-\epsilon}u^n\Big|_0^1 + \int_0^1 du\ \frac{2}{3}(n-1-\epsilon)(1-u)^{3/2}u^{-1-\epsilon}u^{n-1} \\ &= \int_0^1 du\ \frac{2}{3}(n-1-\epsilon)\sqrt{1-u}(1-u)u^{-1-\epsilon}u^{n-1} \\ &= \frac{2}{3}(n-1-\epsilon)(\text{Bub}^{(n-1)} - \text{Bub}^{(n)}).\end{aligned}$$

The boundary term vanishes because $\text{Re}(\epsilon) < 0$. From this we get the following recursion relation.

$$\text{Bub}^{(n)} = \frac{(n-1-\epsilon)}{(n+\frac{1}{2}-\epsilon)}\text{Bub}^{(n-1)}. \tag{3.2}$$

This recursion is easily solved. We write the solution in the form

$$\text{Bub}^{(n)} = F_{2\to 2}^{(n)}\text{Bub}^{(0)}, \tag{3.3}$$

where the form factor is

$$F_{2\to 2}^{(n)} = \frac{\Gamma(3/2-\epsilon)\Gamma(n-\epsilon)}{\Gamma(-\epsilon)\Gamma(n+3/2-\epsilon)}. \tag{3.4}$$

Notice that this form factor does not depend on any kinematical variables.

There is another expression for $\text{Bub}^{(n)}$, obtained by a different choice of integration by parts.

$$\text{Bub}^{(n)} = \frac{u^{n-\epsilon}}{n-\epsilon}\sqrt{1-u}\Big|_0^1 + \frac{1}{2(n-\epsilon)}\int_0^1 du\ \frac{u^{n-\epsilon}}{\sqrt{1-u}} = \frac{1}{2(n-\epsilon)}\int_0^1 du\ \frac{u^{n-\epsilon}}{\sqrt{1-u}}. \tag{3.5}$$

It is useful to be able to recognize this alternative expression when it shows up as an integrand. It is the same integral found in one-mass and two-mass triangles (for details, see appendix B).

3.2 Cut triangles

We label the triangle such that the cut momentum is $K = K_1$. Then the cut-integrand is given by

$$\frac{\delta(\widetilde{\ell}^2-\mu^2)\delta((\widetilde{\ell}-K_1)^2-\mu^2)}{((\widetilde{\ell}+K_3)^2-\mu^2)}.$$

Using the general integration measure of (2.18), we get

$$C[I_3(K_1;K_3)] = \int_0^1 du\ u^{-1-\epsilon}\int\langle\ell\ d\ell\rangle\,[\ell\ d\ell]\int dt\ \frac{t\ \delta((1-2z)K_1^2 + t\,\langle\ell|K_1|\ell]\rangle}{K_3^2 + 2zK_1\cdot K_3 - t\,\langle\ell|K_3|\ell]}.$$

After t-integration we get

$$C[I_3(K_1;K_3)] = -\int_0^1 du\ u^{-1-\epsilon}\sqrt{1-u}\int\langle\ell\ d\ell\rangle\,[\ell\ d\ell]\frac{1}{\langle\ell|K_1|\ell]\,\langle\ell|P_1|\ell]} \tag{3.6}$$

$$= -\int_0^1 du\ u^{-1-\epsilon}\sqrt{1-u}\int_0^1 dx\frac{1}{P^2}, \tag{3.7}$$

with

$$P_1 = \frac{K_3^2 + 2zK_1 \cdot K_3}{K_1^2} K_1 + (1-2z)K_3, \tag{3.8}$$

$$P = xP_1 - (1-x)K_1 \tag{3.9}$$

After some algebraic manipulations we reach

$$C[I_3(K_1; K_3)] = -\int_0^1 du\ u^{-1-\epsilon} \frac{1}{\sqrt{\Delta_3}} \ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right), \tag{3.10}$$

with

$$Z = -\frac{K_1 \cdot K_3 + K_3^2}{\sqrt{(K_1 \cdot K_3)^2 - K_1^2 K_3^2}}, \qquad \Delta_3 = 4[(K_1 \cdot K_3)^2 - K_1^2 K_3^2]. \tag{3.11}$$

It can be shown that in our kinematic region, in which only $K_1^2 > 0$ and all other momentum invariants are negative, we will have $Z \geq 1$.

It is hard to evaluate the integral over u for (3.10), but our strategy is that we *never need to evaluate it.* While keeping it in integral form, we will be able to reduce general integrands by our recursion and reduction formulas.

Recursion and reduction for triangles: we define the following dimensionless integrals for all nonnegative integers n:

$$\mathrm{Tri}^{(n)}(Z) \equiv \int_0^1 du\ u^{-1-\epsilon} u^n \ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right). \tag{3.12}$$

The physical cut is

$$C[I_3(K_1, K_3)] = -\frac{1}{\sqrt{\Delta_3}} \mathrm{Tri}^{(0)}(Z), \tag{3.13}$$

if we take the Z and Δ_3 defined in (3.11). The definition (3.12) was chosen because it is free of dimensional factors. For $n \geq 1$ we can do the following integration by parts.

$$\begin{aligned}\mathrm{Tri}^{(n)}(Z) = {} & u^{n-1-\epsilon}\left((Z^2-1+u)\ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right) - 2Z\sqrt{1-u}\right)\Big|_0^1 \\ & - \int_0^1 du\ u^{n-2-\epsilon}(n-1-\epsilon)\left((Z^2-1+u)\ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right) - 2Z\sqrt{1-u}\right).\end{aligned}$$

From this we derive the following recursion and reduction relation.

$$\mathrm{Tri}^{(n)}(Z) = -\frac{(Z^2-1)(n-1-\epsilon)}{(n-\epsilon)} \mathrm{Tri}^{(n-1)}(Z) + \frac{2Z(n-1-\epsilon)}{(n-\epsilon)} \mathrm{Bub}^{(n-1)}. \tag{3.14}$$

Here the last term in (3.14) is the same one defined in (3.1), which is related to cut bubbles.

The result (3.14) plays two roles. First, it is the recursion formula for coefficients of triangles. After n steps in the recursion we arrive at $n = 0$, which is related to the cut of the scalar triangle by the factor $-1/\sqrt{\Delta_3}$. Second, it establishes the reduction relation for

tensor triangles to scalar bubbles. For a given triangle, there is only one bubble that can result from reduction, consistent with a given cut-momentum (here K_1).

Now we solve (3.14) and get

$$\mathrm{Tri}^{(n)}(Z) = F^{(n)}_{3\to 3}(Z)\mathrm{Tri}^{(0)}(Z) + \widetilde{F}^{(n)}_{3\to 2}(Z)\mathrm{Bub}^{(0)}, \tag{3.15}$$

in terms of two form factors, which are functions of only one variable Z, the identifier of a given triangle. Explicitly, these form factors are given by

$$F^{(n)}_{3\to 3}(Z) = \frac{-\epsilon}{n-\epsilon}(1-Z^2)^n, \tag{3.16}$$

$$\widetilde{F}^{(n)}_{3\to 2}(Z) = \frac{1}{n-\epsilon}\frac{\Gamma(3/2-\epsilon)}{\Gamma(-\epsilon)}\sum_{k=1}^{n} 2Z(1-Z^2)^{n-k}\frac{\Gamma(k-\epsilon)}{\Gamma(k+1/2-\epsilon)}. \tag{3.17}$$

Equation (3.15) is not in the exact form that we want. We need to return to the language of physical cuts by including the factor $-1/\sqrt{\Delta_3}$ from (3.13). The recursion/reduction formula that we need is thus:

$$\int_0^1 du\; u^{-1-\epsilon}u^n\left[-\frac{1}{\sqrt{\Delta_3}}\ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right)\right] = F^{(n)}_{3\to 3}(Z)C[I_3(K_1,K_3)] + F^{(n)}_{3\to 2}(K_1,K_3)C[I_2(K_1)], \tag{3.18}$$

where

$$F^{(n)}_{3\to 2}(K_1,K_3) = -\frac{1}{\sqrt{\Delta_3}}\widetilde{F}^{(n)}_{3\to 2}(Z). \tag{3.19}$$

The relation (3.18) is the main result of this subsection. Let us comment briefly on how it will be used. Our d-dimensional unitarity cut method separates the complete cut integration into the u-integral, $\int_0^1 duu^{-1-\epsilon}$, and the massive 4D part. After doing the 4D integral, we come to an expression of the form $\int_0^1 duu^{-1-\epsilon}\sum_{i\in basis} f_i(u)C[I_i]$, where $C[I_i]$ is the cut of master integral I_i and $f_i(u) = \sum_i a_i u^i$ is the polynomial of u. Then we know immediately that this term will contribute $\sum_i a_i F^{(i)}_{3\to 2}(K_1,K_3)$ to the bubble coefficient and $\sum_i a_i F^{(i)}_{3\to 3}(Z)$ to the coefficient of the triangle.

As in the bubble case, we derive a useful identity by integrating by parts in a different way:

$$\mathrm{Tri}^{(n)}(Z) = \frac{u^{-1-\epsilon}u^{n+1}}{n-\epsilon}\ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right)\Bigg|_0^1 - \int_0^1 du\;\frac{u^{-1-\epsilon}u^{n+1}}{n-\epsilon}\frac{Z}{\sqrt{1-u}(1-u-Z^2)}.$$

Again, since we have $Re(-\epsilon) > 0$, the boundary contribution is zero, so we end up with

$$\mathrm{Tri}^{(n)}(Z) = -\frac{Z}{n-\epsilon}\int_0^1 du\; u^{-1-\epsilon}\frac{u^{n+1}}{\sqrt{1-u}(1-u-Z^2)}. \tag{3.20}$$

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3.3 Cut boxes

In this subsection we will deal with box functions. There are several different cuts, but we would like to simplify calculations by representing them collectively by the same expression:

$$\frac{\delta(\widetilde{\ell}^2-\mu^2)\delta((\widetilde{\ell}-K)^2-\mu^2)}{((\widetilde{\ell}-P_1)^2-\mu^2)((\widetilde{\ell}-P_2)^2-\mu^2)},$$

where for different cuts we need to take different values of P_1, P_2. To be clear, we list the six possible cuts of a box in table (3.21), with K_1, K_2, K_3, K_4 in clockwise ordering. There will be two cut triangles related to each cut box consistent with the cut momentum K; these are indicated here as well, for use in the reduction relations.

Box Cut K	P_1	P_2	*Triangle One's* (K_1,K_3)	*Triangle Two's* (K_1,K_3)
K_1	K_{12}	$-K_4$	(K_1,K_{34})	(K_1,K_4)
K_2	K_{23}	$-K_1$	(K_2,K_{41})	(K_2,K_1)
K_3	K_{34}	$-K_2$	(K_3,K_{12})	(K_3,K_2)
K_4	K_{41}	$-K_3$	(K_4,K_{23})	(K_4,K_3)
K_{12}	K_1	$-K_4$	(K_{34},K_2)	(K_{12},K_4)
K_{23}	K_2	$-K_1$	(K_{41},K_3)	(K_{23},K_1)

(3.21)

After performing the t-integration we get

$$\begin{aligned} C[I_4(K;P_1,P_2)] &= \int_0^1 du\; u^{-1-\epsilon}\frac{(1-2z)}{K^2}\int \langle \ell\; d\ell\rangle\,[\ell\; d\ell]\frac{1}{\langle\ell|Q_1|\ell]\,\langle\ell|Q_2|\ell]} \\ &= \int_0^1 du\; u^{-1-\epsilon}\frac{\sqrt{1-u}}{K^2}\int_0^1 dx\frac{1}{Q^2}, \end{aligned}$$

where

$$Q = xQ_1 + (1-x)Q_2 \tag{3.22}$$

and

$$Q_i = -(1-2z)P_i + \frac{P_i^2 - 2zP_i\cdot K}{K^2}K. \tag{3.23}$$

For future convenience let us give the names R_1, R_2 to these vectors at the point $u=0$:

$$R_i \equiv -P_i + \frac{P_i^2}{K^2}K. \tag{3.24}$$

We define some additional variables:

$$\begin{aligned} &\alpha_i \equiv R_i\cdot K, && \beta_i \equiv P_i^2 - \frac{(P_i\cdot K)^2}{K^2} \\ &A \equiv -\frac{1}{K^2}\det\begin{pmatrix} P_1^2 & P_1\cdot P_2 & P_1\cdot K \\ P_1\cdot P_2 & P_2^2 & P_2\cdot K \\ P_1\cdot K & P_2\cdot K & K^2 \end{pmatrix}, && C \equiv \frac{1}{K^2}\det\begin{pmatrix} P_1\cdot P_2 & P_1\cdot K \\ P_2\cdot K & K^2 \end{pmatrix}, \\ &B \equiv -\det\begin{pmatrix} R_1^2 & R_1\cdot R_2 \\ R_1\cdot R_2 & R_2^2 \end{pmatrix}, && D \equiv R_1\cdot R_2. \end{aligned} \tag{3.25}$$

Using the identities

$$\begin{aligned} Q_i^2 &= (1-u)\beta_i + \frac{\alpha_i^2}{K^2}, \\ Q_1 \cdot Q_2 &= (1-u)C + \frac{\alpha_1\alpha_2}{K^2}\,, \\ (Q_1\cdot Q_2)^2 - Q_1^2 Q_2^2 &= (1-u)(B-Au), \end{aligned} \tag{3.26}$$

we can derive

$$\int_0^1 dx \frac{1}{Q^2} = \frac{1}{2\sqrt{(Q_1\cdot Q_2)^2 - Q_1^2Q_2^2}} \ln \frac{(Q_1\cdot Q_2) + \sqrt{(Q_1\cdot Q_2)^2 - Q_1^2Q_2^2}}{(Q_1\cdot Q_2) - \sqrt{(Q_1\cdot Q_2)^2 - Q_1^2Q_2^2}}, \tag{3.27}$$

$$= \frac{1}{2\sqrt{1-u}\sqrt{B-Au}} \ln\left(\frac{D - Cu + \sqrt{1-u}\sqrt{B-Au}}{D - Cu - \sqrt{1-u}\sqrt{B-Au}}\right), \tag{3.28}$$

where from (3.27) to (3.28) we have worked out the u-dependence.

Finally we arrive at the expression

$$C[I_4(K;P_1,P_2)] = \frac{1}{2K^2}\int_0^1 du\; u^{-1-\epsilon} \frac{1}{\sqrt{B-Au}} \ln\left(\frac{D-Cu+\sqrt{1-u}\sqrt{B-Au}}{D-Cu-\sqrt{1-u}\sqrt{B-Au}}\right). \tag{3.29}$$

Recursion and reduction for boxes: we define the following dimensionless integrals for all nonnegative integers n:

$$\mathrm{Box}^{(n)}(A,B,C,D) \equiv \int_0^1 du\; u^{-1-\epsilon} \frac{u^n}{\sqrt{B-Au}} \ln\left(\frac{D - Cu + \sqrt{1-u}\sqrt{B-Au}}{D - Cu - \sqrt{1-u}\sqrt{B-Au}}\right). \tag{3.30}$$

The physical cut is

$$C[I_4(K;P_1,P_2)] = \frac{1}{2K^2}\mathrm{Box}^{(0)}(A,B,C,D), \tag{3.31}$$

if A, B, C, D are defined as in (3.25).

To derive recursion and reduction relations, we select the factor $1/\sqrt{B-Au}$ and integrate by parts. The boundary term is zero if $n \geq 1$; thus we have

$$\mathrm{Box}^{(n)}(A,B,C,D) = \frac{2(n-1-\epsilon)}{A}(B\ \mathrm{Box}^{(n-1)}(A,B,C,D) - A\ \mathrm{Box}^{(n)}(A,B,C,D)) + T, \tag{3.32}$$

with

$$T = -\frac{2}{A}\int_0^1 du\; \frac{u^{n-1-\epsilon}}{\sqrt{1-u}} \frac{(AD-2BC+BD) - u(2AD-AD-BC)}{(D-Cu)^2 - (1-u)(B-Au)}. \tag{3.33}$$

This quantity T is related to triangle integrals. To see this, factorize the quadratic polynomial in the denominator:

$$(D-Cu)^2 - (1-u)(B-Au) = \beta_1\beta_2(1-u-Z_1^2)(1-u-Z_2^2),$$

with

$$Z_i^2 = -\frac{\alpha_i^2}{\beta_i K^2}. \tag{3.34}$$

Comparing with (3.11) we can see that these Z_1^2 and Z_2^2 correspond to the Z^2 of some triangle functions. Here in particular,

$$Z_i = -\frac{R_i \cdot K}{\sqrt{(P_i \cdot K)^2 - P_i^2 K^2}}. \tag{3.35}$$

In table (3.21) we have listed which kinds of triangles a box with given cut would reduce to.

Using the above result with the cut-triangle expression from (3.20) we get

$$\begin{aligned} T &= \frac{2}{A}\int_0^1 du\ \frac{u^{n-1-\epsilon}}{\sqrt{1-u}}\left(\frac{C_{Z_1}}{1-u-Z_1^2}+\frac{C_{Z_2}}{1-u-Z_2^2}\right) \\ &= -\frac{2C_{Z_1}(n-1-\epsilon)}{Z_1 A}\mathrm{Tri}^{(n-1)}(Z_1) - \frac{2C_{Z_2}(n-1-\epsilon)}{Z_2 A}\mathrm{Tri}^{(n-1)}(Z_2), \end{aligned} \tag{3.36}$$

where Z_1, Z_2 should be derived from (3.35) with reference to table (3.21), and C_{Z_1} and C_{Z_2} are given by

$$C_{Z_i} = D + (Z_i^2 - 1)C. \tag{3.37}$$

Putting it all together we find the recursion and reduction formulas.

$$\mathrm{Box}^{(n)}(A,B,C,D) = \frac{(n-1-\epsilon)}{(n-\frac{1}{2}-\epsilon)}\frac{B}{A}\mathrm{Box}^{(n-1)}(A,B,C,D) + \frac{1}{2(n-\frac{1}{2}-\epsilon)}T \tag{3.38}$$

$$\begin{aligned} &= \frac{(n-1-\epsilon)}{(n-\frac{1}{2}-\epsilon)}\frac{B}{A}\mathrm{Box}^{(n-1)}(A,B,C,D) \\ &\quad -\frac{(n-1-\epsilon)C_{Z_1}}{(n-\frac{1}{2}-\epsilon)A\ Z_1}\mathrm{Tri}^{(n-1)}(Z_1) - \frac{(n-1-\epsilon)C_{Z_2}}{(n-\frac{1}{2}-\epsilon)A\ Z_2}\mathrm{Tri}^{(n-1)}(Z_2). \end{aligned} \tag{3.39}$$

Solving (3.39) we get the following final result for the recursion and reduction relation. To use this formula, it is of course necessary to understand the cut box integrals as being defined in terms of the underlying arguments K, P_1, P_2, so that it is possible to find the necessary Z_1, Z_2 for reduction relations.

$$\begin{aligned} \mathrm{Box}^{(n)}(A,B,C,D) &= F_{4\to4}^{(n)}(A,B)\mathrm{Box}^{(0)}(A,B,C,D) + \widetilde{F}_{4\to3}^{(n)}(A,B,C,D;Z_1)\mathrm{Tri}^{(0)}(Z_1) \\ &\quad + \widetilde{F}_{4\to3}^{(n)}(A,B,C,D;Z_2)\mathrm{Tri}^{(0)}(Z_2) + \widetilde{F}_{4\to2}^{(n)}(A,B,C,D;Z_i)\mathrm{Bub}^{(0)}, \end{aligned} \tag{3.40}$$

where these form factors are given by

$$F_{4\to4}^{(n)}(A,B) = \frac{\Gamma(1/2-\epsilon)\Gamma(n-\epsilon)}{\Gamma(-\epsilon)\Gamma(n+1/2-\epsilon)}\left(\frac{B}{A}\right)^n, \tag{3.41}$$

$$\widetilde{F}_{4\to3}^{(n)}(A,B,C,D;Z_i) = -\frac{\Gamma(n-\epsilon)}{\Gamma(n+1/2-\epsilon)}\frac{C_{Z_i}}{A\ Z_i}\sum_{k=1}^{n}\frac{\Gamma(k-1/2-\epsilon)}{\Gamma(k-1-\epsilon)}\left(\frac{B}{A}\right)^{n-k}F_{3\to3}^{(k-1)}(Z_i), \tag{3.42}$$

$$\begin{aligned} \widetilde{F}_{4\to2}^{(n)}(A,B,C,D;Z_i) &= -\frac{\Gamma(n-\epsilon)}{\Gamma(n+1/2-\epsilon)}\frac{1}{A} \\ &\quad \times\sum_{k=1}^{n}\frac{\Gamma(k-1/2-\epsilon)}{\Gamma(k-1-\epsilon)}\left(\frac{B}{A}\right)^{n-k}\left(\frac{C_{Z_1}}{Z_1}\widetilde{F}_{3\to2}^{(k-1)}(Z_1)+\frac{C_{Z_2}}{Z_2}\widetilde{F}_{3\to2}^{(k-1)}(Z_2)\right). \end{aligned} \tag{3.43}$$

Again (3.40) is not the final formula we are after. To get the proper physical result, we need to replace the kinematic factor $1/2K^2$. The result is

$$\int_0^1 du\; u^{-1-\epsilon}u^n \left[\frac{1}{2K^2\sqrt{B-Au}}\ln\left(\frac{D-Cu+\sqrt{1-u}\sqrt{B-Au}}{D-Cu-\sqrt{1-u}\sqrt{B-Au}}\right)\right] = \tag{3.44}$$
$$F^{(n)}_{4\to4}(A,B)C[I_4(K;P_1,P_2)] + \sum_{i=1}^{2} F^{(n)}_{4\to3}(A,B,C,D;Z_i)C[I_3(K_1^{(i)},K_3^{(i)})] +$$
$$+F^{(n)}_{4\to2}(A,B,C,D;Z_i)C[I_2(K)],$$

where for the triangles, $K_1^{(i)}$ and $K_3^{(i)}$ are given by table (3.21), and the form factors are

$$F^{(n)}_{4\to3}(A,B,C,D;Z_i) = -\frac{\sqrt{\Delta_3^{(i)}}}{2K^2}\;\widetilde{F}^{(n)}_{4\to3}(A,B,C,D;Z_i),$$
$$F^{(n)}_{4\to2}(A,B,C,D;Z_i) = \frac{1}{2K^2}\;\widetilde{F}^{(n)}_{4\to2}(A,B,C,D;Z_i). \tag{3.45}$$

3.4 Cut pentagons

The double cut will be

$$\frac{\delta(\widetilde{\ell}^2-\mu^2)\delta((\widetilde{\ell}-K)^2-\mu^2)}{((\widetilde{\ell}-P_1)^2-\mu^2)((\widetilde{\ell}-P_2)^2-\mu^2)((\widetilde{\ell}-P_3)^2-\mu^2)}.$$

Using $\widetilde{\ell}=\ell+zK$ and doing the t-integration we reach

$$C[I_5(K;P_1,P_2,P_3)] = \int_0^1 du\; u^{-1-\epsilon}\int \langle \ell\; d\ell\rangle\,[\ell\; d\ell]\frac{(1-2z)}{(K^2)^2}\frac{\langle\ell|K|\ell]}{\langle\ell|Q_1|\ell]\,\langle\ell|Q_2|\ell]\,\langle\ell|Q_3|\ell]}, \tag{3.46}$$

with

$$Q_i = -(1-2z)P_i + \frac{P_i^2-2zP_i\cdot K}{K^2}K. \tag{3.47}$$

Now we do the splitting and get

$$C[I_5(K;P_1,P_2,P_3)] = \int_0^1 du\; u^{-1-\epsilon}\int \langle \ell\; d\ell\rangle\,[\ell\; d\ell]\frac{(1-2z)}{(K^2)^2}(I_1+I_2+I_3)$$
$$I_1 = \frac{\langle\ell|KQ_1|\ell\rangle^2}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_1|\ell\rangle}\frac{1}{\langle\ell|K|\ell]\,\langle\ell|Q_1|\ell]}$$
$$I_2 = -\frac{\langle\ell|KQ_2|\ell\rangle^2}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_2|\ell\rangle}\frac{1}{\langle\ell|K|\ell]\,\langle\ell|Q_2|\ell]}$$
$$I_3 = \frac{\langle\ell|KQ_3|\ell\rangle^2}{\langle\ell|Q_3Q_2|\ell\rangle\,\langle\ell|Q_3Q_1|\ell\rangle}\frac{1}{\langle\ell|K|\ell]\,\langle\ell|Q_3|\ell]}.$$

First we use Feynman parametrization and then write the integrand as a total derivative [8]. Next we do the Feynman parameter integration and finally we read out the pole contribution.[6]

[6] In general we need to be careful about changing the order of taking residues and doing the Feynman parameter integration. One can check that in the case here, it is legitimate to do the Feynman parameter integral first.

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The general integration has been done in appendix C (equation (C.20)). The result can be summarized as the following replacement:

$$\frac{1}{\langle\ell|K|\ell]\,\langle\ell|Q|\ell]} \to -\frac{1}{\langle\ell|QK|\ell\rangle}\ln\left(-\frac{x_+\,\langle\ell|K|\ell]}{\langle\ell|Q|\ell]}\right),$$

where x_+ is one solution to the equation $(Q+xK)^2=0$. First, notice that after summing up residues of all poles, the term with $\ln(-x_+)$ will not contribute, because the sum of all residues of a holomorphic function is zero. After dropping it we have

$$\begin{aligned}
C[I_5(K;P_1,P_2,P_3)] &= \int_0^1 du\; u^{-1-\epsilon}\frac{(1-2z)}{(K^2)^2}\;(I_1+I_2+I_3)|_{\text{residue}}\,,\\
I_1 &= \frac{\langle\ell|KQ_1|\ell\rangle}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_1|\ell\rangle}\ln\left(\frac{\langle\ell|K|\ell]}{\langle\ell|Q_1|\ell]}\right),\\
I_2 &= -\frac{\langle\ell|KQ_2|\ell\rangle}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_2|\ell\rangle}\ln\left(\frac{\langle\ell|K|\ell]}{\langle\ell|Q_2|\ell]}\right),\\
I_3 &= \frac{\langle\ell|KQ_3|\ell\rangle}{\langle\ell|Q_3Q_2|\ell\rangle\,\langle\ell|Q_3Q_1|\ell\rangle}\ln\left(\frac{\langle\ell|K|\ell]}{\langle\ell|Q_3|\ell]}\right),
\end{aligned}$$

where $|_{\text{residue}}$ means that we take residues of the two poles in $\langle\ell|Q_iQ_j|\ell\rangle$. This type of pole is discussed in detail in appendix C. Here, we apply the trick given by (C.4). More concretely, we have

$$\begin{aligned}
\langle\ell|Q_iQ_j|\ell\rangle &= \frac{\langle\ell\ P_{ij,+}\rangle\,\langle\ell\ P_{ij,-}\rangle\,[P_{ij,+}\ P_{ij,-}]}{x_{ij,+}-x_{ij,-}},\\
x_{ij,\pm} &= \frac{-Q_i\cdot Q_j\pm\sqrt{(Q_i\cdot Q_j)^2-Q_i^2Q_j^2}}{Q_j^2}.
\end{aligned}\tag{3.48}$$

Thus after taking the residue, we will have the following replacement

$$\frac{1}{\langle\ell|Q_iQ_j|\ell\rangle}\to\frac{(x_{ij,+}-x_{ij,-})}{[P_{ij,+}\ P_{ij,-}]\,\langle P_{ij,+}\ P_{ij,-}\rangle}=-\frac{1}{2\sqrt{(Q_i\cdot Q_j)^2-Q_i^2Q_j^2}}.\tag{3.49}$$

Now, applying it to the poles from $\langle\ell|Q_2Q_1|\ell\rangle$, we need to add the two contributions from I_1 and I_2. Defining

$$F_{12}(\ell)=\frac{\langle\ell|KQ_1|\ell\rangle}{\langle\ell|Q_3Q_1|\ell\rangle},\qquad \widetilde{F}_{12}(\ell)=-\frac{\langle\ell|KQ_2|\ell\rangle}{\langle\ell|Q_3Q_2|\ell\rangle},$$

it is easy to check that $F_{12}(P_{12})+\widetilde{F}_{12}(P_{12})=0$. This identity is important for the cancellation of unphysical singularities.

The sum of the I_1 and I_2 contributions for the poles from $\langle \ell | Q_2 Q_1 | \ell \rangle$ is given by

$$
\begin{aligned}
-\frac{1}{2\sqrt{(Q_2\cdot Q_1)^2-Q_2^2Q_1^2}}\Bigg\{&\frac{F_{12}(P_{12,+})+F_{12}(P_{12,-})}{2}\ln\left(\frac{\langle P_{12,+}|K|P_{12,+}]}{\langle P_{12,+}|Q_1|P_{12,+}]}\frac{\langle P_{12,-}|Q_1|P_{12,-}]}{\langle P_{12,-}|K|P_{12,-}]}\right)\\
&+\frac{F_{12}(P_{12,+})-F_{12}(P_{12,-})}{2}\ln\left(\frac{\langle P_{12,+}|K|P_{12,+}]}{\langle P_{12,+}|Q_1|P_{12,+}]}\frac{\langle P_{12,-}|K|P_{12,-}]}{\langle P_{12,-}|Q_1|P_{12,-}]}\right)\\
&+\frac{\widetilde{F}_{12}(P_{12,+})+\widetilde{F}_{12}(P_{12,-})}{2}\ln\left(\frac{\langle P_{12,+}|K|P_{12,+}]}{\langle P_{12,+}|Q_2|P_{12,+}]}\frac{\langle P_{12,-}|Q_2|P_{12,-}]}{\langle P_{12,-}|K|P_{12,-}]}\right)\\
&+\frac{\widetilde{F}_{12}(P_{12,+})-\widetilde{F}_{12}(P_{12,-})}{2}\ln\left(\frac{\langle P_{12,+}|K|P_{12,+}]}{\langle P_{12,+}|Q_2|P_{12,+}]}\frac{\langle P_{12,-}|K|P_{12,-}]}{\langle P_{12,-}|Q_2|P_{12,-}]}\right)\Bigg\}.
\end{aligned}
$$

Using the relation $F_{12}(P_{12})+\widetilde{F}_{12}(P_{12})=0$ we can combine the first term and third term as well as second term and fourth term. Next we use (C.13) and (C.14) to get the final result

$$
\begin{aligned}
-\frac{1}{2\sqrt{(Q_2\cdot Q_1)^2-Q_2^2Q_1^2}}\Bigg(&\frac{F_{12}(P_{12,+})+F_{12}(P_{12,-})}{2}\ln\frac{Q_2\cdot Q_1-\sqrt{(Q_2\cdot Q_1)^2-Q_2^2Q_1^2}}{Q_2\cdot Q_1+\sqrt{(Q_2\cdot Q_1)^2-Q_2^2Q_1^2}}\\
&+\frac{F_{12}(P_{12,+})-F_{12}(P_{12,-})}{2}\ln\left(\frac{Q_2^2}{Q_1^2}\right)\Bigg),
\end{aligned}
$$

where the first term has the right physical singularity while the second term does not. However, we can rewrite the second term as

$$
\begin{aligned}
-\frac{1}{2\sqrt{(Q_2\cdot Q_1)^2-Q_2^2Q_1^2}}\Bigg(&\frac{F_{12}(P_{12,+})-F_{12}(P_{12,-})}{2}\ln(Q_1^2)^{-1}\\
&+\frac{\widetilde{F}_{12}(P_{12,+})-\widetilde{F}_{12}(P_{12,-})}{2}\ln(Q_2^2)^{-1}\Bigg).
\end{aligned}
$$

Now we recognize that it is the residue of the following expression:

$$
\left(\frac{\langle \ell|KQ_1|\ell\rangle}{\langle \ell|Q_2Q_1|\ell\rangle\,\langle \ell|Q_3Q_1|\ell\rangle}\ln(Q_1^2)^{-1}-\frac{\langle \ell|KQ_2|\ell\rangle}{\langle \ell|Q_2Q_1|\ell\rangle\,\langle \ell|Q_3Q_2|\ell\rangle}\ln(Q_2^2)^{-1}\right)\Bigg|_{\text{residues from }\langle \ell|Q_2Q_1|\ell\rangle}.
$$

Because it is a holomorphic function, when we sum up all residues we get zero. This illustrates how the unphysical singularities cancel out in the final result, so that we are left with only the physical cut structure.

By similar manipulations we identify the pole contribution of $\langle \ell | Q_3 Q_1 | \ell \rangle$ as

$$
-\frac{1}{2\sqrt{(Q_3\cdot Q_1)^2-Q_3^2Q_1^2}}\frac{F_{13}(P_{13,+})+F_{13}(P_{13,-})}{2}\ln\frac{Q_3\cdot Q_1-\sqrt{(Q_3\cdot Q_1)^2-Q_3^2Q_1^2}}{Q_3\cdot Q_1+\sqrt{(Q_3\cdot Q_1)^2-Q_3^2Q_1^2}},
$$

and the contribution of $\langle \ell | Q_3 Q_2 | \ell \rangle$ as

$$
-\frac{1}{2\sqrt{(Q_3\cdot Q_2)^2-Q_3^2Q_2^2}}\frac{F_{32}(P_{32,+})+F_{32}(P_{32,-})}{2}\ln\frac{Q_3\cdot Q_2-\sqrt{(Q_3\cdot Q_2)^2-Q_3^2Q_2^2}}{Q_3\cdot Q_2+\sqrt{(Q_3\cdot Q_2)^2-Q_3^2Q_2^2}},
$$

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where

$$F_{13}(\ell) = \frac{\langle \ell | K Q_1 | \ell \rangle}{\langle \ell | Q_2 Q_1 | \ell \rangle}, \qquad F_{32}(\ell) = -\frac{\langle \ell | K Q_2 | \ell \rangle}{\langle \ell | Q_2 Q_1 | \ell \rangle}.$$

Collecting everything together we have

$$\begin{aligned} C[I_5(K;P_1,P_2,P_3)] = &-\int_0^1 du\ u^{-1-\epsilon} \frac{(1-2z)}{(K^2)^2} \\ &\left(\frac{1}{2\sqrt{(Q_3\cdot Q_2)^2 - Q_3^2 Q_2^2}} \frac{F_{32}(P_{32,+}) + F_{32}(P_{32,-})}{2} \ln \frac{Q_3\cdot Q_2 - \sqrt{(Q_3\cdot Q_2)^2 - Q_3^2 Q_2^2}}{Q_3\cdot Q_2 + \sqrt{(Q_3\cdot Q_2)^2 - Q_3^2 Q_2^2}} \right. \\ &+ \frac{1}{2\sqrt{(Q_3\cdot Q_1)^2 - Q_3^2 Q_1^2}} \frac{F_{13}(P_{13,+}) + F_{13}(P_{13,-})}{2} \ln \frac{Q_3\cdot Q_1 - \sqrt{(Q_3\cdot Q_1)^2 - Q_3^2 Q_1^2}}{Q_3\cdot Q_1 + \sqrt{(Q_3\cdot Q_1)^2 - Q_3^2 Q_1^2}} \\ &\left. + \frac{1}{2\sqrt{(Q_2\cdot Q_1)^2 - Q_2^2 Q_1^2}} \frac{F_{12}(P_{12,+}) + F_{12}(P_{12,-})}{2} \ln \frac{Q_2\cdot Q_1 - \sqrt{(Q_2\cdot Q_1)^2 - Q_2^2 Q_1^2}}{Q_2\cdot Q_1 + \sqrt{(Q_2\cdot Q_1)^2 - Q_2^2 Q_1^2}} \right). \end{aligned}$$

To write the result in a good form, refer to the spinor algebra in appendix C. There, in equation (C.17), we define the following function:

$$S[Q_i, Q_j, Q_k, K] = \frac{T_1}{T_2}, \tag{3.50}$$

with

$$T_1 = -8 \det \begin{pmatrix} K\cdot Q_k & Q_i\cdot K & Q_j\cdot K \\ Q_i\cdot Q_k & Q_i^2 & Q_i\cdot Q_j \\ Q_j\cdot Q_k & Q_i\cdot Q_j & Q_j^2 \end{pmatrix};\ T_2 = -4 \det \begin{pmatrix} Q_k^2 & Q_i\cdot Q_k & Q_j\cdot Q_k \\ Q_i\cdot Q_k & Q_i^2 & Q_i\cdot Q_j \\ Q_j\cdot Q_k & Q_i\cdot Q_j & Q_j^2 \end{pmatrix}. \tag{3.51}$$

Our result can then be written as

$$\begin{aligned} C[I_5(K;P_1,P_2,P_3)] = &-\int_0^1 du\ u^{-1-\epsilon} \frac{\sqrt{1-u}}{(K^2)^2} \\ &\left(\frac{S[Q_3,Q_2,Q_1,K]}{4\sqrt{(Q_3\cdot Q_2)^2 - Q_3^2 Q_2^2}} \ln \frac{Q_3\cdot Q_2 - \sqrt{(Q_3\cdot Q_2)^2 - Q_3^2 Q_2^2}}{Q_3\cdot Q_2 + \sqrt{(Q_3\cdot Q_2)^2 - Q_3^2 Q_2^2}} \right. \\ &+ \frac{S[Q_3,Q_1,Q_2,K]}{4\sqrt{(Q_3\cdot Q_1)^2 - Q_3^2 Q_1^2}} \ln \frac{Q_3\cdot Q_1 - \sqrt{(Q_3\cdot Q_1)^2 - Q_3^2 Q_1^2}}{Q_3\cdot Q_1 + \sqrt{(Q_3\cdot Q_1)^2 - Q_3^2 Q_1^2}} \\ &\left. + \frac{S[Q_2,Q_1,Q_3,K]}{4\sqrt{(Q_2\cdot Q_1)^2 - Q_2^2 Q_1^2}} \ln \frac{Q_2\cdot Q_1 - \sqrt{(Q_2\cdot Q_1)^2 - Q_2^2 Q_1^2}}{Q_2\cdot Q_1 + \sqrt{(Q_2\cdot Q_1)^2 - Q_2^2 Q_1^2}} \right). \end{aligned} \tag{3.52}$$

It is important to notice that $C[I_5]$ is given by three different box cuts $C[I_4]$ multiplying corresponding factors $\frac{S[\bullet]}{2K^2}$. Thus the factor $S[\bullet]$, especially its denominator T_2, which is the same for all three $S[\bullet]$, can be considered as another signature of a pentagon.

The reduction of a pentagon is very easy. Suppose that we find the integral

$$\int_0^1 du\ u^{-1-\epsilon} u^n \sum_{i=1}^{3} \frac{S[i]}{2K^2} C[I_4^{(i)}].$$

Then what we need to do is expand $\frac{u^n}{T_2}$ (where T_2 is the denominator of $S[\bullet]$) in the form $f[u] + \frac{A}{T_2}$, where $f[u]$ is a polynomial in u and A is constant in u. That is, we write

$$\int_0^1 du\ u^{-1-\epsilon} u^n \sum_{i=1}^{3} \frac{S[i]}{2K^2} C[I_4^{(i)}] = \int_0^1 du\ u^{-1-\epsilon} u^n \sum_{i=1}^{3} \frac{T_1[i]}{2K^2 T_2} C[I_4^{(i)}]$$
$$= \int_0^1 du\ u^{-1-\epsilon} \sum_{i=1}^{3} \frac{f(u) T_1[i]}{2K^2} C[I_4^{(i)}] + A \sum_{i=1}^{3} \frac{S[i]}{2K^2} C[I_4^{(i)}].$$

Then A is the coefficient of the true pentagon, and $f(u)T_1[i]/2K^2$ are the reductive coefficients of the corresponding boxes.

4. Examples

Here we present a few basic examples to illustrate the main points of our technique. We begin with the case of five gluons of positive helicity. Later we list the u-integrals for four-gluon amplitudes, just to show the structures that arise. These amplitudes were first computed to all orders in ϵ in [27].

4.1 Five gluons of positive helicity

In this section, we demonstrate the cut integration for the all-plus helicity configuration of the five-gluon amplitude. All cuts are of course trivially related by permutation symmetry. Here we work with the cut C_{12}. We show the calculation in some detail to illustrate our method.

The cut momentum is K_{12}, and we begin with the integrand

$$I = 2A_L((-\ell_1), 1, 2, (-\ell_2)) A_R(\ell_2, 3, 4, 5, \ell_1)$$
$$= 2 \frac{\mu^2 [1\ 2]}{\langle 1\ 2\rangle\, ((-\ell_1 + k_1)^2 - \mu^2)} \frac{\mu^2 [5|K_{345}\ell_2|3]}{\langle 3\ 4\rangle\, \langle 4\ 5\rangle\, ((\ell_2 + k_3)^2 - \mu^2)((\ell_1 + k_5)^2 - \mu^2)}.$$

The letter I will actually represent the full integral; we neglect to write the integral signs and measures while we follow the steps. Notice that with our choice of direction of the propagator momentum, we have $\ell_2 = K_{12} - \ell_1$, in keeping with the convention in (2.17). After performing the t-integration and substituting $\mu^2 = z(1-z)s_{12}$, we have

$$I = -\frac{2z^2(1-z)^2(1-2z)[1\ 2]^2}{\langle 3\ 4\rangle\, \langle 4\ 5\rangle} (I_1 + I_2), \tag{4.1}$$

$$I_1 = \frac{z[5\ 3]\, \langle \ell | K_{12} | \ell]}{\langle \ell | Q_1 | \ell]\, \langle \ell | Q_2 | \ell]\, \langle \ell | Q_3 | \ell]}, \tag{4.2}$$

$$I_2 = -\frac{(1-2z)\, \langle \ell | K_{12} | 3]\, [\ell\ 5]}{\langle \ell | Q_1 | \ell]\, \langle \ell | Q_2 | \ell]\, \langle \ell | Q_3 | \ell]}, \tag{4.3}$$

with

$$Q_1 = (1-2z)k_1 + zK_{12}, \tag{4.4}$$

$$Q_2 = (1-2z)k_5 + \frac{2z(K_{12}\cdot k_5)}{K_{12}^2}K_{12}, \tag{4.5}$$

$$Q_3 = -(1-2z)k_3 + \frac{(1-z)2k_3\cdot K_{12}}{K_{12}^2}K_{12}. \tag{4.6}$$

After splitting the denominator factors with partial fractions, we can write the integral as a sum of three terms, related to one another by permuting Q_1, Q_2, Q_3.

$$\begin{aligned}
I &= P_1 + P_2 + P_3,\\
P_1 &= -\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\langle 3\ 4\rangle\langle 4\ 5\rangle}\frac{\langle\ell|K_{12}Q_1|\ell\rangle\,(z[5\ 3]\,\langle\ell|K_{12}Q_1|\ell\rangle + (1-2z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_1|5])}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_1|\ell\rangle}\times\\
&\qquad\times\frac{1}{\langle\ell|K_{12}|\ell]\,\langle\ell|Q_1|\ell]},\\
P_2 &= \frac{u^2\sqrt{1-u}[1\ 2]^2}{8\langle 3\ 4\rangle\langle 4\ 5\rangle}\frac{\langle\ell|K_{12}Q_2|\ell\rangle\,(z[5\ 3]\,\langle\ell|K_{12}Q_2|\ell\rangle + (1-2z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_2|5])}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_2|\ell\rangle}\times\\
&\qquad\times\frac{1}{\langle\ell|K_{12}|\ell]\,\langle\ell|Q_2|\ell]},\\
P_3 &= -\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\langle 3\ 4\rangle\langle 4\ 5\rangle}\frac{\langle\ell|K_{12}Q_3|\ell\rangle\,(z[5\ 3]\,\langle\ell|K_{12}Q_3|\ell\rangle + (1-2z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_3|5])}{\langle\ell|Q_3Q_1|\ell\rangle\,\langle\ell|Q_3Q_2|\ell\rangle}\times\\
&\qquad\times\frac{1}{\langle\ell|K_{12}|\ell]\,\langle\ell|Q_3|\ell]}.
\end{aligned}$$

4.1.1 Spinor integration

Let us start with P_1. Upon writing it as a total derivative and choosing the auxiliary spinor to be λ_1, we get

$$\begin{aligned}
P_1 = -\int_0^1 dx\frac{u^2(1-u)[1\ 2]^2}{8\langle 3\ 4\rangle\langle 4\ 5\rangle}&\frac{(z[5\ 3]\,\langle\ell|K_{12}Q_1|\ell\rangle + (1-2z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_1|5])}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_1|\ell\rangle}\times\\
&\times\frac{[1\ \ell]\,\langle 1\ \ell\rangle}{\langle\ell|R_1|\ell]\,(xz+1-x)},
\end{aligned}$$

with

$$R_1 = xQ_1 + (1-x)K_{12}.$$

There are four single poles from the factors $\langle\ell|Q_2Q_1|\ell\rangle$ and $\langle\ell|Q_3Q_1|\ell\rangle$. We can do the x-integration first. Then we get

$$\begin{aligned}
P_1 &= -\frac{u^2(1-u)[1\ 2]^2}{8\langle 3\ 4\rangle\langle 4\ 5\rangle}\frac{(z[5\ 3]\,\langle\ell|K_{12}Q_1|\ell\rangle + (1-2z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_1|5])}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_1|\ell\rangle}\times\\
&\qquad\times\frac{[1\ \ell]\,\langle 1\ \ell\rangle}{\sqrt{1-u}\,\langle\ell|k_1|\ell]}\left(\ln(z) - \ln\frac{\langle\ell|Q_1|\ell]}{\langle\ell|K_{12}|\ell]}\right)\\
&= \frac{u^2\sqrt{1-u}[1\ 2]K_{12}^2}{8\langle 1\ 2\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle}\frac{(z[5\ 3]\,\langle\ell|K_{12}Q_1|\ell\rangle + (1-2z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_1|5])}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_1|\ell\rangle}\ln\frac{z\,\langle\ell|K_{12}|\ell]}{\langle\ell|Q_1|\ell]}.
\end{aligned}$$

The next thing is to read out the residues of these four single poles. Notice that since we can write

$$\ln \frac{z\,\langle \ell|K_{12}|\ell]}{\langle \ell|Q_1|\ell]} = \ln z + \ln \frac{\langle \ell|K_{12}|\ell]}{\langle \ell|Q_1|\ell]},$$

and since the sum of residues of a holomorphic function is zero, we get

$$P_1 = \left(\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle 3\ 4\rangle\,\langle 4\ 5\rangle} \frac{(z[5\ 3]\,\langle \ell|K_{12}Q_1|\ell\rangle + (1-2z)\,\langle \ell|K_{12}|3]\,\langle \ell|Q_1|5])}{\langle \ell|Q_2Q_1|\ell\rangle\,\langle \ell|Q_3Q_1|\ell\rangle} \ln \frac{\langle \ell|K_{12}|\ell]}{\langle \ell|Q_1|\ell]} \right)\Big|_{\text{residues}}.$$

Similar calculations give

$$P_2 = \left(-\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle 3\ 4\rangle\,\langle 4\ 5\rangle} \frac{(z[5\ 3]\,\langle \ell|K_{12}Q_2|\ell\rangle + (1-2z)\,\langle \ell|K_{12}|3]\,\langle \ell|Q_2|5])}{\langle \ell|Q_2Q_1|\ell\rangle\,\langle \ell|Q_3Q_2|\ell\rangle} \ln \frac{\langle \ell|K_{12}|\ell]}{\langle \ell|Q_2|\ell]} \right)\Big|_{\text{residues}}$$

and

$$P_3 = \left(\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle 3\ 4\rangle\,\langle 4\ 5\rangle} \frac{(z[5\ 3]\,\langle \ell|K_{12}Q_3|\ell\rangle + (1-2z)\,\langle \ell|K_{12}|3]\,\langle \ell|Q_3|5])}{\langle \ell|Q_3Q_1|\ell\rangle\,\langle \ell|Q_3Q_2|\ell\rangle} \ln \frac{\langle \ell|K_{12}|\ell]}{\langle \ell|Q_3|\ell]} \right)\Big|_{\text{residues}}.$$

4.1.2 Taking the residues

Now we need to take the residues of P_i. Again we start with P_1. The pole contribution of $\langle \ell|Q_2Q_1|\ell\rangle$ can be written as

$$\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_2Q_1|2\rangle} \frac{1}{\langle \eta_1\ \eta_2\rangle} \left(F_1(\eta_1) \ln \frac{\langle \eta_1|K_{12}|\eta_1]}{\langle \eta_1|Q_1|\eta_1]} - F_1(\eta_2) \ln \frac{\langle \eta_2|K_{12}|\eta_2]}{\langle \eta_2|Q_1|\eta_2]} \right),$$

where η_1, η_2 are the two solutions of $\langle \ell|Q_2Q_1|\ell\rangle = 0$ (see appendix C), and $F_1(\eta)$ is defined as

$$F_1(\eta) = \frac{(z[5\ 3]\,\langle \eta|K_{12}Q_1|\eta\rangle + (1-2z)\,\langle \eta|K_{12}|3]\,\langle \eta|Q_1|5])}{\langle \eta|Q_3Q_1|\eta\rangle}. \tag{4.7}$$

Decompose $F(\eta)$ into two pieces that are respectively symmetric and antisymmetric under the exchange $\eta_1 \leftrightarrow \eta_2$:

$$F_1(\eta_1) = F_1^S + F_1^A, \qquad F_1(\eta_2) = F_1^S - F_1^A, \tag{4.8}$$

and the pole contribution can be written as

$$\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_2Q_1|2\rangle} \frac{1}{\langle \eta_1\ \eta_2\rangle} \left(F_1^S \ln \frac{\langle \eta_1|K_{12}|\eta_1]}{\langle \eta_1|Q_1|\eta_1]} \frac{\langle \eta_2|Q_1|\eta_2]}{\langle \eta_2|K_{12}|\eta_2]} + F_1^A \ln \frac{\langle \eta_1|K_{12}|\eta_1]}{\langle \eta_1|Q_1|\eta_1]} \frac{\langle \eta_2|K_{12}|\eta_2]}{\langle \eta_2|Q_1|\eta_2]} \right).$$

We substitute the solutions η_1, η_2 and define

$$\widetilde{A} \equiv \frac{s_{25}}{s_{15}}. \tag{4.9}$$

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Then we find

$$\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_2Q_1|2\rangle}\frac{1}{\langle \eta_1\ \eta_2\rangle}\left(F_1^S\ln\frac{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}+F_1^A\ln\frac{4(1+\widetilde{A})}{u(1+\widetilde{A}u)}\right)=$$
$$\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_2Q_1|2\rangle}\frac{1}{\langle \eta_1\ \eta_2\rangle}\Bigg(\frac{F_1(\eta_1)+F_1(\eta_2)}{2}\ln\frac{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}$$
$$+\frac{F_1(\eta_1)-F_1(\eta_2)}{2}\ln\frac{4(1+\widetilde{A})}{u(1+\widetilde{A}u)}\Bigg).$$

Notice that the second term can be interpreted as the pole contribution of $\langle\ell|Q_2Q_1|\ell\rangle$. This observation will be useful to prove that the sum of all pole contributions is zero.

Now we consider the contribution of $\langle\ell|Q_3Q_1|\ell\rangle$. Using η_3,η_4 as the solutions with

$$\widetilde{B}\equiv\left(\frac{s_{13}}{s_{23}}\right)^2, \tag{4.10}$$

and the definition

$$F_2(\eta)\equiv\frac{z[5\ 3]\,\langle\eta|K_{12}Q_1|\eta\rangle+(1-2z)\,\langle\eta|K_{12}|3]\,\langle\eta|Q_1|5]}{\langle\eta|Q_2Q_1|\eta\rangle}, \tag{4.11}$$

we get

$$\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_3Q_1|2\rangle}\frac{1}{\langle \eta_3\ \eta_4\rangle}\Bigg(\frac{F_2(\eta_3)+F_2(\eta_4)}{2}\ln\frac{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}}+$$
$$\frac{F_2(\eta_3)-F_2(\eta_4)}{2}\ln\frac{4(1+\widetilde{B})}{u(1+\widetilde{B}u)}\Bigg). \tag{4.12}$$

Putting everything together, we have

$$P_1=\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle 3\ 4\rangle\,\langle 4\ 5\rangle}$$
$$\Bigg(\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_2Q_1|2\rangle}\frac{1}{\langle \eta_1\ \eta_2\rangle}\Bigg(\frac{F_1(\eta_1)+F_1(\eta_2)}{2}\ln\frac{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}+$$
$$\frac{F_1(\eta_1)-F_1(\eta_2)}{2}\ln\frac{4(1+\widetilde{A})}{u(1+\widetilde{A}u)}\Bigg)$$
$$+\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_3Q_1|2\rangle}\frac{1}{\langle \eta_3\ \eta_4\rangle}\Bigg(\frac{F_2(\eta_3)+F_2(\eta_4)}{2}\ln\frac{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}}+$$
$$\frac{F_2(\eta_3)-F_2(\eta_4)}{2}\ln\frac{4(1+\widetilde{B})}{u(1+\widetilde{B}u)}\Bigg)\Bigg).$$

Let us examine the various terms in this expression. Some of the singularities are spurious. First, the terms with $\ln\frac{4}{u}$ can be written as

$$\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle 3\ 4\rangle\,\langle 4\ 5\rangle}\ln\frac{4}{u}\left(\frac{z[5\ 3]\,\langle\ell|K_{12}Q_1|\ell\rangle+(1-2z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_1|5]}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_1|\ell\rangle}\right)\Bigg|_{\text{residues}}.$$

Since the function is holomorphic, the sum of the residues of all poles will be zero, so these terms can be discarded collectively.

Then we are left with

$$
\begin{aligned}
P_1 = & \frac{u^2\sqrt{1-u}[1\ 2]^2}{8\langle 3\ 4\rangle\langle 4\ 5\rangle}\left(\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_2Q_1|2\rangle}\frac{1}{\langle \eta_1\ \eta_2\rangle}\frac{F_1(\eta_1)+F_1(\eta_2)}{2}\ln\frac{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}\right. \\
& +\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_3Q_1|2\rangle}\frac{1}{\langle \eta_3\ \eta_4\rangle}\frac{F_2(\eta_3)+F_2(\eta_4)}{2}\ln\frac{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}} \\
& +\frac{1}{2}\left(\frac{z[5\ 3]\langle\ell|K_{12}Q_1|\ell\rangle+(1-2z)\langle\ell|K_{12}|3]\langle\ell|Q_1|5]}{\langle\ell|Q_3Q_1|\ell\rangle\langle\ell|Q_2Q_1|\ell\rangle}\right)\Big|_{\langle\ell|Q_2Q_1|\ell\rangle=0}\ln\left(\frac{1+\widetilde{A}}{1+\widetilde{A}u}\right) \\
& \left.+\frac{1}{2}\left(\frac{z[5\ 3]\langle\ell|K_{12}Q_1|\ell\rangle+(1-2z)\langle\ell|K_{12}|3]\langle\ell|Q_1|5]}{\langle\ell|Q_3Q_1|\ell\rangle\langle\ell|Q_2Q_1|\ell\rangle}\right)\Big|_{\langle\ell|Q_3Q_1|\ell\rangle=0}\ln\left(\frac{1+\widetilde{B}}{1+\widetilde{B}u}\right)\right).
\end{aligned}
\tag{4.13}
$$

In this expression we have written the last two terms in a form where the poles need to be substituted.

Similarly we have

$$
\begin{aligned}
P_2 = & -\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\langle 3\ 4\rangle\langle 4\ 5\rangle}\left(\frac{\langle 3\ 5\rangle^2}{\langle 5|Q_3Q_2|5\rangle}\frac{1}{\langle \eta_5\ \eta_6\rangle}\frac{F_4(\eta_5)+F_4(\eta_6)}{2}\ln\frac{\sqrt{1+\widetilde{C}u}+\sqrt{1-u}}{\sqrt{1+\widetilde{C}u}-\sqrt{1-u}}\right. \\
& +\frac{\langle 1\ 2\rangle^2}{\langle 2|Q_2Q_1|2\rangle}\frac{1}{\langle \eta_1\ \eta_2\rangle}\frac{F_3(\eta_1)+F_3(\eta_2)}{2}\ln\frac{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}} \\
& +\frac{1}{2}\left(\frac{z[5\ 3]\langle\ell|K_{12}Q_2|\ell\rangle+(1-2z)\langle\ell|K_{12}|3]\langle\ell|Q_2|5]}{\langle\ell|Q_2Q_1|\ell\rangle\langle\ell|Q_3Q_2|\ell\rangle}\right)\Big|_{\langle\ell|Q_3Q_2|\ell\rangle=0}\ln\left(\frac{1+\widetilde{C}}{1+\widetilde{C}u}\right) \\
& \left.+\frac{1}{2}\left(\frac{z[5\ 3]\langle\ell|K_{12}Q_2|\ell\rangle+(1-2z)\langle\ell|K_{12}|3]\langle\ell|Q_2|5]}{\langle\ell|Q_2Q_1|\ell\rangle\langle\ell|Q_3Q_2|\ell\rangle}\right)\Big|_{\langle\ell|Q_2Q_1|\ell\rangle=0}\ln\left(\frac{1+\widetilde{A}}{1+\widetilde{A}u}\right)\right),
\end{aligned}
\tag{4.14}
$$

where

$$
\widetilde{C}\equiv\frac{s_{12}s_{35}}{s_{34}s_{45}} \tag{4.15}
$$

and

$$
F_3(\eta)=\frac{z[5\ 3]\langle\eta|K_{12}Q_2|\eta\rangle+(1-2z)\langle\eta|K_{12}|3]\langle\eta|Q_2|5]}{\langle\eta|Q_3Q_2|\eta\rangle}, \tag{4.16}
$$

$$
F_4(\eta)=\frac{z[5\ 3]\langle\eta|K_{12}Q_2|\eta\rangle+(1-2z)\langle\eta|K_{12}|3]\langle\eta|Q_2|5]}{\langle\eta|Q_2Q_1|\eta\rangle}. \tag{4.17}
$$

Similarly again for P_3, we have

$$
\begin{aligned}
P_3 = \frac{u^2\sqrt{1-u}[1\ 2]^2}{8\langle 3\ 4\rangle\langle 4\ 5\rangle}\Bigg(&\frac{\langle 3\ 5\rangle^2}{\langle 5|Q_3Q_2|5\rangle}\frac{1}{\langle \eta_5\ \eta_6\rangle}\frac{F_6(\eta_5)+F_6(\eta_6)}{2}\ln\frac{\sqrt{1+\widetilde{C}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{C}u}+\sqrt{1-u}} \\
&+\frac{\langle 1\ 2\rangle^2}{\langle \ell|Q_3Q_1|\ell\rangle}\frac{1}{\langle \eta_3\ \eta_4\rangle}\frac{F_5(\eta_3)+F_5(\eta_4)}{2}\ln\frac{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}} \\
&+\frac{1}{2}\left(\frac{z[5\ 3]\langle \ell|K_{12}Q_3|\ell\rangle+(1-2z)\langle \ell|K_{12}|3]\langle \ell|Q_3|5]}{\langle \ell|Q_3Q_1|\ell\rangle\langle \ell|Q_3Q_2|\ell\rangle}\right)\Bigg|_{\langle \ell|Q_3Q_1|\ell\rangle=0}\ln\left(\frac{1+\widetilde{B}}{1+\widetilde{B}u}\right) \\
&+\frac{1}{2}\left(\frac{z[5\ 3]\langle \ell|K_{12}Q_3|\ell\rangle+(1-2z)\langle \ell|K_{12}|3]\langle \ell|Q_3|5]}{\langle \ell|Q_3Q_1|\ell\rangle\langle \ell|Q_3Q_2|\ell\rangle}\right)\Bigg|_{\langle \ell|Q_3Q_2|\ell\rangle=0}\ln\left(\frac{1+\widetilde{C}}{1+\widetilde{C}u}\right)\Bigg),
\end{aligned}
\tag{4.18}
$$

where

$$
F_5(\eta)\equiv\frac{z[5\ 3]\langle \eta|K_{12}Q_3|\eta\rangle+(1-2z)\langle \eta|K_{12}|3]\langle \eta|Q_3|5]}{\langle \eta|Q_3Q_2|\eta\rangle},
\tag{4.19}
$$

$$
F_6(\eta)\equiv\frac{z[5\ 3]\langle \eta|K_{12}Q_3|\eta\rangle+(1-2z)\langle \eta|K_{12}|3]\langle \eta|Q_3|5]}{\langle \eta|Q_3Q_1|\eta\rangle}.
\tag{4.20}
$$

4.1.3 Summing up the result

Now we sum up P_1, P_2, P_3. First we check that the spurious singularities cancel out. For $\ln\frac{1+\widetilde{A}}{1+\widetilde{A}u}$ we get

$$
\begin{aligned}
&\left(\frac{z[5\ 3]\langle \ell|K_{12}Q_1|\ell\rangle+(1-2z)\langle \ell|K_{12}|3]\langle \ell|Q_1|5]}{\langle \ell|Q_3Q_1|\ell\rangle\langle \ell|Q_2Q_1|\ell\rangle}\right) \\
&\quad-\left(\frac{z[5\ 3]\langle \ell|K_{12}Q_2|\ell\rangle+(1-2z)\langle \ell|K_{12}|3]\langle \ell|Q_2|5]}{\langle \ell|Q_2Q_1|\ell\rangle\langle \ell|Q_3Q_2|\ell\rangle}\right) \\
&\qquad=\frac{z[5\ 3]\langle \ell|K_{12}Q_3|\ell\rangle\langle \ell|Q_1Q_2|\ell\rangle+(1-2z)\langle \ell|K_{12}|3]\langle \ell|Q_3|5]\langle \ell|Q_1Q_2|\ell\rangle}{\langle \ell|Q_3Q_1|\ell\rangle\langle \ell|Q_2Q_1|\ell\rangle\langle \ell|Q_3Q_2|\ell\rangle},
\end{aligned}
$$

where we should calculate only the pole contribution from $\langle \ell|Q_2Q_1|\ell\rangle$. However, the factor $\langle \ell|Q_1Q_2|\ell\rangle$ in the numerator shows us that the contribution is zero. Thus the singularity in $\ln\left(\frac{1+\widetilde{A}}{1+\widetilde{A}u}\right)$ disappears from the final result.

For $\ln\left(\frac{1+\widetilde{B}}{1+\widetilde{B}u}\right)$, we have

$$
\begin{aligned}
&\left(\frac{z[5\ 3]\langle \ell|K_{12}Q_1|\ell\rangle+(1-2z)\langle \ell|K_{12}|3]\langle \ell|Q_1|5]}{\langle \ell|Q_3Q_1|\ell\rangle\langle \ell|Q_2Q_1|\ell\rangle}\right)+ \\
&\quad\left(\frac{z[5\ 3]\langle \ell|K_{12}Q_3|\ell\rangle+(1-2z)\langle \ell|K_{12}|3]\langle \ell|Q_3|5]}{\langle \ell|Q_3Q_1|\ell\rangle\langle \ell|Q_3Q_2|\ell\rangle}\right) \\
&\qquad=\frac{\langle \ell|Q_3Q_1|\ell\rangle\left(z[5\ 3]\langle \ell|K_{12}Q_2|\ell\rangle+(1-z)\langle \ell|K_{12}|3]\langle \ell|Q_2|5]\right)}{\langle \ell|Q_3Q_1|\ell\rangle\langle \ell|Q_2Q_1|\ell\rangle\langle \ell|Q_3Q_2|\ell\rangle}.
\end{aligned}
$$

Again, the numerator factor $\langle \ell|Q_3Q_1|\ell\rangle$ tells us the sum is zero.

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For $\ln\left(\frac{1+\widetilde{C}}{1+\widetilde{C}u}\right)$, we have

$$\left(\frac{z[5\ 3]\,\langle\ell|K_{12}Q_2|\ell\rangle+(1-2z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_2|5]}{\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_2|\ell\rangle}\right)$$
$$+\left(\frac{z[5\ 3]\,\langle\ell|K_{12}Q_3|\ell\rangle+(1-2z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_3|5]}{\langle\ell|Q_3Q_1|\ell\rangle\,\langle\ell|Q_3Q_2|\ell\rangle}\right)$$
$$=-\frac{\langle\ell|Q_3Q_2|\ell\rangle\,(z[5\ 3]\,\langle\ell|K_{12}Q_1|\ell\rangle+(1-z)\,\langle\ell|K_{12}|3]\,\langle\ell|Q_1|5])}{\langle\ell|Q_3Q_1|\ell\rangle\,\langle\ell|Q_2Q_1|\ell\rangle\,\langle\ell|Q_3Q_2|\ell\rangle}.$$

Again, the numerator factor $\langle\ell|Q_3Q_2|\ell\rangle$ tells us the sum is zero.

Now we consider the remaining singularities. For the first factor we have

$$\begin{aligned}
I=&\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle3\ 4\rangle\,\langle4\ 5\rangle}\frac{\langle1\ 2\rangle^2}{\langle2|Q_2Q_1|2\rangle}\frac{1}{\langle\eta_1\ \eta_2\rangle}\times\\
&\qquad\times\frac{F_1(\eta_1)+F_1(\eta_2)+F_3(\eta_1)+F_3(\eta_2)}{2}\ln\frac{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}\\
&+\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle3\ 4\rangle\,\langle4\ 5\rangle}\frac{\langle1\ 2\rangle^2}{\langle2|Q_3Q_1|2\rangle}\frac{1}{\langle\eta_3\ \eta_4\rangle}\times\\
&\qquad\times\frac{F_2(\eta_3)+F_2(\eta_4)-F_5(\eta_3)+F_5(\eta_4)}{2}\ln\frac{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}}\\
&+\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle3\ 4\rangle\,\langle4\ 5\rangle}\frac{\langle3\ 5\rangle^2}{\langle5|Q_3Q_2|5\rangle}\frac{1}{\langle\eta_5\ \eta_6\rangle}\times\\
&\qquad\times\frac{F_6(\eta_5)+F_6(\eta_6)+F_4(\eta_5)+F_4(\eta_6)}{2}\ln\frac{\sqrt{1+\widetilde{C}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{C}u}+\sqrt{1-u}}\\
=&\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle3\ 4\rangle\,\langle4\ 5\rangle}\frac{\langle1\ 2\rangle^2}{\langle2|Q_2Q_1|2\rangle}\frac{1}{\langle\eta_1\ \eta_2\rangle}(F_1(\eta_1)+F_1(\eta_2))\ln\frac{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}\\
&+\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle3\ 4\rangle\,\langle4\ 5\rangle}\frac{\langle1\ 2\rangle^2}{\langle2|Q_3Q_1|2\rangle}\frac{1}{\langle\eta_3\ \eta_4\rangle}(F_2(\eta_3)+F_2(\eta_4))\ln\frac{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}}\\
&+\frac{u^2\sqrt{1-u}[1\ 2]^2}{8\,\langle3\ 4\rangle\,\langle4\ 5\rangle}\frac{\langle3\ 5\rangle^2}{\langle5|Q_3Q_2|5\rangle}\frac{1}{\langle\eta_5\ \eta_6\rangle}(F_4(\eta_5)+F_4(\eta_6))\ln\frac{\sqrt{1+\widetilde{C}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{C}u}+\sqrt{1-u}}.
\end{aligned}$$

It is easy to check that $F_1(\eta_{1,2})=F_3(\eta_{1,2})$ up to the term $\langle\ell|Q_2Q_1|\ell\rangle$ which is zero in our case. Similarly $F_2(\eta_{3,4})=-F_5(\eta_{3,4})$ and $F_4(\eta_{5,6})=F_6(\eta_{5,6})$. We need to carry out the summation, especially to show that the factor $\sqrt{1-u}$ cancels out.

The summation can be carried out using the technique presented in appendix C, and we get

$$I=\frac{s_{12}^2}{8\,\langle1\ 2\rangle\,\langle2\ 3\rangle\,\langle3\ 4\rangle\,\langle4\ 5\rangle\,\langle5\ 1\rangle}(u^2T+u^3\,\langle2|k_3k_1k_5-k_5k_1k_3|2]\,U)$$

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$$
\begin{aligned}
T \equiv & \frac{1}{\sqrt{1+u\widetilde{A}}} \ln \frac{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}} - \frac{1}{\sqrt{1+u\widetilde{B}}} \ln \frac{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}} \\
& - \frac{1}{\sqrt{1+\widetilde{C}u}} \ln \frac{\sqrt{1+\widetilde{C}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{C}u}+\sqrt{1-u}} \\
U \equiv & \frac{\langle 2|k_5k_4k_3+k_3k_4k_5|2]}{4s_{51}s_{23}s_{34}s_{45}-\langle 2|k_3k_1k_5-k_5k_1k_3|2]^2 u} \frac{1}{\sqrt{1+u\widetilde{A}}} \ln \frac{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}} \\
& - \frac{\langle 3|k_4k_5k_1+k_1k_5k_4|3]}{4s_{51}s_{23}s_{34}s_{45}-\langle 2|k_3k_1k_5-k_5k_1k_3|2]^2 u} \frac{1}{\sqrt{1+u\widetilde{B}}} \ln \frac{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}} \\
& - \frac{\langle 5|k_1k_2k_3+k_3k_2k_1|5]}{4s_{51}s_{23}s_{34}s_{45}-\langle 2|k_3k_1k_5-k_5k_1k_3|2]^2 u} \frac{1}{\sqrt{1+\widetilde{C}u}} \ln \frac{\sqrt{1+\widetilde{C}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{C}u}+\sqrt{1-u}}.
\end{aligned}
$$

It is easy to see that T is a pure box contribution and U is the exact expression for the pentagon. The coefficient u^3 in front of U is easy to deal with. Since there is a common denominator factor in the three terms of U, we write

$$
u^3 \to \left(\left(u - \frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5-k_5k_1k_3|2]^2} \right) + \frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5-k_5k_1k_3|2]^2} \right)^3, \tag{4.21}
$$

and make the expansion. Some terms go to boxes, and the remainder is the pure pentagon contribution.

To finish the program and read out the exact coefficients, we need to identify the cut boxes exactly for this amplitude. They are the following (See also the subsection on one-mass boxes in appendix B).

- (1) For box $(12|3|4|5)$, we have $K = K_{12}$, $P_1 = -k_5$ and $P_2 = -K_{45}$. Thus $A = -\frac{s_{45}s_{34}s_{35}}{4s_{12}} > 0$, $D = \frac{s_{34}s_{45}}{2s_{12}} > 0$ and $B = D^2$. Notice that here A, B, C, D are defined as in (3.25), and the quantities $\widetilde{A}, \widetilde{B}, \widetilde{C}$ we have defined in this section are just $-A/B$ in various cuts. Then we find

$$
C[I_{12|3|4|5}] = \frac{2}{s_{34}s_{45}} \int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{1+\widetilde{C}u}} \ln \frac{\sqrt{1+\widetilde{C}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{C}u}+\sqrt{1-u}}. \tag{4.22}
$$

- (2) For $(1|2|3|45)$, we have $K = -K_{12}$, $P_1 = k_3$ and $P_2 = -k_2$, thus by (3.25) we have $D = -\frac{s_{23}}{2} > 0$, $A = -\frac{s_{23}s_{13}}{4} < 0$ and $B = D^2$. Thus

$$
C[I_{1|2|3|45}] = \frac{2}{s_{12}s_{23}} \int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{1+\widetilde{B}u}} \ln \frac{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}}. \tag{4.23}
$$

- (6) For $(1|2|34|5)$, we have $P_1 = k_1$, $P_2 = -k_5$ and $K = K_{12}$, thus by (3.25) we have $D = \frac{s_{51}}{2} > 0$, $A = -\frac{s_{51}s_{52}}{4} < 0$ and $B = D^2$. Thus

$$
C[I_{1|2|34|5}] = -\frac{2}{s_{12}s_{51}} \int_0^1 du u^{-1-\epsilon} \frac{1}{\sqrt{1+Au}} \ln \frac{\sqrt{1+Au}-\sqrt{1-u}}{\sqrt{1+Au}+\sqrt{1-u}}. \tag{4.24}
$$

Collecting all results, we find the following coefficients. (These are the integrands for $\int_0^1 du u^{-1-\epsilon}$.)

$$C_{\text{pentagon}} = -\frac{s_{12}^3 \langle 2|k_3k_1k_5 - k_5k_1k_3|2]}{32\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}\left(\frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)^3$$

$$C_{1|2|34|5} = -\frac{s_{12}^3 s_{51}}{16\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}\left(u^2 - \frac{\langle 2|k_5k_4k_3 + k_3k_4k_5|2]}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]}\right.$$
$$\left(\left(u - \frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)^2 + 3\left(u - \frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)\right.$$
$$\left.\left.\frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2} + 3\left(\frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)^2\right)\right)$$

$$C_{1|2|3|45} = -\frac{s_{12}^3 s_{23}}{16\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}\left(u^2 - \frac{\langle 3|k_4k_5k_1 + k_1k_5k_4|3]}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]}\right.$$
$$\left(\left(u - \frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)^2 + 3\left(u - \frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)\right.$$
$$\left.\left.\frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2} + 3\left(\frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)^2\right)\right)$$

$$C_{12|3|4|5} = -\frac{s_{12}^2 s_{34}s_{45}}{16\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}\left(u^2 - \frac{\langle 5|k_1k_2k_3 + k_3k_2k_1|5]}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]}\right.$$
$$\left(\left(u - \frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)^2 + 3\left(u - \frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)\right.$$
$$\left.\left.\frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2} + 3\left(\frac{4s_{51}s_{23}s_{34}s_{45}}{\langle 2|k_3k_1k_5 - k_5k_1k_3|2]^2}\right)^2\right)\right)$$

The coefficients given above are not the true coefficients yet (except for C_{pentagon}), since of course we need to use the recursion/reduction formula to get the complete ϵ dependence of the coefficients. However, this is easy to do by replacing u^n with the corresponding form factors defined in section 3.3 with the parameters A, B, C, D given above.

At that point, the non-symmetric expression given above will also become symmetric (the pentagon coefficient is already symmetric, as it should be). For example, the u^2 term coefficient in $C_{1|2|34|5}$ is given by $-\frac{s_{12}^3 s_{51}}{16\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}$ while in $C_{1|2|3|45}$ it is given by $-\frac{s_{12}^3 s_{23}}{16\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}$. After using the appropriate form factor, the true coefficient for the box $(1|2|34|5)$ may be expressed as $-\alpha(\epsilon)\frac{s_{12}^3 s_{51}^3}{16 s_{25}^2\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}$, and for the box $(1|2|3|45)$ as $-\alpha(\epsilon)\frac{s_{12}^3 s_{23}^3}{16 s_{13}^2\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}$. The latter is related to the former by index shifting $i \to i+1$, as it must be.

4.1.4 Confirmation of the result

Now we compare our result against [29, 30], where the basis is dimensionally shifted. From our result we see immediately that the part of the amplitude that is reconstructed from

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the cut C_{12} is

$$\frac{(K_{12}^2)^2}{8\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}\langle 2|k_3k_1k_5 - k_5k_1k_3|2]\,\frac{s_{12}}{4}\frac{4^3}{s_{12}^3}I_5[\mu^6]$$
$$-\frac{1}{\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}(s_{51}s_{12}I_{1|2|34|5}[\mu^4] + s_{12}s_{23}I_{1|2|3|45}[\mu^4] + s_{34}s_{45}I_{12|3|4|5}[\mu^4],$$

where we have used $u = \frac{4\mu^2}{s_{12}}$ and the dimensionally shifted basis. To compare with equation (15) of [29] (or equation (4.1) of [30]) we need to use $I_4[\mu^4] = -\epsilon(1-\epsilon)I_4^{8-2\epsilon}$, $I_5[\mu^6] = -\epsilon(1-\epsilon)(2-\epsilon)I_5^{10-2\epsilon}$ as well as

$$tr[\gamma_5\ \not{k}_1\ \not{k}_2\ \not{k}_3\ \not{k}_4] = \langle 2|k_3k_4k_1 - k_1k_4k_3|2] = \langle 2|k_3k_1k_5 - k_5k_1k_3|2]\,. \tag{4.25}$$

We see that our result agrees exactly with the equation (15) of [29].[7]

4.2 Four gluons

In this part, we give only final results (as u-integrals) for four-gluon amplitudes, since the method has already been elaborated in the previous five-gluon example. In principle one then applies our recursion and reduction formulas of section 3 to find the coefficients. Here, we choose instead to confirm our results against those in the literature, which are also given in terms of the final μ-integral, so we do not write the coefficients explicitly.

To begin with, we must establish our basis. For details, see appendix B. First, for the zero-mass box, we have for example with the cut K_{12} that $K = K_{12}$, $P_1 = K_1$ and $P_2 = -K_4$, thus

$$A = s_{13}s_{41}/4, \quad B = D^2, \quad C = -s_{41}/2 - s_{12}/4, \quad D = -s_{41}/2, \tag{4.26}$$

and so

$$C[I_{4;12}^{(0m)}] = -\frac{2}{s_{41}s_{12}}\int_0^1 du\ u^{-1-\epsilon}\frac{1}{\sqrt{1+\widetilde{A}u}}\ln\left(\frac{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}\right),\ \widetilde{A} = \frac{s_{13}}{s_{23}}. \tag{4.27}$$

Second, there are only one-mass triangles. For $(12|3|4)$ with the cut K_{12} we have the expression

$$C[I_3(K_{12},K_4)] = -\frac{1}{s_{12}}\int_0^1 du\ u^{-1-\epsilon}\ln\left(\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}\right). \tag{4.28}$$

Bubbles are simply $\sqrt{1-u}$ in all cases. We will compare our results with known results given first by [27], in the form given in [30].

- (1) For the helicity configuration $(++++)$ and cut C_{12} we find

$$C_{12} = \frac{s_{12}^2[1\ 2][3\ 4]}{8\langle 1\ 2\rangle\langle 3\ 4\rangle}\left(-\frac{2}{s_{41}s_{12}}\int_0^1 du\ u^{-1-\epsilon}u^2\frac{1}{\sqrt{1+\widetilde{A}u}}\ln\left(\frac{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}\right)\right), \tag{4.29}$$

  The integral in parentheses, with its additional factor of u^2, is related to the box integral K_4 of [30]. Using $u = \frac{4\mu^2}{s_{12}}$, we get immediately $\frac{2[1\ 2][3\ 4]}{\langle 1\ 2\rangle\langle 3\ 4\rangle}K_4$.

[7] There is a relative minus sign for the $I_5^{10-2\epsilon}$ term because our definition of master integrals does not include the $(-1)^n$ used in [29].

- (2) For $(--++)$ with cut C_{41} we have

$$I_{\rm cut} = \frac{[2\ 3]^2[4\ 3]^2}{4s_{12}[1\ 3]^2}\int_0^1 du\ u^{-1-\epsilon}\left(-\frac{2s_{13}(s_{13}-s_{12})}{s_{41}^2}u\sqrt{1-u}\right.$$
$$\left.+\frac{2s_{12}(s_{41}-s_{13})}{s_{41}^2}u\ln\left(\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}\right)+\frac{(2+\widetilde{A}u)u}{\sqrt{1+\widetilde{A}u}}\ln\left(\frac{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}\right)\right),$$
$$\widetilde{A} = \frac{s_{13}}{s_{12}}. \tag{4.30}$$

To compare with the results in the literature, we must change coordinates via $u = \frac{4\mu^2}{s_{41}}$. We end up with

$$-\frac{2[2\ 3]^2[3\ 4]^2}{[1\ 3]^2}\frac{s_{13}(s_{13}-s_{12})}{s_{12}s_{41}^3}J_2(s_{41}) - \frac{[2\ 3]^2[3\ 4]^2}{[1\ 3]^2}\left(J_4+\frac{2s_{13}}{s_{41}s_{12}}K_4\right)$$
$$+\frac{2[2\ 3]^2[3\ 4]^2}{[1\ 3]^2}\frac{(s_{41}-s_{13})}{s_{41}^2}J_3(s_{41}).$$

We find complete agreement with equation (3.17) of [30].[8]

- (3) For $(--++)$, the cut C_{12} is almost the same as for $(++++)$, just multiplied by a factor of $\frac{\langle 1\ 2\rangle^2}{[1\ 2]^2}$. This is enough to get the correct box coefficient. For the cut C_{41} we get

$$I = \frac{\langle 1\ 2\rangle^2[3\ 4]^2}{s_{41}s_{12}}\int_0^1 du\ u^{-1-\epsilon}$$
$$\left[\frac{u^2(1+\widetilde{A})^2}{4\sqrt{1+\widetilde{A}u}}\ln\left(\frac{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}\right)+\frac{\sqrt{1-u}}{6}(2-5u-3\widetilde{A}u)\right],$$
$$\widetilde{A} = \frac{s_{13}}{s_{12}}. \tag{4.31}$$

It is straightforward to check this result against [30]: the term with the logarithm translates to K_4, the simple bubble is I_2, and the terms in the brackets with $\sqrt{1-u}(u)$ translate to $J_2(s_{41})$. Again we confirm agreement.[9]

[8] To confirm agreement of these formulas, one must remember the relative minus sign for the triangle integral in the basis of [30].

[9] We need to use the following dimensional shift identities for the basis of [30]:

$$I_4^{6-2\epsilon} = -2J_4 - \frac{st}{2u}I_4 - \frac{s}{u}I_3(s) - \frac{t}{u}I_3(t),$$
$$I_3^{6-2\epsilon}(s) = \frac{1}{2}I_2(s) - J_3(s),$$
$$I_2^{6-2\epsilon}(s) = -\frac{2}{3}J_2(s) + \frac{s}{6}I_2(s)$$

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- (4) For $(-++-+)$ with cut C_{41} we have

$$I = \frac{2\langle 1\ 3\rangle^2 s_{13}}{\langle 2\ 4\rangle^2 s_{41}} \int_0^1 du\ u^{-1-\epsilon} \left[\frac{\sqrt{1-u}(12+3\widetilde{A}(6+u)+\widetilde{A}^2(4+5u))}{12\widetilde{A}^2} \right.$$
$$+\frac{(1+\widetilde{A})^2(8+8\widetilde{A}u+\widetilde{A}^2u^2)}{8\widetilde{A}^3\sqrt{1+\widetilde{A}u}} \ln\left(\frac{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}\right)$$
$$\left. -\frac{(1+\widetilde{A})^2(2+\widetilde{A}u)}{2\widetilde{A}^3} \ln\left(\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}\right)\right],$$
$$\widetilde{A} = \frac{s_{13}}{s_{12}}. \tag{4.32}$$

This integral agrees with equation (3.69) of [30] after accounting for differences of convention.

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A. Kinematics

In this paper we analyze unitarity cuts in Minkowski space with signature $(+,-,-,-)$. The kinematic region in question is the one where $K^2 > 0$ and all other invariants are negative. Let us study the consequences of these conditions in terms of the four-dimensional momenta $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ of the cut propagators. These vectors satisfy

$$\widetilde{\ell}_1^2 = \widetilde{\ell}_2^2 = \mu^2, \qquad K - \widetilde{\ell}_1 = \widetilde{\ell}_2. \tag{A.1}$$

First, we can choose a frame such that $\vec{K} = (K,0,0,0)$ and $\widetilde{\ell}_1 = (a,b,0,0)$. Then, $\widetilde{\ell}_2 = (K-a,-b,0,0)$. The mass-shell conditions become $a^2 - b^2 = \mu^2 = (K-a)^2 - b^2$, so $a = K/2$ and $b^2 = K^2/4 - \mu^2$. Since b is real, $b^2 \geq 0$, so we draw the following important conclusion:

$$\mu^2 \leq \frac{K^2}{4}, \tag{A.2}$$

or equivalently,

$$u \leq 1. \tag{A.3}$$

In the procedure described in this paper, we decompose $\widetilde{\ell}_1 = \ell_1 + zK$ with $\ell_1^2 = 0$. Under this decomposition, we can write $\widetilde{\ell}_1 = (b+zK, \alpha b, 0, 0)$ with $\alpha = \pm 1$. Using $b^2 = K^2/4 - \mu^2$,

$a = b + zK = K/2$, we can get $z = (1 \pm \sqrt{1-u})/2$. Furthermore, since we choose the positive light cone with $\delta^+(\ell^2)$, i.e., $b > 0$, we have our second important conclusion: if $K > 0$ we need to choose $z = (1 - \sqrt{1-u})/2$, but if $K < 0$ we need to choose $z = (1 + \sqrt{1-u})/2$. Throughout the paper, we will always assume $K > 0$, thus

$$z = \frac{1 - \sqrt{1-u}}{2}. \tag{A.4}$$

The choice of this solution does not affect our discussion.

B. Special cases of master integrals

B.1 One-mass and two-mass triangles

Consider a cut triangle in the massless limit where $K_3^2 = 0$ (so it is a one-mass or two-mass triangle). From (3.11), we see that $Z = 1$ and $\sqrt{\Delta_3} = -(2K_1 \cdot K_3) = K_1^2 - K_2^2$. Thus we have

$$C[I_3^{1m/2m}(K_1)] = -\int_0^1 du\; u^{-1-\epsilon} \frac{1}{\sqrt{\Delta_3}} \ln\left(\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}\right). \tag{B.1}$$

We can integrate by parts to get a different expression:

$$\begin{aligned}\sqrt{\Delta_3} C[I_{3,cut}^{1m/2m}] &= \frac{u^{-\epsilon}}{\epsilon} \ln\left(\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}\right)\Big|_0^1 + \int_0^1 du\; \frac{u^{-\epsilon}}{\epsilon} \frac{1}{u\sqrt{1-u}} \\ &= \frac{1}{\epsilon}\int_0^1 du\; u^{-1-\epsilon} \frac{1}{\sqrt{1-u}}.\end{aligned} \tag{B.2}$$

Comparing this formula with (3.5), we see that the form is the same. In fact, if we allow coefficients of scalar functions to be general functions of ϵ, then there is no need to distinguish one-mass and two-mass triangles from bubbles. Thus, if one likes, one can think in terms of keeping only bubble functions in the basis and discarding both one-mass triangles and two-mass triangles.[10]

It is, of course, easy to carry out the u integral in (B.2) explicitly and check it against the known expressions for one- and two-mass triangles after restoring the correct normalization factors.

B.2 Some boxes with massless legs

Here we discuss some special cases of boxes with massless legs. We follow all the conventions of section 3. Suppose that P_1 is the momentum of a massless leg, so $P_1^2 = 0$. Then, with the definitions (3.25), we find Thus for the special case where $P_1^2 = 0$, we have $B = D^2$, and thus

$$D - Cu + \sqrt{1-u}\sqrt{B - Au} = \frac{D}{2}\left(2 - u\frac{2C}{D} + \mathrm{sign}(D)\sqrt{1-u}\sqrt{1 - u\frac{A}{D^2}}\right).$$

[10] More concretely, we know that scalar bubbles, one-mass triangles and two-mass triangles all have the form $c(\epsilon)(-K^2)^{-\epsilon}$ where $c(\epsilon)$ is a function of ϵ. This same "modified basis" has been used in [6].

The expression in parentheses is a complete square, if

$$1 + \frac{A}{D^2} = \frac{2C}{D},$$

or, equivalently,

$$D^2 + A - 2CD = 0.$$

With the definitions (3.25) subject to $P_1^2 = 0$,

$$D^2 + A - 2CD = \frac{(P_1 \cdot K)^2 P_2^2 (K - P_2)^2}{(K^2)^2}. \tag{B.3}$$

We see that this expression vanishes if $P_2^2 = 0$ or $(K - P_2)^2 = 0$. Under this condition, the cut (3.29) takes the following special form.

$$C[I_4(K; P_1, P_2)] = \frac{1}{K^2 D} \int_0^1 du\ u^{-1-\epsilon} \frac{1}{\sqrt{1 - u\frac{A}{D^2}}} \ln \left(\frac{\sqrt{1 - u\frac{A}{D^2}} + \sqrt{1-u}}{\sqrt{1 - u\frac{A}{D^2}} - \sqrt{1-u}} \right). \tag{B.4}$$

Here we needed the conditions

$$P_1^2 = P_2^2 = 0, \quad \textbf{or} \quad P_1^2 = (K - P_2)^2 = 0. \tag{B.5}$$

Zero-mass box function: for this case we have all $K_i^2 = 0$, so there are only two cuts, K_{12} and K_{23}. These are trivially related by index permutation. For cut K_{12} we have $K = K_{12}$, $P_1 = K_1$ and $P_2 = -K_4$. Define

$$\alpha = \frac{K_{13}^2}{K_{23}^2}. \tag{B.6}$$

We can see that

$$\begin{aligned} B - Au &= \left(-\frac{K_{41}^2}{2} \right)^2 (1 + \alpha u) \\ D - Cu &= \left(-\frac{K_{41}^2}{2} \right) \left(1 - u(1 + \frac{K_{12}^2}{2K_{41}^2}) \right), \end{aligned}$$

thus

$$D - Cu \pm \sqrt{1-u}\sqrt{B - Au} = -\frac{K_{41}^2}{4} \left(\sqrt{1 + \alpha u} \pm \sqrt{1-u} \right)^2.$$

Using this we have

$$C[I_{4,0m}(K_{12}; K_1, -K_4)] = -\frac{2}{K_{41}^2 K_{12}^2} \int_0^1 du\ u^{-1-\epsilon} \frac{1}{\sqrt{1+\alpha u}} \ln \left(\frac{\sqrt{1+\alpha u} + \sqrt{1-u}}{\sqrt{1+\alpha u} - \sqrt{1-u}} \right). \tag{B.7}$$

This is exactly the expression that we find in the four-gluon examples (4.27).

One-mass box function: we assume that $K_1^2 \neq 0$, so there are three cuts, K_1, K_{34} and K_{23}. We will neglect details and give only results.

For cut K_1 we have

$$C[I_{4,1m}(K_1;K_{12},-K_4)] = \frac{2}{K_{34}^2 K_{23}^2}\int_0^1 du\; u^{-1-\epsilon}\frac{1}{\sqrt{1+\alpha u}}\ln\left(\frac{\sqrt{1+\alpha u}+\sqrt{1-u}}{\sqrt{1+\alpha u}-\sqrt{1-u}}\right),$$
$$\alpha = \frac{K_1^2 K_{24}^2}{K_{23}^2 K_{34}^2}. \tag{B.8}$$

For cut K_{34} we have

$$C[I_{4,1m}(K_{34};K_3,-K_2)] = -\frac{2}{K_{34}^2 K_{23}^2}\int_0^1 du\; u^{-1-\epsilon}\frac{1}{\sqrt{1+\alpha u}}\ln\left(\frac{\sqrt{1+\alpha u}+\sqrt{1-u}}{\sqrt{1+\alpha u}-\sqrt{1-u}}\right),$$
$$\alpha = \frac{K_{24}^2}{K_{23}^2}. \tag{B.9}$$

For cut K_{41} we have

$$C[I_{4,1m}(K_{41};K_4,-K_3)] = -\frac{2}{K_{34}^2 K_{23}^2}\int_0^1 du\; u^{-1-\epsilon}\frac{1}{\sqrt{1+\alpha u}}\ln\left(\frac{\sqrt{1+\alpha u}+\sqrt{1-u}}{\sqrt{1+\alpha u}-\sqrt{1-u}}\right),$$
$$\alpha = \frac{K_{24}^2}{K_{34}^2}. \tag{B.10}$$

Two mass easy box functions: We assume $K_1^2 \neq 0$ and $K_3^2 \neq 0$. Then there are four possible cuts. For each one, it is possible to choose P_1, P_2 such that the condition (B.5) is satisfied, as shown in the following table.

Box Cut K	P_1	P_2
K_1	$-K_4$	K_{12}
K_3	$-K_2$	K_{34}
K_{12}	$-K_4$	K_1
K_{23}	K_2	$-K_1$

(B.11)

B.3 Zero-mass pentagon

Here we evaluate (3.52) for the zero-mass pentagon under the cut K_{12}. It is

$$C[I_{5,0m}(K_{12};K_1,-K_{45},-K_5)] = \int_0^1 du u^{-1-\epsilon}\frac{1}{K_{12}^2}\times \tag{B.12}$$
$$\left(-\frac{4\langle 2|k_3k_4k_5+k_5k_4k_3|2]}{4K_{51}^2K_{23}^2K_{34}^2K_{45}^2-\langle 2|k_3k_1k_5-k_5k_1k_3|2]^2 u}\frac{1}{\sqrt{1+\widetilde{A}u}}\ln\frac{\sqrt{1+\widetilde{A}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{A}u}+\sqrt{1-u}}\right.$$
$$+\frac{4\langle 3|k_4k_5k_1+k_1k_5k_4|3]}{4K_{51}^2K_{23}^2K_{34}^2K_{45}^2-\langle 2|k_3k_1k_5-k_5k_1k_3|2]^2 u}\frac{1}{\sqrt{1+\widetilde{B}u}}\ln\frac{\sqrt{1+\widetilde{B}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{B}u}+\sqrt{1-u}}$$
$$\left.+\frac{4\langle 5|k_1k_2k_3+k_3k_2k_1|5]}{4K_{51}^2K_{23}^2K_{34}^2K_{45}^2-\langle 2|k_3k_1k_5-k_5k_1k_3|2]^2 u}\frac{1}{\sqrt{1+\widetilde{C}u}}\ln\frac{\sqrt{1+\widetilde{C}u}-\sqrt{1-u}}{\sqrt{1+\widetilde{C}u}+\sqrt{1-u}}\right),$$

where

$$\widetilde{A} = \frac{K_{52}^2}{K_{51}^2}, \quad \widetilde{B} = \frac{K_{13}^2}{K_{23}^2}, \quad \widetilde{C} = \frac{K_{12}^2 K_{35}^2}{K_{34}^2 K_{45}^2}. \tag{B.13}$$

B.4 Hexagons and beyond are not independent

The integrand of double cut of hexagon is of the form

$$\frac{\delta(\widetilde{\ell}^2 - \mu^2)\delta((\widetilde{\ell} - K)^2 - \mu^2)}{((\widetilde{\ell} - P_1)^2 - \mu^2)((\widetilde{\ell} - P_2)^2 - \mu^2)((\widetilde{\ell} - P_3)^2 - \mu^2)((\widetilde{\ell} - P_4)^2 - \mu^2)}.$$

The momentum vectors K, P_i as well as $\widetilde{\ell}$ are four-dimensional, and moreover the four P_i are linearly independent in general. Therefore we can express K as a linear combination of the P_i:

$$K = \sum_i \alpha_i P_i. \tag{B.14}$$

Within the integral we may make the following substitutions:

$$\sum_i \alpha_i((\widetilde{\ell} - P_i)^2 - \mu^2) = \sum_i \alpha_i(P_i^2 - 2P_i \cdot \widetilde{\ell}) = \sum_i \alpha_i P_i^2 - 2K \cdot \widetilde{\ell} = \sum_i \alpha_i P_i^2 - K^2.$$

In the first step we used the delta function $\delta(\widetilde{\ell}^2 - \mu^2)$. In the second step we used (B.14) while in the third step we have used the second delta function $\delta((\widetilde{\ell} - K)^2 - \mu^2) = \delta(K^2 - 2K \cdot \widetilde{\ell})$.

Using this result we can write

$$\begin{aligned}
&\left(\frac{\sum_i \alpha_i P_i^2 - K^2}{\sum_i \alpha_i P_i^2 - K^2}\right) \frac{\delta(\widetilde{\ell}^2 - \mu^2)\delta((\widetilde{\ell} - K)^2 - \mu^2)}{((\widetilde{\ell} - P_1)^2 - \mu^2)((\widetilde{\ell} - P_2)^2 - \mu^2)((\widetilde{\ell} - P_3)^2 - \mu^2)((\widetilde{\ell} - P_4)^2 - \mu^2)} \\
&\quad = \frac{1}{\sum_i \alpha_i P_i^2 - K^2} \frac{\delta(\widetilde{\ell}^2 - \mu^2)\delta((\widetilde{\ell} - K)^2 - \mu^2) \sum_i \alpha_i((\widetilde{\ell} - P_i)^2 - \mu^2)}{((\widetilde{\ell} - P_1)^2 - \mu^2)((\widetilde{\ell} - P_2)^2 - \mu^2)((\widetilde{\ell} - P_3)^2 - \mu^2)((\widetilde{\ell} - P_4)^2 - \mu^2)} \\
&\quad = \frac{1}{\sum_i \alpha_i P_i^2 - K^2} \sum_i \alpha_i \frac{\delta(\widetilde{\ell}^2 - \mu^2)\delta((\widetilde{\ell} - K)^2 - \mu^2)}{\prod_{j \neq i}((\widetilde{\ell} - P_j)^2 - \mu^2)}.
\end{aligned}$$

Now each term is seen to be a cut pentagon. The lesson is that there are no further independent cuts of scalar functions beyond pentagons.

C. Factors of the form $\langle \ell | QP | \ell \rangle$

In spinor manipulation, we repeatedly encounter factors like $\langle \ell | Q_i K | \ell \rangle$ and $\langle \ell | Q_i Q_j | \ell \rangle$. It is worth developing a systematic approach to deal with these factors. Let us consider a general factor of this type, written as $\langle \ell | QP | \ell \rangle$.

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Method one: spinor basis. One way to find the poles from this factor is by expansion in a basis of any two independent spinors:

$$|\ell\rangle = |a\rangle + y\,|b\rangle\,. \tag{C.1}$$

Then the roots of the equation $0 = \langle\ell|QP|\ell\rangle$ lie at the solutions to the quadratic equation

$$0 = \langle a|QP|a\rangle + y(\langle a|QP|b\rangle + \langle b|QP|a\rangle) + y^2\,\langle b|QP|b\rangle\,,$$

which are

$$y_\pm = \frac{-(\langle a|QP|b\rangle + \langle b|QP|a\rangle) \pm \langle a\ b\rangle\sqrt{\Delta}}{2\,\langle b|QP|b\rangle}, \tag{C.2}$$

where

$$\Delta = 4[(Q\cdot P)^2 - Q^2P^2].$$

With these two solutions $|\ell_+\rangle\,, |\ell_-\rangle$ we have

$$\langle\ell|QP|\ell\rangle = \langle\ell\ \ell_+\rangle\,\langle\ell\ \ell_-\rangle\,\frac{\langle b|QP|b\rangle}{\langle a\ b\rangle^2}. \tag{C.3}$$

Method two: vector solutions. Here we describe a second approach, which avoids having to choose basis spinors and helps manipulate a variety of expressions.

Given two massive momenta Q, P we can construct two lightlike momenta P_+, P_- by solving

$$(Q+xP)^2 = 0, \quad \Longrightarrow x_\pm = \frac{-2Q\cdot P \pm \sqrt{4((Q\cdot P)^2 - Q^2P^2)}}{2P^2} = \frac{-2Q\cdot P \pm \sqrt{\Delta}}{2P^2}. \tag{C.4}$$

We have the following relations among these variables:

$$P_\pm = Q + x_\pm P, \tag{C.5}$$

$$P = \frac{P_+ - P_-}{(x_+ - x_-)}, \qquad Q = \frac{-x_-P_+ + x_+P_-}{(x_+ - x_-)} \tag{C.6}$$

$$x_+x_- = \frac{Q^2}{P^2}, \qquad x_+ + x_- = -\frac{2Q\cdot P}{P^2}, \quad x_+ - x_- = x_+ - x_- = \frac{\sqrt{\Delta}}{P^2}, \tag{C.7}$$

$$\langle P_+|Q|P_+] = \frac{x_+}{(x_+ - x_-)}(-2P_+\cdot P_-) = \frac{x_+}{(x_+ - x_-)}\frac{\Delta}{P^2}, \tag{C.8}$$

$$\langle P_+|P|P_+] = -\frac{1}{(x_+ - x_-)}(-2P_+\cdot P_-) = -\frac{1}{(x_+ - x_-)}\frac{\Delta}{P^2}, \tag{C.9}$$

$$\langle P_-|Q|P_-] = -\frac{x_-}{(x_+ - x_-)}(-2P_+\cdot P_-) = -\frac{x_-}{(x_+ - x_-)}\frac{\Delta}{P^2}, \tag{C.10}$$

$$\langle P_-|P|P_-] = \frac{1}{(x_+ - x_-)}(-2P_+\cdot P_-) = \frac{1}{(x_+ - x_-)}\frac{\Delta}{P^2}. \tag{C.11}$$

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C.1 Application of second method

It is easy to check that

$$\langle \ell | QP | \ell \rangle = \frac{1}{(x_+ - x_-)} \langle \ell \; P_+ \rangle \, [P_+ \; P_-] \, \langle \ell \; P_- \rangle \, . \tag{C.12}$$

This means, in particular, that P_+, P_- are exactly the two poles within the factor $\langle \ell | QP | \ell \rangle$.

When we try to identify the structure of the logarithmic part, we often encounter the following two combinations.

$$\left(\frac{\langle P_+ | Q | P_+]}{\langle P_+ | P | P_+]} \right) \left(\frac{\langle P_- | Q | P_-]}{\langle P_- | P | P_-]} \right) = x_+ x_- = \frac{Q^2}{P^2}, \tag{C.13}$$

$$\left(\frac{\langle P_+ | Q | P_+]}{\langle P_+ | P | P_+]} \right) \left(\frac{\langle P_- | Q | P_-]}{\langle P_- | P | P_-]} \right)^{-1} = \frac{x_+}{x_-} = \frac{Q \cdot P - \sqrt{(Q \cdot P)^2 - Q^2 P^2}}{Q \cdot P + \sqrt{(Q \cdot P)^2 - Q^2 P^2}}. \tag{C.14}$$

Of these two arguments of logarithms, the one given in (C.13) is unphysical and so must drop out of the final result, while the one given in (C.14) is the physical singularity identifying a given triangle, box or pentagon.

Sometimes we need to use the spinor components of P_+, P_-. For this we can expand in a basis of two arbitrary spinors,

$$\lambda_P = \frac{|a\rangle + w \, |b\rangle}{\sqrt{t}}, \qquad \widetilde{\lambda}_P = \frac{|a] + \overline{w} |b]}{\sqrt{t}},$$

where t is a normalization factor. We can then solve to find

$$t = \frac{\langle b | a | b]}{\langle b | P | b]}, \quad w = -\frac{\langle a | P | b]}{\langle b | P | b]} \Longrightarrow \lambda_P = -\frac{P | b] \, \langle a \; b \rangle}{\langle b | P | b] \, \sqrt{t}}.$$

We must also consider factors such as

$$\langle P_+ | RP | P_+ \rangle = \left\langle P_+ | R \frac{P_+ - P_-}{x_+ - x_-} | P_+ \right\rangle = -\frac{\langle P_- \; P_+ \rangle}{x_+ - x_-} \langle P_+ | R | P_-] \, , \tag{C.15}$$

$$\langle P_- | RP | P_- \rangle = \left\langle P_- | R \frac{P_+ - P_-}{x_+ - x_-} | P_- \right\rangle = -\frac{\langle P_- \; P_+ \rangle}{x_+ - x_-} \langle P_- | R | P_+] \, . \tag{C.16}$$

Now we can do the following sum, which is the pattern we encounter in cut pentagons.

$$\begin{aligned} S[Q,P,S,R] &\equiv \frac{\langle P_+ | RP | P_+ \rangle}{\langle P_+ | SP | P_+ \rangle} + \frac{\langle P_- | RP | P_- \rangle}{\langle P_- | SP | P_- \rangle} = \frac{\langle P_+ | R | P_-]}{\langle P_+ | S | P_-]} + \frac{\langle P_- | R | P_+]}{\langle P_- | S | P_+]} \\ &= \frac{\langle P_+ | R | P_-] \, \langle P_- | S | P_+] + \langle P_+ | S | P_-] \, \langle P_- | R | P_+]}{\langle P_+ | S | P_-] \, \langle P_- | S | P_+]} \\ &= \frac{(2P_+ \cdot R)(2P_- \cdot S) + (2P_- \cdot R)(2P_+ \cdot S) - (2P_+ \cdot P_-)(2R \cdot S)}{(2P_+ \cdot S)(2P_- \cdot S) - S^2 (2P_+ \cdot P_-)}. \end{aligned}$$

If we expand P_+, P_- in terms of Q, P, then we find that this quantity can be expressed as follows:

$$S[Q,P,S,R] = \frac{T_1}{T_2}, \tag{C.17}$$

with

$$T_1 = -8\det\begin{pmatrix} R\cdot S & Q\cdot R & P\cdot R \\ Q\cdot S & Q^2 & Q\cdot P \\ P\cdot S & Q\cdot P & P^2 \end{pmatrix}; \qquad T_2 = -4\det\begin{pmatrix} S^2 & Q\cdot S & P\cdot S \\ Q\cdot S & Q^2 & Q\cdot P \\ P\cdot S & Q\cdot P & P^2 \end{pmatrix}. \tag{C.18}$$

The function T_1 is symmetric under exchange of the first two, or the last two, arguments of S. The function T_2 depends only on Q, P, S and is symmetric in all three.

C.2 Spinor integral formulas

Here we derive some useful spinor integral formulas.

First let us consider

$$\int \langle \ell\ d\ell\rangle\,[\ell\ d\ell]\frac{1}{\langle\ell|K|\ell]\,\langle\ell|Q|\ell]} = \int_0^1 dx \int \langle \ell\ d\ell\rangle\,[d\ell\ \partial_\ell]\left(\frac{[\eta\ \ell]}{\langle\ell|R|\ell]\,\langle\ell|R|\eta]}\right),$$
$$R = xQ + (1-x)K.$$

Now we take η to be one solution of $\langle\ell|QK|\ell\rangle$, i.e, the solution (C.4) in which we have substituted $P \to K$, so

$$R = x\frac{-x_-P_+ + x_+P_-}{x_+ - x_-} + (1-x)\frac{P_+ - P_-}{x_+ - x_-},$$

where $x_\pm$ are defined as in (C.4) and $P_\pm$ as in (C.5). Taking η to be P_+ we have

$$\langle\ell|R|\eta] = \langle\ell|R|P_+] = \langle\ell\ P_-\rangle\,[P_-\ P_+]\left(x\frac{(x_+ + 1)}{(x_+ - x_-)} - \frac{1}{(x_+ - x_-)}\right).$$

Now using

$$\int_0^1 dx \frac{1}{(xc+d)(xa+b)} = -\frac{1}{ad-bc}\left(\ln\frac{c+d}{d} - \ln\frac{a+b}{b}\right), \tag{C.19}$$

we get

$$\begin{aligned}\int_0^1 dx\left(\frac{[\eta\ \ell]}{\langle\ell|R|\ell]\,\langle\ell|R|\eta]}\right) &= \frac{(x_+ - x_-)[P_+\ \ell]}{\langle\ell\ P_-\rangle\,[P_-\ P_+]\,\langle\ell|P_+|\ell]}\ln\left(\frac{-x_+\,\langle\ell|K|\ell]}{\langle\ell|Q|\ell]}\right) \\ &= -\frac{1}{\langle\ell|QK|\ell\rangle}\ln\left(\frac{-x_+\,\langle\ell|K|\ell]}{\langle\ell|Q|\ell]}\right),\end{aligned} \tag{C.20}$$

where we have used the formula (C.12) to simplify the result.[11]

[11] It is important to realize that in principle we should also take the pole contribution of $\langle\ell\ P_-\rangle$ from the middle equation of (C.20). However, in many examples, there is a factor of $\langle\ell|QK|\ell\rangle$ in the numerator, so this pole has zero residue.

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