High-voltage engineering

E. Gaxiola
CERN, Geneva, Switzerland

Abstract

High-voltage engineering covers the application, the useful use and proper working of high voltages and high fields. Here we give some introductory examples, i.e., 'septa' and 'kicker' at the Large Hadron Collider (14 TeV), the Super Proton Synchrotron (450 GeV) and the Proton Synchrotron (26 GeV) accelerators as found at the European Orginization for Nuclear Research (CERN) today.

We briefly cover the theoretical foundation (Maxwell equations) and aspects of numerical field simulation methods. Concepts relating to electrical fields, insulation geometry and medium and breakdown are introduced. We discuss ways of generating high voltages with examples of AC sources (50/60 Hz), DC sources, and pulse sources.

Insulation and breakdown in gases, liquids, solids and vacuum are presented, including Paschen's law (breakdown field and streamer breakdown). Applications of the above are discussed, in particular the general application of a transformer.

We briefly discuss measurement techniques of partial discharges and loss angle $\tan(\delta)$.

The many basic high-voltage engineering technology aspects — high-voltage generation, field calculations, and discharge phenomena — are shown in practical accelerator environments: vacuum feed through (triple points), breakdown field strength in air 10 kV/cm, and challenging calculations for real practical geometries.

Exceptionally, the series editor decided that this contribution could be published in the form of a reproduction of the author's transparencies.

High Voltage Engineering

Enrique Gaxiola
Many thanks to the Electrical Power Systems Group, Eindhoven University of Technology, The Netherlands
& CERN AB-BT Group colleagues

Introductory examples

Theoretical foundation and numerical field simulation methods

Generation of high voltages

Insulation and Breakdown

Measurement techniques

CAS on Small Accelerators

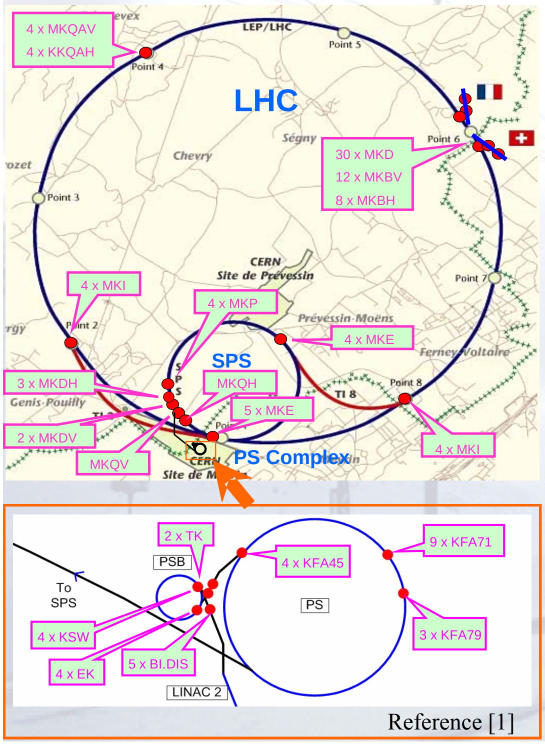
Introduction E.Gaxiola:

Studied Power Engineering
Ph.D. on Dielectric Breakdown in Insulating Gases;
Non-Uniform Fields and Space Charge Effects
Industry R&D on Plasma Physics / Gas Discharges
CERN Accelerators & Beam, Beam Transfer,
Kicker Innovations:

- Electromagnetism
- Beam impedance reduction
- Vacuum high voltage breakdown in traveling wave structures.
- Pulsed power semiconductor applications

CAS on Small Accelerators

CERN Septa and Kicker examples



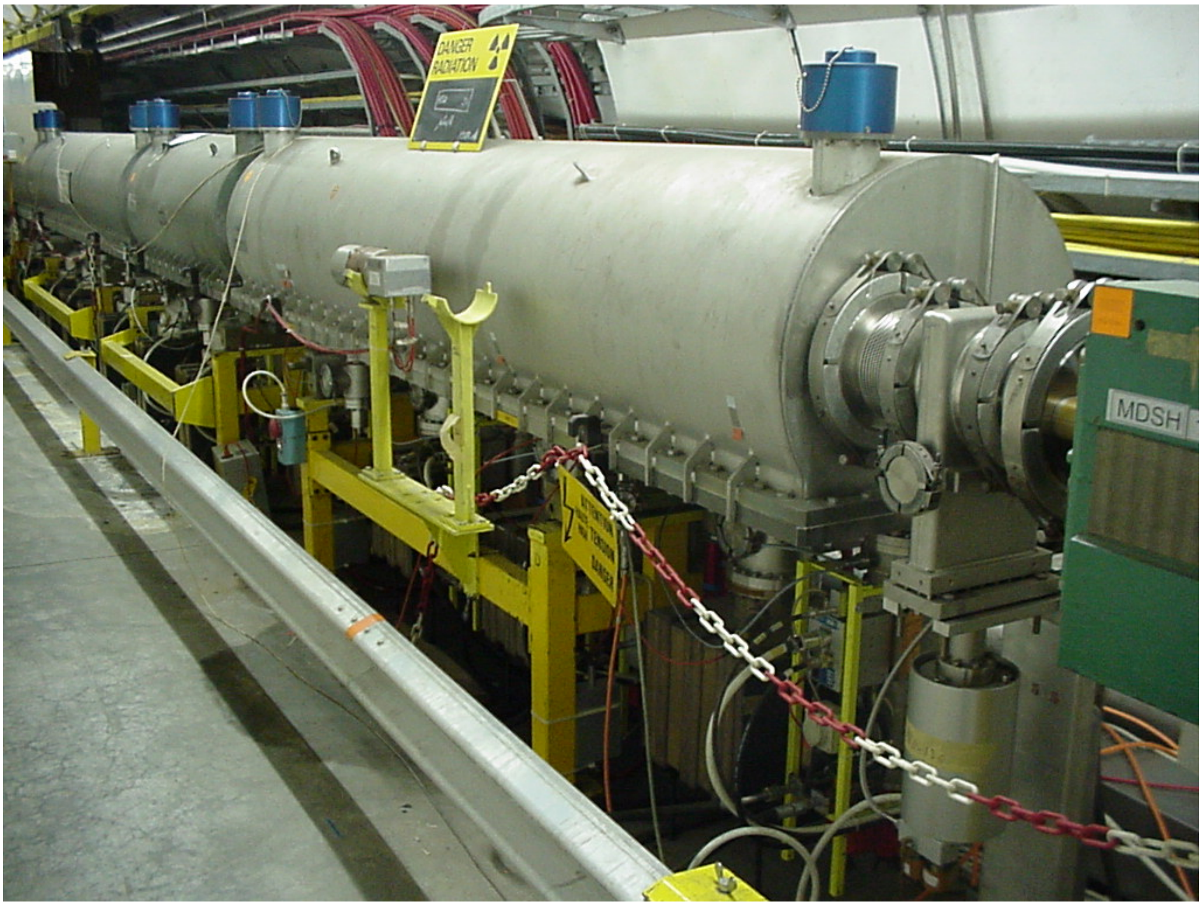
Reference [1]

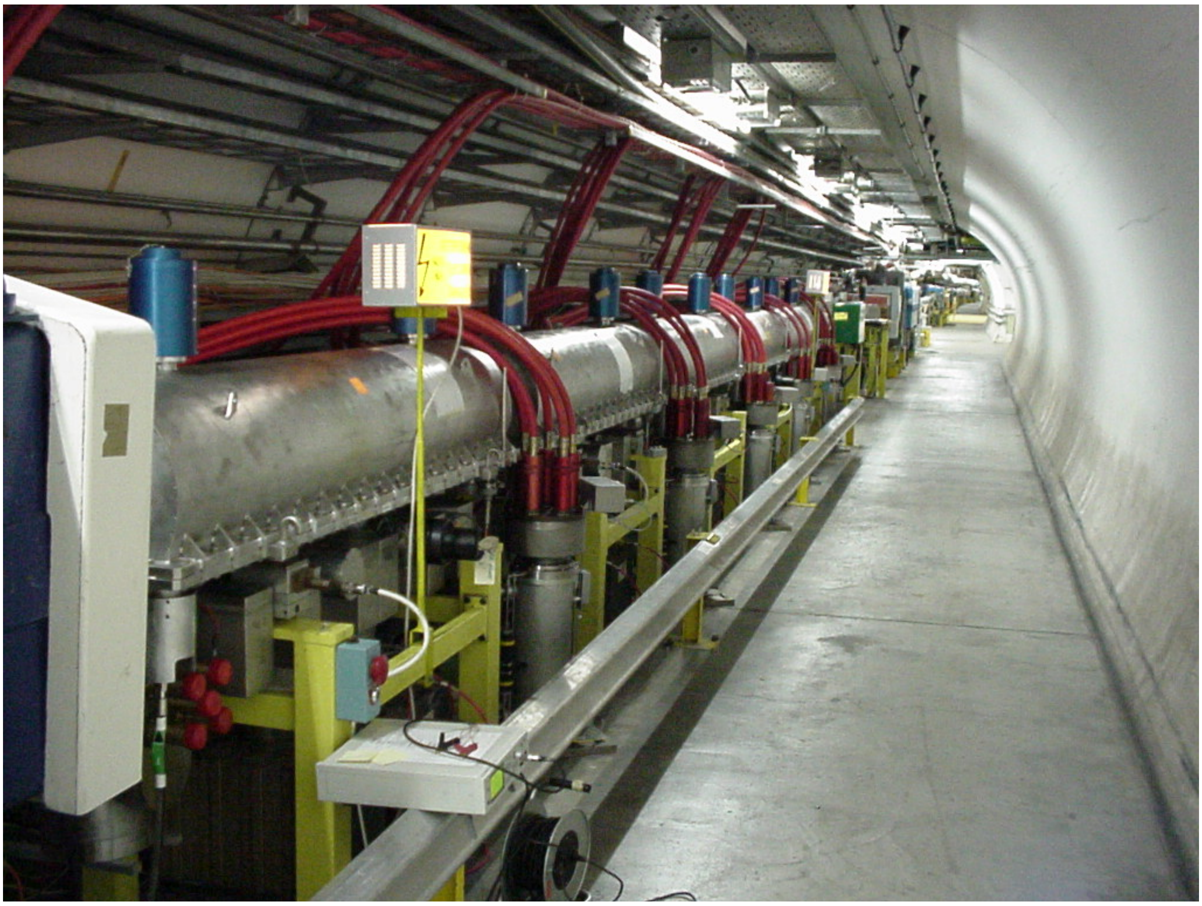
- Large Hadron Collider
  14 TeV
- Super Proton Synchrotron
  450 GeV
- Proton Synchrotron
  26 GeV

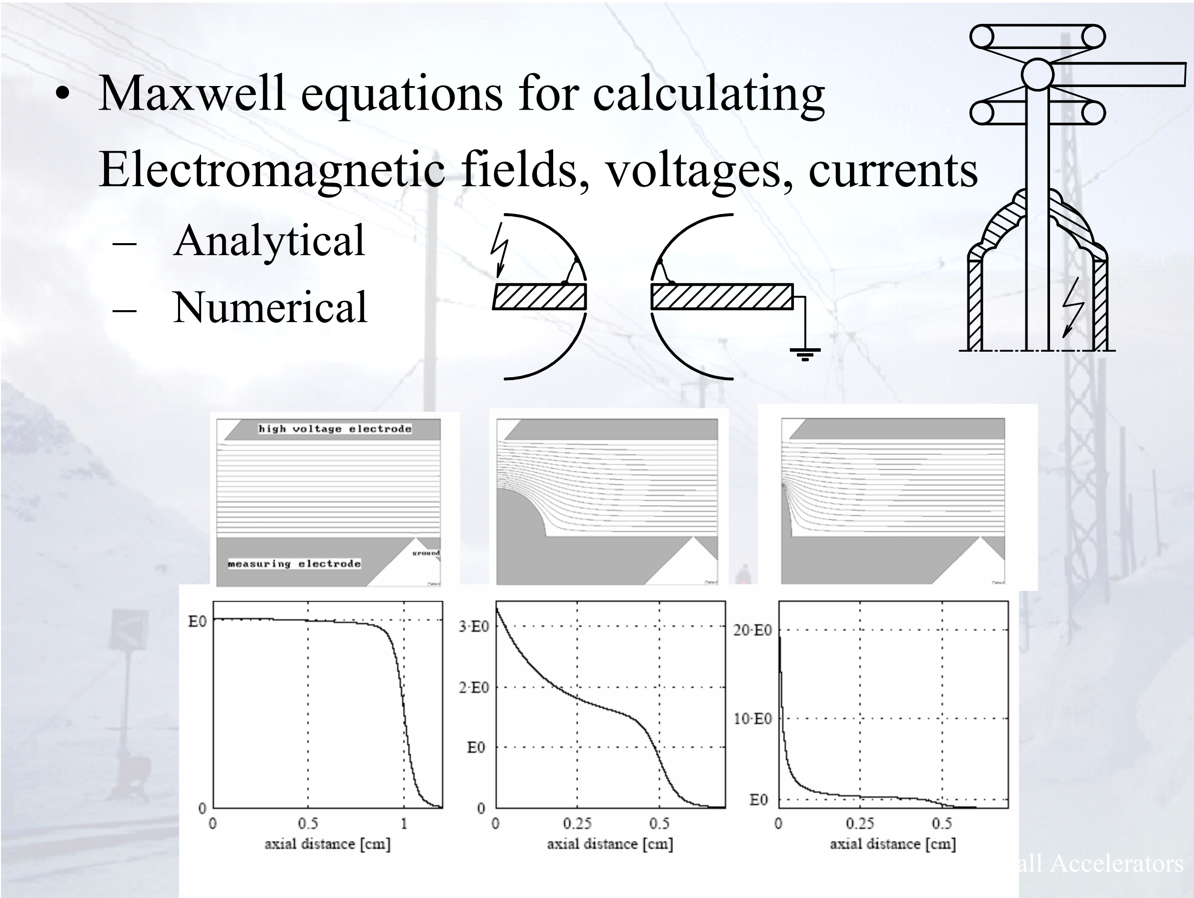
Septum: E ≤ 12 MV/m T = d.c.
l= 0.8 – 15m

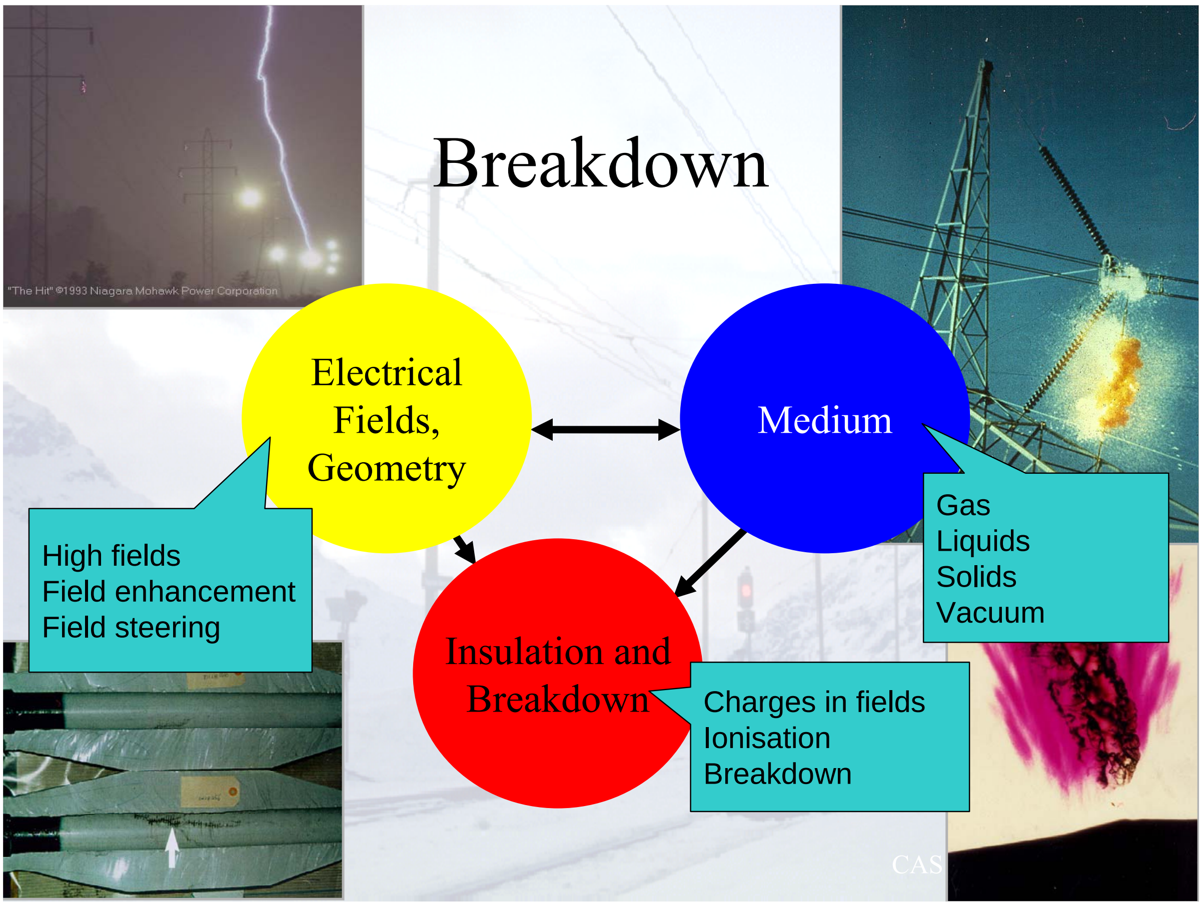
Kicker: V=80kV
B = 0.1-0.3 T T = 10 ns - 200µs
l=0.2 – 16m

RF cavities: High gradients, E ≤ 150MV/m


DANGER
RADIATION
MDSH




• Maxwell equations for calculating
Electromagnetic fields, voltages, currents
– Analytical
– Numerical
high voltage electrode
measuring electrode
ground
E0
0
0
0.5
1
axial distance [cm]
3·E0
2·E0
E0
0
0
0.25
0.5
axial distance [cm]
20·E0
10·E0
E0
0
0.25
0.5
axial distance [cm]
all Accelerators


Breakdown
"The Hit" ©1993 Niagara Mohawk Power Corporation
Electrical Fields, Geometry
Medium
Insulation and Breakdown
High fields
Field enhancement
Field steering
Gas
Liquids
Solids
Vacuum
Charges in fields
Ionisation
Breakdown
CAS

NUMERICAL FIELD SIMULATION METHODS

-CSM (Charge Simulation Method): (Coulomb)

Electrode configuration is replaced by a set of discrete charges

- **FDM** (Finite Difference Method):

Laplace equation is discretised on a rectangular grid

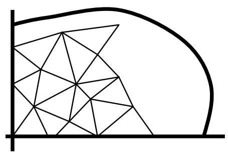
- **FEM (Finite Element Method):** Vector Fields (Opera, Tosca), Ansys, Ansoft

Potential distribution corresponds with minimum electric field energy (w=½εE²)

- **BEM (Boundary Element Method):** IES (Electro, Oersted)

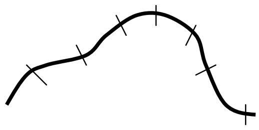
Potential and its derivative in normal direction on boundary are sufficient

Procedure FEM



1. Generate mesh of triangles:
2. Calculate matrix coefficients: $[S]_{ij} = (\nabla \alpha_i \cdot \nabla \alpha_j) A$
3. Solve matrix equation: $\begin{bmatrix} S_{kf} & S_{kp} \end{bmatrix} \begin{bmatrix} U_f \\ U_p \end{bmatrix} = 0$
4. Determine equipotential lines and/or field lines

Procedure BEM



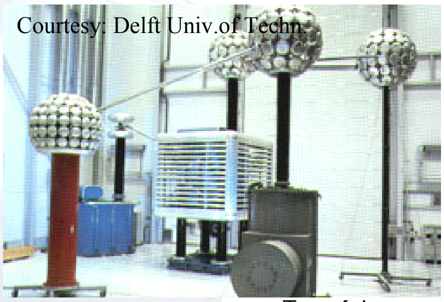
1. Generate elements along interfaces
2. Generate matrix coefficients: $H_{ij} = \int_{S_j} \frac{\partial \ln r_i}{\partial n} ds\,, \quad G_{ij} = \int_{S_j} \ln r_i ds$
3. Solve matrix equation: $\sum_{j=1}^{n} (H_{ij} - \pi \delta_{ij}) U_j = \sum_{j=1}^{n} G_{ij} Q_j$
4. Determine potential on abritary position:

$$U(x_0, y_0) = \frac{1}{2\pi} \left(\sum_{j=1}^{n} U_j \int_{S_j} \frac{\partial \ln r}{\partial n} ds - \sum_{j=1}^{n} Q_j \int_{S_j} \ln r ds \right)$$

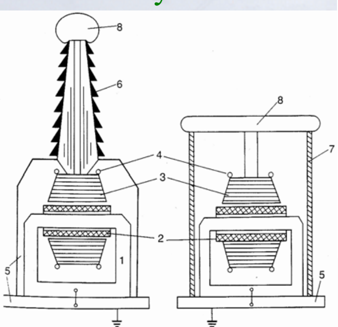
Generation of High Voltages

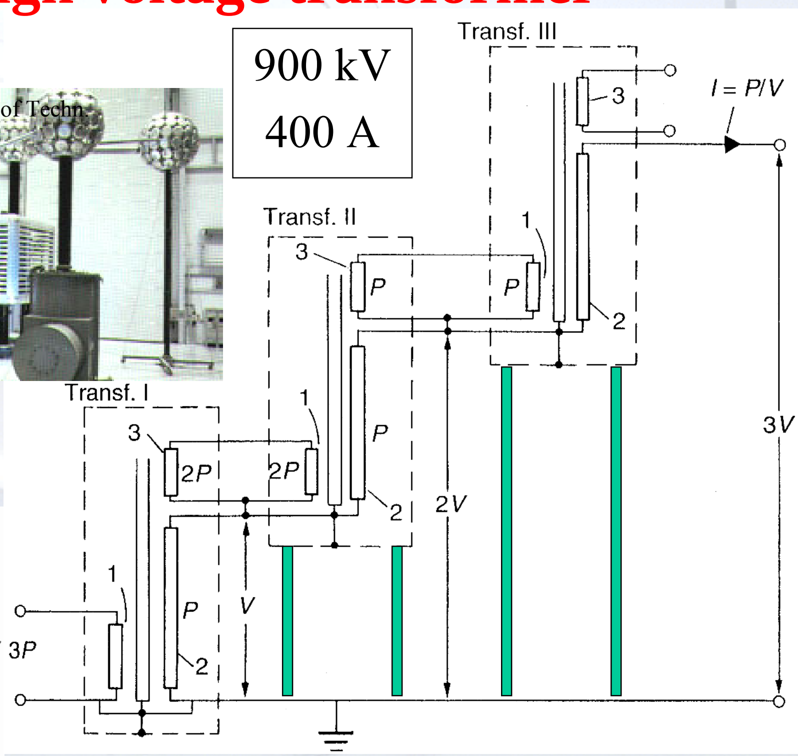
- AC Sources (50/60 Hz)
  - High voltage transformer (one coil; divided coils; cascade)
  - Resonance source (series; parallel)
- DC Sources
  - Rectifier circuits (single stage; cascade)
  - Electrostatic generator (van de Graaff generator)
- Pulse sources
  - Pulse circuits (single stage; cascade; pulse transformer)
  - Traveling wave generators (PFL; PFN; transmission line transformer)

Cascaded High voltage transformer

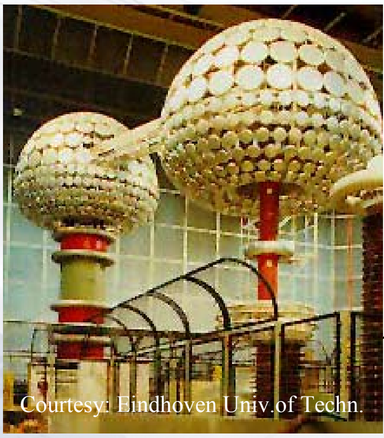


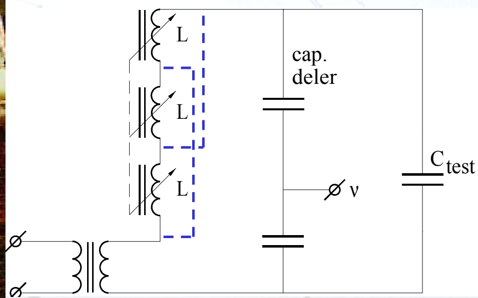
1: primary coil
2: secondary coil
3: tertiary coil



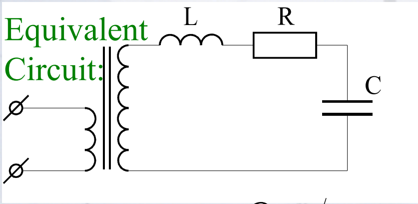


Resonance Source





Equivalent Circuit:



$$|H(\omega)| = \frac{Q\,\omega_0/\omega}{\sqrt{1+Q^2(\omega_0/\omega-\omega/\omega_0)^2}}$$

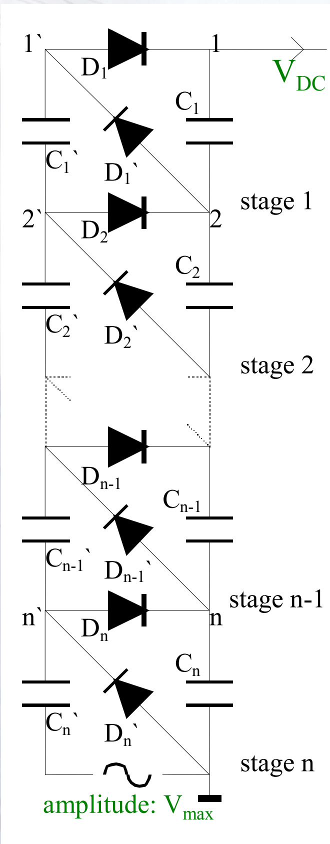
$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

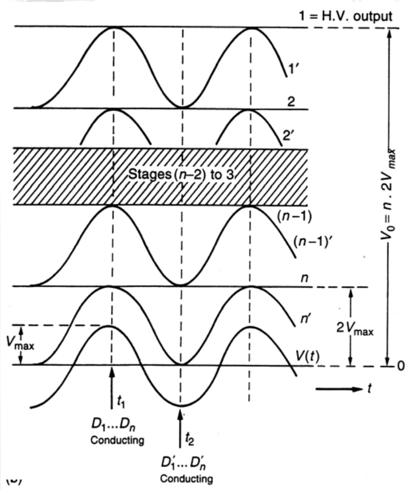
+ Waveform: almost perfect sinusoidal
+ Power: 1/Q of “normal” transformer
+ Short circuit: Q→0 results in V→0
- No resistive load

900 kV
100 mA

Cascaded Rectifier

(Greinacher; Cockcroft - Walton)

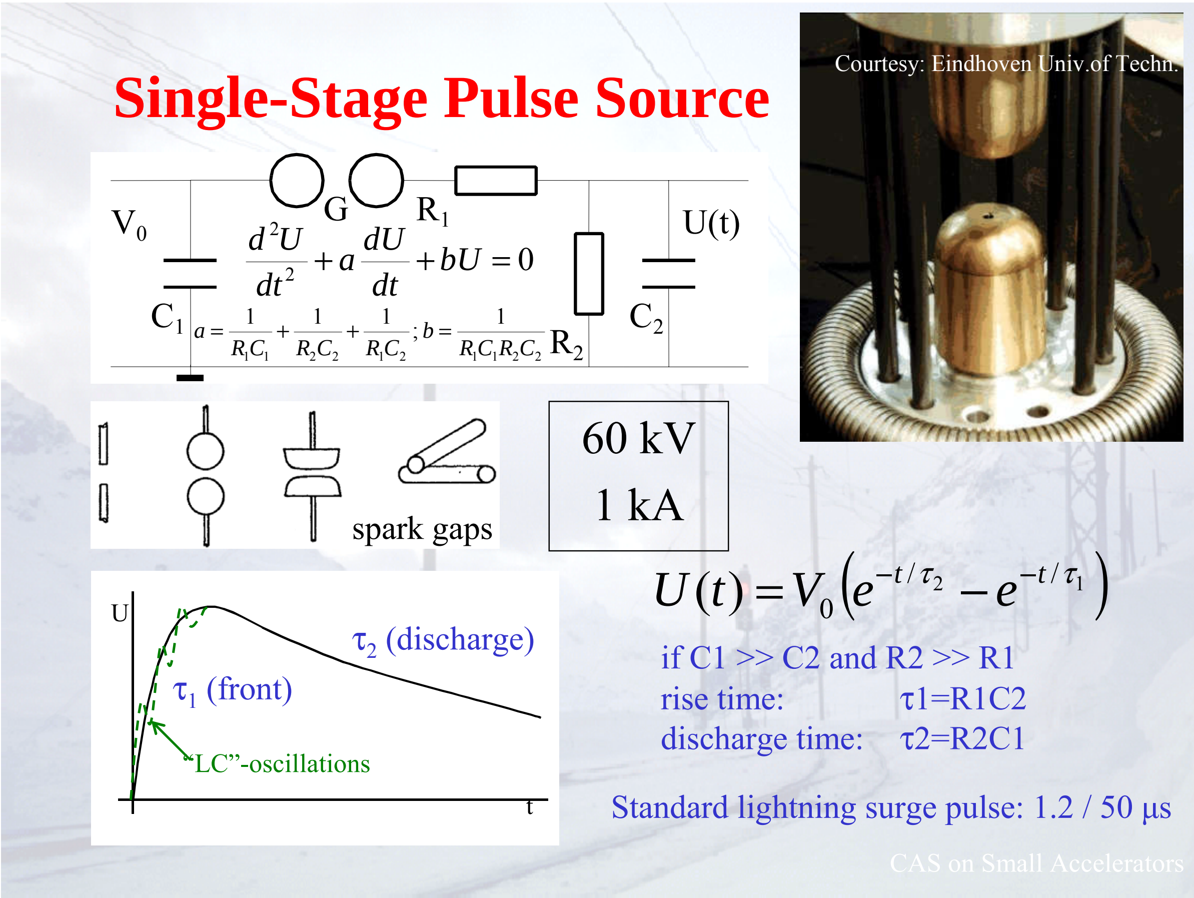


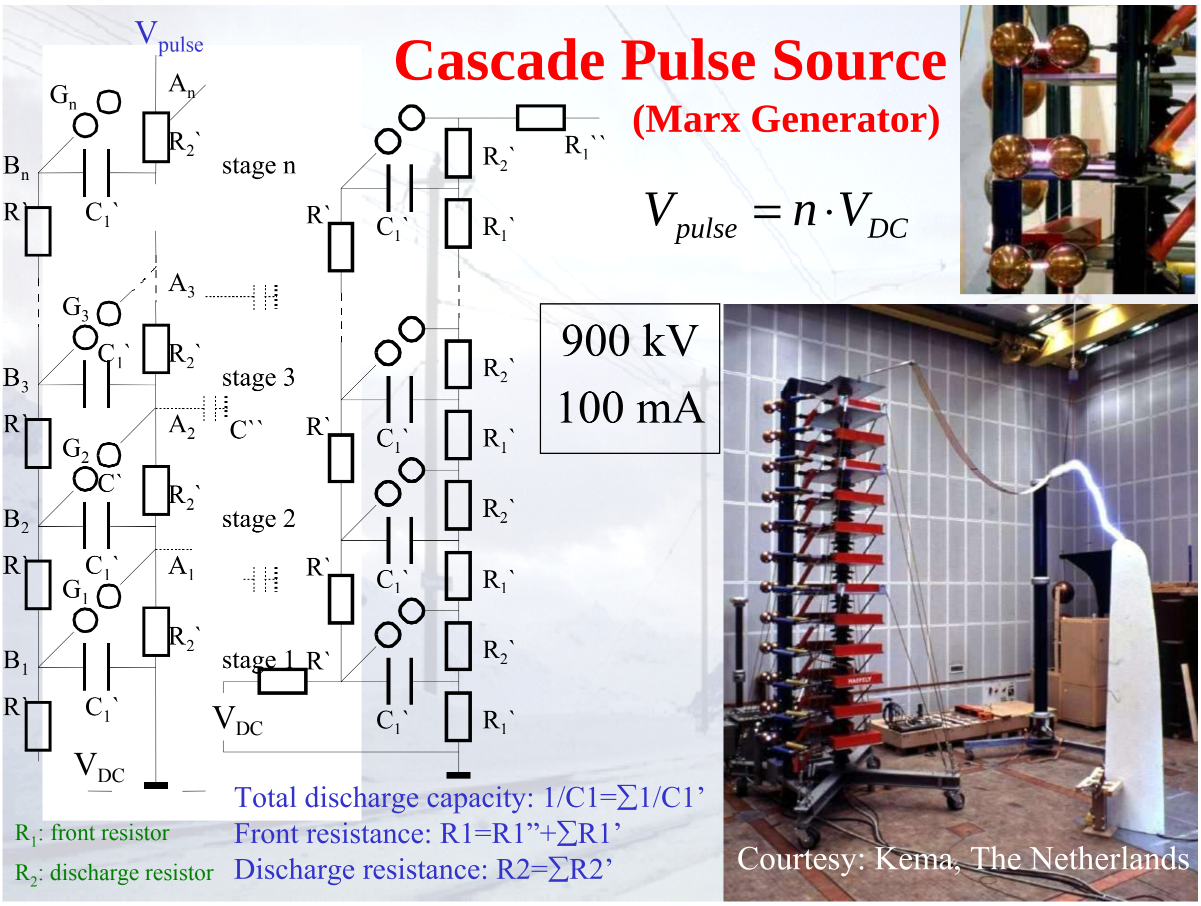


$$V_{DC} = 2nV_{max}$$

Voltage: 2 MV

Reduce δV (~n^2) and ΔV (~n^3) by:
larger C’s (more energy in cascade)
higher f (up to tens of kilohertz)

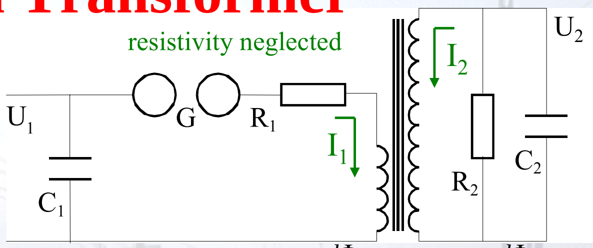

Single-Stage Pulse Source
Courtesy: Eindhoven Univ.of Techn.
V_0
G
R_1
U(t)
$\frac{d^2U}{dt^2} + a\frac{dU}{dt} + bU = 0$
C_1
$a = \frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_1C_2}; b = \frac{1}{R_1C_1R_2C_2}$
R_2
C_2
spark gaps
60 kV
1 kA
U
τ_2 (discharge)
τ_1 (front)
"LC"-oscillations
t
$U(t) = V_0\left(e^{-t/\tau_2} - e^{-t/\tau_1}\right)$
if C1 >> C2 and R2 >> R1
rise time: τ1=R1C2
discharge time: τ2=R2C1
Standard lightning surge pulse: 1.2 / 50 µs
CAS on Small Accelerators


Cascade Pulse Source
(Marx Generator)
V_{pulse}
G_n
A_n
B_n
R_2`
C_1`
stage n
R_1``
R_1`
$V_{pulse} = n \cdot V_{DC}$
A_3
G_3
B_3
stage 3
A_2
C``
G_2
B_2
stage 2
A_1
G_1
B_1
stage 1
V_{DC}
900 kV
100 mA
Total discharge capacity: 1/C1=∑1/C1'
Front resistance: R1=R1"+∑R1'
Discharge resistance: R2=∑R2'
R_1: front resistor
R_2: discharge resistor
Courtesy: Kema, The Netherlands

Voltage: 300 kV

Pulse Source with Transformer

resistivity neglected



$$\text{primary}: \quad L_1C_1\frac{d^2I_1}{dt^2} - MC_1\frac{d^2I_2}{dt^2} + I_1 = 0$$

$$\text{secondary}: \quad L_2C_2\frac{d^2I_2}{dt^2} - MC_2\frac{d^2I_1}{dt^2} + I_2 = 0$$

$$U_1(t) = \frac{d\Phi_1}{dt} \qquad U_2(t) = \frac{d\Phi_2}{dt}$$

Eigen frequencies from characteristic equation:

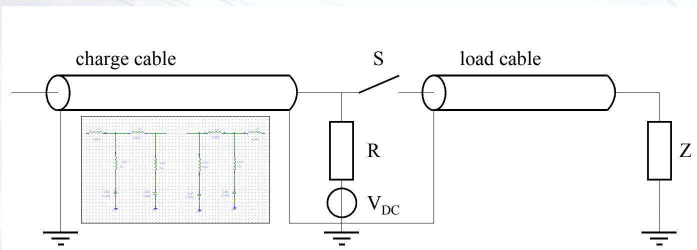
$$\begin{pmatrix} L_1C_1 & -MC_1 \\ -MC_2 & L_2C_2 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \frac{1}{\omega^2}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \Rightarrow \left(L_1C_1 - \frac{1}{\omega^2}\right)\left(L_2C_2 - \frac{1}{\omega^2}\right) - M^2C_1C_2 = 0$$

Approximation: transformer almost ideal: $k=M/\sqrt{(L_1L_2)}\rightarrow 1$

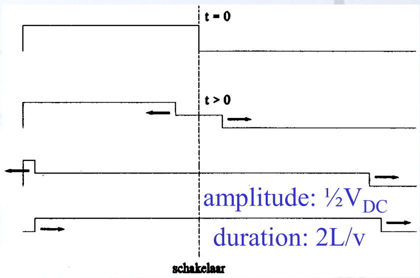
$$\omega_1 \approx \frac{1}{\sqrt{L_1C_1 + L_2C_2}} = \frac{1}{\sqrt{L_1(C_1 + C_2^{'})}} \quad , \quad \omega_2 \approx \frac{1}{\sqrt{1-k^2}}\sqrt{\frac{1}{L_1C_1} + \frac{1}{L_2C_2}} = \frac{1}{\sqrt{L_{eq}(C_1 // C_2^{'})}}$$

slow oscillation — fast oscillation

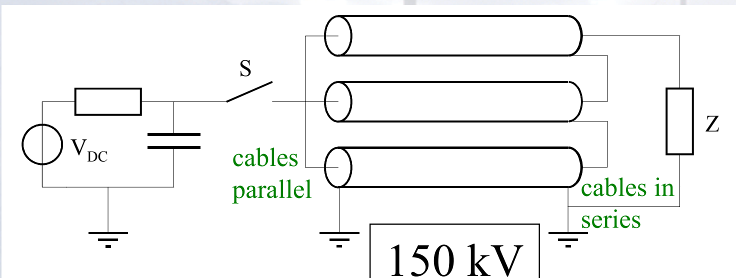
Pulse Forming Line / Network



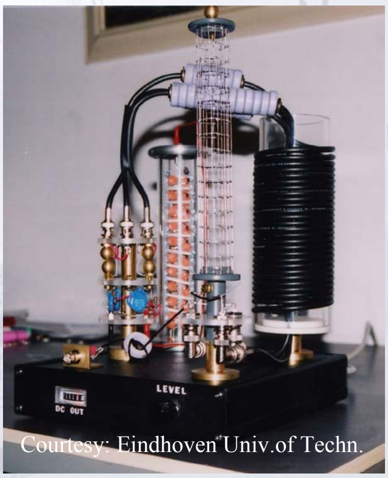
80 kV, 10 kA, T=20ns - 10µs



Transmission Line Transformer



150 kV
1 kA


Courtesy: Eindhoven Univ.of Techn.

Insulation and Breakdown

- In Gases
  - Ionisation and Avalanche Formation
  - Townsend and Streamer Breakdown
  - Paschen Law: Gas Type
  - Breakdown Along Insulator
  - Inhomogeneous Fields, Pulsed Voltages, Corona
- Insulating Liquids
- Solid Insulation
  - Breakdown types, Surface tracking, Partial discharges, Polarisation, tan δ
- Vacuum Insulation
  - Applications, Breakdown, Cathode Triple-Point, Insulator Surface Charging, Conditioning

CAS on Small Accelerators


400kV


400kV Geertruidenberg, The Netherlands


800kV South Africa

1^{st} free electron

- Cosmic radiation
- Shortwave UV
- Radio active isotopes

In air:
≈ 2.5×10^{19} molecules/cm^3
≈ 1000 ions/cm^3
≈ 10 electrons/cm^3

Free path, effective cross-section

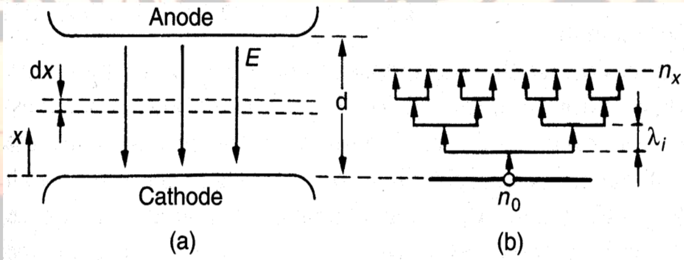
Townsend's 1^{st} ionisation coefficient α

One electron creates α new electrons per unit length

$A + e \rightarrow A^+ + 2e$

$n_e(x=d) = n_0 e^{\alpha d}$

$\alpha/p = f(E/P)$



CAS on Small Accelerators

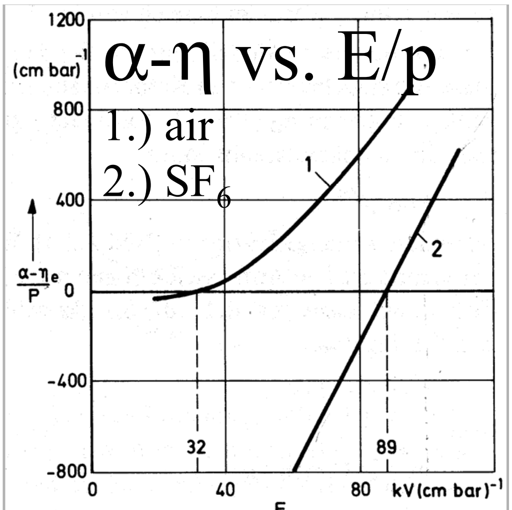
- Electro-negative gasses

Attachment η of electrons to ions

electrons: $n_e(x=d) = n_0 e^{(\alpha - \eta)d}$

negative ions:

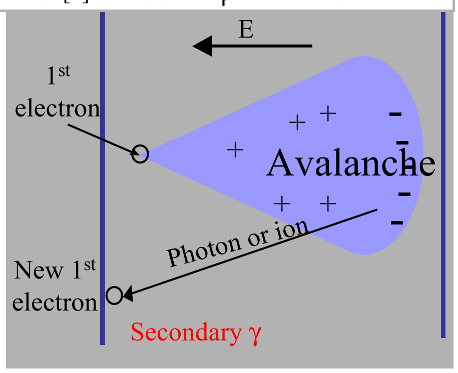
$$n_{-}(x=d) = \frac{n_0 \eta}{\alpha - \eta}[e^{(\alpha-\eta)d} - 1]$$



Reference [3]

Avalanche ≠ Breakdown; creation of secondaries

Townsend's 2^{nd} ionisation coefficient γ
one ion or photon creates γ new electrons
at cathode $n_e = \gamma n_0(e^{\alpha d}-1)$



Breakdown if: # secondary electrons ≥ n_0
$\alpha d \geq \ln(1/\gamma + 1)$

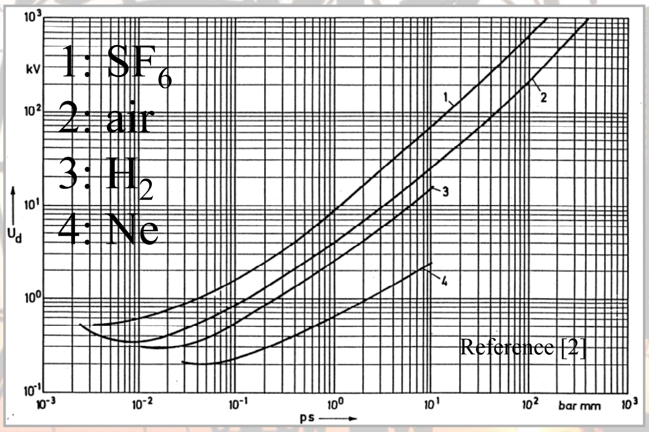
steep function of E/p → $e^{\alpha d}$ very steep → $(E/p)_{critical}$ and V_d well defined → γ of weak influence

Paschen law / breakdown field

- Townsend breakdown criterion $\alpha d = K$:

$$\frac{E_d}{p} = \frac{B}{\ln(Apd/K)} \qquad V_d = \frac{Bpd}{\ln(Apd/K)} \qquad \text{with} \quad A = \sigma_I/kT, \quad B = V_i \sigma_I/kT$$

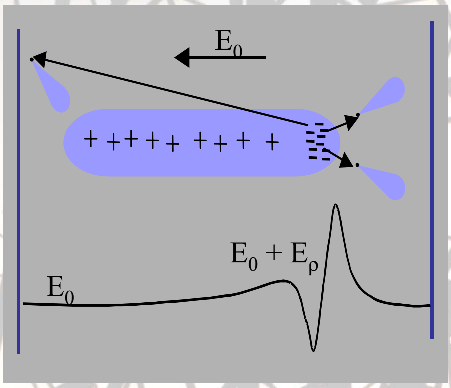
→ E_d and V_d depend only on p*d   p: pressure   d: gap length



Typically practically
E_{bd} = 10 kV/cm
at 1 bar in air

$V_{bd,\text{Paschen min, air}} \approx 300$ V

- Small p*d, d<< λ: few collisions, high field required for ionisation
- Large p*d, d>> λ: collision dominated, small energy build-up, high V_d



Streamer breakdown

Space charge field $E_\rho \approx E_0$

- Field enhancement
  extra ionising collisions (α↑)
- High excitation ⇒ UV photons

when 1 electron grows into ca. 10^8
then E_ρ large enough for streamer breakdown ($n_e \approx 2 \cdot 10^8$ in avalanche head)

Result:

- Secondary avalanches, directional effect (channel formation)
- Grows out into a breakdown within 1 gap crossing (anode and/or cathode directed)

Characteristic:

- Very fast
- Independent of electrodes (no need for electrode surface secondaries)
- Important at large distances (lightning)

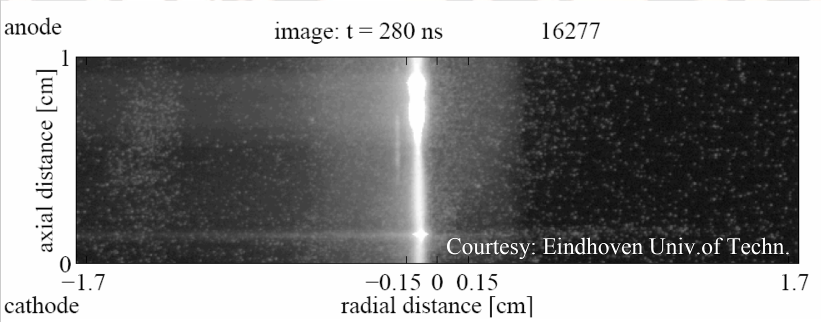
→Townsend, unless:

- Townsend: $\alpha d \geq \ln(1/\gamma + 1) \approx 7...9$ $(\gamma \approx 10^{-4}...10^{-3})$
- Streamer: $\alpha d \geq 18...20$

- Strong non-uniform field (small electrodes, few secondary electrons)

- Pulsed voltages
  - Townsend slow, ion drift, subsequent gap transitions
  - Streamer fast, photons, 1 gap transition

- High pressure
  - Less diffusion E_ρ high
  - photons absorbed in front of cathode
  - positive ions slower

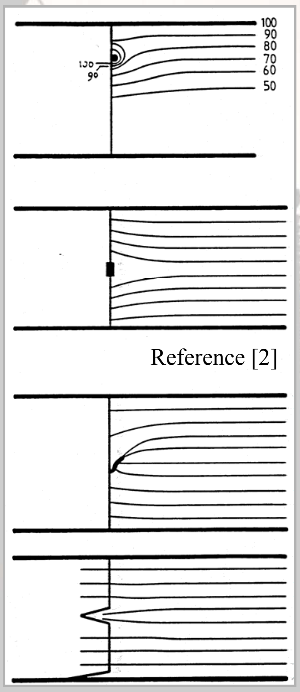


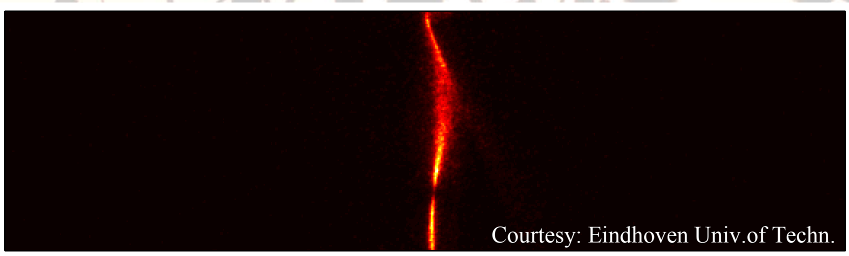
Laser-induced streamer breakdown in air.

Breakdown along insulator

- Surface charge
- (Non-regular) surface conduction
- Particles / contaminations on surface
- Non-regularities (scratches, ridges)

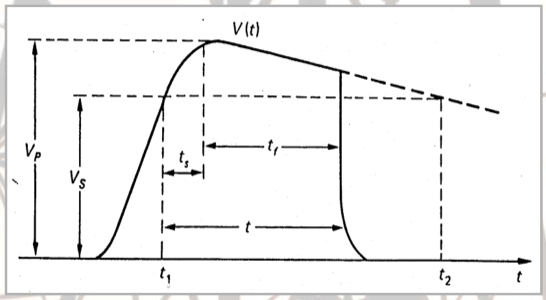
⇒Field enhancement
⇒Increased breakdown probability

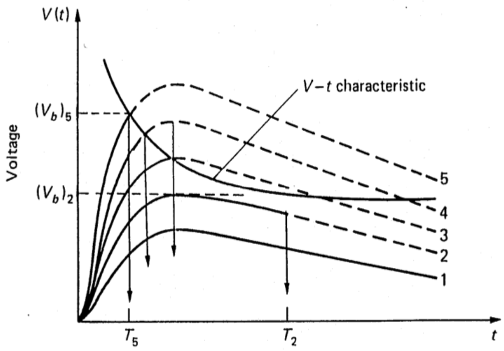


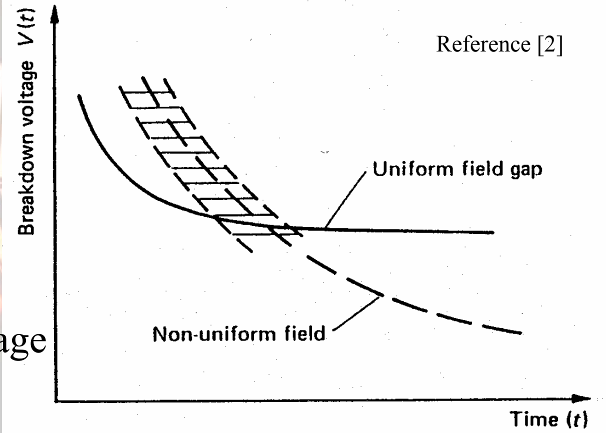


Prebreakdown along insulatior in air.

Breakdown at pulse voltages; time-lag







Reference [2]

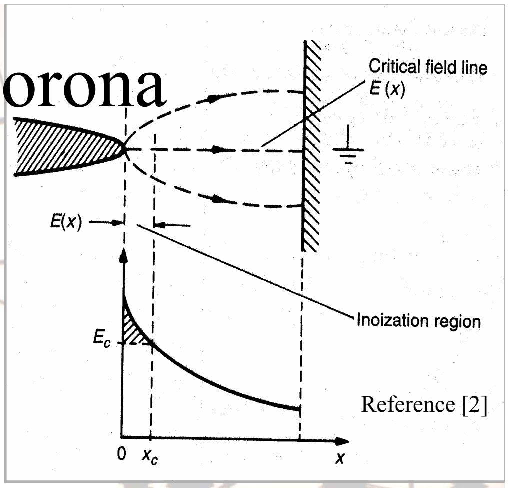
t_s, wait for first elektron

t_f, breakdown formation

- Townsend or Streamer

Short pulses, high breakdown voltage

Non-uniform fields; Corona



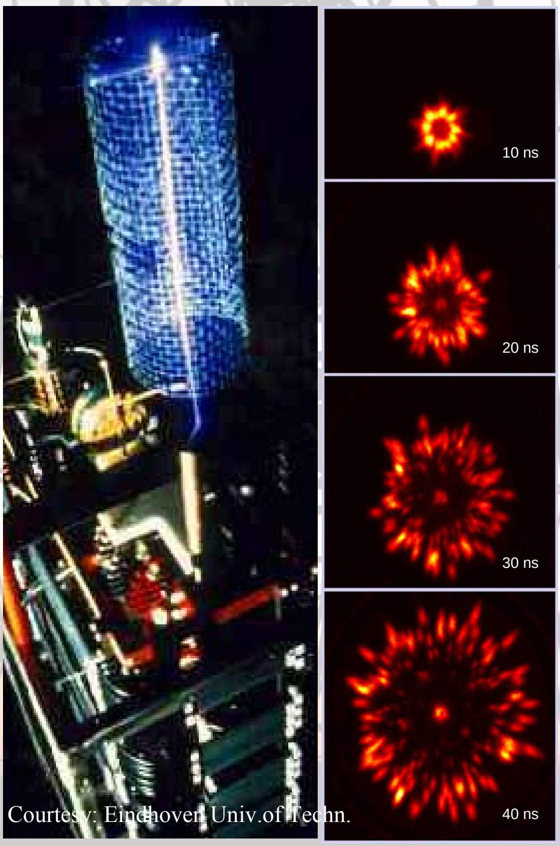
Reference [2]

Breakdown conditions:

- Global → Full breakdown
- Local → Streamer breakdown
  → Partial discharge

Non-uniform field:

- Discharge starts in high field region
- ....and “extinguishes” in low field region



Transmission line transformer

Corona

- Power loss; EM noise
- Chemical corrosion
+ Useful applications

Pulsed corona discharges:

- Fast, short duration HV pulses
- Many streamers, high density
- Generation of electrons, radicals, excited molecules, UV
- E.g. Flue gas cleaning

CAS on Small Accelerators

Solid insulation

Breakdown field strength:

- Very clean (lab): high
- Practical: lower due to imperfections
  - Voids
  - Absorbed water
  - Contaminations
  - Structural deformations

Anorganic Natural Synthetic	Quartz,mica,glas Porcelain Al_2O_3	Disc insulator Feedthrough Spacer
Paper + Oil		Cable Capacitor
Synthetic Organic Polymerisation	Polyethelene HD,LD,XL – PE Teflon Polystyrene, PVC, polypropene,etc	Spec. properties: Moisture content high T losses bonding
Epoxy	Hardener Filler	Moulding in mold

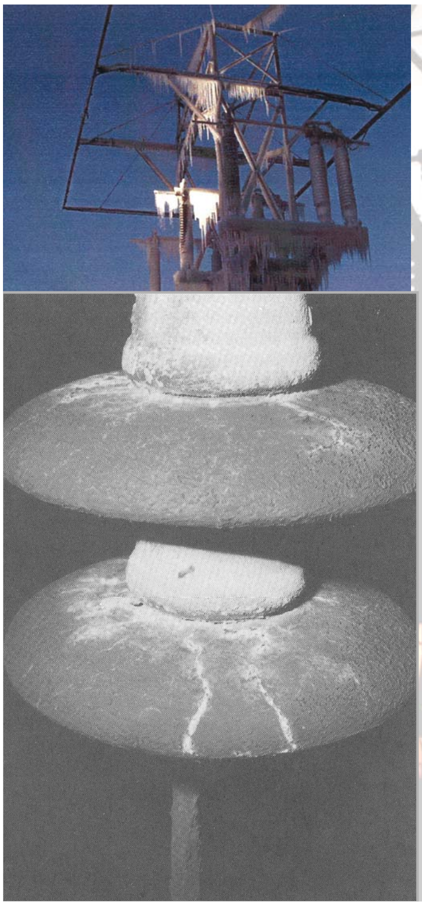
Requirements:

- Mechanical strength
- Contact with electrodes and semiconducting layers
- Resistant to high T, UV, dirt, contamination, rain, ice, desert sand

Problems:

- Surface tracking
- Partial discharges
  - In voids (in material or at electrodes, often created at production).

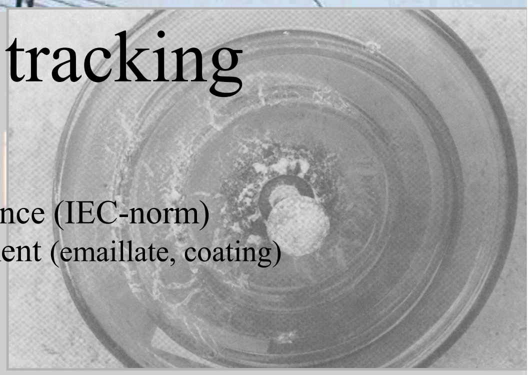
Types of solid insulation materials




Surface tracking

Remedies:

- Geometry
- Creeping distance (IEC-norm)
- Surface treatment (emaillate, coating)
- Washing



Vacuum insulation

Applications:

- Vacuum circuit breaker
- Cathode Ray Tubes / accelerators
- Elektron microscope
- X-ray tube
- Transceiver tube

Advantages:

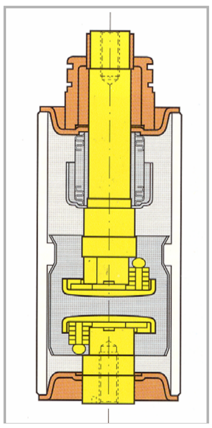
- "Self healing"
- No dielectric losses
- High breakdown fieldstrength
- Non flammable
- Non toxic, non contaminating

What is vacuum?

- "Pressure at which no collisions for Brownian "temperature" movements of electrons"
- $\lambda \gg$ characteristic distances
- E.g. $p = 10^{-6}$ bar, $\lambda = 400$ m

Disadvantages:

- Requires hermetic containment and mechanical support
- Quality determind by:
  - electrodes and insulators
  - Material choice, machining
  - Contaminations, conditioning



Characteristics of vacuum breakdown

No 1st electron from "gas"

- Cathode emission
  - primary: photoemission, thermic emission, field emission, Schottky-emission
  - secondary: e.g. e^- bombarded anode → $^+$ion collides at cathode → e^-

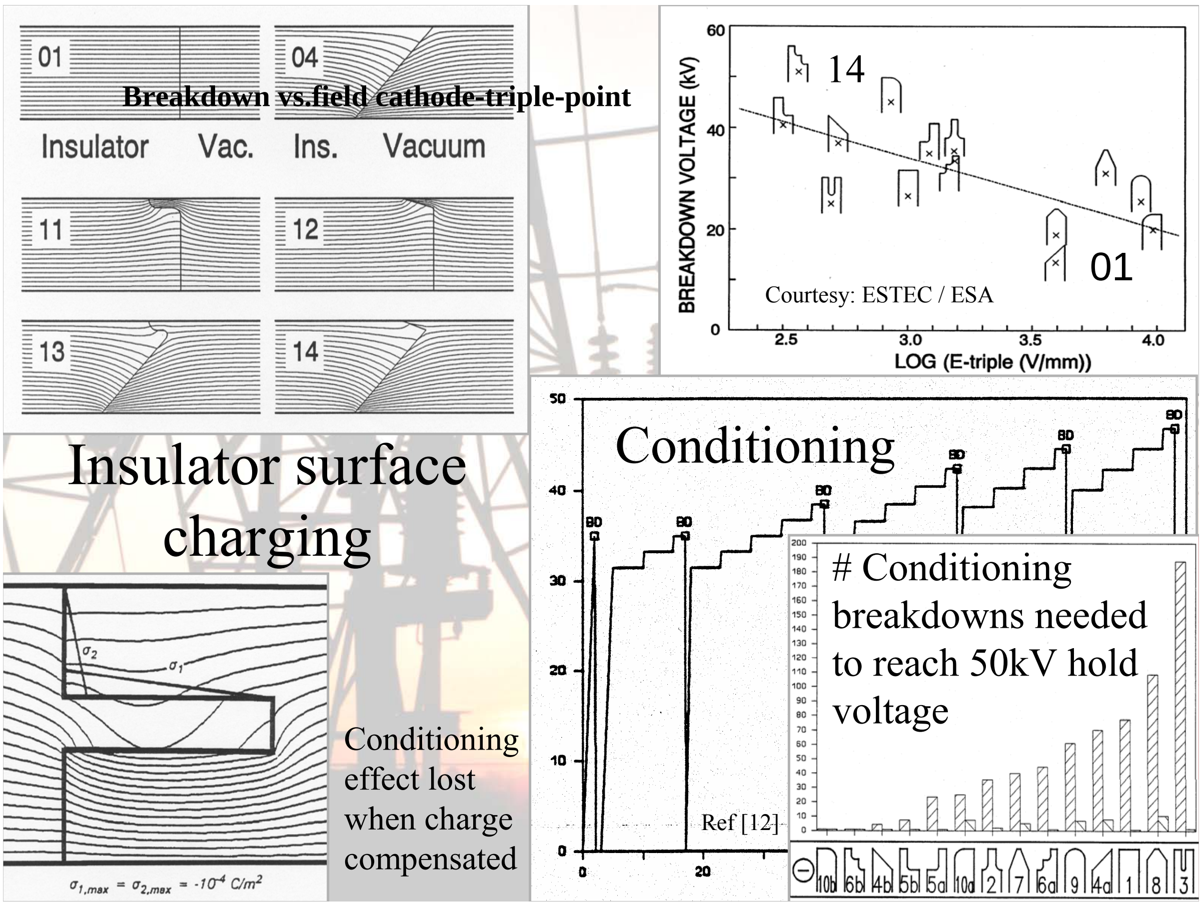
No breakdown medium

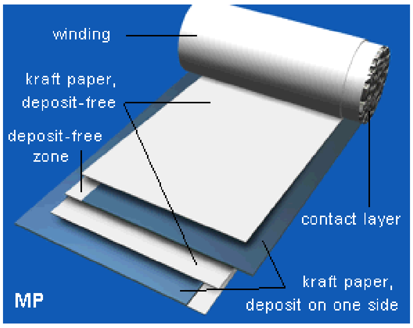
- No multiplication through collision ionisation
- Medium in which the breakdown occurs has to be created ("evaporated" from electrodes, insulators)

Important: prevent field emission

- Keep field at cathode and "cathode triple point" as low as possible
- Insulator surface charging, conditioning

CAS on Small Accelerators



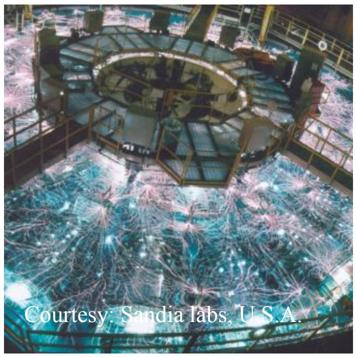


Insulating liquids

Applications:
- Transformers
- Cables
- Capacitors
- Bushings

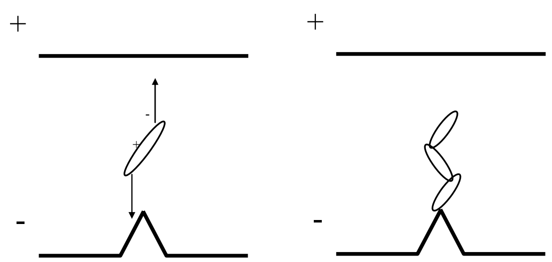
Requirements:
- Pure, dry and free of gases
- ε_r (high for C's, low for trafo) (demi water $\varepsilon_{r,d.c.} = 80$)
- Stable (T), non-flammable, non toxic (pcb's), ageing, viscosity

- No interface problems
- Combined cooling/insulation
- "Cheap" (no mould)
- Liquid tight housing



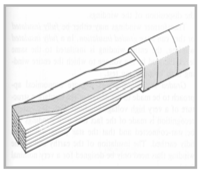
Breakdown fieldstrength:

- **Very clean (lab): high 1 - 4 MV/cm (In practice much lower)**
- **Important at production:outgassing, filtering, drying**
- **Mineral oil** ("old" time application, cheap, flammable)
- **Synthetic oil** (purer, specifically made, more expensive)
  - **Silicon oil** (very stable up to high T, non-toxic, expensive)
- **Liquid H2, N2, Ar, He** (supra-conductors)
- **Demi-water** (incidental applications, pulsed power)
- **Limitation V_{bd}**:
  - Inclusions: Partial discharges → Oil decomposition → Breakdown
  - Growth (pressure increase)
  - "extension" in field direction"
- **Particles drift** to region with highest E → bridge formation → breakdown



CAS on Small Accelerators

Transformer:



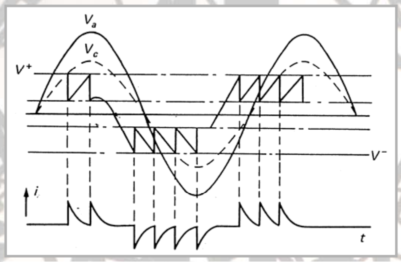
- **Mineral oil**: Insulation and cooling
- **Paper**: Barrier for charge carriers and chain formation
  - Mechanical strength
- **Ageing**
  - Thermical and electrical (partial discharges)
  - Lifetime: 30 years, strongly dependent on temperature, short-circuits, over-loading , over-voltages
  - Breakage of oil moleculs, Creation of gasses, Concentration of various gas components indication for exceeded temperature

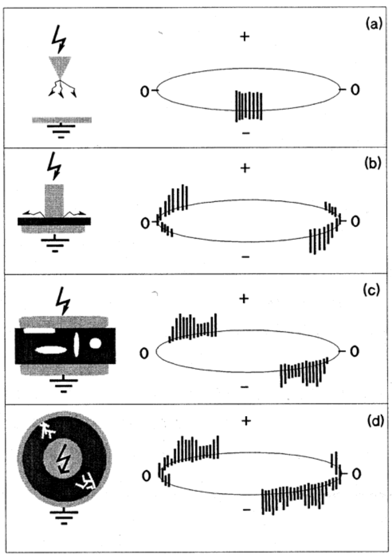
    (as specified in IEC599)

- **Lifetime**
  - Time in which paper looses 50 % of its mechanical strenght
  - Strongly dependent on:
    - Moisture (from 0.2 % to 2 % accelerated ageing factor 20)
    - Oxygen (presence accelerates ageing by a factor 2)

Measurement techniques

Partial discharges





- UV, fast electrons, ions, heat
- Deterioration void:
  - Oxidation, degradation through ion-impact
  - “Pitting”, followed by treeing
- Eventually breakdown

Acceptable lifetime? Preferably no partial discharges.

- High sensitivity measurements on often large objects
- $Q_{app.} \neq Q_{real}$, still useful, because measure for dissipated energy, thereby for induced damage
- relative measurement

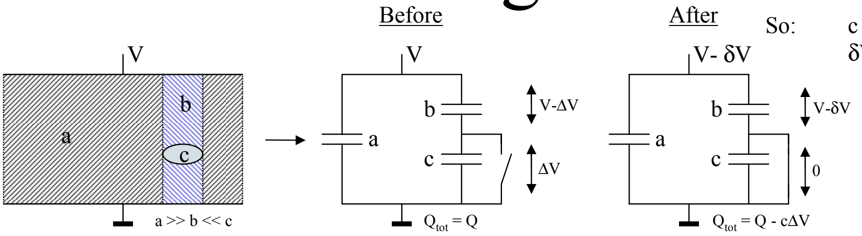
AC voltage phase resolved discharge pattern detection → Type of defect

CAS on Small Accelerators

Partial discharges

Before: Q = aV + b (V - ΔV) + c ΔV
After: Q - c ΔV = (a + b)(V - δV)

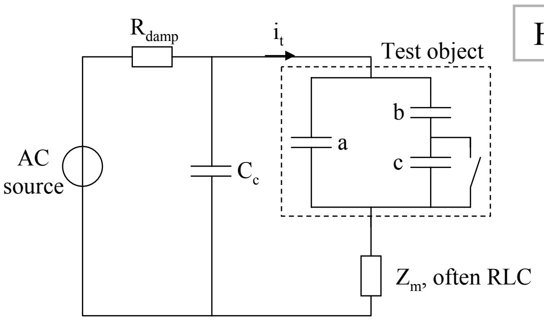
So: c ΔV = a δV + b (δV - ΔV) + c ΔV
δV = ΔV b/(a+b) ≈ ΔV b/a

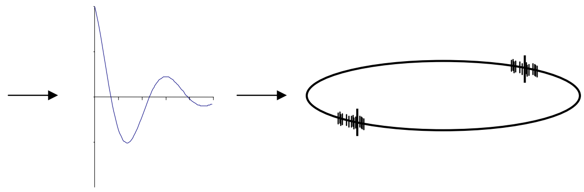


Apparent charge:
$C_{tot}\,\delta V = (a + bc/(b+c))\,\delta V$
$\approx (a+b)\,\delta V \approx a\delta V$

$$\frac{Q_{app}}{Q_{real}} = \frac{a\delta V}{c\Delta V} = \frac{b\Delta V}{c\Delta V} = \frac{b}{c} \approx \frac{\varepsilon_r . d_{void}}{d_{insul}}$$

High sensitivity measurement because b/c << 1





- Q_{app} gives i_t ($Q_{app} = \int i_t dt$)
- C_c gives i_t if $C_c >> C_{object}$
- Calibration through injecting known charge

- Measure with resonant RLC circuit:
  - Excitation by short pulse i_t
  - No 50 Hz problem
  - $V = q/C \exp(-\alpha t) \{ \cos\beta t - \alpha/\beta \sin\beta t \}$
    $\alpha=1/(2RC)$ $\beta=[1/(LC) - \alpha^2]^{-1/2}$

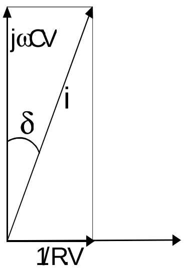
Loss angle, tan(δ)

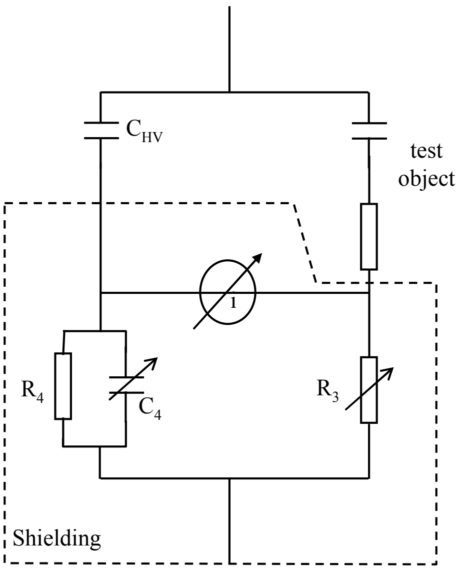
Sources:

- Conduction σ (for DC or LF)
- Partial discharges
- Polarisation

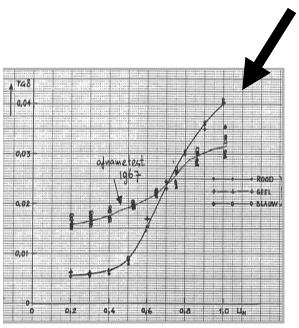
Schering bridge:

- i=0, $RC=R_4C_4$
- Gives: tan(δ)
  - parallel: 1/ωRC
  - serie: ωRC

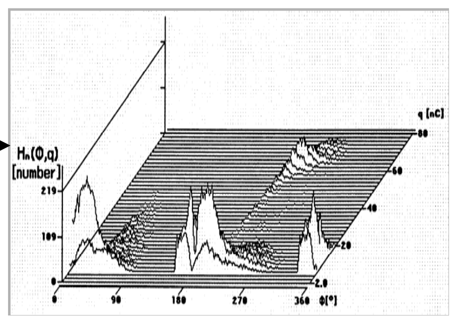




Tan δ:
- "Bulk" parameter
- No difference between phases



PD:
- Detection of weakest spot
- Largest activity and asymmetry in "blue" phase (ridge discharges)



Summary

Seen many basic high voltage engineering technology aspects here:

- High voltage generation
- Field calculations
- Discharge phenomena

The above to be applied in your practical accelerator environments as needed:

- Vacuum feed through: Triple points
- Breakdown field strength in air 10kV/cm
- Challenging calculations for real practical geometries.

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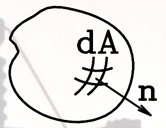
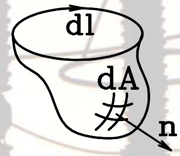
Appendix I

Maxwell equations in integral form

	Electrostatic		
$\oint\!\!\oint \vec{D}\cdot d\vec{A} = \iiint \rho\, dV = Q_{omsl.}$	$\oint\!\!\oint \vec{D}\cdot d\vec{A} = Q_{omsl.}$	(1)	
$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{A} = -\frac{d\phi_{omsl.}}{dt}$	$\oint \vec{E}\cdot d\vec{l} = 0$	(2)	
	Magnetostatic		
$\oint\!\!\oint \vec{B}\cdot d\vec{A} = 0$	$\oint\!\!\oint \vec{B}\cdot d\vec{A} = 0$	(3)	
$\oint \vec{H}\cdot d\vec{l} = \iint (\vec{J} + \frac{\partial \vec{D}}{\partial t})\cdot d\vec{A}$	$\oint \vec{H}\cdot d\vec{l} = I_{omsl.}$	(4)	





Maxwell equations in differential form

	Electrostatic	No space charge	
$\vec{\nabla}\cdot\vec{D} = \rho$	$\vec{\nabla}\cdot\vec{D} = \rho$	$\vec{\nabla}\cdot\vec{D} = 0$	(5)
$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\vec{\nabla}\times\vec{E} = 0$	$\vec{\nabla}\times\vec{E} = 0$	(6)
	Magnetostatic	**In area without source**	
$\vec{\nabla}\cdot\vec{B} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$	(7)
$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\vec{\nabla}\times\vec{H} = \vec{J}$	$\vec{\nabla}\times\vec{H} = 0$	(8)

Appendix III Finite Element Method (FEM)

Field energy minimal inside each closed region G:

$$W = \int_G \tfrac{1}{2}\varepsilon\left|\vec{E}\right|^2 dV = \int_G \tfrac{1}{2}\varepsilon\left|\nabla U\right|^2 dV$$

Assume U satisfies Laplace equation, but U' does not, then

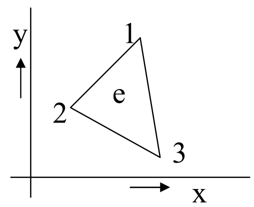
$$W_{U'} - W_U \geq 0:$$

$$W_{U'} - W_U = \tfrac{1}{2}\varepsilon \iiint_G \left(\left|\nabla U'\right|^2 - \left|\nabla U\right|^2\right) dV = \cdots = \tfrac{1}{2}\varepsilon \iiint_G \left|\nabla U' - \nabla U\right|^2 dV \geq 0$$

Field energy for one element (2-dim)

Potential is linear inside element: $U = a + bx + cy = \begin{pmatrix}1 & x & y\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}$

on corners: $\begin{pmatrix}U_1\\ U_2\\ U_3\end{pmatrix} = \begin{pmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}$



Potential can be written as:

$$U = \begin{pmatrix}1 & x & y\end{pmatrix}\begin{pmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{pmatrix}^{-1}\begin{pmatrix}U_1\\ U_2\\ U_3\end{pmatrix} = \sum_{i=1}^{3} U_i \alpha_i(x,y)$$

Field energy in element (e):

$$W^{(e)} = \tfrac{1}{2}\varepsilon U^T \left[S^{(e)}\right] U \quad \text{with} \quad S_{ij}^{(e)} = \iint \left(\nabla\alpha_i \cdot \nabla\alpha_j\right) dx\,dy$$

α's are linear in x and y ⇒ ∇α is constant: $S_{ij} = (\nabla\alpha_i \cdot \nabla\alpha_j)\, A^{(e)}$

Total field energy of n elements

All elements together: $U^T = (U_1 \quad \cdots \quad U_m \quad U_{m+1} \quad \cdots \quad U_n) \equiv (U_f \quad U_p)$

free *prescribed*

free: potential values to be determined

prescribed: potential according to boundary conditions

$$W = \tfrac{1}{2}\varepsilon U^T [S] U = \tfrac{1}{2}\varepsilon \left[U_{f'}^T \quad U_{p'}^T\right] \begin{bmatrix} S_{f'f} & S_{f'p} \\ S_{p'f} & S_{p'p} \end{bmatrix} \begin{bmatrix} U_f \\ U_p \end{bmatrix}$$

Partial derivatives of W to U_k are zero for $1 \le k \le m$ (m equations):

$$\frac{\partial W}{\partial U_k} = 0 \quad \Rightarrow \quad \begin{bmatrix} S_{kf} & S_{kp} \end{bmatrix} \begin{bmatrix} U_f \\ U_p \end{bmatrix} = 0$$

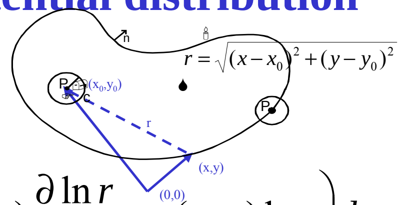
Boundary Element Method (BEM)

Boundaries uniquely prescribe potential distribution

Laplace equation : $\Delta u = 0$

border $\Gamma_1 : u(x,y)$ (Dirichlet)

border $\Gamma_2 : q(x,y) \equiv \dfrac{\partial u}{\partial n}$ (Neumann)



$P_0=(x_0,y_0)$ inside Γ: $$u(x_0,y_0) = \frac{1}{2\pi} \oint_\Gamma \left(u(x,y) \frac{\partial \ln r}{\partial n} - q(x,y) \ln r \right) ds$$

Problem: $u(x,y)$ and $q(x,y)$ not both known at the same time

Green II (2-dim): $$\oint (u\nabla v - v\nabla u)\cdot \hat{n} ds = \iint (u\Delta v - v\Delta u) dxdy$$

Choose $v(x,y)=\ln(1/r)$ $\Delta v(x,y)=0$ for $P \neq P_0(x_0,y_0)$

Exclude region σ around P_0 by means of circle c

$$\oint_{\Gamma+c} \left(u \frac{\partial \ln r^{-1}}{\partial n} - \ln r^{-1} \frac{\partial u}{\partial n}\right) ds = \iint_{\Omega-\sigma} (u \Delta \ln r^{-1} - \ln r^{-1} \Delta u) dxdy = 0$$

$$-\oint_\Gamma \left(u \frac{\partial \ln r}{\partial n} - \ln r \frac{\partial u}{\partial n}\right) ds + \oint_c \left(u \frac{\partial \ln r^{-1}}{\partial n} - \ln r^{-1} \frac{\partial u}{\partial n}\right) ds = 0$$

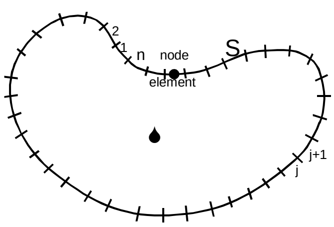
$$\lim_{\varepsilon \downarrow 0} \int_0^{2\pi} \left(u \frac{1}{\varepsilon} + \frac{\partial u}{\partial n} \ln \varepsilon \right) \varepsilon d\vartheta = 2\pi u(x_0,y_0)$$

$P_i=(x_i,y_i)$ on border Γ: $$u(x_i,y_i) = \frac{1}{\pi} \oint_\Gamma \left(u(x,y) \frac{\partial \ln r}{\partial n} - q(x,y) \ln r \right) ds$$

Discretisation:

$$\pi U(x_i,y_i) = \sum_{j=1}^{n} U_j \int_{S_j} \frac{\partial \ln r_{ij}}{\partial n} ds - \sum_{j=1}^{n} Q_j \int_{S_j} \ln r_{ij} ds$$

H $_{ij}$ G $_{ij}$



In matrix notation:

$$\sum_{j=1}^{n} (H_{ij} - \pi\delta_{ij}) U_j = \sum_{j=1}^{n} G_{ij} Q_j$$

Generates missing information

Appendix II Electrical Fields

Vacuum and matter

Dielectric displacement **D** | Electric field strength **E**
Magnetic induction **B** | **M**agnetic field strength **H**

Ohm's law: relation between current density **J** in a conductor and specific conduction σ_s (=1/ρ_s) and electrical field **E**

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \quad ; \quad \vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) = \mu_0 \mu_r \vec{H} = \mu \vec{H} \quad ; \quad \vec{J} = \sigma_s \vec{E} \qquad (9,10,11)$$

Polarisation **P**
Magnetisation **M**

(Static) boundary conditions for normal (index n) and tangential (index t) field components in terms of surface charge and surface current:

$$\oiint \vec{D} \cdot \vec{dA} = Q_{omsl.} \quad \Rightarrow \quad D_{1n} - D_{2n} = \sigma \qquad (12)$$

$$\oint \vec{E} \cdot \vec{dl} = 0 \quad \Rightarrow \quad E_{1t} - E_{2t} = 0 \qquad (13)$$

$$\oiint \vec{B} \cdot \vec{dA} = 0 \quad \Rightarrow \quad B_{1n} - B_{2n} = 0 \qquad (14)$$

$$\oint \vec{H} \cdot \vec{dl} = I_{omsl.} \quad \Rightarrow \quad H_{1t} - H_{2t} = J^* \qquad (15)$$

Conservation of charge / continuity of current

Ampere's law (8) in differential form gives:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{H} \right) = \vec{\nabla} \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (16)$$

Gauss Theorem gives the integral form:

$$\oiint (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \cdot \vec{dA} = 0 \quad \text{or} \quad \oiint \vec{J} \cdot \vec{dA} + \frac{\partial Q_{omsl.}}{\partial t} = 0 \qquad (17)$$

Electrical potential

(6) Gradient or (scalar) potential U:

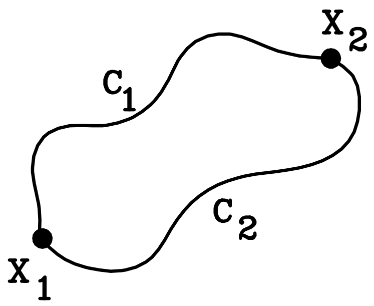
$$\vec{\nabla} \times \vec{E} = \vec{0} \quad \text{define } \vec{E} = -\vec{\nabla} U \quad \Rightarrow \quad U(x) = -\int_{x'(U=0)}^{x} \vec{E} \cdot \vec{dl} \qquad (21)$$

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot \vec{\nabla} U = -\Delta U \quad \text{so} \quad \Delta U = -\frac{\rho}{\varepsilon} \quad \text{(Poisson)} \qquad (22)$$

$$\text{Without space charge: } \Delta U = 0 \quad \text{(Laplace)} \qquad (23)$$

Equations define every position in space between potential and charge distribution, given the boundary conditions.

Definition of potential only valid in the absence of varying magnetic induction. If $\partial B / \partial t \neq 0$ no more (scalar) potentials, only voltage differences, which have become dependent on the path (non-conservative field):



$$\int_{C_1} \vec{E} \cdot \vec{dl} - \int_{C_2} \vec{E} \cdot \vec{dl} = \oint \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \iint \vec{B} \cdot \vec{dA} \quad (24)$$

or

$$[V_1 - V_2]_{via\ C_1} - [V_1 - V_2]_{via\ C_2} \neq 0 \quad (25)$$

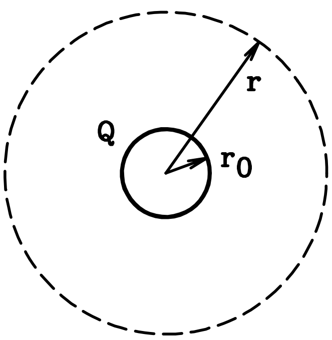
Analytical methods

Gauss' law in integral form (1), charges → E-field (and potential) for configuration.
Potential equations (22) or (23) for given boundary conditions → potential and E-field.

Fields from charges			Fields from Potentials		
Q and Gauss' law	⇒	E	Laplace / Poisson + boundary (V)	⇒	U
Integration of E	⇒	U	Differentiation of U	⇒	E
U on boundary	⇒	V	E and Gauss' law	⇒	Q
V and Q	⇒	C	Q and V	⇒	C

Fields from charges

Gauss' law, electrical field coupled to charge.
Two concentric spheres:



$$\oiint \varepsilon_0 \varepsilon_r \vec{E} \cdot \vec{dA} = Q \quad \Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_o \varepsilon_r r^2} \quad (26)$$

$$U(r) = -\int E(r)\,dr = \frac{Q}{4\pi\varepsilon_0 \varepsilon_r r} + Integrationconstant \quad (27)$$

Potential in infinity zero →

$$U(r) = \frac{Q}{4\pi\varepsilon_0 \varepsilon_r r}$$

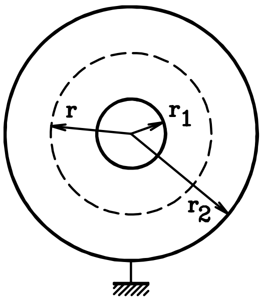
Capacity of sphere with radius r_0:

$$C = \frac{Q}{U(r_0)} = \frac{Q}{V} = 4\pi\varepsilon_0 \varepsilon_r r_0 \quad (29)$$

Field maximum on sphere surface:

$$E_{\max} = E(r_0) = \frac{Q}{4\pi\varepsilon_0\varepsilon_r r_0^2} = \frac{V}{r_0} \tag{30}$$

Concentric metal spheres



$$U(r) = \frac{Q}{4\pi\varepsilon_0\varepsilon_r}\left[\frac{r_2 - r}{r_2\, r}\right] \quad (r_1 \le r \le r_2) \tag{31}$$

$$C = 4\pi\varepsilon_0\varepsilon_r \frac{r_1\, r_2}{r_2 - r_1} \tag{32}$$

Maximum field on surface inner sphere:

$$E_{\max} = E(r_1) = \frac{V}{r_1(1 - r_1/r_2)} \tag{33}$$

Concentric cylinders

Two concentric cylinders with inner radius r_1 and outer radius r_2:

$$E(r) = \frac{Q/l}{2\pi\varepsilon_0\varepsilon_r r} \quad \Rightarrow \quad U(r) = \frac{Q/l}{2\pi\varepsilon_0\varepsilon_r}\cdot\ln(r_2/r) \tag{34}$$

$$C/l = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln(r_2/r_1)} \tag{35}$$

$$E_{\max} = \frac{V}{r_1 \ln(r_2/r_1)} \tag{36}$$

Fields from potential equations

Poisson equation or (without space charge, ρ = 0) Laplace:

$$\begin{aligned} &\text{Poisson:} \quad \Delta U = -\frac{\rho}{\varepsilon} \\ &\text{Laplace:} \quad \Delta U = 0 \end{aligned} \tag{37}$$

General solution + boundary conditions → specific solution.
Differentiation gives E-field, applying Gauss gives the charge. From charge and voltage → capacity.

Laplace equation in Cartesian, cylindrical or spherical coordinates:

$$\text{Carthesian}: \quad \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

$$\text{Cylindrical}: \quad \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial U}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 U}{\partial \vartheta^2} + \frac{\partial^2 U}{\partial z^2} = 0 \tag{38}$$

$$\text{Sphere}: \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial U}{\partial r}\right] + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left[\sin\vartheta\frac{\partial U}{\partial \vartheta}\right] + \frac{1}{r^2 \sin\vartheta}\frac{\partial^2 U}{\partial \varphi^2} = 0$$

Concentric spheres

Two concentric metal spheres. Only r-dependent (38):

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial U}{\partial r}\right] = 0 \quad \Rightarrow \quad \frac{\partial U}{\partial r} = \frac{C_1}{r^2} \quad \Rightarrow \quad U = -\frac{C_1}{r} + C_2$$
$$\text{and} \quad E = -\frac{C_1}{r^2} \tag{45}$$

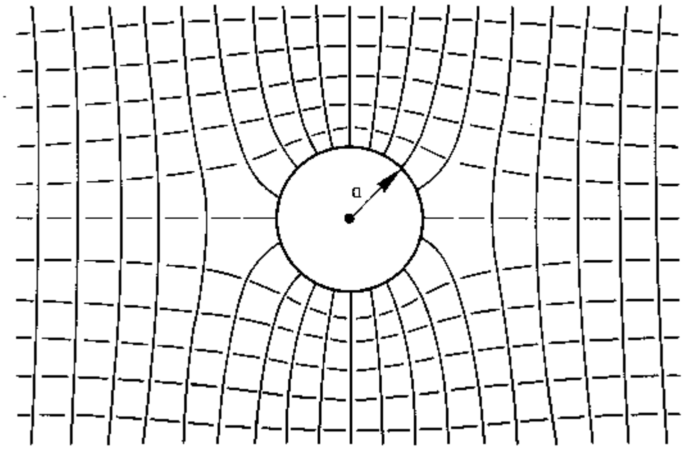
→ Result: spherical equipotential surfaces. With boundary conditions on inner sphere $U(r_1) = V$ and on outer sphere $U(r_2) = 0$:

$$U = \frac{r_1}{r}\left(\frac{r_2 - r}{r_2 - r_1}\right)V \quad \text{and} \quad E = \left(\frac{r_1 r_2}{r_2 - r_1}\right)\frac{V}{r^2} \tag{46}$$

Charge on the sphere Q and C = Q/V:

$$Q = \oiint \varepsilon_0 \varepsilon_r \vec{E} \cdot \overrightarrow{dA} = 4\pi\varepsilon_0\varepsilon_r\left(\frac{r_1 r_2}{r_2 - r_1}\right)V \quad \text{and} \quad C = 4\pi\varepsilon_0\varepsilon_r\left(\frac{r_1 r_2}{r_2 - r_1}\right) \tag{47}$$

Cylinder in a uniform field



$$U = -E_0\, r \cos\vartheta = -E_0 x \qquad (48)$$

$$U(r) = -E_0 r \cos\vartheta + \sum_{m=1}^{\infty} d_m r^{-m}\left(g_m \cos(m\vartheta) + h_m \sin(m\vartheta)\right) \qquad (49)$$

$$U(r,\vartheta) = -E_0\left(r - \frac{a^2}{r}\right)\cos\vartheta \qquad (50)$$

$$E_r = -\frac{\partial U}{\partial r} = E_0\left(1 + \frac{a^2}{r^2}\right)\cos\vartheta \quad and \quad E_\vartheta = -\frac{1}{r}\frac{\partial U}{\partial\vartheta} = -E_0\left(1 - \frac{a^2}{r^2}\right)\sin\vartheta \qquad (51)$$

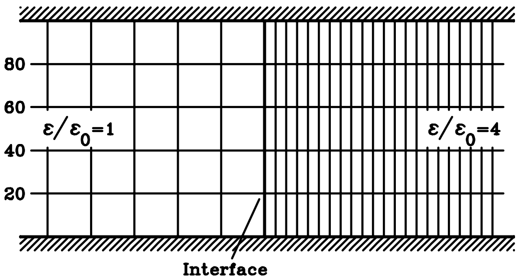
Maximum field $2 \cdot E_0$ at $(r,\theta) = (a,0)$ and the tangential field is zero on $r = a$.

Fields from transformations

Graphical method

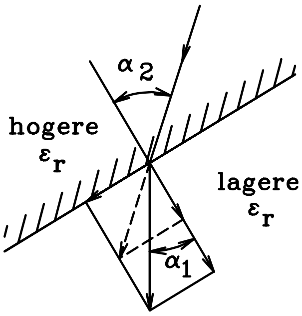
The graphical method uses properties as described more extensively in Alston [4].
Example:

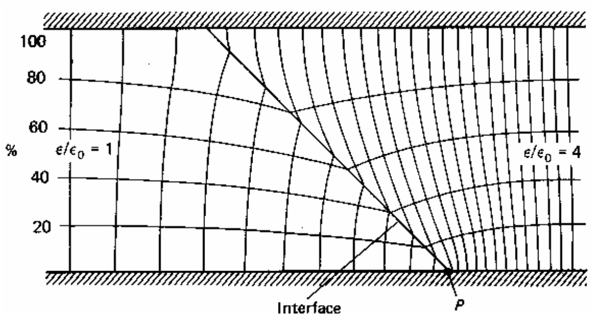
1. Draw E-lines and U-lines in uniform area.
2. Divide U in equal partitions
3. Choose a “mesh factor”.
4. Draw the “guessed” U-lines in the non-uniform area, which fit continuously to those in the uniform area.
5. Now draw the E-lines such that the mesh factor is respected.
6. Correct where necessary the U-lines, repeat the procedure with the E-lines, and refine if needed.



For areas with different dielectrics ε_r, the mesh factor changes (see example above). Since the field lines are drawn on equal flux distances $\delta\psi$ we in fact draw D-lines.
On an interface between dielectrics we in addition get dielectric refraction.
On the interface E_t and D_n are continuous →

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \tag{55}$$

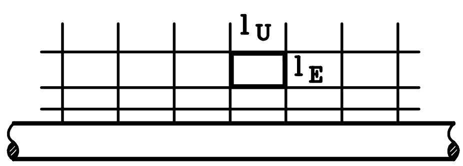




The field becomes non-uniform, with field enhancement at point P.

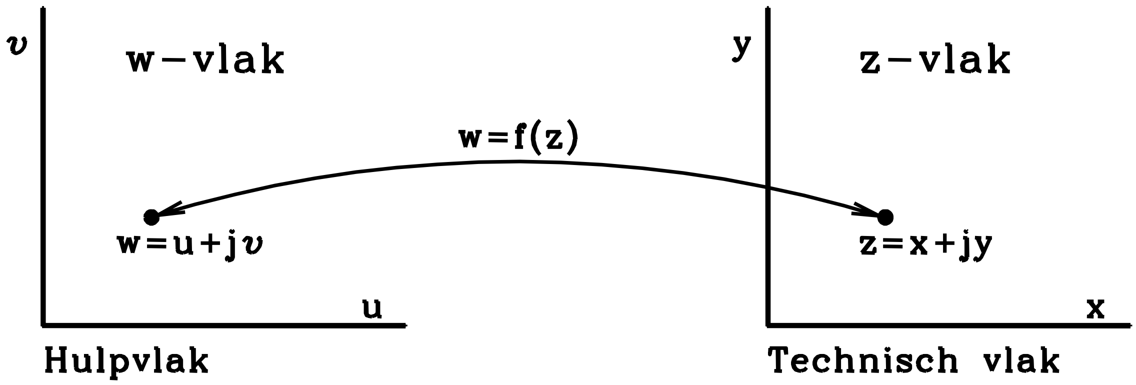
For rotational symmetry we have to replace de constant l_z by 2πr:

$$\frac{\varepsilon_r l_U r}{l_E} = \frac{1}{2\pi\varepsilon_0}\frac{\delta\psi}{\delta U} = constant \tag{56}$$



Conformal mapping

Conformal mapping is a mathematical method to determine E-fields in two dimensional, z-independent cases (see Binns [5] or Feynman [6]). We consider a complex analytical function f(z) of the complex variable z; i.e. a function for which in an area both f(z) and its derivative f′(z) exist. Complex function theory shows the transformation defined by f(z) to be conform, except for those points where the derivative f′(z) equals zero (see e.g. Kreyszig [7]). Conformal mapping assumes the transformed angles to be "conform". We will use this property to transform a E-U field to a different pattern which again we can interpret as a E-U plot.



A mathematical help surface, the w-surface with coordinates u and v, is projected onto the "technical" z-surface, with coordinates x and y, for which we want to obtain a field.
Consider the coordinates: z = x + jy and w = f(z) = u + jv, with x and y on the technical surface and u and v on the mathematical surface.
For a function f(z) with which the z-surface is projected onto the w-surface we can show:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} \quad and \quad f'(z) = -j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad (59)$$

The first for Δz→0 for Δy=0 and Δx→0. The second for Δx=0 and Δy→0. Both derivatives are equal for an analytical function:

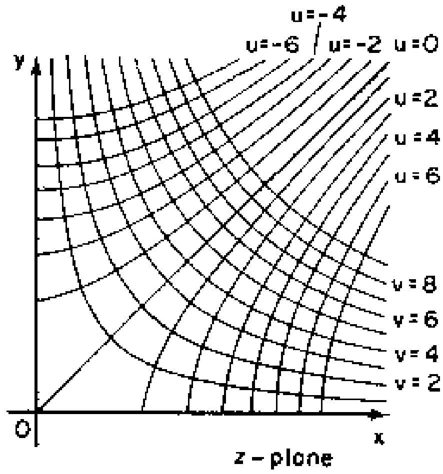
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{en} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (60)$$

The so called Cauchy-Riemann equations.
It can be shown, for all functions that satisfy these criteria to be analytical. Both the function u(x,y) as well as v(x,y) fulfill the Laplace equation, as can be seen when applying Cauchy-Riemann:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \quad ; \qquad \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0 \qquad (61)$$

We can interpret u as potential and v as field or vice versa. We now choose in the w-surface the simplest field projection meeting the Laplace requirement, e.g. equipotential lines u = cst and E-lines v = cst. Every transformation of the z-surface gives a projection meeting Laplace's equation, and which we can interpret as field projection. The projection of u = cst is again to be interpreted as equipotential line, and the projection of v = cst as E-line, or vice versa.

The generated field projection only depends on the transformation function w = f(z). See Binns [5] or Bewley [8] for a more systematical method for determining such functions, the so called Schwartz-Christoffel transformation method.



Shown here is an example for the function w = f(z) = z^2.

$$u + jv = (x + jy)^2 = x^2 - y^2 + 2jxy \qquad (62)$$

Transformation of u = cst and v = cst gives the contours:

$$u = x^2 - y^2 = cst \quad and \quad v = xy = cst \qquad (63)$$

These are hyperboles crossing each other under a 90° angle.
To be interpreted as two electrodes in one quadrant, under an angle of 90°: x- and y-axes together form one electrode (projection of v = 0), the other electrode is the projection of any other line with v = constant.