Beam lines

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Abstract
In this paper we describe the linearized beam dynamics in transfer lines in the context of the CAS School for Small Accelerators.

1 Introduction

Accelerator facilities, big or small, typically employ one or more acceleration stages, and transfer lines are needed to transport the beam for the particle source to the accelerator(s) and from the accelerator to the users. Often transfer lines have a more active role, they are used to adapt the beam from one stage of acceleration to the other, to host diagnostics or safety elements, or to get rid of unwanted particles in dedicated spots in order to avoid activation of downstream accelerators or contamination of the user experiments.

The design and the study of transfer lines involves the study of the dynamics of a beam of charged particles under the influence of magnetic and electric fields. In this lecture the self-fields (e.g. space charge) are not treated and the interested reader can consult Ref. [1].

2 Emittance

In this paragraph we recapitulate on the definition of the emittance of a beam of particles. For a complete definition see Ref. [2].

Acceleration is a controlled manipulation of an ensemble of charge particles. In order to control the process of acceleration it is necessary to know the law that relates the six coordinates of each particle at any time. These coordinates are the position and momentum in each of the three planes and are denoted as x, p_x, y, p_y, z, p_z, and are measured in metres and eV/c respectively. The motion of the beam can be represented in a 6D space, called phase space. The projection of the hyper-ellipsoid on each of the three planes (x, p_x or y, p_y or z, p_z) will describe the motion in each of the three directions. A typical beam of particles is composed of bunches containing some 10^6–10^{10} particles. It is impractical — if not impossible — to follow the motion of each individual point and therefore it is necessary to define a global statistical variable representative of the status of the beam at any time.

2.1 r.m.s. definition:

As we said before, the motion of each particle is defined by the evolution in time of the classical coordinates (position, momentum):

$$\vec{x} = (x, p_x, y, p_y, z, p_z)\,.$$

In accelerator physics it is more convenient is to use position and normalized divergence, in units of metre and 'radians'

$$\vec{x} = \left(x, x_p = \operatorname{arctg}\left(\beta\gamma \frac{p_x}{p_z}\right), y, y_p = \operatorname{arctg}\left(\beta\gamma \frac{p_y}{p_z}\right), z, \frac{\Delta p}{p} \right)$$

where $\beta\gamma$ are the relativistic parameters.

In paraxial approximation the above reduces to

$$\vec{x} = \left(x, x_{\mathrm{p}} = \beta\gamma \frac{p_{\mathrm{x}}}{p_{\mathrm{z}}}, y, y_{\mathrm{p}} = \beta\gamma \frac{p_{\mathrm{y}}}{p_{\mathrm{z}}}, z, \frac{\Delta p}{p} \right)$$

In each of the three planes the area occupied in the phase space can be related to a statistical quantity (emittance) defined as:

$$E_{\mathrm{rms}} = \pi \sqrt{\langle x^2 \rangle \cdot \langle x_{\mathrm{p}}^2 \rangle - \langle x \cdot x_{\mathrm{p}} \rangle^2}\ \text{metres}$$

Where <> is the average over the distribution.

2.2 Courant-Snyder definition

Another definition of the emittance has been put forward by Courant and Snyder [3]. It is based on the consideration that under the influence of *linear forces* the trajectory of a particle in phase space (e.g., x, x_{p}) follows an elliptical path and can be characterized by three parameters (α, β, γ[1]) which follow the relations:

$$\gamma \cdot x^2 + 2 \cdot \alpha \cdot x \cdot x_{\mathrm{p}} + \beta \cdot x_{\mathrm{p}}^2 = \frac{E}{\pi} = \varepsilon$$

$$\beta\gamma - \alpha^2 = 1$$

where ε is constant during the motion and represents the emittance of the beam.

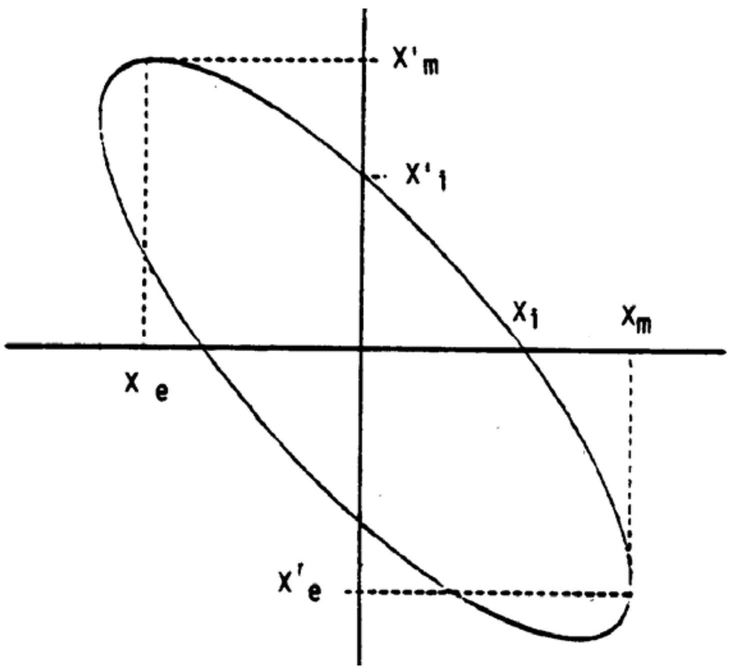

Fig. 1: Courant–Snyder ellipse and related beam parameters, from Ref. [4]

[1] Please note that these parameters are NOT related to the relativistic parameter $\beta\gamma$.

Figure 1 shows the relation between the Courant–Snyder parameters (*α,β,γ*) and the points at which the ellipse intersects the axis in the plane *x*, x_p. The relations are reported in the following:

$$x_e = -\alpha\sqrt{\varepsilon/\gamma} \;\; x_i = \sqrt{\varepsilon/\gamma} \quad x_m = -\alpha\sqrt{\beta\varepsilon}$$

$$x'_e = -\alpha\sqrt{\varepsilon/\gamma} \;\; x'_i = \sqrt{\varepsilon/\gamma} \quad x'_m = -\alpha\sqrt{\beta\varepsilon}\ .$$

A practical use of the Courant–Snyder parameters can be found in Fig. 2. A sample beam of particles, containing 30 000 macro-particles, has been tracked through a beam line with the code TRACEWIN [5]. The output results are analyzed and the Courant–Snyder ellipse is drawn over the cloud of particles. The alpha and beta parameters are calculated. The ellipse on the left-hand side has β = 6.37 mm/π.mrad, α = –2.88. The ellipse on the right-hand side has β = 1.80 mm/π.mrad, α = 0.83. The sign of alphas tells whether the beam is converging towards the axis (alpha positive) or diverging (alpha negative).

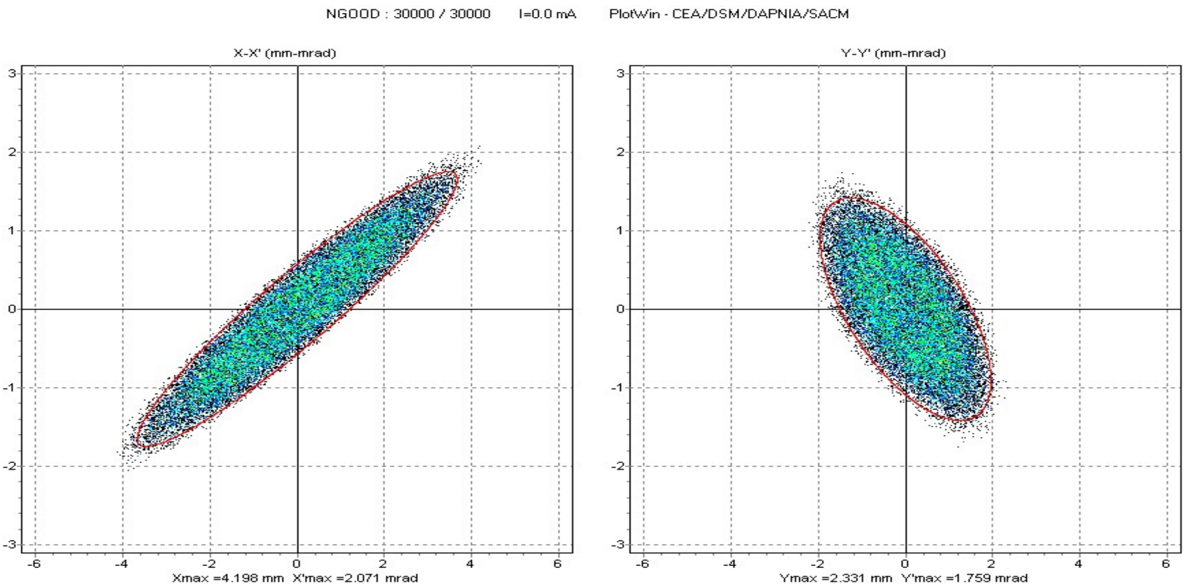

Fig. 2: Example of transverse phase plots

2.3 Matching the beam

Matching the beam means to prepare the beam for the focusing structure of the accelerator. In an accelerator one tries to give energy to the beam while keeping both the transverse dimensions confined. From the transverse point of view the accelerator is built up as a (super) periodic structure where the envelope of the beam oscillates periodically and the normalized phase space portrait is identical after each period. Once the accelerator is designed and built there is only one orientation of the input phase space for which the above conditions are met. Bringing the beam at the input of the accelerator with the correct α and β, i.e., the correct Courant–Snyder parameters, is called 'matching the beam' and it is done with magnetic elements in a beam line.

If the matching conditions are not met, i.e., the beam is mis-matched, then the envelope oscillation are not regular and in extreme limits not bound. This can cause emittance growth, and beam losses inside the accelerator. An example of a matched and mismatched beam is shown in Fig. 3.

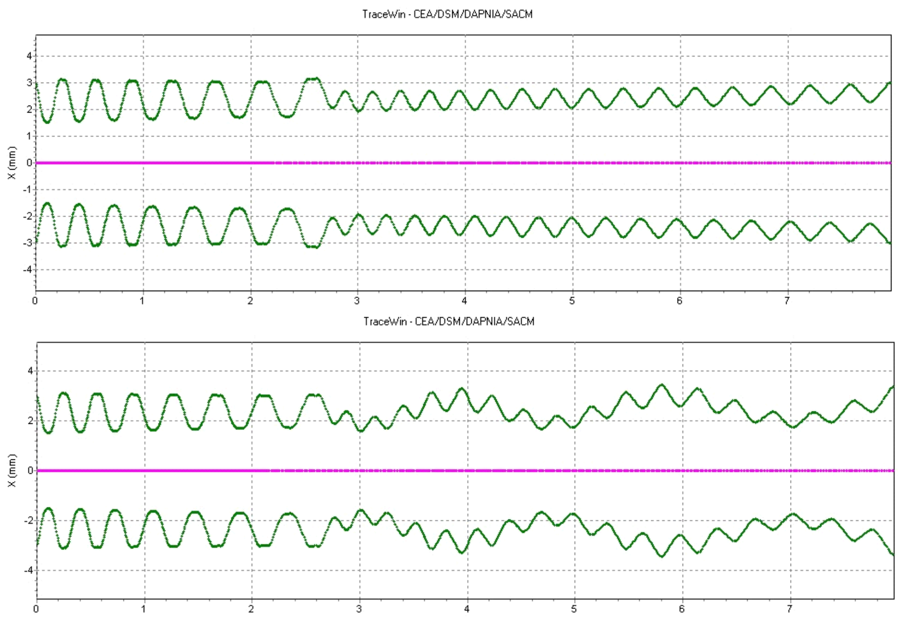

Fig. 3: Beam envelope in a Drift Tube Linac accelerator. The envelope is matched to the focusing structure (top) and mismatched (bottom).

3 Transfer lines

Transfer lines are sections hosting magnetic and electric elements that, generally, do not change the energy of the beam but are used to transport the beam between different stages of acceleration: there are lines from the particle source to the first RF accelerator, from linear accelerator to linear accelerator, and also injection lines from a linear accelerator in a circular accelerator. The last topic is dealt with in Ref. [6].

3.1 Calculation of the dynamic of a single particle in a transfer line

Let us assume a transfer line as sketched in Fig 4: the beam travels from left to right (point 1 to point 2). Under the influence of a linear force the coordinates at a point 2 are a linear combination of the coordinates at point1. It can therefore be written in the following matrix form:

$$\overrightarrow{x_2} = R \cdot \overrightarrow{x_1}$$

where R is the transfer matrix of the line.

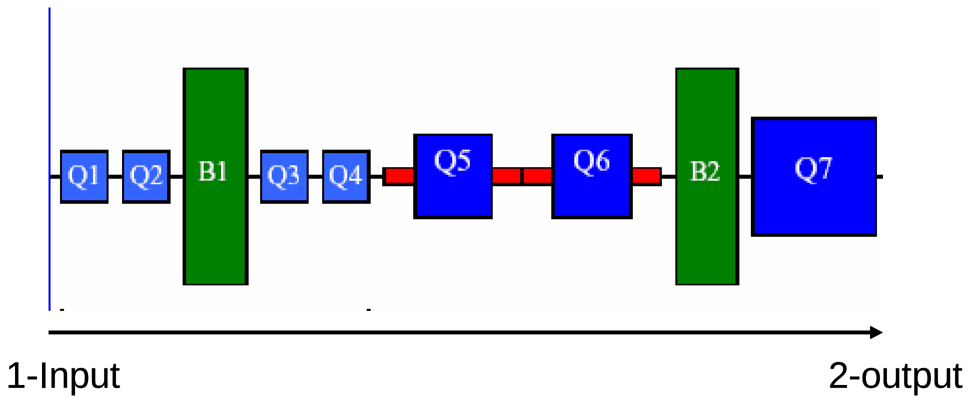

Fig. 4: Sketch of a transfer line. Q denotes a quadrupole, B a buncher cavity.

If the line is composed of several elements, like, for example, quadrupoles (Q), bunchers (B), the expression above can be applied for each element and therefore the matrix of the system is the product of the matrix of each element:

$$R = R_7 R_{67} R_6 R_{56} \ldots\ldots$$

where R_i is the matrix of the active element i whereas R_{ij} is the matrix representing the drift between active elements.

The R matrix is a six-by-six matrix with units of m or m^{-1}:

$$\begin{bmatrix} 1 & m & 1 & m & 1 & m \\ m^{-1} & 1 & m^{-1} & 1 & m^{-1} & 1 \\ 1 & m & 1 & m & 1 & m \\ m^{-1} & 1 & m^{-1} & 1 & m^{-1} & 1 \\ 1 & m & 1 & m & 1 & m \\ m^{-1} & 1 & m^{-1} & 1 & m^{-1} & 1 \end{bmatrix}$$

It contains the linear coefficient that gives the output coordinates of a particle as a linear combination of its input coordinates. The matrix can be divided into sub-matrices:

$$R = \begin{matrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{matrix}$$

In an un-coupled, non-dispersive system only the terms R_{xx}, R_{yy} and R_{zz} are non-zero. In this case the motion in each of the three directions is independent of the other two and the dynamics can be decoupled.

Once the layout of the line is defined and the parameters of the line are known, the R matrix is fully determined. Using the relation, discussed above,

$$\overrightarrow{x_2} = R \cdot \overrightarrow{x_1}$$

three types of computation are possible.

- Given the input condition $\vec{x}_1$ it is possible to calculate directly the output beam $\vec{x}_2$.
- Choosing the output conditions $\vec{x}_2$ it is possible, by inverting the R matrix, to calculate $\vec{x}_1$.
- It is possible to modify the element of the R matrix, i.e., the settings of the line, so that for a given $\vec{x}_1$ it is possible to obtain a wanted $\vec{x}_2$. This operation, generally performed with some minimization technique by a computer, allows matching the beam, i.e., to take the beam from one specific accelerator and adapt it to the (focusing) system of the next one.

3.2 Matrix of some common elements in a transfer line

In this section the matrices of some key elements are derived or described. A more detailed description can be found in Ref. [4]. In this lecture we limit the description to transverse effects, therefore from now on we will be drop the terms containing the longitudinal coordinates and reduce the R matrix to:

$$R = \begin{matrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{matrix} .$$

3.2.1 Drift

In an empty space of length L, i.e., a drift space, the divergence of a particle will stay constant (no forces are acting) and the position will evolve under the effect of a divergence as sketched in Fig. 5.

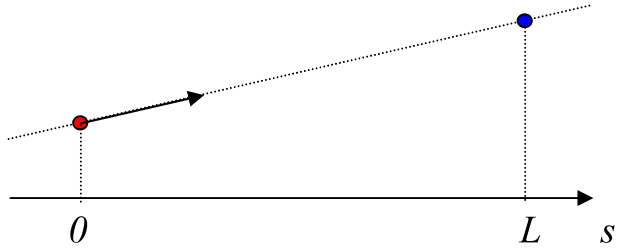

Fig. 5: Transverse motion of a particle in a drift

The output coordinates of the particle in the x plane can be related to the input coordinates by the relations:

$$x_2 = x_1 + x_{p1} \cdot L$$
$$x_{p2} = x_{p1}$$

and equivalently for the other transverse plane. Therefore the matrix of a drift is

$$R = \begin{matrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{matrix}$$

3.2.2 Quadrupole

A quadrupole is a magnetic element presenting a four-fold symmetry. Its field, concentrated between the four magnetic poles is sketched in Fig. 6. A particle travelling into the sheet off-axis with respect to the centre of the quadrupole, will experience a force towards the centre in the y direction and away from the centre in the x direction (arrows of Fig. 6). If the polarity was reversed, the forces will have opposite sign. This force is proportional to the distance from the axis (linear focusing) and proportional to the field and length of the quadrupole. In the focusing plane the motion is represented in Fig. 7.

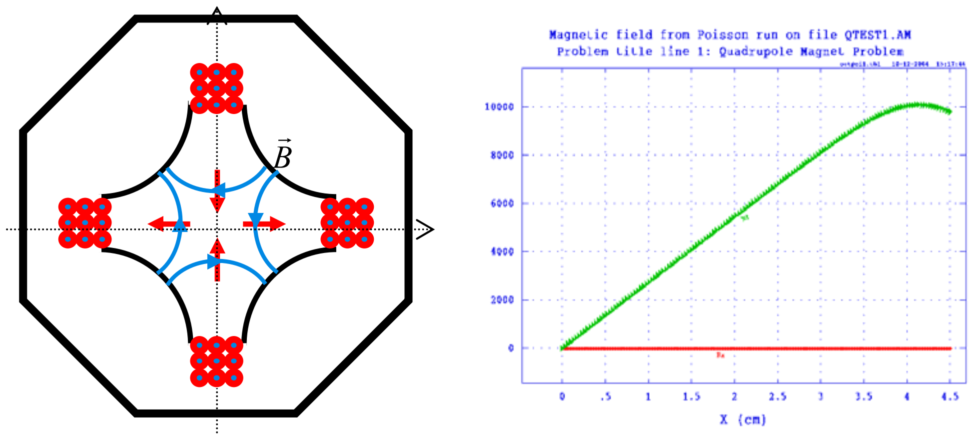

Fig. 6: Magnetic quadrupole and transverse components of the field along the main axis

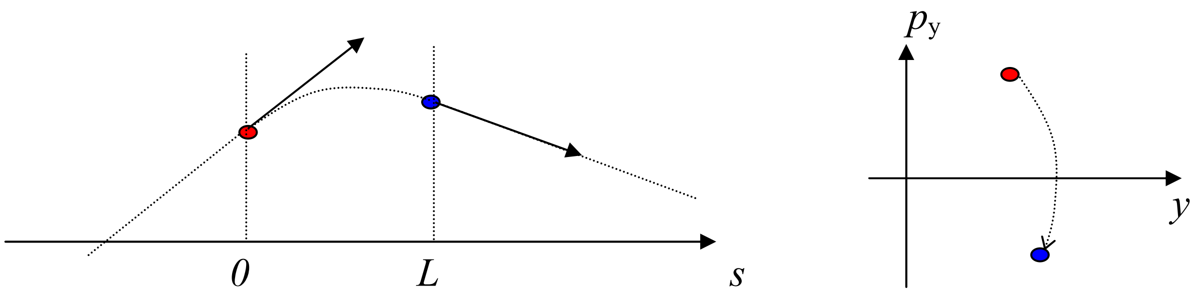

Fig. 7: Effect of the quadrupole of length L on the transverse motion of a particle (focusing plane)

To quantify the effect of the quadrupole and to derive the R matrix we need to introduce the following quantities:

The magnetic rigidity $B\rho = \frac{m_0 c\beta\gamma}{q} = \frac{momentum}{charge}$ measured in Tesla metres

and the normalized gradient, k^2 measured in 1/metres:

$$k = \left[\left|\frac{B'}{B_\rho}\right|\right]^{\frac{1}{2}}$$

where B' is the field gradient, i.e., the field on the pole tip divided by the bore radius.

The transfer matrix of a quadrupole of length L can then be written as

$$R_{yy} = \begin{pmatrix} \cos(k\cdot L) & \frac{1}{k}\cdot\sin(k\cdot L) \\ -k\cdot\sin(k\cdot L) & \cos(k\cdot L) \end{pmatrix} \qquad R_{xx} = \begin{pmatrix} \cosh(k\cdot L) & \frac{1}{k}\cdot\sinh(k\cdot L) \\ k\cdot\sinh(k\cdot L) & \cosh(k\cdot L) \end{pmatrix}$$

where we have respected the polarity of Fig. 6 and the quadrupole is focusing in y and defocusing in x.

As we have seen, a single quadrupole can focus a beam in one plane but it will defocus in the other. In order to keep the beam confined in both transverse planes, it is necessary to have a sequence of quadrupoles with alternate polarity. This arrangement is called a FODO cell and a sequence of FODO cells is called a FODO channel: this is the most used focusing system for hadron accelerators. The envelope in a FODO channel oscillates with the period of the cell and the beam 4D phase space is identical after each period. The equation of motion in a periodic channel (Hill's equation) has a solution which can be expressed as a function of the emittance (ε), the beta function (β) and the phase advance (σ):

$$x(s) = \sqrt{\varepsilon\beta(s)}\cdot\cos(\sigma(s))\pi$$

where

$$\sigma(s) = \int_0^s \frac{ds}{\beta(s)}\,.$$

3.2.3 Solenoid

A solenoid consists of a winding of current, possibly around an iron core, which generates a field perpendicular to the windings themselves. A sketch of a solenoid is shown in Fig. 8; the corresponding magnetic field on Fig. 9. As the field lines are closed around the windings, there is a transverse radial component of the magnetic field at the entrance/exit of the solenoid. This is called the fringe field. A particle entering the solenoid on-axis and with zero transverse velocity will not experience any force. A particle entering the solenoid off-axis with a given longitudinal velocity will receive an azimuthal force proportional to its velocity, to the distance from the axis and the field of the solenoid. After crossing the fringe field and entering the solenoid itself, the transverse velocity, generated in the fringe field area, will couple to the longitudinal field and will result in a focusing force towards the axis in both planes. The overall resulting force of the solenoid is therefore proportional to the distance from the axis (linear focusing) and proportional to the square of the B field. The longitudinal field couples the motion in the two transverse planes.

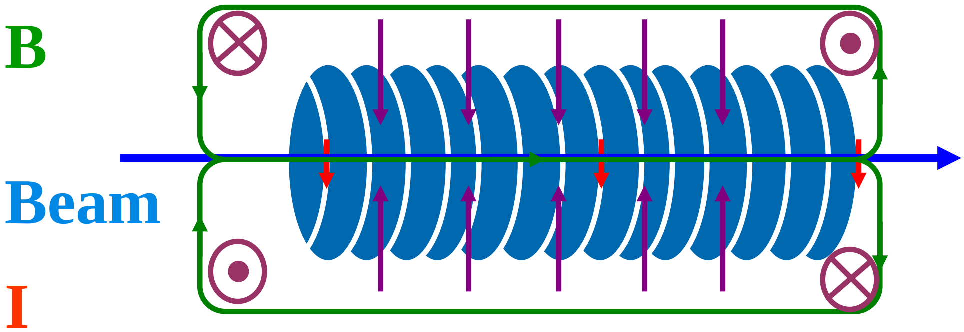

Fig. 8: Solenoid. The field lines are closed around the windings.

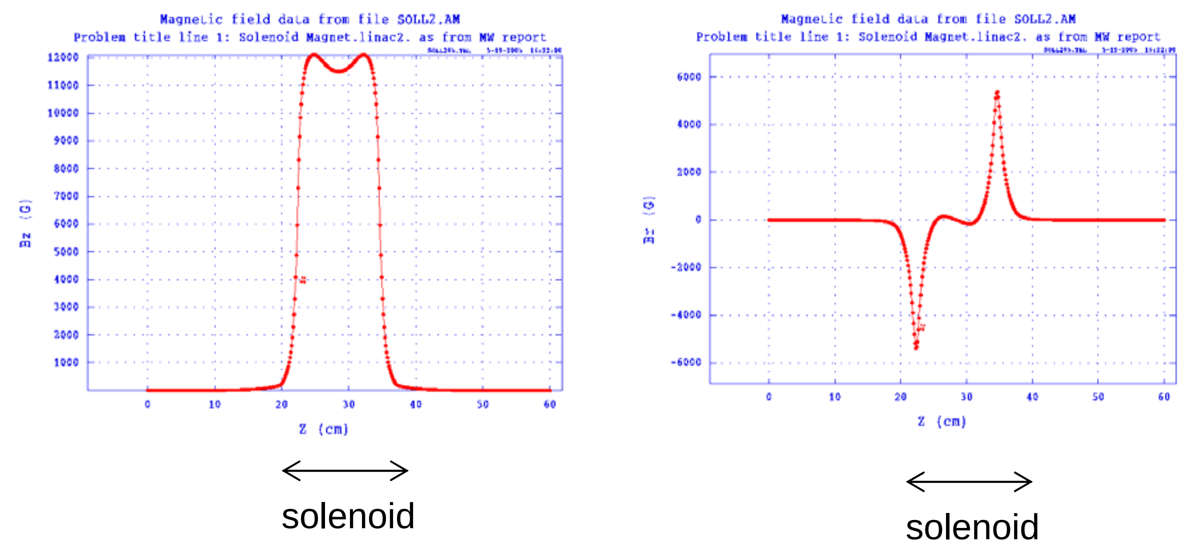

Fig. 9: Longitudinal and radial field along the longitudinal axis

The matrix of a solenoid can be expressed as a function of

$$k = \frac{B}{2B\rho} \qquad S = \sin(kL) \qquad C = \cos(kL)$$

where B is the field in the solenoid and L its length:

$$R_{xx} = R_{yy} = \begin{pmatrix} C^2 & \frac{1}{k}\cdot SC \\ -k\cdot SC & C^2 \end{pmatrix} \qquad R_{xy} = -R_{yx} = \begin{pmatrix} SC & \frac{1}{k}\cdot S^2 \\ -k\cdot S^2 & SC \end{pmatrix}$$

It can be observed that in a solenoid both transverse planes are focused simultaneously but the two are also coupled.

A sequence of solenoids used for focusing a beam in an accelerator is called a FOFO. The solenoid is generally used for very-low-energy hadrons or for electrons.

Acknowledgements

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References

[1] L. Palumbo, these proceedings.

[2] D. Moehl, these proceedings.

[3] E.D. Courant and H.S. Snyder *Ann. Phys.* (*N.Y.*) **3** (1958) 1–48.

[4] K.R. Crandall, TRACE3d documentation, Los Alamos Accelerator Code Group, LA-UR-90-4146.

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