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As in the past, the interests of the group have been divided about equally between the development of analog computer components and the exploration of practical methods of network synthesis.

A. ANALOG COMPUTER DEVELOPMENT

1. Cathode Ray Tube Multiplier

This multiplier works on the principle that the area of a rectangle is proportional to the product of its base times its height. A square pattern is obtained on the face of a cathode-ray tube by means of a television type of raster. Four separate photomultiplier tubes pick up the light from the four quadrants of the pattern, as shown in Fig. XVIII-1. The outputs of quadrants ABCD are combined as (A + D) - (B + C). Low-frequency signals E_x and E_y are then used to move the raster, and the output voltage is proportional to the product $E_x E_y$, as can be readily seen from the geometry of the figure. The advantage of this scheme over the one proposed by E. J. Angelo in his doctoral thesis, M.I.T., 1952, is that both channels are separately linear, and feedback can be applied to them.

The major troubles encountered in the experimental device were associated with dc drifts in the external amplifiers.

M. M. Cerier

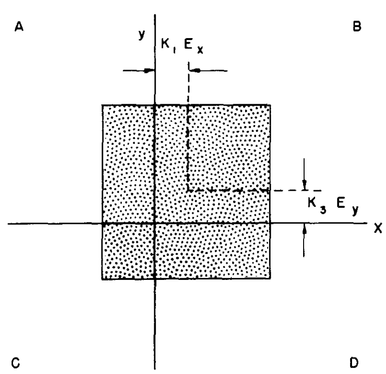

Fig. XVIII-1
Cathode ray tube multiplier.

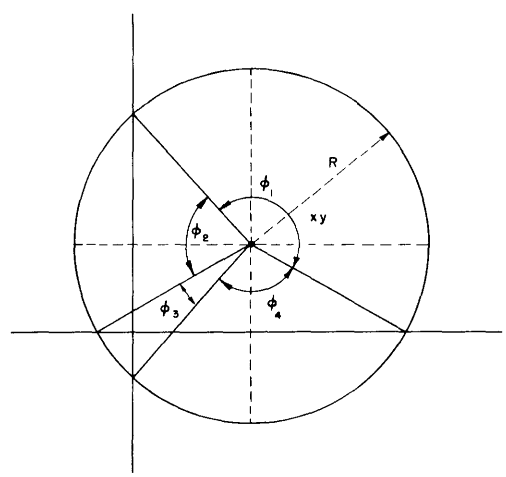

Fig. XVIII-2
Inverse trigonometric generator.

2. A Square-Law Multiplier

A germanium-diode, square-law device, with a static accuracy of 1 percent and a bandwidth of 100 kc/sec has been built and tested. An "absolute-value" circuit which allows four-quadrant operation of a square-law multiplier with only two squaring circuits has also been built.

J. L. Perry

3. A Probability Multiplier

The probability multiplier uses the mathematical principle that the probability of a group of independent events occurring simultaneously is equal to the product of the probability of each event occurring by itself. Consider a series of similar pulses in which the period of each pulse is B and the duration of each pulse is A. The probability of a pulse occurring at any particular instant is equal to the duty ratio A/B. Two such trains of pulses are used, and coincidence between them is detected. With carrier pulses at approximately 50 kc/sec, the frequency response is good only to 40 cps. The accuracy is approximately 3 percent. The frequency response is the main limitation of this type of multiplier.

A. N. Spector

4. High-Speed Analog Computer

For a given accuracy of solution the gain required in an analog computer drops as the solution speed is increased. A first approximation to the theoretical analysis shows that for an amplifier with a given gain bandwidth product the error is constant regardless of the solution time. To test these conclusions an amplifier with a bandwidth of 10 Mc/sec has been constructed and used in a computer with a repetition rate of 1000 cps.

An integrator, designed around this amplifier, had a phase error which was less than 2° from 1000 cps to more than 100, 000 cps. The corresponding transient errors would be in the neighborhood of 1.5 percent.

R. M. Beckett

5. Solution of Simultaneous Equations on an Analog Computer

An investigation was conducted on the practicability of solving simultaneous algebraic equations with a standard analog computer.

The first result of the investigation showed that much more satisfactory operation could be obtained by using integrators than by using adders. With integrators the equations are solved directly by the method of steepest descent.

A second and perhaps more surprising result was concerned with solving matrices whose determinants approach zero. It was found experimentally that the percentage of error in the results decreased as the determinant decreased in size, right to the point of instability. This result indicates that machine errors are more important in influencing the results than the size of the determinant.

F. R. Hultberg

6. Inverse Trigonometric Function Generator

A device has been built that generates the inverse trigonometric functions of an analog input. The principle used is the measurement of the arc lengths in various quadrants of a circle whose axes are directly dependent on the value of the input. The basic diagram is shown in Fig. XVIII-2. From the geometry it is easily seen that

$$\phi_2 + \phi_3 = 2 \cos^{-1} \frac{x}{R} \tag{1}$$

and

$$(\phi_1 + \phi_4) - (\phi_2 + \phi_3) = 4 \sin^{-1} \frac{y}{R} \tag{2}$$

The circle is obtained by means of a circular sweep on a cathode-ray tube. The arc length is measured by means of the output from two phototubes separated by a partition along the y-axis of the cathode-ray tube.

The accuracy of the first crude device was approximately 5 percent; the frequency response was better than 1 kc/sec. These figures can be improved.

R. J. Szmerda

7. Diode Network Synthesis

As is well known, networks containing diodes plus linear circuit elements are commonly used to attain various nonlinear transfer characteristics such as rectification, gating, limiting, formation of piecewise linear functions of a single variable, and so on.

The common characteristic of all diode networks is their piecewise linear behavior. The following is a description of methods by which diode networks may be used to produce arbitrary piecewise linear functions of two variables. Investigation is now in progress on the feasibility of extending this to arbitrary functions of n variables, and applying some general techniques of diode network synthesis to other problems in nonlinear network analysis and synthesis.

a. Piecewise linear function of two variables

Suppose that we wish to make an approximation of a function

$$y = f(x_1, x_2)$$

which is given in tabular form for various increments of the independent variables. For simplicity it will be assumed that the function is tabulated at equal increments of the independent variables, Δ being the interval separating each point. The tabulation can then be represented as a matrix whose general term is $[\alpha_{ij}]$, the value of the function at $x_1 = i\Delta$, $x_2 = j\Delta$.

The function can be represented as a surface whose height above the $x_1 - x_2$ plane is equal to y. Since the use of diodes limits us to piecewise linear approximations, it is clear that any surfaces we may construct will be described by the linear equation

$$y = Ax_1 + Bx_2 + C$$

over each region of analyticity; that is, the approximating surface will be piecewise planar. Figure XVIII-3 is an example of a portion of a surface approximated in this manner. The points of intersection are the given tabulated points. (Note: In general, an arbitrary surface must necessarily be approximated by triangular segments of planes.)

This complex surface can be synthesized by the superposition of many simpler surfaces. For example, the addition of "step" surfaces like the one shown in Fig. XVIII-4 produces a resultant surface similar to that shown in Fig. XVIII-3. In order for the resultant surface to coincide with the elements of the $[\alpha_{ij}]$ matrix, it must be constituted of the summation of mn steplike elements (mn is the product of the dimensions of the α matrix), each step being centered over a different tabulated point. The height, A_{ij}, of a typical element centered over $x_1 = i\Delta$, $x_2 = j\Delta$, must be

$$A_{ij} = \alpha_{ij} + \alpha_{i-1,\,j-1} - \alpha_{i-1,\,j} - \alpha_{i,\,j-1}$$

The circuit of Fig. XVIII-5 is one method of realizing a steplike element. An alternative method of synthesizing a piecewise planar surface is the use of a summation of square-pyramidal elements.

As one generalizes to functions of more than two variables, this procedure becomes more involved. For example, in dealing with functions of three variables, we find that each region of analyticity must be a tetrahedronal element of volume bounded by four planes rather than a triangular element of a plane bounded by three lines. In going to still more dimensions, the bounding surfaces become hyperplanes, and visualization becomes impossible.

b. Algebra of piecewise linear functions

The synthesis techniques mentioned above are primarily based upon the formulation of an algebraic notation that enables one to express the complete behavior of a piecewise linear, or more generally, piecewise analytic, continuous function as a single equation, rather than as a set of equations coupled with inequalities to

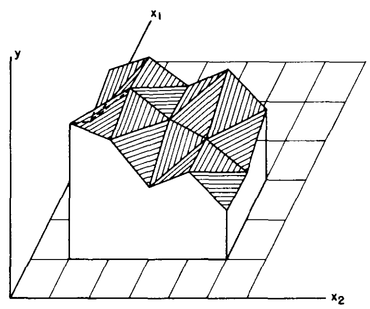

Fig. XVIII-3
Section of a piecewise planar surface.

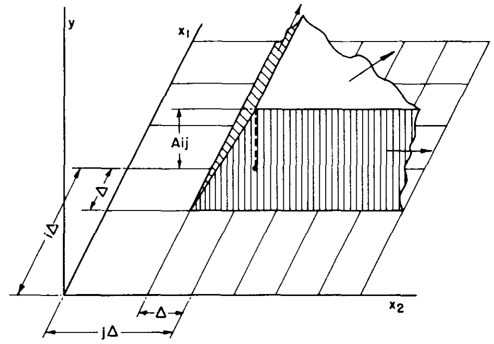

Fig. XVIII-4
Steplike element.

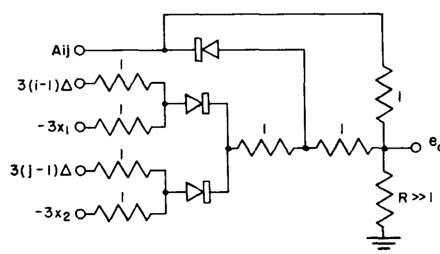

Fig. XVIII-5
Realization of a steplike element.

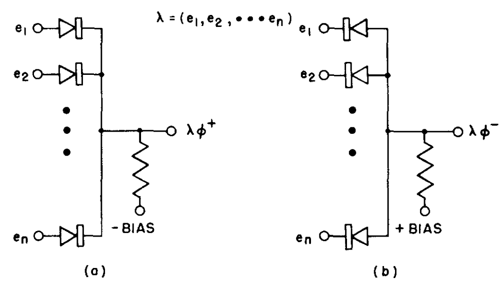

Fig. XVIII-6
Realization of transformations.

show their regions of validity.

To accomplish this it is convenient to define two types of algebraic transformations: ϕ^+ and ϕ^-.

Let $\lambda = (a, b, \ldots, n)$ where p is the greatest element in λ and q is the least element in λ. (λ can be any set of elements of a well-ordered integral domain.) Then we define: $\lambda\phi^+ = p$; $\lambda\phi^- = q$.

By the use of suitable combinations of these transformations it is possible to eliminate the need of the conventional inequality relations that inevitably complicate the analysis of systems that are piecewise analytic in behavior.

These particular transformations are easily realizable electronically through the use of diodes, as illustrated in Fig. XVIII-6. (Note that the bias voltage must be greater in magnitude than the greatest of the elements of λ.) Thus, once a piecewise linear function is represented in terms of these transformations, it is, at least in theory, a relatively simple matter to realize it electronically.

Investigation is in progress on the development of a suitable algebra for these sets of elements and their transformations. The algebra is being developed along the lines of the algebra of vector spaces.

T. E. Stern

8. The Saturable Reactor in Analog Computers

In the effort to construct a driftless, high-gain dc amplifier, a type of modulated circuits have been developed (1). For modulation and demodulation, synchronous vibrators (choppers) or variable capacitors are used. The objection to these samplers is their dependence on a mechanical operation. For certain applications, a nonmechanical chopper has been developed whose operation relies on the resistance changes of a photosensitive crystal. In quest of other nonmechanical choppers, the possibilities of the saturable reactor were investigated, and some preliminary experiments have been performed on two types of saturable reactor circuits, with promising results.

The first circuit consists of a series-connected, balanced saturable reactor (Fig. XVIII-7); the second, of a parallel-connected saturable reactor (Fig. XVIII-8). Both are driven from a square-wave voltage generator with very low internal impedance. With the former, a pulse-modulated signal having an amplitude proportional to the input bias is obtained at the output; with the latter, a pulse having a width proportional to the input is obtained. It should be noted that the sign-sensitive circuit that is required is obtained with a single core. However, for better waveshapes parallel-connected cores should be used; then the circuit is not sign-sensitive. To obtain the sign sensitivity two more parallel-connected cores are required, together with some additional circuitry, as shown in Fig. XVIII-9. The corresponding waveforms are shown in Fig. XVIII-10. The output is filtered (1), giving a positive or negative dc voltage proportional to the input offset voltage of the dc amplifier. The over-all gain (input-filtered output) should be of the order of 250, 000. Our circuit falls short of this number by a factor of about 15. Better design in the saturable reactor might give the required gain.

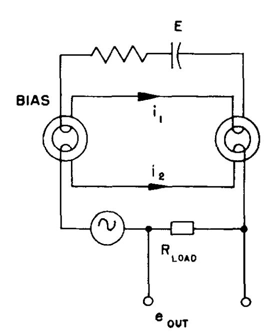

Fig. XVIII-7
Series-connected saturable reactors.

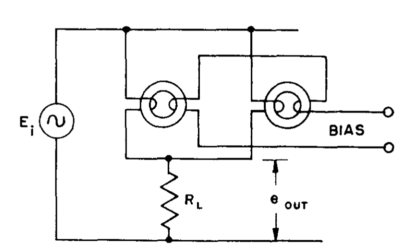

Fig. XVIII-8
Parallel-connected saturable reactors.

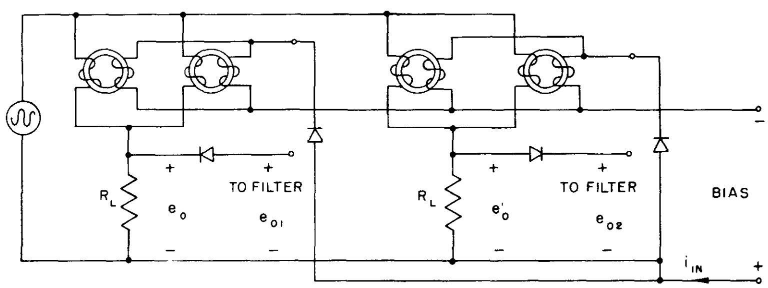

Fig. XVIII-9
Circuit for the sign-sensitive magnetic chopper.

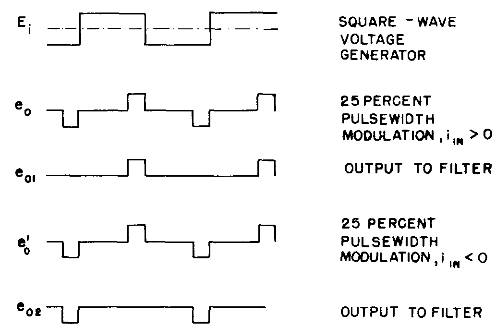

Fig. XVIII-10
Waveforms for the circuit of Fig. XVIII-9.

Comparing the series and parallel circuits we realized that each has its own advantages. The first does not require sharp knee magnetization curves for the cores, operates effectively on rather high input currents, but would require complex circuitry for sign sensitivity. The second relies upon very sharp knee magnetization curves, operates on small input currents, requires rather simple circuitry for sign sensitivity, and can accommodate a predetermined maximum offset input voltage, since for this maximum we have a 100 percent pulsewidth modulation. A further development of the second type may easily compete with the mechanical chopper.

M. S. Macrakis

References

1. G. A. Korn and T. M. Korn, Electronic Analog Computers (McGraw-Hill Book Co., Inc., New York, 1952).

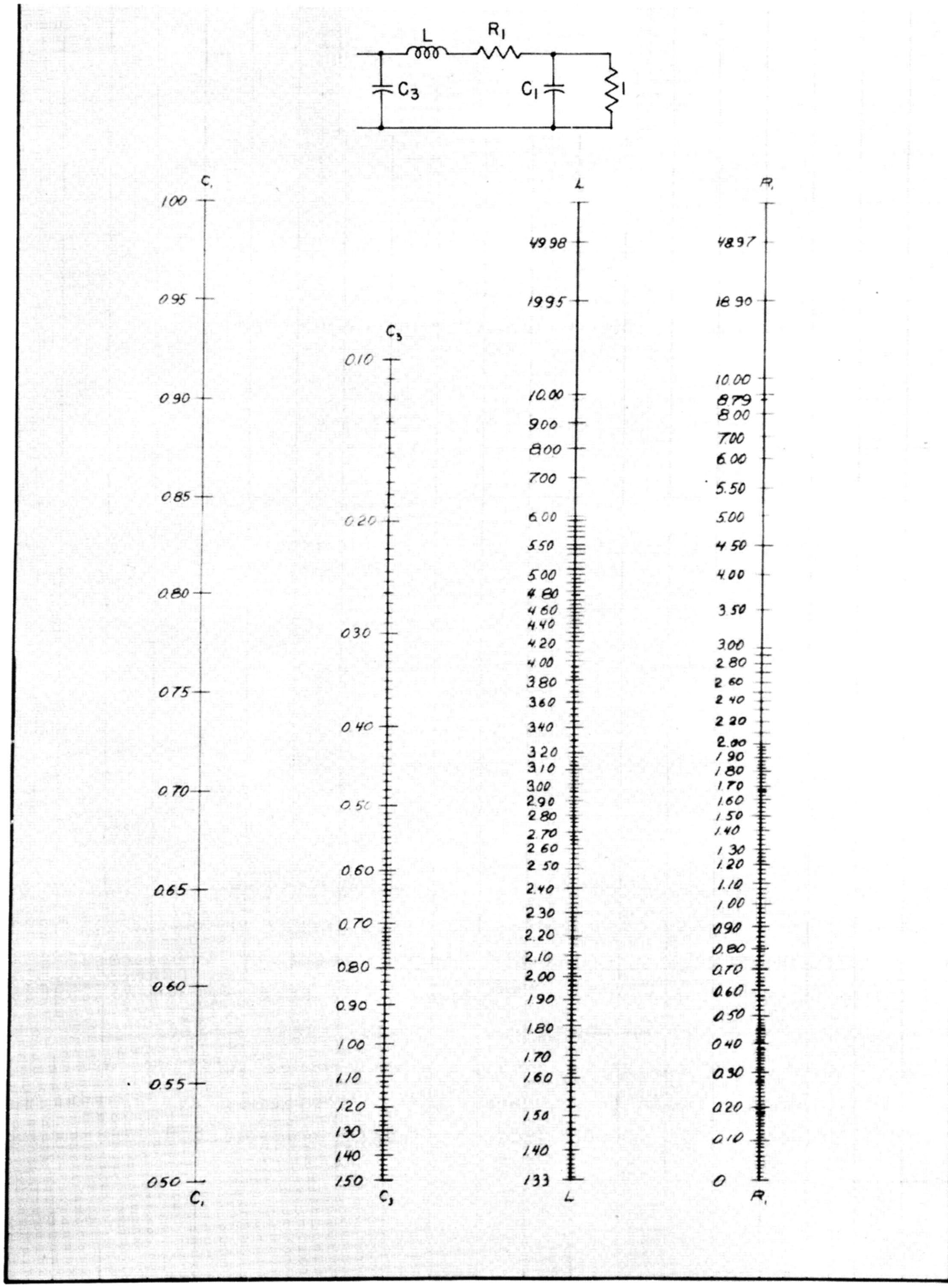

Fig. XVIII-11

Nomogram for third-order Butterworth function with loss.

B. APPLIED NETWORK THEORY

1. Complex Frequency Plane Synthesizer

A device has been constructed which has for its transfer function a pair of conjugate-complex poles in the complex-frequency plane. The real and imaginary coordinates of these poles are independently and continuously adjustable by means of a system of ganged potentiometers. A switching system is provided so that the device can be used with the poles in either half-plane. The basic elements of the device are two integrators of the "clamped" analog computer type.

Several of these elements can be cascaded for experimental synthesis of networks and the steady-state and transient responses are both available.

L. A. Irish

2. A Nomogram for a Butterworth Filter with Internal Loss

All of the modern methods of network synthesis yield lossless networks terminated in a single resistor. The addition of internal loss gives added flexibility in the design and is often desirable for itself. Scott's iterative method has been applied to the third-order Butterworth filter. A nomogram of the results is shown in Fig. XVIII-11.

J. P. Blake

3. An Approximate Method for Frequency Time Domain Transformation

In system or network analysis and synthesis, one is often faced with the problem of evaluating inverse L-transforms of the general type,

$$F(s) = \frac{b_0 + b_1 s + b_2 s^2 + \dots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \dots + a_n s^n} \tag{1}$$

The classical approaches to this problem by integral equation or partial fraction expansion are usually lengthy and often very difficult to carry out for $n > 4$.

There exist several approximate semigraphical methods for this frequency time domain transform as described in references (1) and (2). The method (3) proposed here is entirely analytical and involves little or no graphical manipulation. Basically, one proceeds by replacing every s term in Eq. 1 by $(1 - \epsilon^{-sT}/T)$. Thus

$$F(s) \to F^*(\epsilon^{-sT}) = \frac{d_0 + d_1\epsilon^{-sT} + \dots + d_m\epsilon^{-msT}}{c_0 + c_1\epsilon^{-sT} + c_2\epsilon^{-2sT} + \dots + c_n\epsilon^{-nsT}} = \frac{\sum_{k=0}^{m} d_k\epsilon^{-ksT}}{\sum_{k=0}^{n} c_k\epsilon^{-ksT}} \tag{2}$$

by straightforward long division

$$F^*(\epsilon^{-sT}) = \frac{d_0}{c_0} + \left(d_1 - c_1 \frac{d_0}{c_0}\right)/c_0 \, \epsilon^{-sT} + \ldots = f_0 + f_1\epsilon^{-sT} + f_2\epsilon^{-2sT} + \ldots$$

$$+ f_k\epsilon^{-ksT} + \ldots = \sum_{f=0}^{\infty} f_k\epsilon^{-ksT}$$

This form is then easily transformed into time domain. The coefficient f_k becomes the ordinate of $f(t)$ at time kT.

The selection of the interval T in the approximation $s \approx (1 - \epsilon^{-sT})/T$ is naturally important since it directly determines the accuracy. It is necessary to have some idea about the order of magnitude of the highest significant frequency component in the resultant time function before one can select T intelligently. If s_{max} is the highest significant frequency component, then T_{max} should be

$$T_{max} \approx \frac{1}{40\, f_{max}} \approx \frac{1}{6.5\, s_{max}} \tag{3}$$

corresponds to 5° of phase shift.

However, the percentage error increases for $s^2 \approx (1-\epsilon^{-sT})^2/T^2$, $s^3 \approx (1-\epsilon^{-sT})^3/T^3$, and so on when the same T is used. Furthermore, all the error caused by approximating the different s terms in the function may add up as well as cancel out. Thus, as a first approximation,

$$T_{max} \approx \frac{1}{6.5\, n^2 s_{max}} \qquad n = \text{order of the s transform} \tag{4}$$

The advantage of this method lies chiefly in the fact that once the s-function is transformed into the so-called z-domain, all the sampled-data analysis techniques (4) already developed are applicable. In many cases, analysis procedures are more easily handled or become intuitively obvious when they are in sampled-data form. For example, in sampled-data language, one knows that the approximation $s^2 \approx (1 - \epsilon^{-sT})^2/(T\epsilon^{-sT})$ is much better than $s^2 \approx (1 - \epsilon^{-sT})^2/T^2$. In the example shown in Fig. XVIII-12, $f(t)$ is computed from $F(s)$ using the above-mentioned method but with different approximation for s^2. The accuracy increases by three times in one case. Furthermore, once $F^*(\epsilon^{-sT})$ is obtained it is not necessary to determine f_0, f_1, f_2, ..., f_{k-1}, before f_k can finally be computed. One merely has to throw away successively the even or odd parts of $F^*(\epsilon^{-sT})$, $F^*(\epsilon^{-2sT})$, $F^*(\epsilon^{-4sT})$, etc. For example, if it is necessary to determine f_4 for $F^*(\epsilon^{-sT})$ one proceeds by throwing away first the odd part of $F^*(\epsilon^{-sT})$ leaving $F^*(\epsilon^{-2sT})$ which is rational in ϵ^{-2sT}. This procedure can be repeated to obtain $F^*(\epsilon^{-4sT})$ which contains only terms rational in ϵ^{-4sT}

$$F^*(\epsilon^{-sT}) \rightarrow F^*(\epsilon^{-4sT}) = \frac{\sum\limits_{k=0}^{m/4} d_{4k}\, \epsilon^{-4ksT}}{\sum\limits_{k=0}^{n/4} c_{4k}\, \epsilon^{-4ksT}} \tag{5}$$

With one division, one obtains f_4.

In short it is interesting to note how the various sampled-data techniques fit into this method of approach. Further investigation into the practical application of this method to system design is under way.

Y. C. Ho

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1. G. S. Brown and D. P. Campbell, Principles of Servomechanisms (John Wiley and Sons, Inc., New York, 1948) Chapter 11.
2. E. A. Guillemin, Technical Report No. 268, Research Laboratory of Electronics, M.I.T., Sept. 2, 1953.
3. A paper of similar nature, entitled "A mathematical technique for the analysis of linear systems," was published during this period of research by J. R. Ragazzini and A. R. Bergen of Columbia University.
4. For detailed descriptions of these techniques see Whirlwind Report R-222 by W. K. Linvill and R. Sittler and other papers by W. K. Linvill.

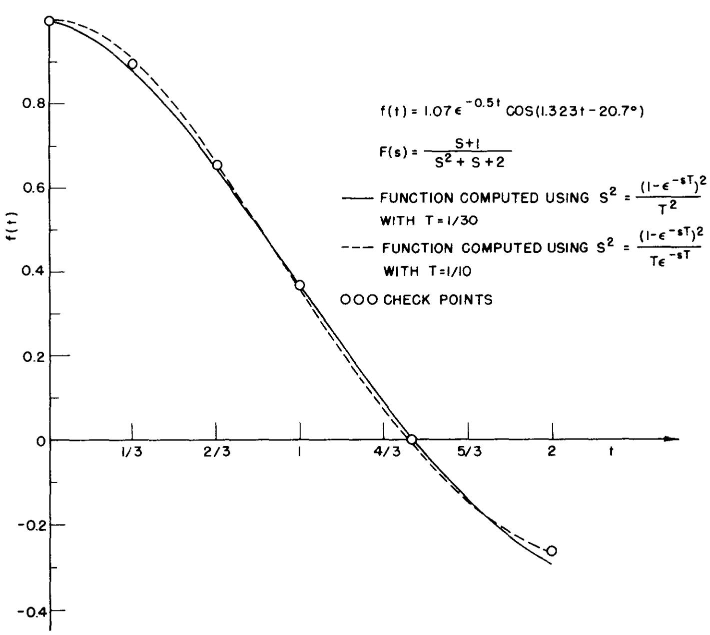

Fig. XVIII-12
Plot of computed f(t).

4. Duality of Ideal Transformers

In the Quarterly Progress Report of January 15, 1954, the analysis of ideal transformers was based on a pair-terminal view; that is, using input and output voltages and currents with no attention to the specific geometric configuration of the transformer. The ideal transformer was defined as a specified ratio of a number of voltages somewhere in the system. Windings were used only as a means of graphical presentation.

In practice, the best that one can do is to approximate this concept with actual windings; and the topological aspects of this physical system play an important role in the over-all analysis. The problem of duality of ideal transformers is considered on a purely topological basis.

Duality of two electrical networks is here defined by the relations: $V \backsim I$; $I \backsim V$; where the symbol "$\backsim$" is used for "dual of."

On examining the elementary circuit of a two-winding transformer, one sees instantly that the quantities flux ϕ and mmf F are directly related to the topological nature of the system. They are related to the voltages and currents as follows:

$$F = NI \qquad \lambda = N\phi = \int E\,dt \tag{1}$$

$$F \leftrightarrow I \qquad \phi \leftrightarrow QE \tag{2}$$

where the symbol $\leftrightarrow$ indicates "topologically equal." The letter Q indicates an operation with no topological consequences. Equations 2 have led a number of writers to consider F and ϕ as excitation and response quantities, respectively, or vice versa. The main point is that these quantities are interrelated through the electrical and topological properties of the "magnetic" medium.

To represent the topological properties of a transformer one needs to make a "ring" diagram where the current and flux paths are designated with solid and broken rings, respectively. This type of diagram preserves only the "linked" topological dependence of the physical quantities F and ϕ. The scheme is best illustrated by the examples shown in Figs. XVIII-13 and XVIII-14. In Fig. XVIII-14 the parallel lines in the

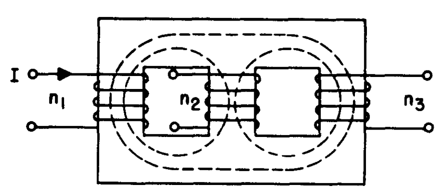

Fig. XVIII-13
A transformer.

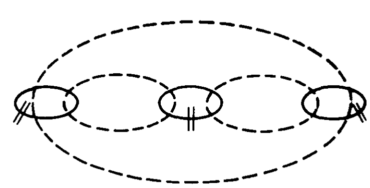

Fig. XVIII-14
A ring diagram.

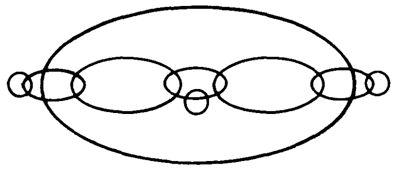

Fig. XVIII-15
Dual ring diagram.

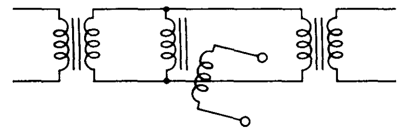

Fig. XVIII-16
Transformer equivalent of Fig. XVIII-15.

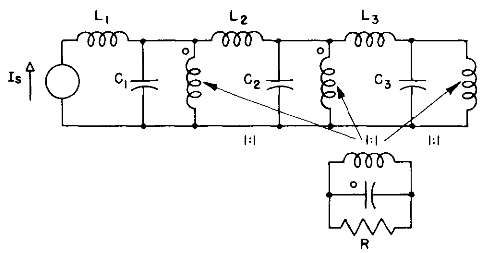

Fig. XVIII-17
An electrical analog.

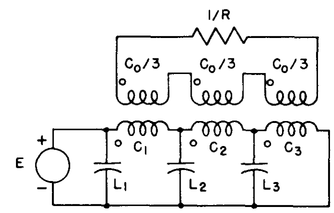

Fig. XVIII-18
The dual circuit.

solid rings were used to indicate that energy is "fed" or "dissipated" by a "cut" in that link.

To obtain the purely topological dual of a "ring" diagram one simply has to interchange the role of the solid and broken rings, and replace the current "cuts" by voltage "rings." Thus for the dual of Fig. XVIII-14 one obtains Fig. XVIII-15. From Fig. XVIII-15 the actual circuit of Fig. XVIII-16 is easily constructed.

The "ring" diagram holds true only if no energy loss or storage is associated with the topological quantities of solid and broken rings. If one wishes to consider such a problem it will be necessary to represent such effects by lumped equivalent elements outside the ideal transformer. It is true that in many instances such a presentation will result in a good approximation at the best; it is also true that the sense of duality in these instances is not well satisfied by using purely physical quantities; e.g., "mutual charge flux" as the dual to the "magnetic mutual flux."

A second example is shown in Fig. XVIII-17. It is a type that occurs in electrical analogs of mechanical systems. The dual of the network in Fig. XVIII-17 is shown in Fig. XVIII-18. In effect the circuit of Fig. XVIII-18 is more practical because, since an inductance is associated with each of the windings of the transformers, unity coupled coils instead of ideal transformers are required.

Summary. A fundamental theorem has been derived which allows one to obtain the dual of any linear passive and bilateral electrical network by a simple graphical method. Justification of the theorem is based on both algebraic and topological analysis.

N. DeClaris