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Black hole entropy and quantum information [1]

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ABSTRACT

We review some recently established connections between the mathematics of black hole entropy in string theory and that of multipartite entanglement in quantum information theory. In the case of $N = 2$ black holes and the entanglement of three qubits, the quartic $[SL(2)]^3$ invariant, Cayley's hyperdeterminant, provides both the black hole entropy and the measure of tripartite entanglement. In the case of $N = 8$ black holes and the entanglement of seven qubits, the quartic E_7 invariant of Cartan provides both the black hole entropy and the measure of a particular tripartite entanglement encoded in the Fano plane.

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Contents

1 Black holes and qubits

It sometimes happens that two very different areas of theoretical physics share the same mathematics. This may eventually lead to the realisation that they are, in fact, dual descriptions of the same physical phenomena, or it may not. Either way, it frequently leads to new insights in both areas. In this paper the two areas in question are black hole entropy in string theory and qubit entanglement in quantum information theory. Going one way, we shall learn that the entropy of the so-called STU $N = 2$ black hole is given by the "hyperdeterminant", a quantity first introduced by Cayley in 1845 and which describes the tripartite entanglement of three qubits [1, 2, 3]. Going the other way, we discover that the exceptional group E_7, the U-duality group of $N = 8$ supergravity, plays a part in the tripartite entanglement of seven qubits [4, 5].

We begin in section 2 with an interesting subsector of string compactification to four dimensions which is provided by the STU model whose low energy limit is described by $N = 2$ supergravity coupled to three vector multiplets. One may regard it as a truncation of an $N = 4$ theory obtained by compactifying the heterotic string on T^6 where S, T, U correspond to the dilaton/axion, complex Kahler form and complex structure fields respectively. It exhibits an $SL(2, Z)_S$ strong/weak coupling duality and an $SL(2, Z)_T \times SL(2, Z)_U$ target space duality. By string/string duality, this is equivalent to a Type IIA string on $K3 \times T^2$ with S and T exchanging roles [6, 7, 8]. Moreover, by mirror symmetry this is in turn equivalent to a Type IIB string on the mirror manifold with T and U exchanging roles. Another way to obtain this model is by truncation of the N=8 theory that results from T^7 compactification of M-theory. Either way, the truncated theory has a combined $[SL(2, Z)]^3$ duality and complete $S - T - U$ triality symmetry [9]. Alternatively, one may simply start with this $N = 2$ theory directly as an interesting four-dimensional supergravity in its own right, as described in section 2.

The model admits extremal black holes solutions carrying four electric and magnetic charges and we organize these 8 charges into the $2 \times 2 \times 2$ *hypermatrix*, a_{ABC}, and display the $S - T - U$ symmetric Bogomolnyi mass formula [9]. Associated with this hypermatrix is a *hyperdeterminant*, Det a_{ABC}, discussed in section 3, first introduced by Cayley in 1845 [10]. The black hole entropy, first calculated in [11], is quartic in the charges and must be invariant under $[SL(2, Z)]^3$ and under triality. The main result of section 4, is to show [1] that this entropy given by the square root of Cayley's hyperdeterminant:

$$S = \pi\sqrt{|\text{Det } a_{ABC}|}. \tag{1.1}$$

The hyperdeterminant also makes it appearance in quantum information theory [13]. Let the three qubit system ABC (Alice, Bob amd Charlie) be in a pure state $|\Psi\rangle$, and let the components of $|\Psi\rangle$ in the standard basis be a_{ABC}:

$$|\Psi\rangle = a_{ABC}|ABC\rangle \tag{1.2}$$

or

$$\begin{aligned}|\Psi\rangle = a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle \\ +a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle\end{aligned} \tag{1.3}$$

Then the three way entanglement of the three qubits A, B and C is given by the *3-tangle* [12]

$$\tau_3(ABC) = 4|\text{Det } a_{ABC}|. \tag{1.4}$$

The 3-tangle is maximal for the GHZ state $|000\rangle + |111\rangle$ [28] and vanishes for the states $p|100\rangle + q|010\rangle + r|001\rangle$. The relation between three qubit quantum entanglement and the Cayley hyperdeterminant was pointed out by Miyake and Wadati [13].

As far as we can tell [1], the appearance of the Cayley hyperdeterminant in these two different contexts of stringy black hole entropy (where the a_{ABC} are integers and the symmetry is $[SL(2,Z)]^3$) and three-qubit quantum entanglement (where the a_{ABC} are complex numbers and the symmetry is $[SL(2,C]^3)$ is a purely mathematical coincidence. Nevertheless, it has already provided fascinating new insights [1, 2, 3, 4, 5] into the connections between strings, black holes, and quantum information[4].

In section 6 we extend the argument to the $N = 8$ case and, noting that

$$E_{7(7)}(Z) \supset [SL(2,Z)]^7 \tag{1.5}$$

and

$$E_7(C) \supset [SL(2,C)]^7, \tag{1.6}$$

show that the corresponding system in quantum information theory is that of seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George). However, the larger symmetry requires that they undergo at most tripartite entanglement of a very specific kind. As discussed in section 8, the entanglement measure will be given by the quartic Cartan $E_7(C)$ invariant [16, 17, 18, 19]. The entanglement may be represented by the Fano plane [15] which also provides the multiplication table of the split octonions. See also the interesting paper by Levay [5] who noted independently the connection to the Fano plane.

2 The N=2 STU model

Consider the three complex scalars axion/dilaton field S, the complex Kahler form field T and the complex structure field U

$$\begin{aligned} S &= S_1 + iS_2 \\ T &= T_1 + iT_2 \\ U &= U_1 + iU_2 . \end{aligned} \tag{2.1}$$

This complex parameterization allows for a natural transformation under the various $SL(2,Z)$ symmetries. The action of $SL(2,Z)_S$ is given by

$$S \rightarrow \frac{aS+b}{cS+d} , \tag{2.2}$$

[4] A third application [14], not considered in this paper, is the Nambu-Goto string whose action is also given by $\sqrt{|\text{Det } a_{ABC}|}$ in spacetime signature $(2,2)$.

where a,b,c,d are integers satisfying $ad-bc=1$, with similar expressions for $SL(2,Z)_T$ and $SL(2,Z)_U$. Defining the matrices $\mathcal{M}_S$, $\mathcal{M}_T$ and $\mathcal{M}_U$ via

$$\mathcal{M}_S = \frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix} , \tag{2.3}$$

the action of $SL(2,Z)_S$ now takes the form

$$\mathcal{M}_S \to {\omega_S}^T \mathcal{M}_S \omega_S \ , \tag{2.4}$$

where

$$\omega_S = \begin{pmatrix} d & b \\ c & a \end{pmatrix} , \tag{2.5}$$

with similar expressions for $\mathcal{M}_T$ and $\mathcal{M}_U$. We also define the $SL(2,Z)$ invariant tensors

$$\epsilon_S = \epsilon_T = \epsilon_U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.6}$$

Starting from the heterotic string,the bosonic action for the graviton $g_{\mu\nu}$, dilaton η, two-form $B_{\mu\nu}$ four $U(1)$ gauge fields A_S^a and two complex scalars T and U is [9]

$$\begin{aligned} I_{STU} \;=\; \frac{1}{16\pi G}\int d^4x\sqrt{-g}e^{-\eta}\Big[R_g &+ g^{\mu\nu}\partial_\mu\eta\partial_\nu\eta - \frac{1}{12}g^{\mu\lambda}g^{\nu\tau}g^{\rho\sigma}H_{\mu\nu\rho}H_{\lambda\tau\sigma} \\ &+\frac{1}{4}\mathrm{Tr}(\partial{\mathcal{M}_T}^{-1}\partial\mathcal{M}_T) + \frac{1}{4}\mathrm{Tr}(\partial{\mathcal{M}_U}^{-1}\partial\mathcal{M}_U) \\ &-\frac{1}{4}{F_{S\mu\nu}}^T(\mathcal{M}_T\times\mathcal{M}_U){F_S}^{\mu\nu}\Big] . \end{aligned} \tag{2.7}$$

where the metric $g_{\mu\nu}$ is related to the four-dimensional canonical Einstein metric $g^c_{\mu\nu}$ by $g_{\mu\nu} = e^\eta {g^c}_{\mu\nu}$ and where

$$H_{\mu\nu\rho} = 3(\partial_{[\mu}B_{\nu\rho]} - \tfrac{1}{2}{A_{S[\mu}}^T(\epsilon_T\times\epsilon_U)F_{S\nu\rho]}). \tag{2.8}$$

This action is manifestly invariant under T-duality and U-duality, with

$$F_{S\mu\nu} \to ({\omega_T}^{-1}\times{\omega_U}^{-1})F_{S\mu\nu} \ , \qquad \mathcal{M}_{T/U} \to \omega^T_{T/U}\,\mathcal{M}_{T/U}\,\omega_{T/U} \ , \tag{2.9}$$

and with η, $g_{\mu\nu}$ and $B_{\mu\nu}$ inert. Its equations of motion and Bianchi identities (but not the action itself) are also invariant under S-duality (2.2), with T and ${g^c}_{\mu\nu}$ inert and with

$$\begin{pmatrix} {F_{S\mu\nu}}^a \\ {\widetilde{F}_{S\mu\nu}}{}^a \end{pmatrix} \to \omega_S^{-1}\begin{pmatrix} {F_{S\mu\nu}}^a \\ {\widetilde{F}_{S\mu\nu}}{}^a \end{pmatrix} , \tag{2.10}$$

where

$${\widetilde{F}_{S\mu\nu}}{}^a = -S_2[({\mathcal{M}_T}^{-1}\times{\mathcal{M}_U}^{-1})(\epsilon_T\times\epsilon_U)]^a{}_b * {F_{S\mu\nu}}^b - S_1{F_{S\mu\nu}}^a \ , \tag{2.11}$$

where the axion field a is defined by

$$\epsilon^{\mu\nu\rho\sigma}\partial_\sigma a = \sqrt{-g}e^{-\eta}g^{\mu\sigma}g^{\nu\lambda}g^{\rho\tau}H_{\sigma\lambda\tau} \ , \tag{2.12}$$

and where $S = S_1 + iS_2 = a + ie^{-\eta}$.

Thus T-duality transforms Kaluza-Klein electric charges $({F_S}^3, {F_S}^4)$ into winding electric charges $({F_S}^1, {F_S}^2)$ (and Kaluza-Klein magnetic charges into winding magnetic charges), U-duality transforms the Kaluza-Klein and winding electric charge of one circle $({F_S}^3, {F_S}^2)$ into those of the other $({F_S}^4, {F_S}^1)$ (and similarly for the magnetic charges) but S-duality transforms Kaluza-Klein electric charge $({F_S}^3, {F_S}^4)$ into winding magnetic charge $(\tilde{F_S}^3, \tilde{F_S}^4)$ (and winding electric charge into Kaluza-Klein magnetic charge). In summary we have $SL(2,Z)_T \times SL(2,Z)_U$ and $T \leftrightarrow U$ off-shell but $SL(2,Z)_S \times SL(2,Z)_T \times SL(2,Z)_U$ and an S–T–U interchange on-shell.

One may also consider the Type IIA action I_{TUS} and the Type IIB action I_{UST} obtained by cyclic permutation of the fields S, T, U. Finally, one may consider an action [11] where the S, T and U fields enter democratically with a prepotential

$$F = STU \tag{2.13}$$

which off-shell has the full STU interchange but none of the $SL(2,Z)$. All four versions are on-shell equivalent.

Following [9], it is now straightforward to write down an S–T–U symmetric Bogomolnyi mass formula. Let us define electric and magnetic charge vectors α_S^a and β_S^a associated with the field strengths ${F_S}^a$ and $\tilde{F_S}^a$ in the standard way. The electric and magnetic charges Q_S^a and P_S^a are given by

$$F_{S\,0r}^{\;a} \sim \frac{Q_S^a}{r^2} \qquad * F_{S\,0r}^{\;a} \sim \frac{P_S^a}{r^2}\ , \tag{2.14}$$

giving rise to the charge vectors

$$\begin{pmatrix} \alpha_S^a \\ \beta_S^a \end{pmatrix} = \begin{pmatrix} S_2^{(0)} \mathcal{M}_T^{-1} \times \mathcal{M}_U^{-1} & S_1^{(0)} \epsilon_T \times \epsilon_U \\ 0 & -\epsilon_T \times \epsilon_U \end{pmatrix}^{ab} \begin{pmatrix} Q_S^b \\ P_S^b \end{pmatrix} . \tag{2.15}$$

For our purpose it is useful to define a $2 \times 2 \times 2$ array $a_{AA'A''}$ via

$$\begin{pmatrix} a_{000} \\ a_{001} \\ a_{010} \\ a_{011} \\ a_{100} \\ a_{101} \\ a_{110} \\ a_{111} \end{pmatrix} = \begin{pmatrix} -\beta_S^1 \\ -\beta_S^2 \\ -\beta_S^3 \\ -\beta_S^4 \\ \alpha_S^1 \\ \alpha_S^2 \\ \alpha_S^3 \\ \alpha_S^4 \end{pmatrix} , \tag{2.16}$$

transforming as

$$a^{AA'A''} \rightarrow {\omega_S}^A{}_B {\omega_T}^{A'}{}_{B'} {\omega_U}^{A''}{}_{B''} a^{ABC}\ . \tag{2.17}$$

Then the mass formula is

$$m^2 = \frac{1}{16} a^T ({\mathcal{M}_S}^{-1} {\mathcal{M}_T}^{-1} {\mathcal{M}_U}^{-1} - {\mathcal{M}_S}^{-1} \epsilon_T \epsilon_U - \epsilon_S {\mathcal{M}_T}^{-1} \epsilon_U - \epsilon_S \epsilon_T {\mathcal{M}_U}^{-1}) a\ . \tag{2.18}$$

This is consistent with the general $N = 2$ Bogomolnyi formula [34]. Although all theories have the same mass spectrum, there is clearly a difference of interpretation with electrically charged elementary states in one picture being solitonic monopole or dyon states in the other.

This $2 \times 2 \times 2$ array $a_{AA'A''}$ is an example a "hypermatrix", a term coined by Cayley in 1845 [10] where he also introduced a "hyperdeterminant".

3 Cayley's hyperdeterminant

In 1845 Cayley [10] generalized the determinant of a 2×2 matrix $a_{AA'}$

$$\begin{aligned} \det a &= \frac{1}{2}\epsilon^{AB}\epsilon^{A'B'}a_{AA'}a_{BB'} \\ &= a_{00}a_{11} - a_{01}a_{10} \end{aligned} \tag{3.1}$$

to the *hyperdeterminant* of a $2 \times 2 \times 2$ *hypermatrix* $a_{AA'A''}$

$$\begin{aligned} \mathrm{Det}\ a &= -\frac{1}{2}\epsilon^{AB}\epsilon^{A'B'}\epsilon^{CD}\epsilon^{C'D'}\epsilon^{A''C''}\epsilon^{B''D''}a_{AA'A''}a_{BB'B''}a_{CC'C''}a_{DD'D''} \\ &= a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{100}^2a_{011}^2 \\ &\quad -2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} \\ &\quad +a_{000}a_{100}a_{011}a_{111} + a_{001}a_{010}a_{101}a_{110} \\ &\quad +a_{001}a_{100}a_{011}a_{110} + a_{010}a_{100}a_{011}a_{101}) \\ &\quad +4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}) \end{aligned} \tag{3.2}$$

$$\begin{aligned} &= a_0^2a_7^2 + a_1^2a_6^2 + a_2^2a_5^2 + a_3^2a_4^2 \\ &-2(a_0a_1a_6a_7 + a_0a_2a_5a_7 + a_0a_4a_3a_7 + a_1a_2a_5a_6 + a_1a_3a_4a_6 + a_2a_3a_4a_5) \\ &+4(a_0a_3a_5a_6 + a_1a_2a_4a_7) \end{aligned} \tag{3.3}$$

where we have made the binary conversion $0, 1, 2, 3, 4, 5, 6, 7$ for $000, 001, 010, 011, 100, 101, 110, 111$.

The hyperdeterminant vanishes iff the following system of equations in six unknowns $p^A, q^{A'}, r^{A''}$ has a nontrivial solution, not allowing any of the pairs to be both zero:

$$\begin{aligned} a_{AA'A''}p^Aq^{A'} &= 0 \\ a_{AA'A''}p^Ar^{A''} &= 0 \\ a_{AA'A''}q^{A'}r^{A''} &= 0 \end{aligned} \tag{3.4}$$

For our purposes, the important properties of the hyperdeterminant are that it is a quartic invariant under $[SL(2)]^3$ and under a triality that interchanges A, A' and A''. These properties are valid whether the $a_{AA'A''}$ are complex, real or integer.

One way to understand this triality is to think of having three different metrics (Alice, Bob and Charlie)

$$\alpha_{AB} = \epsilon^{A'B'}\epsilon^{A''B''}a_{AA'A''}a_{BB'B''}$$

$$
\begin{aligned}
\beta_{A'B'} &= \epsilon^{A''B''}\epsilon^{AB}a_{AA'A''}a_{BB'B''} \\
\gamma_{A''B''} &= \epsilon^{AB}\epsilon^{A'B'}a_{AA'A''}a_{BB'B''}
\end{aligned} \tag{3.5}
$$

Explicitly,

$$
\gamma = \begin{pmatrix} 2(a_0a_6 - a_2a_4) & a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4 \\ a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4 & 2(a_1a_7 - a_3a_5) \end{pmatrix} \tag{3.6}
$$

$$
\beta = \begin{pmatrix} 2(a_0a_3 - a_1a_2) & a_0a_7 - a_1a_6 + a_4a_3 - a_5a_2 \\ a_0a_7 - a_1a_6 + a_4a_3 - a_5a_2 & 2(a_4a_7 - a_5a_6) \end{pmatrix} \tag{3.7}
$$

$$
\alpha = \begin{pmatrix} 2(a_0a_5 - a_4a_1) & a_0a_7 - a_4a_3 + a_2a_5 - a_6a_1 \\ a_0a_7 - a_4a_3 + a_2a_5 - a_6a_1 & 2(a_2a_7 - a_6a_3) \end{pmatrix} \tag{3.8}
$$

All are equivalent, however, since

$$
det\ \alpha = det\ \beta = det\ \gamma = -\mathrm{Det}\ a \tag{3.9}
$$

If we make the identifications

$$
\begin{aligned}
a_0 &= \tfrac{1}{\sqrt{2}}(-P^0 + P^2) \\
a_1 &= \tfrac{1}{\sqrt{2}}(-Q^0 + Q^2) \\
a_2 &= \tfrac{1}{\sqrt{2}}(\ P^1 - P^3) \\
a_3 &= \tfrac{1}{\sqrt{2}}(\ Q^1 - Q^3) \\
a_4 &= \tfrac{1}{\sqrt{2}}(-P^1 - P^3) \\
a_5 &= \tfrac{1}{\sqrt{2}}(-Q^1 - Q^3) \\
a_6 &= \tfrac{1}{\sqrt{2}}(-P^0 - P^2) \\
a_7 &= \tfrac{1}{\sqrt{2}}(-Q^0 - Q^2)
\end{aligned} \tag{3.10}
$$

then we find the $O(2,2)$ scalar products

$$
2(a_0a_6 - a_2a_4) = (P^0)^2 + (P^1)^2 - (P^2)^2 - (P^3)^2 = P^2
$$

$$
\begin{aligned}
2(a_1a_7 - a_3a_5) &= (Q_0)^2 + (Q_1)^2 - (Q_2)^2 - (Q_3)^2 = Q^2 \\
a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4 &= (P^0Q_0) + (P^1Q_1) + (P^2Q_2) + (P^3Q_3) = P.Q
\end{aligned}
$$

so

$$
\gamma = \begin{pmatrix} P^2 & P.Q \\ P.Q & Q^2 \end{pmatrix} \tag{3.11}
$$

and

$$
-Det\ a = P^2Q^2 - (P.Q)^2
$$

4 Black hole entropy

The STU model admits extremal black hole solutions satisfying the Bogomolnyi mass formula. As usual, their entropy is given by one quarter the area of the event horizon. However, to calculate this area requires evaluating the mass not with the asymptotic values of the moduli, but with their frozen values on the horizon which are fixed in terms of the charges [27]. This ensures that the entropy is moduli-independent, as it should be. The relevant calculation was carried out in [11] for the model with the STU prepotential. The electric and magnetic charges of that paper are denoted (p^0, q_0), (p^1, q_1), (p^2, q_2), (p^3, q_3). In these variables, the entropy is given by

$$S = \pi \left(W(p^\Lambda, q_\Lambda) \right)^{1/2} \tag{4.1}$$

where

$$W(p^\Lambda, q_\Lambda) = -(p \cdot q)^2 + 4((p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^3 q_3)(p^2 q_2)) - 4p^0 q_1 q_2 q_3 + 4q_0 p^1 p^2 p^3 \ . \tag{4.2}$$

The function $W(p^\Lambda, q_\Lambda)$ is symmetric under transformations: $p^1 \leftrightarrow p^2 \leftrightarrow p^3$ and $q_1 \leftrightarrow q_2 \leftrightarrow q_3$. For the solution to be BPS we have to require $W > 0$.

If we now make the identifications

$$\begin{pmatrix} a_{000} \\ a_{001} \\ a_{010} \\ a_{011} \\ a_{100} \\ a_{101} \\ a_{110} \\ a_{111} \end{pmatrix} = \begin{pmatrix} p^0 \\ -p^1 \\ -p^2 \\ q^3 \\ -p^3 \\ q_2 \\ q_1 \\ q_0 \end{pmatrix} , \tag{4.3}$$

we recognize from (3) that

$$W = -\text{Det } a \tag{4.4}$$

and hence the black hole entropy is given by

$$S = \pi \sqrt{-\text{Det } a} \tag{4.5}$$

Some examples of supersymmetric black hole solutions [29] are provided by the electric Kaluza-Klein black hole with $\alpha = (1, 0, 0, 0)$ and $\beta = (0, 0, 0, 0)$; the electric winding black hole with $\alpha = (0, 0, 0, -1)$ and $\beta = (0, 0, 0, 0)$; the magnetic Kaluza-Klein black hole with $\alpha = (0, 0, 0, 0)$ and $\beta = (0, -1, 0, 0)$; the magnetic winding black hole with $\alpha = (0, 0, 0, 0)$ and $\beta = (0, 0, -1, 0)$. These are characterized by a scalar-Maxwell coupling parameter $a = \sqrt{3}$. By combining these 1-particle states, we may build up 2-, 3- and 4-particle bound states at threshold [29, 9]. For example $\alpha = (1, 0, 0, -1)$ and $\beta = (0, 0, 0, 0)$ with $a = 1$; $\alpha = (1, 0, 0, -1)$ and $\beta = (0, -1, 0, 0)$ with $a = 1/\sqrt{3}$; $\alpha = (1, 0, 0, -1)$ and $\beta = (0, -1, -1, 0)$ with $a = 0$. The 1-, 2- and 3-particle states all yield vanishing contributions to Det a. A non-zero value is obtained for the 4-particle example, however, which is just the Reissner-Nordstrom black hole.

5 The N=8 generalization

The black holes described by Cayley's hyperdeterminant are those of $N = 2$ supergravity coupled to three vector multiplets, where the symmetry is $[SL(2,Z)]^3$. One might therefore ask whether the black hole/information theory correspondence could be generalized. There are three generalizations we might consider:

1) $N = 2$ supergravity coupled to l vector multiplets where the symmetry is $SL(2,Z) \times SO(l-1,2,Z)$ and the black holes carry charges belonging to the $(2, l+1)$ representation ($l+1$ electric plus $l+1$ magnetic).

2) $N = 4$ supergravity coupled to m vector multiplets where the symmetry is $SL(2,Z) \times SO(6, 6+m, Z)$ where the black holes carry charges belonging to the $(2, 12+m)$ representation ($m + 12$ electric plus $m + 12$ magnetic).

3) $N = 8$ supergravity where the symmetry is the non-compact exceptional group $E_{7(7)}(Z)$ and the black holes carry charges belonging to the fundamental 56-dimensional representation (28 electric plus 28 magnetic).

In all three case there exit quartic invariants akin to Cayley's hyperdeterminant whose square root yields the corresponding black hole entropy. If there is to be a quantum information theoretic interpretation, however, it cannot just be random entanglement of more qubits, because the general n qubit entanglement is described by the group $[SL(2,C)]^n$, which, even after replacing Z by C, differs from the above symmetries (except when $n = 3$, which correspond to case (1) above with $l = 3$, the case we already know.).

We note, however, that

$$E_{7(7)}(Z) \supset [SL(2,Z)]^7 \tag{5.1}$$

and

$$E_7(C) \supset [SL(2,C)]^7, \tag{5.2}$$

We shall now show that the corresponding system in quantum information theory is that of seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George). However, the larger symmetry requires that they undergo at most tripartite entanglement of a very specific kind. The entanglement measure will be given by the quartic Cartan $E_7(C)$ invariant [16, 17, 18, 19].

6 Decomposition of $E_{7(7)}$

Consider the decomposition of the fundamental 56-dimensional representation of $E_{7(7)}$ under its maximal subgroup

$$E_{7(7)} \supset SL(2)_A \times SO(6,6)$$

$$56 \rightarrow (2,12) + (1,32) \tag{6.1}$$

Further decomposing $SO(6,6)$,

$$SL(2)_A \times SO(6,6) \supset SL(2)_A \times SL(2)_B \times SL(2)_D \times SO(4,4)$$

$$(2,12) + (1,32) \rightarrow (2,2,2,1)$$

$$+(2,1,1,8_v)+(1,2,1,8_s)+(1,1,2,8_c) \tag{6.2}$$

Further decomposing $SO(4,4)$,

$$SL(2)_A\times SL(2)_B\times SL(2)_D\times SO(4,4)\supset SL(2)_A\times SL(2)_B\times SL(2)_D$$

$$\times SO(2,2)\times SO(2,2)$$

$$(2,2,2,1)+(2,1,1,8_v)+(1,2,1,8_s)+(1,1,2,8_c)\to$$

$$(2,2,2,1,1)+(2,1,1,4,1)+(2,1,1,1,4)$$

$$+(1,2,1,2,2)+(1,2,1,2,2)+(1,1,2,2,2)+(1,1,2,2,2) \tag{6.3}$$

Finally, further decomposing each $SO(2,2)$

$$SL(2)_A\times SL(2)_B\times SL(2)_D\times SO(2,2)\times SO(2,2)\supset$$

$$SL(2)_A\times SL(2)_B\times SL(2)_D\times SL(2)_C\times SL(2)_G\times SL(2)_F\times SL(2)_E$$

$$(2,2,2,1,1)+(2,1,1,4,1)+(2,1,1,1,4)$$

$$+(1,2,1,2,2)+(1,2,1,2,2)+(1,1,2,2,2)+(1,1,2,2,2)\to$$

$$(2,2,2,1,1,1,1)+(2,1,1,2,2,1,1)+(2,1,1,1,1,2,2)+$$

$$(1,2,1,2,1,1,2)+(1,2,1,1,2,2,1)+(1,1,2,2,1,2,1)+(1,1,2,1,2,1,2)$$

In summary,

$$E_{7(7)}\supset SL(2)_A\times SL(2)_B\times SL(2)_C\times SL(2)_D\times SL(2)_E\times SL(2)_F\times SL(2)_G \tag{6.4}$$

and the 56 decomposes as

$$\begin{aligned}&56\to\\&(2,2,1,2,1,1,1)\\&+(1,2,2,1,2,1,1)\\&+(1,1,2,2,1,2,1)\\&+(1,1,1,2,2,1,2)\\&+(2,1,1,1,2,2,1)\\&+(1,2,1,1,1,2,2)\\&+(2,1,2,1,1,1,2)\end{aligned} \tag{6.5}$$

An analogous decomposition holds for

$$E_7(C)\supset[SL(2,C)]^7. \tag{6.6}$$

7 Tripartite entanglement of 7 qubits

We have seen that in the case of three qubits, the tripartite entanglement is described by $[SL(2,C)]^3$ and that the entanglement measure is given by Cayley's hyperdeterminant. Now we consider seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George) but where Alice has tripartite entanglement not only with Bob/Daisy but also with Emma/Fred and also George/Charlie, and similarly for the other six individuals. So, in fact, each person has tripartite entanglement with each of the remaining three couples:

$$\begin{aligned}
&|\Psi\rangle = \\
&a_{ABD}|ABD\rangle \\
&+b_{BCE}|BCE\rangle \\
&+c_{CDF}|CDF\rangle \\
&+d_{DEG}|DEG\rangle \\
&+e_{EFA}|EFA\rangle \\
&+f_{FGB}|FGB\rangle \\
&+g_{GAC}|GAC\rangle
\end{aligned} \tag{7.1}$$

Note that:
1) Any pair of states has an individual in common
2) Each individual is excluded from four out of the seven states
3) Two given individuals are excluded from two out of the seven states
4) Three given individuals are never excluded

The entanglement may be represented by a heptagon with vertices A,B,C,D,E,F,G and seven triangles ABD, BCE, CDF, DEG, EFA, FGB, and GAC. See Figure 1. Alternatively, we can use the Fano plane. See Figure 2. The Fano plane corresponds to the multiplication table of the split octonions as may be seen from the description of the state $|\Psi\rangle$ given in Table 1.

Each of the seven states transforms as a $(2,2,2)$ under three of the $SL(2)$'s and are singlets under the remaining four. Note that from (6.2) we see that the A-B-C triality of section 3 is linked with the $8_v - 8_s - 8_c$ triality of the $SO(4,4)$. For example, interchanging A and B leaves $|\Psi\rangle$ invariant provided we also interchange C and F. Individually, therefore, the tripartite entanglement of each of the seven states is given by Cayley's hyperdeterminant. Taken together however, we see from (6.5) that they transform as a complex 56 of $E_7(C)$. Their tripartite entanglement must be is given by an expression that is quartic in the coefficients a, b, c, d, e, f, g and invariant under $E_7(C)$. The unique possibility is the Cartan invariant J_4, and so the 3-tangle is given by

$$\tau_3(ABCDEFG) = 4|J_4| \tag{7.2}$$

If the wave-function (7.1) is normalized, then $0 \leq \tau_3(ABCDEFG) \leq 1$.

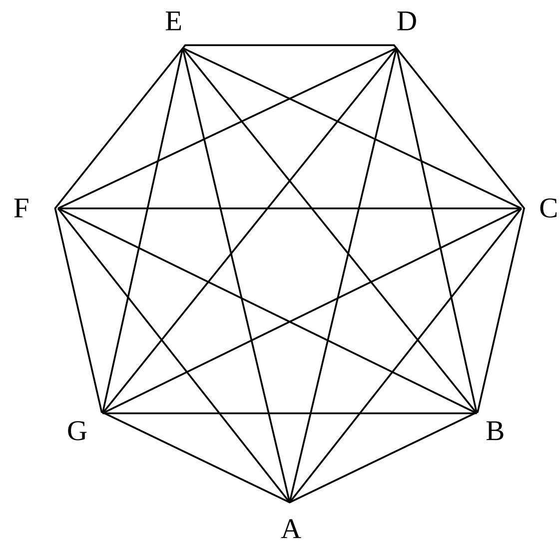

Figure 1: The E_7 entanglement diagram. Each of the seven vertices A,B,C,D,E,F,G represents a qubit and each of the seven triangles ABD, BCE, CDF, DEG, EFA, FGB, GAC describes a tripartite entanglement.

8 Cartan's $E_{7(7)}$ invariant

The Cremmer-Julia [17] form of the Cartan $E_{7(7)}$ invariant may be written as

$$J_4 = \mathrm{Tr}(Z\bar{Z})^2 - \tfrac{1}{4}(\mathrm{Tr}\, Z\bar{Z})^2 + 4(\mathrm{Pf}\ Z + \mathrm{Pf}\ \bar{Z}\,)\ , \tag{8.1}$$

and the Cartan form [16] may be written as

$$J_4 = -\mathrm{Tr}(\,x\,y)^2 + \tfrac{1}{4}(\mathrm{Tr}\,x\,y)^2 - 4(\mathrm{Pf}\ x + \mathrm{Pf}\ y\,)\ . \tag{8.2}$$

Here

$$Z_{AB} = -\frac{1}{4\sqrt{2}}(x^{ab} + iy_{ab})(\Gamma^{ab})_{AB} \tag{8.3}$$

and

$$x^{ab} + iy_{ab} = -\frac{\sqrt{2}}{4} Z_{AB}(\Gamma^{AB})_{ab} \tag{8.4}$$

The matrices of the $SO(8)$ algebra are $(\Gamma^{ab})_{AB}$ where $(a\,b)$ are the 8 vector indices and (A, B) are the 8 spinor indices. The $(\Gamma^{ab})_{AB}$ matrices can be considered also as $(\Gamma^{AB})_{ab}$ matrices due to equivalence of the vector and spinor representations of the $SO(8)$ Lie algebra. The exact relation between the Cartan invariant in (8.2) and Cremmer-Julia invariant [17] in (8.1) was established in [20, 21]. The quartic invariant J_4 of $E_{7(7)}$ is also related to the octonionic Jordan algebra J_3^O [19].

In the stringy black hole context, Z_{AB} is the central charge matrix and (x, y) are the quantized charges of the black hole (28 electric and 28 magnetic). The relation between the entropy of stringy black holes and the Cartan-Cremmer-Julia $E_{7(7)}$ invariant was established

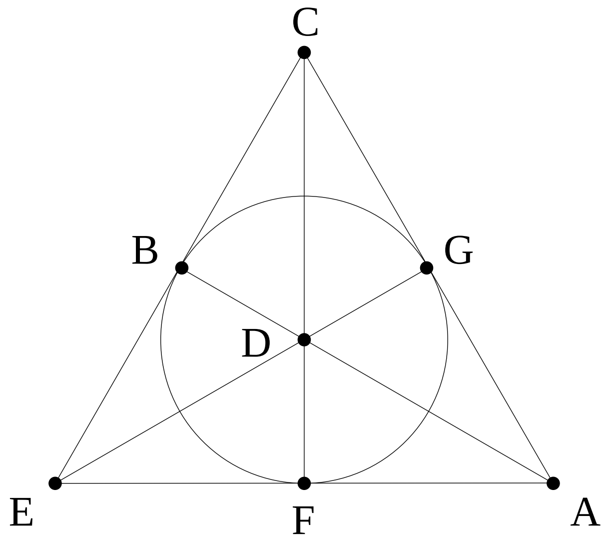

Figure 2: The Fano plane has seven points, representing the seven qubits, and seven lines (the circle counts as a line) with three points on every line, representing the tripartite entanglement, and three lines through every point.

in [18]. The central charge matrix Z_{AB} can be brought to the canonical basis for the skew-symmetric matrix using an $SU(8)$ transformation:

$$Z_{ab} = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8.5}$$

where $z_i = \rho_i e^{i\varphi_i}$ are complex. In this way the number of entries is reduced from 56 to 8. In a systematic treatment in [22], the meaning of these parameters was clarified. From 4 complex values of $z_i = \rho_i e^{i\varphi_i}$ one can remove 3 phases by an $SU(8)$ rotation, but the overall phase cannot be removed; it is related to an extra parameter in the class of black hole solutions [23, 24]. In this basis, the quartic invariant takes the form [18]

$$\begin{aligned} J_4 &= \sum_i |z_i|^4 - 2\sum_{i<j} |z_i|^2|z_j|^2 + 4(z_1 z_2 z_3 z_4 + \bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{z}_4) \\ &= (\rho_1 + \rho_2 + \rho_3 + \rho_4)(\rho_1 + \rho_2 - \rho_3 - \rho_4)(\rho_1 - \rho_2 + \rho_3 - \rho_4)(\rho_1 - \rho_2 - \rho_3 + \rho_4) \\ &\quad + 8\rho_1\rho_2\rho_3\rho_4(\cos\varphi - 1) \end{aligned} \tag{8.6}$$

Therefore a 5-parameter solution is called a generating solution for other black holes in N=8 supergravity/M-theory. The expression for their entropy is always given by

$$S = \pi\sqrt{|J_4|} \tag{8.7}$$

for some subset of 5 of the 8 parameters mentioned above. Recently a new class of solutions was discovered, describing black rings. The maximal number of parameters for the known

	A	B	C	D	E	F	G
A		D	G	-B	F	-E	-C
B	-D		E	A	-C	G	-F
C	-G	-E		F	B	-D	A
D	B	-A	-F		G	C	-E
E	-F	C	-B	-G		A	D
F	E	-G	D	-C	-A		B
G	C	F	-A	E	-D	-B	

Table 1: The entanglement of the state $|\Psi\rangle$ coincides with the multiplication table of the split octonions.

solutions is 7. The entropy of black ring solutions found so far was identified in [25, 26] with the expression (8.7) for a subset of 7 out of 8 parameters mentioned above.

Kallosh and Linde have shown that J_4 depending on 4 complex eigenvalues can be represented as Cayley's hyperdeterminant of a hypermatrix a_{ABC}. To see this, we that in x, y basis only the $SO(8)$ symmetry is manifest, which means that every term in (8.2) is invariant only under $SO(8)$ symmetry. However, it was proved in [16] and [17] that the sum of all terms in (8.2) is invariant under the full $SU(8)$ symmetry, which acts as follows

$$\delta(x^{ab} \pm i y_{ab}) = (2\Lambda^{[a}{}_{[c}\delta^{b]}{}_{d]} \pm i\Sigma_{abcd})(x^{cd} \mp i y_{cd}) \ . \tag{8.8}$$

The total number of parameters is 63, where 28 are from the manifest $SO(8)$ and 35 from the antisymmetric self-dual $\Sigma_{abcd} = {}^*\Sigma^{abcd}$. Thus one can use the $SU(8)$ transformation of the complex matrix $x^{ab} + iy_{ab}$ and bring it to the canonical form with some complex eigenvalues $\lambda_I, I = 1, 2, 3, 4$. The value of the quartic invariant (8.2) will not change.

$$(x^{ab} + iy_{ab})_{\rm can} = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_4 & 0 \end{pmatrix} \tag{8.9}$$

The relation between the complex coefficients λ_I, the parameters x_{ij} and y^{kl}, the matrix a_{ABC} and the black hole charges p^i and q_k is given by the following dictionary:

$$\lambda_1 \ = \ x_{12} + iy^{12} \ = \ a_{111} + ia_{000} \ = \ q_0 + ip^0$$

$$\begin{array}{lllllll}\lambda_2 &=& x_{34}+iy^{34} &=& a_{001}+ia_{110} &=-p^1+iq_1 \\ \lambda_3 &=& x_{56}+iy^{56} &=& a_{010}+ia_{101} &=-p^2+iq_2 \\ \lambda_4 &=& x_{78}+iy^{78} &=& a_{100}+ia_{011} &=-p^3+iq_3\end{array} \tag{8.10}$$

If we now write the quartic $E_{7(7)}$ Cartan invariant in the canonical basis (x_{ij}, y^{ij}), $i,j = 1,...,8$:

$$\begin{aligned} J_4 \quad &= -(x_{12}y^{12}+x_{34}y^{34}+x_{56}y^{56}+x_{78}y^{78})^2 - 4(x_{12}x_{34}x_{56}x_{78}+y^{12}y^{34}y^{56}y^{78}) \\ &+4(x_{12}x_{34}y^{12}y^{34}+x_{12}x_{56}y^{12}y^{56}+x_{34}x_{56}y^{34}y^{56}+x_{12}x_{78}y^{12}y^{78}+x_{34}x_{78}y^{34}y^{78} \\ &+x_{56}x_{78}y^{56}y^{78})\ . \end{aligned} \tag{8.11}$$

then it may now be compared to Cayley's hyperdeterminant (3.2). We find

$$J_4 = -\mathrm{Det}\ a \tag{8.12}$$

The above discussion of $E_{7(7)}$ also applies, mutatis mutandis, to $E_7(C)$.

To understand better the entanglement we note that, as a result of (6.5), Cartan's invariant contains not one Cayley hyperdeterminant but seven! It may be written as the sum of seven terms each of which is invariant under $[SL(2)]^3$ plus cross terms. To see this, denote a 2 in one of the seven entries in (6.5) by A, B, C, D, E, F, G. So we may rewrite (6.5) as

$$56 = (ABD)+(BCE)+(CDF)+(DEG)+(EFA)+(FGB)+(GAC) \tag{8.13}$$

or symbolically

$$56 = a+b+c+d+e+f+g \tag{8.14}$$

Then J_4 is the singlet in $56\times56\times56\times56$:

$$\begin{aligned} J_4 \sim\ & a^4+b^4+c^4+d^4+e^4+f^4+g^4+ \\ & 2[a^2b^2+b^2c^2+c^2d^2+d^2e^2+e^2f^2+f^2g^2+g^2a^2+ \\ & a^2c^2+b^2d^2+c^2e^2+d^2f^2+e^2g^2+f^2a^2+g^2b^2+ \\ & a^2d^2+b^2e^2+c^2f^2+d^2g^2+e^2a^2+f^2b^2+g^2c^2] \\ & +8[bcdf+cdeg+defa+efgb+fgac+gabd+abce] \end{aligned} \tag{8.15}$$

where products like

$$\begin{aligned} a^4 &= (ABD)(ABD)(ABD)(ABD) \\ &= \epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{D_2D_3}a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4} \end{aligned} \tag{8.16}$$

exclude four individuals (here Charlie, Emma, Fred and George), products like

$$\begin{aligned} a^2b^2 &= (ABD)(ABD)(FGB)(FGB) \\ &= \epsilon^{A_1A_2}\epsilon^{B_1B_3}\epsilon^{D_1D_2}\epsilon^{F_3F_4}\epsilon^{G_3G_4}\epsilon^{B_2B_4}a_{A_1B_1D_1}a_{A_2B_2D_2}b_{F_3G_3B_3}b_{F_4G_4B_4} \end{aligned} \tag{8.17}$$

exclude two individuals (here Charlie and Emma), and products like

$$\begin{aligned} abce &= (ABD)(BCE)(CDF)(EFA) \\ &= \epsilon^{A_1A_4}\epsilon^{B_1B_2}\epsilon^{D_1D_3}\epsilon^{C_3C_4}\epsilon^{G_2G_4}\epsilon^{F_2F_3}a_{A_1B_1D_1}b_{B_2C_2E_2}c_{C_3D_3F_3}e_{E_4F_4A_4} \end{aligned} \tag{8.18}$$

exclude one individual (here George).

9 The black hole analogy

In the STU stringy black hole context [1, 9, 11, 2] the a_{ABC} are integers (corresponding to quantized charges) and hence the symmetry group is $[SL(2,Z)]^3$ rather than $[SL(2,C)]^3$. However, as discussed by Levay [3], there is a branch of quantum information theory which concerns itself with real qubits, called *rebits*, for which the a_{ABC} are real. (One difference remains, however: one may normalize the wave function, whereas for black holes there is no such restriction on the charges a_{ABC}.) It turns out that there are three reality classes which can be characterized by the hyperdeterminant

$$\begin{aligned} &1)\ \mathrm{Det}\ a < 0 \\ &2)\ \mathrm{Det}\ a = 0 \\ &3)\ \mathrm{Det}\ a > 0 \end{aligned} \tag{9.1}$$

Case (1) corresponds to the non-separable or GHZ class [28], for example,

$$|\Psi\rangle = \frac{1}{2}(-|000\rangle + |011\rangle + |101\rangle + |110\rangle) \tag{9.2}$$

Case (2) corresponds to the separable (A-B-C, A-BC, B-CA, C-AB) and Werner classes, for example

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \tag{9.3}$$

In the string/supergravity interpretation [1], cases (1) and (2) were shown to correspond to BPS black holes, for which half of the supersymmetry is preserved. Case (1) has non-zero horizon area and entropy ("large" black holes), and case (2) to vanishing horizon area and entropy ("small" black holes), at least at the semi-classical level. However, small black holes may acquire a non-zero entropy through higher order quantum effects. This entropy also has a quantum information interpretation involving bipartite entanglement of the three qubits [2].

Case (3) is also GHZ, for example the above GHZ state (9.2) with a sign flip

$$|\Psi\rangle = \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) \tag{9.4}$$

In the string/supergravity interpretation, case (3) corresponds to non-BPS black holes [2]. With four non-zero charges (q_0, p^1, p^2, p^3) in (8.10), for example, an extreme but non-BPS black hole [29] may be obtained by flipping the sign [30] of one of the charges. The canonical GHZ state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|111\rangle + \frac{1}{\sqrt{2}}|000\rangle \tag{9.5}$$

also belongs to case (3).

In the $N = 8$ theory, "large" and "small" black holes are classified by the sign of J_4:

$$\begin{aligned} &1)\ J_4 > 0 \\ &2)\ J_4 = 0 \end{aligned}$$

$$3)\ J_4 < 0 \tag{9.6}$$

Once again, non-zero J_4 corresponds to large black holes, which are BPS for $J_4 > 0$ and non-BPS for $J_4 < 0$, and vanishing J_4 to small black holes. However, in contrast to $N = 2$, case (1) requires that only 1/8 of the supersymmetry is preserved, while we may have 1/8, 1/4 or 1/2 for case (2).

It is worth noting that the charge orbits corresponding to non-zero J_4 are associated with the following cosets:

$$\frac{E_{7(7)}}{E_{6(2)}} \tag{9.7}$$

and

$$\frac{E_{7(7)}}{E_{6(6)}} \tag{9.8}$$

The large black hole solutions can be found [31] by solving the $N = 8$ classical attractor equations [27] when at the attractor value the Z_{AB} matrix, in normal form, becomes

$$Z_{AB} = \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9.9}$$

for positive J_4 and

$$Z_{AB} = e^{i\pi/4}|Z| \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \tag{9.10}$$

for negative J_4. These values exhibit the maximal compact symmetries $SU(6) \times SU(2)$ and $USp(8)$ for the positive and negative J_4, respectively.

If the phase in (8.6) vanishes (which is the case if the configuration preserves at least 1/4 supersymmetry [22]), J_4 becomes

$$J_4 = \lambda_1\lambda_2\lambda_3\lambda_4\ , \tag{9.11}$$

where we have defined λ_i by

$$\begin{aligned} \lambda_1 &= \rho_1 + \rho_2 + \rho_3 + \rho_4 \\ \lambda_2 &= \rho_1 + \rho_2 - \rho_3 - \rho_4 \\ \lambda_3 &= \rho_1 - \rho_2 + \rho_3 - \rho_4 \\ \lambda_4 &= \rho_1 - \rho_2 - \rho_3 + \rho_4 \end{aligned} \tag{9.12}$$

and we order the λ_i so that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq |\lambda_4|$. The charge orbits for the small black holes depend on the number of unbroken supersymmetries or the number of vanishing eigenvalues. The orbit is [19, 22, 32]

$$\frac{E_{7(7)}}{H_{1,2,3}} \tag{9.13}$$

where

$$H_1 = F_{4(4)} \ltimes T_{26} \qquad \lambda_1, \lambda_2, \lambda_3 \neq 0, \quad \lambda_4 = 0 \quad (1/8\ BPS)$$

$$H_2 = SO(5,6) \ltimes (T_{32} \times T_1) \qquad \lambda_1, \lambda_2 \neq 0, \quad \lambda_3, \lambda_4 = 0 \quad (1/4\ BPS)$$
$$H_3 = E_{6(6)} \ltimes T_{27} \qquad \lambda_1 \neq 0, \quad \lambda_2, \lambda_3, \lambda_4 = 0 \quad (1/2\ BPS) \tag{9.14}$$

For $N = 8$, as for $N = 2$, the large black holes correspond to the two classes of GHZ-type (entangled) states and small black holes to the separable or Werner class.

10 Subsectors

Having understood the analogy between $N = 8$ black holes and the tripartite entanglement of 7 qubits using $E_{7(7)}$, we may now find the analogy in the $N = 4$ case using $SL(2) \times SO(6,6)$ and the $N = 2$ case using $SL(2) \times SO(2,2)$.

For $N = 4$, as may be seen from (6.2), we still have an $[SL(2)]^7$ subgroup but now there are only 24 states

$$|\Psi\rangle = a_{ABD}|ABD\rangle + e_{EFA}|EFA\rangle + g_{GAC}|GAC\rangle \tag{10.1}$$

So only Alice talks to all the others. This is described by just those three lines passing through A in the Fano plane. Then the equations analagous to (8.13) and (8.14) are

$$(2,12) = (ABD) + (EFA) + (GAC) = a + e + g \tag{10.2}$$

and the corresponding quartic invariant, J_4, reduces to the singlet in $(2,12) \times (2,12) \times (2,12) \times (2,12)$.

$$J_4 \sim a^4 + e^4 + g^4 + 2[e^2g^2 + g^2a^2 + a^2e^2] \tag{10.3}$$

If we identify the 24 numbers $(a_{ABD}, e_{EFA}, g_{GAC})$ with (P^μ, Q_ν) with $\mu, \nu = 1, \ldots 12$ in a way analogous to (3.10), this becomes [9, 23, 24]

$$J_4 = P^2Q^2 - (P.Q)^2 \tag{10.4}$$

For $N = 2$, as may be seen from (6.2), we only an $[SL(2)]^3$ subgroup and there are only 8 states

$$|\Psi\rangle = a_{ABD}|ABD\rangle \tag{10.5}$$

This is described by just the ABD line in the Fano plane. This is simply the usual tripartite entanglement, for which

$$(2,2,2) = (ABD) = a \tag{10.6}$$

and the corresponding quartic invariant

$$J_4 \sim a^4 \tag{10.7}$$

is just Cayley's hyperdeterminant

$$J_4 = -\mathrm{Det}a \tag{10.8}$$

11 Conclusions

The Fano plane also finds application in switching networks that can connect any phone to any other phone. It is the 3-switching network for 7 numbers. However there also exists a 4-switching network for 13 numbers, a 5-switching network for 21 numbers, and generally an $(n+1)$-switching network for (n^2+n+1) numbers corresponding to the projective planes of order n [33]. It would be interesting to explore the corresponding quantum bit entanglements.

Exceptional groups, such as $E_{7(7)}$, have featured in supergravity, string theory, M-theory and other speculative attempts at unification of the fundamental forces. However, as far as we are aware, this is the first time that an exceptional group has appeared in physics as well-established as the Copenhagen interpretation of quantum mechanics. It would be interesting to see whether it can be subject to experimental test.

We believe that that this is just the beginning and that a very fruitful exchange of ideas between the two disciplines lies ahead. For example, one area deserving of further study is the relation between the attractor mechanism for black holes [27] and certain distillation protocols in quantum information theory [3].

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