XI. STATISTICAL COMMUNICATION THEORY

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A. SECOND-ORDER CORRELATION

The second-order autocorrelation functions for two simple random waves have been obtained.

The first random wave under consideration is one that alternates between two possible values, $E + A$ and $-E + A$, as illustrated in Fig. XI-1, with the number of crossings in a given interval of time distributed according to the Poisson law

$$P(n, \tau) = \frac{(k\tau)^n}{n!} e^{-k\tau}$$

In this expression τ is a given interval of time, n is the number of crossings in τ, k is the average number of crossings per second, and $P(n, \tau)$ is the probability of finding n crossings in τ. The second-order autocorrelation function of the wave is found to be

$$\phi_{111}(\tau_1, \tau_2) = \frac{AE^2}{2}\left[e^{-2k|\tau_1|} + e^{-2k|\tau_1+\tau_2|} + e^{-2k|\tau_2|}\right] + A^3 \qquad (1)$$

Figure XI-2 is a plot of this function for $A = E/2$, $k = 1$, and $E = 1$. It is noted that $\phi_{111}(\tau_1, \tau_2) = 0$ for $A = 0$.

The second random wave considered consists of a series of rectangular pulses of height E and duration d with a separation time between pulses equal to the pulse duration as shown in Fig. XI-3. It is assumed that the pulses have an equal probability of being on or off, and that the state of a pulse being on or off is independent of the states of the others. The second-order autocorrelation function is more conveniently described graphically and is given in Fig. XI-4.

J. Y. Hayase

B. EFFECTS OF PERIODIC SAMPLING ON OUTPUT NOISE IN AUTOCORRELATION DETECTION

It was indicated in the Quarterly Progress Report, April 15, 1954, that the output signal-to-noise ratio for autocorrelation detection of a sine wave in noise is given by

$$R_{oa} = 10 \log_{10} \frac{N}{2\rho_i^4 + 4\rho_i^2 + B} \qquad (1)$$

where $R_{oa} \equiv$ output signal-to-noise ratio in db, $N \equiv$ sample size, $\rho_i \equiv$ input noise-to-signal ratio, and

$$B = \frac{1}{N}\left[\frac{\sin\left(2\pi\frac{\omega_1}{\omega_o}\right)N}{\sin\left(2\pi\frac{\omega_1}{\omega_o}\right)}\right]^2 \tag{2}$$

In Eq. 2, ω_1 is the radian frequency of the periodic component of the input function and ω_o is the sampling frequency (of the periodic sampling process).

If the input signal is sampled at random, the output signal-to-noise ratio is given by Eq. 1 with B = 1. Therefore,

$$\left[R_{oa}\right]_{periodic} \geqslant \left[R_{oa}\right]_{random} \qquad \text{for } B \leqslant 1 \tag{3}$$

and

$$\left[R_{oa}\right]_{periodic} < \left[R_{oa}\right]_{random} \qquad \text{for } B > 1 \tag{4}$$

In Fig. XI-5 is shown a plot of B vs the frequency-ratio $x = \omega_1/\omega_o$. The plot is made for N = 10, and serves only to illustrate the general character of B(x).

As can be seen in Fig. XI-5, B(x) is a periodic function of x, of period 0.5, and has the maximum amplitude N and the minimum amplitude zero. The period of the small oscillations is 1/2N. The envelope of the maxima of B(x) is given by

$$\frac{1}{N\sin^2(2\pi x)} \qquad \text{for} \quad x = \frac{2k+1}{4N} \quad (k = 0, 1, 2, \ldots) \tag{5}$$

The envelope of B(x) for N = 1000 is shown plotted on a logarithmic scale in Fig. XI-6.

From the plots of B(x) it follows that optimum performance (maximum signal-to-noise ratio) is obtained for $\omega_1/\omega_o = 1/4$ or $3/4$ or, more generally, $\omega_1/\omega_o = k + 1/4$ or $k + 3/4$, where k is any integer. It should be noted, however, that in practice it is best to make the signal frequency ω_1 and the sampling frequency ω_o about the same order of magnitude because if k is very large a small drift in the signal frequency will be sufficient to change the operation from optimum performance to very bad performance. This is shown very simply with a numerical example.

Suppose the frequencies had originally been adjusted so that $\omega_1 = (k + 1/4)\,\omega_o$. Let $\omega_o = 10$ rad/sec, and k = 10. Then, for $\omega_1 = k\omega_o + (\omega_o/4) = 100 + 10/4 = 102.5$ rad/sec, $B \approx 0$. Now suppose ω_1 has a drift of 2.0 percent, and that it changes to (approximately) $\omega_1 = 100$ rad/sec. Now, $\omega_1/\omega_o = 10$, and B = N, which would yield the worst possible signal-to-noise ratio.

Thus, if there is any possibility of drift, it is important that k be chosen as small as possible.

R. E. Wernikoff

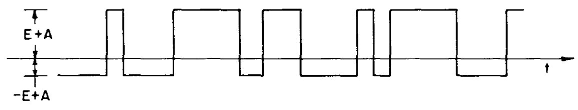

Fig. XI-1

A random wave alternating between E + A and -E + A with the number of crossings in a given interval Poisson-distributed.

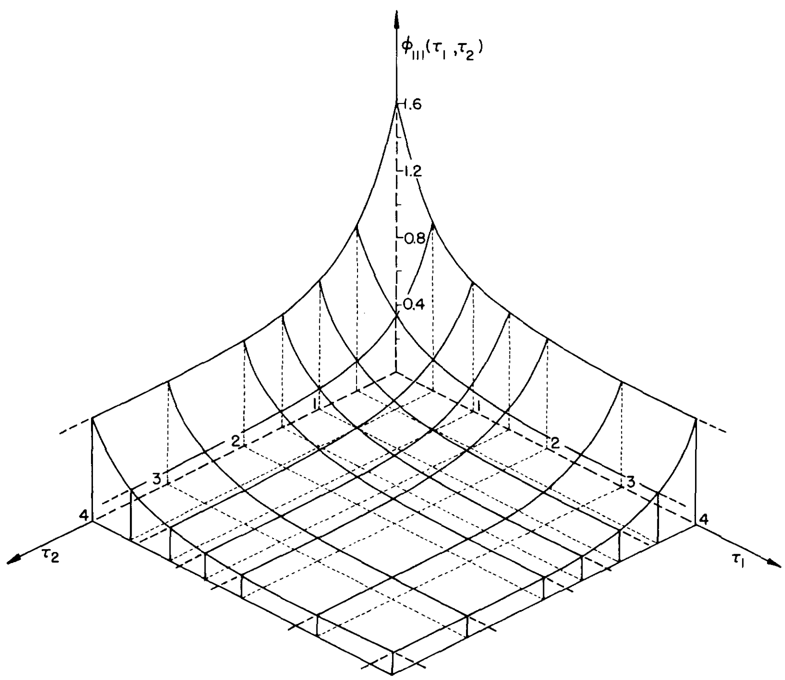

Fig. XI-2

The second-order autocorrelation function of the wave shown in Fig. XI-1 as expressed by Eq. 1. Drawn for k = 1, E = 1, A = E/2.

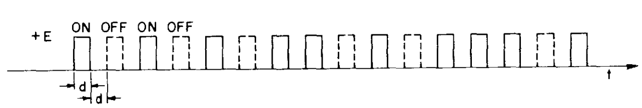

Fig. XI-3

A random wave of rectangular pulses.

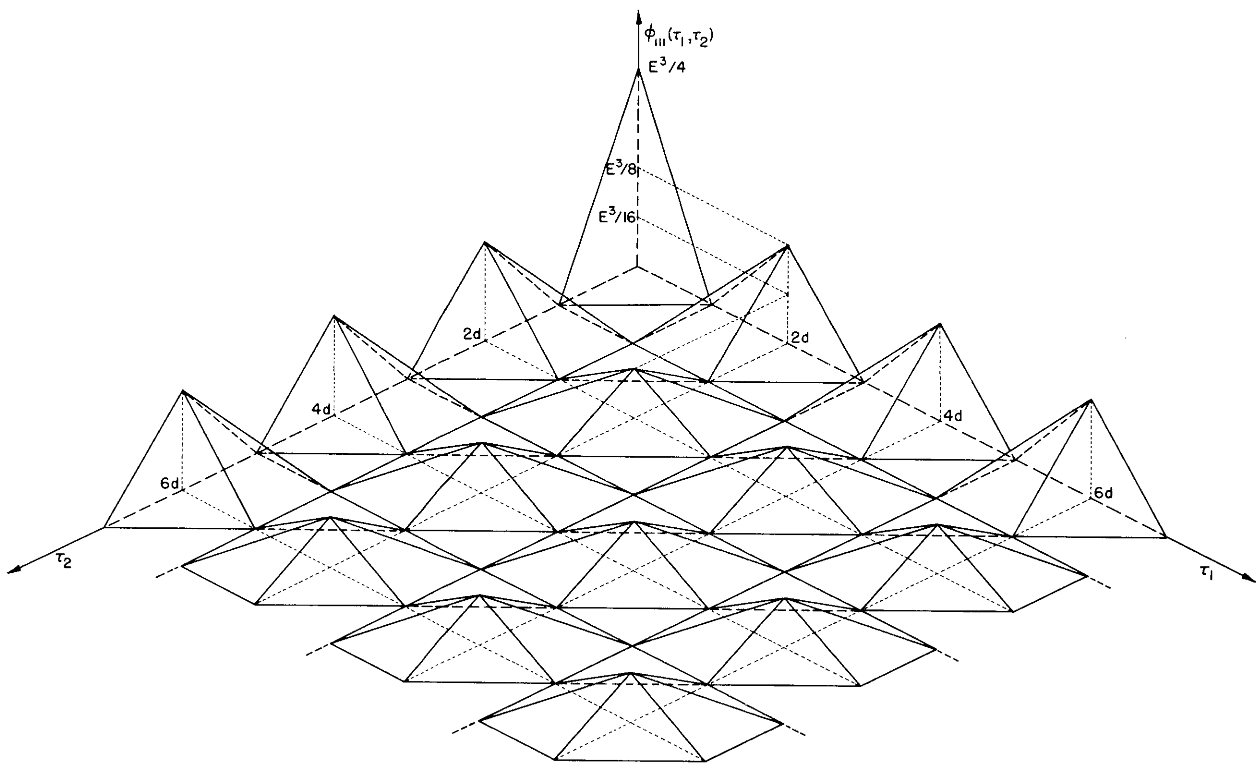

Fig. XI-4

The second-order autocorrelation function of the wave shown in Fig. XI-3.

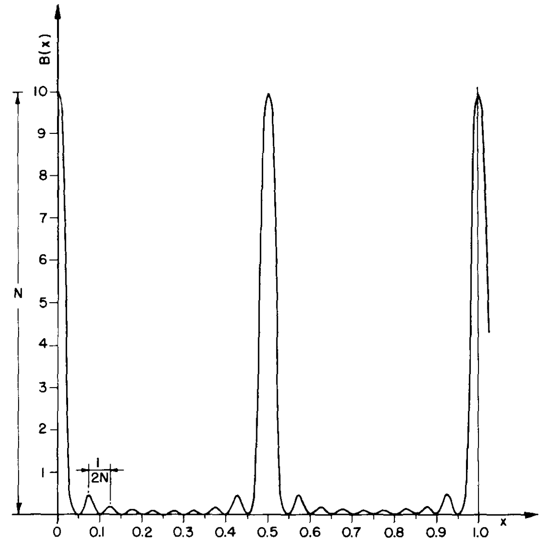

Fig. XI-5

Plot of two cycles of the periodic function

$$B = \frac{1}{N}\left[\frac{\sin(2\pi Nx)}{\sin(2\pi x)}\right]^2 \text{ vs x for N = 10.}$$

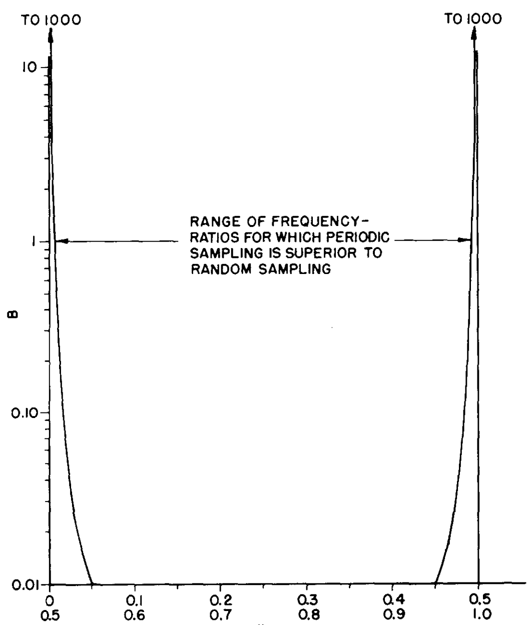

Fig. XI-6

Plot of the envelope of $B = \frac{1}{N}\left[\frac{\sin(2\pi Nx)}{\sin(2\pi x)}\right]^2$ vs x for N = 1000.