Analysis of Dual Consistency for Discontinuous Galerkin Discretizations of Source Terms *

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Abstract

The effects of dual consistency on discontinuous Galerkin (DG) discretizations of solution and solution gradient dependent source terms are examined. Two common discretizations are analyzed: the standard weighting technique for source terms and the mixed formulation. It is shown that if the source term depends on the first derivative of the solution, the standard weighting technique leads to a dual inconsistent scheme. A straightforward procedure for correcting this dual inconsistency and arriving at a dual consistent discretization is demonstrated. The mixed formulation, where the solution gradient in the source term is replaced by an additional variable that is solved for simultaneously with the state, leads to an asymptotically dual consistent discretization. *A priori* error estimates are derived to reveal the effect of dual inconsistent discretization on computed functional outputs. Combined with bounds on the dual consistency error, these estimates show that for a dual consistent discretization or the asymptotically dual consistent discretization resulting from the mixed formulation, $O(h^{2p})$ convergence can be shown for linear problems and linear outputs. For similar but dual inconsistent schemes, only $O(h^p)$ can be shown. Numerical results for a one-dimensional test problem confirm that the dual consistent and asymptotically dual consistent schemes achieve higher asymptotic convergence rates with grid refinement than a similar dual inconsistent scheme for both the primal and adjoint solutions as well as a simple functional output.

1 Introduction

In recent years, the discontinuous Galerkin (DG) finite element method has become a popular tool in the numerical simulation of many complex physical phenomena. In particular, many researchers

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have investigated high-order accurate DG discretizations of the Euler and Navier-Stokes equations for use in computational fluid dynamics [22, 12, 6, 8, 7, 13, 5]. In this context, DG is attractive because it allows the development of high-order accurate discretizations with element-wise compact stencils. These compact stencils simplify the task of achieving high-order accuracy for problems involving complex geometries, where unstructured meshes are often employed, and allow the development of efficient solution methods.

In this paper, high-order accurate DG discretizations of source terms depending on the state and its gradient are examined. Interest in such terms stems from the Reynolds-averaged Navier-Stokes (RANS) equations and, specifically, from the turbulence models used to close the RANS equations. For example, the Spalart-Allmaras turbulence model [25] incorporates state and state derivative dependent source terms to model the production, destruction, and diffusion of turbulent eddy viscosity, and state derivative dependent source terms appear in both the turbulent kinetic energy and dissipation rate equations of the the $k - \epsilon$ turbulence model [28].

The impact of dual consistency on source term discretizations is considered. Dual consistency provides a connection between the continuous and discrete dual problems. In particular, if a discretization is dual consistent, then the exact solution of the strong form of the dual problem satisfies the discrete dual problem taken about the exact solution of the strong form of the primal problem. A more precise definition of dual consistency is given in Section 2.

For many types of discretization, algorithms involving the dual problem have become popular for design optimization, error estimation, and grid adaptation [21, 2, 15, 17, 9, 10, 18, 27]. It is well known that dual consistency can significantly impact the performance of these algorithms. For example, Collis and Heinkenschloss [14] showed that when applying a dual inconsistent streamline upwind/Petrov Galerkin (SUPG) method for linear advection-diffusion to an optimal control problem, superior results are obtained using a direct discretization of the continuous dual problem as opposed to the discrete dual problem derived from the primal discretization. Specifically, both the control function and the adjoint solution converge at a higher rate when the continuous dual problem is discretized directly.

For DG discretizations, Harriman *et al.* [20, 19] examined symmetric and non-symmetric interior penalty (SIPG and NIPG, respectively) DG methods for the solution of Poisson's equation. They showed that to achieve optimal convergence rates for a linear functional output, the dual consistent method (i.e. SIPG) must be used. Lu [23] considered the impact of dual consistency on the accuracy of functional outputs computed using DG discretizations of the Euler and Navier-Stokes equations. He demonstrated the importance of implementing the boundary conditions on the primal problem in a dual consistent manner. In particular, when using dual consistent boundary conditions, superconvergent functional output results were obtained, while, when using a dual inconsistent boundary condition treatment, significant degradation of the output convergence rates was observed.

Furthermore, it is well known that dual consistency can impact the convergence of the error in the primal solution. For example, for many DG discretizations of Poisson's equation, standard proofs of order of accuracy of the solution error in the L^2 norm exist. Typically these proofs rely on the Aubin-Nitsche "duality trick" [26, 24] to obtain an optimal estimate in the L^2 norm given an optimal estimate in the energy norm [11, 4]. The use of this duality argument requires that the scheme be dual consistent. Thus, some dual inconsistent methods—e.g. NIPG and the Baumann-Oden method—do not achieve optimal accuracy in the L^2 norm, and dual inconsistent methods that do achieve optimal accuracy in the L^2 norm are typically super-penalized [4].

The paper begins with a brief review of the definition of dual consistency in Section 2. To demonstrate the importance of dual consistency, *a priori* error estimates for functional outputs for general DG schemes are derived in Section 3. As motivation for the technique used for discretizing source terms, Section 4 considers the implications of dual consistency for DG discretizations of Poisson's equation. In Section 5, DG discretizations of source terms are considered. It is shown that, while a standard weighting treatment of solution derivative dependent source terms leads to a dual inconsistent DG discretization, dual consistency can be achieved by adding terms proportional to the jumps in the solution between elements to the discretization. A mixed formulation for the source term is analyzed in Section 6. The resulting discretization is shown to be asymptotically dual consistent. In Section 7, bounds on the dual consistency error resulting from the standard weighting and mixed formulation discretizations are derived. Finally, numerical results for a simple test problem are shown in Section 8.

2 Dual Consistency Definition and Preliminaries

Let $u \in \mathcal{V}$, where $\mathcal{V}$ is some appropriate function space, be a weak solution of a general PDE. Furthermore, let $\mathcal{V}_h^p$ be a finite dimensional vector space of piecewise polynomial functions of degree at most p on a triangulation, T_h, of the domain of interest, $\Omega \subset \mathbb{R}^n$, into elements, κ, such that $\bar{\Omega} = \cup_{\kappa \in T_h} \bar{\kappa}$. In particular,

$$\mathcal{V}_h^p \equiv \{v \in L^2(\Omega) \,|\, v|_\kappa \in P^p, \,\forall \kappa \in T_h\},$$

where P^p denotes the space of polynomial functions of degree at most p.

Let $\mathcal{W}_h^p \equiv \mathcal{V}_h^p + \mathcal{V}$, where

$$\mathcal{V}_h^p + \mathcal{V} \equiv \{h = f + g \mid f \in \mathcal{V}_h^p, \, g \in \mathcal{V}\}.$$

Then, consider a general DG discretization of the underlying PDE: find $u_h \in \mathcal{V}_h^p$ such that

$$R_h(u_h, v_h) = 0, \quad \forall v_h \in \mathcal{V}_h^p,$$

where $R_h : \mathcal{W}_h^p \times \mathcal{W}_h^p \to \mathbb{R}$ is a semi-linear form—linear in the second argument—derived from the weak form of the underlying governing PDE. Furthermore, let $\mathcal{J}_h : \mathcal{W}_h^p \to \mathbb{R}$ be some discrete functional of interest. Then, the discrete dual problem is given by the following statement: find $\psi_h \in \mathcal{V}_h^p$ such that

$$R_h'[u_h](v_h, \psi_h) = \mathcal{J}_h'[u_h](v_h), \quad \forall v_h \in \mathcal{V}_h^p$$

where $R_h'[u_h](\cdot, \psi_h) : \mathcal{W}_h^p \to \mathbb{R}$ is the linear functional given by evaluating the Frechét derivative of the function $N_{\psi_h} : \mathcal{W}_h^p \to \mathbb{R}$ at u_h, where, for fixed $\psi_h \in \mathcal{W}_h^p$,

$$N_{\psi_h}(w_h) = R_h(w_h, \psi_h), \quad \forall w_h \in \mathcal{W}_h^p.$$

Similarly, $\mathcal{J}_h'[u_h] : \mathcal{W}_h^p \to \mathbb{R}$ is the linear functional given by evaluating the Frechét derivative of $\mathcal{J}_h$ at u_h.

Definition 1. *The discretization defined by the semi-linear form, R_h, and discrete functional, $\mathcal{J}_h$, is said to be* dual consistent *if, given exact solutions $u \in \mathcal{V}$ and $\psi \in \mathcal{V}$ of the continuous primal and dual problems, respectively,*

$$R_h'[u](v, \psi) = \mathcal{J}_h'[u](v), \quad \forall v \in \mathcal{W}_h^p.$$

In addition to dual consistency as defined in Definition 1, a weaker form of dual consistency is also useful.

Definition 2. *The discretization defined by the semi-linear form, R_h, and discrete functional, $\mathcal{J}_h$, is said to be* asymptotically dual consistent *if, given exact solutions $u \in \mathcal{V}$ and $\psi \in \mathcal{V}$ of the continuous primal and dual problems, respectively,*

$$\lim_{h \to 0} \left(\sup_{v \in \mathcal{W}_h^p, \|v\|_{\mathcal{W}_h^p} = 1} |R_h'[u](v, \psi) - \mathcal{J}_h'[u](v)| \right) = 0.$$

Note that all dual consistent discretizations are automatically asymptotically dual consistent. In this work, a discretization will be referred to as asymptotically dual consistent only if it is not also dual consistent.

For a more in depth discussion of the definition of dual consistency as well as the connection between the discrete and continuous dual problems, see [23].

Two addition useful concepts—consistency and boundedness—are defined as follows.

Definition 3. *A semi-linear form, R_h, is* consistent *if, given an exact primal solution, u, and a discrete primal solution, u_h,*

$$R_h(u, v_h) = R_h(u_h, v_h), \quad \forall v_h \in \mathcal{V}_h^p.$$

Definition 4. *A semi-linear form, R_h, is* bounded *if there exists a constant, c_b, such that*

$$R_h(w, v) \leq c_b \, |||w||| \, |||v|||, \quad \forall w, v \in \mathcal{W}_h^p,$$

where $|||\cdot|||$ is an appropriate energy norm.

3 *A Priori* Error Estimation

In this section, *a priori* error estimates for functional outputs are derived. In particular, a bilinear discretization and linear functional output are considered. The analysis and results of this simple case are similar to that in [20], though a more general problem and discretization are considered. Extensions to nonlinear problems and more complicated outputs are left for future work.

As a model problem, let $u \in \mathcal{V}$ be a solution of the following weak form of a general linear PDE:

$$B(u, v) = \ell(v), \quad \forall v \in \mathcal{V},$$

where B is a bilinear form and $\ell : \mathcal{V} \to \mathbb{R}$ is a linear functional. Then, for some linear functional of interest, $\mathcal{J}$, $\psi \in \mathcal{V}$ is a solution of the dual problem:

$$B^*(\psi, v) = B(v, \psi) = \mathcal{J}(v), \quad \forall v \in \mathcal{V}.$$

Proposition 1. *Consider the following DG discretization: find $u_h \in \mathcal{V}_h^p$ such that,*

$$B_h(u_h, v_h) = \ell(v_h), \quad \forall v_h \in \mathcal{V}_h^p,$$

where B_h is a consistent and bounded bilinear form that satisfies

$$B_h(w, v) = B(w, v), \quad \forall w, v \in \mathcal{V}.$$

Furthermore, let $\mathcal{J}_h$ be a discrete functional satisfying $\mathcal{J}_h(w) = \mathcal{J}(w)$ for all $w \in \mathcal{W}_h^p$.

Then, if B_h and $\mathcal{J}_h$ form a dual consistent pair, there exists a constant, c_b, such that

$$|\mathcal{J}(u) - \mathcal{J}_h(u_h)| \leq c_b \, |||u - u_h||| \, |||\psi - \psi_h|||.$$

However, if B_h together with $\mathcal{J}_h$ is not dual consistent, then this inequality may not hold. For the general dual inconsistent case, there exists a constant, c_b, such that

$$|\mathcal{J}(u) - \mathcal{J}_h(u_h)| \leq c_b \, |||u - u_h||| \, |||\psi - \psi_h||| + c_b \, |||u_h||| \, |||\psi - \psi_h|||.$$

Proof. Since $\mathcal{J}_h(w) = \mathcal{J}(w)$, the discrete dual problem is given by the following statement: find $\psi_h \in \mathcal{V}_h^p$ such that,

$$B_h(v_h, \psi_h) = \mathcal{J}(v_h), \quad \forall v_h \in \mathcal{V}_h^p.$$

By the definitions of the dual problems, the error in the functional output of interest is given by

$$\mathcal{J}(u) - \mathcal{J}(u_h) = B(u, \psi) - B_h(u_h, \psi_h). \tag{1}$$

Thus, using the bilinearity of B_h,

$$\begin{aligned} \mathcal{J}(u) - \mathcal{J}(u_h) &= B_h(u, \psi) - B_h(u_h, \psi_h) \\ &= B_h(u - u_h, \psi) + B_h(u_h, \psi) - B_h(u_h, \psi_h) \\ &= B_h(u - u_h, \psi - \psi_h) + B_h(u - u_h, \psi_h) + B_h(u_h, \psi - \psi_h). \end{aligned}$$

Then, by consistency,

$$\mathcal{J}(u) - \mathcal{J}(u_h) = \underbrace{B_h(u - u_h, \psi - \psi_h)}_{\text{Standard error}} + \underbrace{B_h(u_h, \psi - \psi_h).}_{\text{Dual consistency error}}$$

For clarity, the error is split into two contributions: the standard error and the dual consistency error.

If B_h together with $\mathcal{J}_h$ forms a dual consistent discretization, then, by the definition of dual consistency,

$$B_h(u_h, \psi - \psi_h) = 0.$$

Thus, if the scheme is dual consistent, the error in the functional output can be written as a bilinear functional of the primal and adjoint solution errors. Furthermore, the boundedness of B_h implies that

$$|\mathcal{J}(u) - \mathcal{J}(u_h)| \le c_b |||u - u_h||| \, |||\psi - \psi_h|||.$$

However, if the scheme is dual inconsistent, $B_h(u_h, \psi - \psi_h) \neq 0$. Thus, by boundedness,

$$|\mathcal{J}(u) - \mathcal{J}_h(u_h)| \le c_b \, |||u - u_h||| \, |||\psi - \psi_h||| + c_b \, |||u_h||| \, |||\psi - \psi_h|||.$$

□

Remark 1. For a dual consistent scheme, the functional output error convergence rate is determined by the convergence of both the primal and dual solutions in the energy norm. If the scheme attains an order of accuracy of r for the primal solution and q for the dual solution—i.e. there exist constants c_u and c_ψ such that

$$|||u - u_h||| \le c_u h^r, \quad |||\psi - \psi_h||| \le c_\psi h^q,$$

where h is the grid spacing—then, for some constant c_o,

$$|\mathcal{J}(u) - \mathcal{J}_h(u_h)| \le c_o h^{r+q}.$$

Thus, for dual consistent schemes that are optimal for both the primal and dual solutions in the energy norm, i.e. $r = q = p$, the functional output converges at a rate of $2p$.

Remark 2. The estimate for the dual inconsistent scheme implies that the functional output converges at a rate equal to that of the dual solution in the energy norm, i.e.

$$|||\psi - \psi_h||| \le c_\psi h^q \Rightarrow |\mathcal{J}(u) - \mathcal{J}_h(u_h)| \le c_o h^q.$$

In practice, this estimate is often too pessimistic. To get a better estimate, it is necessary to consider the specific form of the dual consistency error. For example, for a dual inconsistent but otherwise optimal scheme for diffusion, a typical result is

$$B_h(u_h, \psi - \psi_h) \le c_{de} |||\psi||| \, |||u - u_h|||.$$

To prove this inequality, see the proof of boundedness given in [4]. Thus, in this case, the convergence of the functional is controlled by the convergence of the primal solution, which, for a dual inconsistent scheme, is expected to be better than the convergence of the dual solution (i.e. $r > q$ is expected). However, usually the convergence rate of the functional output computed using a dual consistent discretization will still be higher.

Error bounds on the dual consistency error for the discretizations to be shown in Sections 5 and 6 are derived in Section 7.

4 Dual Consistency for Linear Diffusion

This section examines the impact of dual consistency on DG discretizations for Poisson's equation. In particular, a family of dual consistent discretizations is derived starting from a family of dual inconsistent schemes. This dual consistent family contains many existing DG schemes for diffusion—e.g. local discontinuous Galerkin (LDG) [13] and the second method of Bassi and Rebay (BR2) [7]—and thus, the objective of this section is not to suggest a new DG discretization of this problem. Rather, it is to demonstrate the effect of requiring dual consistency on potential DG schemes for diffusion and, more importantly, to motivate the technique used in Section 5.

Let $\Omega \subset \mathbb{R}^n$ be the physical domain of interest, and let $u \in \mathcal{V} \equiv H^2(\Omega)$ be a solution of the following scalar problem:

$$
\begin{aligned}
-\nabla \cdot (\nu \nabla u) &= f \quad \text{for } x \in \Omega, \\
u &= 0 \quad \text{for } x \in \partial\Omega,
\end{aligned}
\tag{2}
$$

where $f \in L^2(\Omega)$ and $\nu \in H^1(\Omega)$ are independent of u. For clarity, homogeneous Dirichlet boundary conditions are used. A complete discussion of the impact of boundary conditions on dual consistency for discretizations of elliptic operators is given in [23].

Define the functional of interest as

$$
\mathcal{J}(u, \nabla u) = \int_\Omega J_I(u) + \int_{\partial\Omega} J_{B,1}(u) + J_{B,2}(\nu \nabla u \cdot \vec{n}),
\tag{3}
$$

where $\vec{n}$ is the outward pointing unit normal vector. Then, the adjoint solution $\psi \in H^2(\Omega)$ is specified by the following dual problem:

$$
-\nabla \cdot (\nu \nabla \psi) = J_I'[u], \quad \text{for } x \in \Omega,
\tag{4}
$$

subject to Dirichlet boundary conditions specified weakly by

$$
-\int_{\partial\Omega} \psi q = \int_{\partial\Omega} J_{B,2}'[\nu \nabla u \cdot \vec{n}](q), \quad \forall\, q \in H^{1/2}(\partial\Omega).
$$

To derive a baseline discretization, one might consider using the standard approach for hyperbolic conservation laws. In particular, the strong form of the problem is weighted by a test function and integrated over the triangulation of the domain. Then, integrating by parts gives

$$
\begin{aligned}
-\sum_{\kappa \in T_h} \int_\kappa v_h \nabla \cdot (\nu \nabla u_h) \;&=\; \sum_{\kappa \in T_h} \int_\kappa \nabla v_h \cdot (\nu \nabla u_h) \\
&\quad - \int_{\Gamma_i} (\llbracket v_h \rrbracket \cdot \{\nu \nabla u_h\} + \{v_h\} \llbracket \nu \nabla u_h \rrbracket) - \int_{\partial\Omega} v_h^+ (\nu \nabla u_h)^+ \cdot \vec{n}^+,
\end{aligned}
$$

where Γ_i denotes the union of the interior faces and $\vec{n}^+$ denotes the outward pointing unit normal. Furthermore, the average, $\{\cdot\}$, and jump, $\llbracket \cdot \rrbracket$, operators for interior faces are defined as follows: for scalar quantities,

$$
\{s\} = \frac{1}{2}(s^+ + s^-), \quad \llbracket s \rrbracket = s^+ \vec{n}^+ + s^- \vec{n}^-,
$$

and, for vector quantities,

$$
\{\vec{v}\} = \frac{1}{2}(\vec{v}^+ + \vec{v}^-), \quad \llbracket \vec{v} \rrbracket = \vec{v}^+ \cdot \vec{n}^+ + \vec{v}^- \cdot \vec{n}^-,
$$

where $(\cdot)^+$ and $(\cdot)^-$ represent trace values taken from opposite sides of an interior face, $\vec{n}^+$ points from the $(+)$ side of the face to the $(-)$ side, and $\vec{n}^- = -\vec{n}^+$.

Replacing the trace values of the flux with numerical flux functions, one arrives at the following discretization: find $u_h \in \mathcal{V}_h^p$ such that

$$
R_h(u_h, v_h) = \sum_{\kappa \in T_h} \int_\kappa v_h f, \quad \forall v_h \in \mathcal{V}_h^p,
$$

where

$$
\begin{aligned}
R_h(u_h, v_h) \equiv &\sum_{\kappa \in T_h} \int_\kappa \nabla v_h \cdot (\nu \nabla u_h) \\
&- \int_{\Gamma_i} (\llbracket v_h \rrbracket \cdot \{\vec{q}_i(u_h)\} + \{v_h\} \llbracket \vec{q}_i(u_h) \rrbracket) - \int_{\partial\Omega} v_h^+ \vec{q}_b(u_h) \cdot \vec{n}^+,
\end{aligned} \tag{5}
$$

and $\vec{q}_i$ and $\vec{q}_b$ are numerical flux functions intended to approximate $\nu \nabla u$ for interior and boundary faces, respectively. No assumptions about the form of $\vec{q}_i$ or $\vec{q}_b$ have been made in (5). In general, $\vec{q}_i$ and $\vec{q}_b$ could be functions of both the state, u_h, and the gradient of the state, ∇u_h. The notation $\vec{q}_i(u_h)$ is used only to show that $\vec{q}_i$ is evaluated using the state and its gradient—as opposed to the weight function, v_h, and its gradient—not that $\vec{q}_i$ is only a function of u_h.

For the remainder of the section, it is assumed that $\vec{q}_i$ and $\vec{q}_b$ are linear functions. This assumption simplifies the required notation—and makes sense given that the underlying PDE is linear—but is otherwise irrelevant because the conditions on $\vec{q}_i$ and $\vec{q}_b$ for dual consistency are independent of linearity.

For the discrete functional of interest, consider

$$\mathcal{J}_h(w_h, \nabla w_h) \equiv \mathcal{J}(w_h, \nabla w_h). \tag{6}$$

Proposition 2. *The discretization defined by the bilinear form R_h, defined in* (5)*, together with the discrete functional $\mathcal{J}_h$, defined in* (6)*, is dual inconsistent.*

Proof. To evaluate the dual consistency of the scheme, it is necessary to examine $R_h'[u](v_h, \psi) - \mathcal{J}_h'[u](v_h)$. Since R_h is bilinear, $R_h'[u](\cdot, v_h) = R_h(\cdot, v_h)$. Integrating by parts, one can show that

$$\begin{aligned} R_h(v_h, \psi) &= -\sum_{\kappa \in T_h} \int_\kappa v_h \nabla \cdot (\nu \nabla \psi) + \int_{\Gamma_i} \left([\![v_h]\!] \cdot \{\nu \nabla \psi\} + \{v_h\} [\![\nu \nabla \psi]\!] \right) \\ &\quad - \int_{\Gamma_i} \left([\![\psi]\!] \cdot \{\vec{q}_i(v_h)\} + \{\psi\} [\![\vec{q}_i(v_h)]\!] \right) + \int_{\partial\Omega} v_h^+ (\nu \nabla \psi)^+ \cdot \vec{n}^+ \\ &\quad - \int_{\partial\Omega} \psi^+ \vec{q}_b(v_h) \cdot \vec{n}^+. \end{aligned}$$

Since $\psi \in H^2(\Omega)$ and $\nu \in H^1(\Omega)$, $[\![\psi]\!] = 0$, $\{\psi\} = \psi$, $[\![\nu \nabla \psi]\!] = 0$, and $\{\nu \nabla \psi\} = \nu \nabla \psi$. Thus,

$$\begin{aligned} R_h(v_h, \psi) &= -\sum_{\kappa \in T_h} \int_\kappa v_h \nabla \cdot (\nu \nabla \psi) + \int_{\Gamma_i} \left([\![v_h]\!] \cdot (\nu \nabla \psi) - \psi [\![\vec{q}_i(v_h)]\!] \right) \\ &\quad + \int_{\partial\Omega} \left(v_h^+ (\nu \nabla \psi) \cdot \vec{n}^+ - \psi \vec{q}_b(v_h) \cdot \vec{n}^+ \right). \end{aligned}$$

Finally,

$$R_h'[u](v_h, \psi) - \mathcal{J}_h'[u](v_h) = (\mathcal{L}_I(u, \psi))(v_h) + (\mathcal{L}_B(u, \psi))(v_h), \tag{7}$$

where

$$(\mathcal{L}_I(u, \psi))(v_h) \equiv \int_{\Gamma_i} \left([\![v_h]\!] \cdot (\nu \nabla \psi) - \psi [\![\vec{q}_i(v_h)]\!] \right), \tag{8}$$

$$\begin{aligned} (\mathcal{L}_B(u, \psi))(v_h) &\equiv \int_{\partial\Omega} \left(v_h^+ (\nu \nabla \psi) \cdot \vec{n}^+ - \psi \vec{q}_b(v_h) \cdot \vec{n}^+ \right) \\ &\quad - \int_{\partial\Omega} \left(J_{B,1}'[u](v_h) + J_{B,2}'[\nu \nabla u \cdot \vec{n}](\nu \nabla v_h \cdot \vec{n}^+) \right). \end{aligned} \tag{9}$$

For general $\vec{q}_i$ and $\vec{q}_b$ there exists $v_h \in \mathcal{V}_h^p$ such that the right hand side of (7) does not vanish, which implies that the scheme is dual inconsistent. □

Remark 3. It is possible to construct a dual consistent discretization by subtracting terms from the bilinear form defined in (5) and modifying the discrete functional defined in (6). In particular, define a new bilinear form,

$$R_{h,DC}(w_h, v_h) \equiv R_h(w_h, v_h) - A_{h,I}(w_h, v_h) - A_{h,B}(w_h, v_h)$$

and a new discrete functional $J_{h,DC}$, where $A_{h,I}$ will serve to eliminate the interior face dual inconsistency term, $\mathcal{L}_I$, and $A_{h,B}$ combined with the modifications to the discrete functional will serve to eliminate the boundary face dual inconsistency, $\mathcal{L}_B$.

Begin by considering the dual inconsistency resulting from the interior faces. Two interesting features are apparent. First, note that the $\psi[\![\vec{q}_i]\!]$ term is zero when the numerical flux function is conservative. By definition, a conservative flux function, $\hat{f}$, has the property that $[\![\hat{f}]\!] = 0$. Thus, one contribution to the dual inconsistency may be eliminated by choosing a conservative flux function for $\vec{q}_i$. Second, the $[\![v_h]\!] \cdot (\nu\nabla\psi)$ term results from the absence of a "dual flux" term in the primal bilinear form. To eliminate this contribution to the dual inconsistency, one can subtract a term of the following form from R_h:

$$A_{h,I}(w_h, v_h) = \int_{\Gamma_i} [\![w_h]\!] \cdot \{\vec{q}_{i,dual}(v_h)\},$$

where $\vec{q}_{i,dual}$ is the dual flux function. Dual consistency requires that $\{\vec{q}_{i,dual}(\psi)\} = \nu\nabla\psi$, i.e. $\{\vec{q}_{i,dual}\}$ must be a consistent approximation of the flux for the dual problem.

Now, consider the boundary contribution. To achieve dual consistency, both the bilinear form and the discrete functional are modified. Redefine the discrete functional as

$$\mathcal{J}_{h,DC}(w_h, \nabla w_h) \equiv \int_\Omega J_I(w_h) + \int_{\partial\Omega} J_{B,1}(0) + J_{B,2}(\vec{q}_b(w_h) \cdot \vec{n}). \tag{10}$$

This change is intuitively satisfying given that the solution on the boundary is known exactly and that $\vec{q}_b$ is used to approximate the boundary flux in the bilinear form. Futhermore,

$$\mathcal{J}'_{h,DC}[u](w_h) = \int_\Omega J'_I[u](w_h) + \int_{\partial\Omega} J'_{B,2}[\vec{q}_b(u) \cdot \vec{n}](\vec{q}_b(w_h) \cdot \vec{n}). \tag{11}$$

Then, if $\vec{q}_b(u) = \nu\nabla u$ and assuming $\vec{q}_b(v_h) \cdot \vec{n} \in H^{1/2}(\Omega)$ for all $v_h \in \mathcal{W}_h^p$, the boundary conditions on the dual problem imply that

$$-\int_{\partial\Omega} \psi\vec{q}_b(v_h) \cdot \vec{n} - \int_{\partial\Omega} J'_{B,2}[\vec{q}_b(u) \cdot \vec{n}](\vec{q}_b(v_h) \cdot \vec{n}) = 0.$$

Thus, the boundary contribution to the dual inconsistency is given by

$$R'_h[u](v_h,\psi) - A'_{h,I}[u](v_h,\psi) - \mathcal{J}'_{h,DC}[u](v_h) = \int_{\partial\Omega} v_h^+ \nu\nabla\psi\cdot\vec{n}.$$

To eliminate this term, the following is subtracted from the form R_h:

$$A_{h,B}(w_h,v_h) = \int_{\partial\Omega} w_h^+ \vec{q}_{b,dual}(v_h)\cdot\vec{n},$$

where dual consistency requires that $\vec{q}_{b,dual}(\psi) = \nu\nabla\psi$.

Proposition 3. *Let $\vec{q}_i$ be a linear conservative flux function such that for an exact solution, $u\in H^2(\Omega)$, of (2), $\vec{q}_i(u) = \nu\nabla u$. Let $\vec{q}_b$ be a linear boundary flux function such that $\vec{q}_b(u) = \nu\nabla u$. Let $\vec{q}_{i,dual}$ be a linear dual flux function such that, for an exact solution, $\psi\in H^2(\Omega)$, of (4), $\{\vec{q}_{i,dual}(\psi)\} = \nu\nabla\psi$, and let $\vec{q}_{b,dual}$ be a linear boundary dual flux function such that $\vec{q}_{b,dual}(\psi) = \nu\nabla\psi$. Then, the discretization defined by the bilinear form*

$$\begin{aligned} R_{h,DC}(w_h,v_h) &= \sum_{\kappa\in T_h}\int_\kappa \nabla v_h\cdot(\nu\nabla w_h) - \int_{\Gamma_i}\left([\![v_h]\!]\cdot\vec{q}_i(w_h) + [\![w_h]\!]\cdot\{\vec{q}_{i,dual}(v_h)\}\right) \\ &\quad - \int_{\partial\Omega}\left(v_h^+\vec{q}_b(w_h)\cdot\vec{n}^+ + w_h^+(\vec{q}_{b,dual}(v_h)\cdot\vec{n}^+)\right), \end{aligned}$$

together with the functional $\mathcal{J}_{h,DC}$, defined in (10), is consistent and dual consistent.

Proof. The proof of consistency is trivial, and the proof of dual consistency is clear from Remark 3. ☐

Remark 4. The discretization described in Proposition 3 is consistent, dual consistent, and conservative. However, it may not be symmetric. Symmetry can be achieved by requiring $\vec{q}_{i,dual} = \vec{q}_i$ and $\vec{q}_{b,dual} = \vec{q}_b$. Thus, a consistent, dual consistent, conservative, and symmetric DG discretization of this problem can be written as follows:

$$\begin{aligned} R_{h,DC}(w_h,v_h) &= \sum_{\kappa\in T_h}\int_\kappa \nabla v_h\cdot(\nu\nabla w_h) \\ &\quad - \int_{\Gamma_i}\left([\![v_h]\!]\cdot\vec{q}_i(w_h) + [\![w_h]\!]\cdot\vec{q}_i(v_h)\right) \\ &\quad - \int_{\partial\Omega}\left(v_h^+\vec{q}_b(w_h)\cdot\vec{n}^+ + w_h^+\vec{q}_b(v_h)\cdot\vec{n}^+\right). \end{aligned} \tag{12}$$

For illustrative purposes, consider

$$\vec{q}_i(w_h) = \{\nu\nabla w_h\};\quad \vec{q}_b(w_h) = \nu\nabla w_h^+.$$

Then, (12) becomes

$$R_{h,DC}(w_h, v_h) = \sum_{\kappa \in T_h} \int_{\kappa} \nabla v_h \cdot (\nu \nabla w_h) - \int_{\Gamma_i} \left([\![v_h]\!] \cdot \{\nu \nabla w_h\} + [\![w_h]\!] \cdot \{\nu \nabla v_h\} \right) - \int_{\partial\Omega} \left(v_h^+ \nu \nabla w_h^+ \cdot \vec{n}^+ + w_h^+ \nu \nabla v_h^+ \cdot \vec{n}^+ \right) . \tag{13}$$

Note that (13) is similar to many dual consistent DG schemes, though lacking in stabilization terms [4].

5 Dual Consistent Discretization for Source Terms

This section considers DG discretizations of source terms depending on the state and first derivatives of the state. As will be shown, the simple approach of weighting by a test function and integrating leads to a dual inconsistent scheme for source terms that depend on derivatives of the state. However, a dual consistent discretization can be constructed by extending the ideas described in Section 4.

Let $u \in \mathcal{V} \equiv H^2(\Omega)$ be a solution of the following scalar problem:

$$\begin{aligned} -\nabla \cdot (\nu \nabla u) &= f(u, \nabla u) \quad \text{for } x \in \Omega \subset \mathbb{R}^n \\ u &= 0 \quad \text{for } x \in \partial\Omega, \end{aligned} \tag{14}$$

where $f \in \mathcal{C}^1(\mathbb{R}^{n+1})$ is the source term of interest.

Using the functional of interest defined in (3), the adjoint solution, $\psi \in H^2(\Omega)$, is given by the dual problem

$$-\nabla \cdot (\nu \nabla \psi) - D_1 f(u, \nabla u)\psi + \nabla \cdot (D_{\nabla u} f(u, \nabla u)\psi) = J_I'[u] \quad \text{for } x \in \Omega, \tag{15}$$

where $D_1 f(u, \nabla u)$ is the partial derivative of f with respect to u evaluated at $(u, \nabla u)$ and

$$D_{\nabla u} f(u, \nabla u) = [D_2 f(u, \nabla u), \ldots, D_{n+1} f(u, \nabla u)]^T$$

where $D_i f(u, \nabla u)$ is the partial derivative of f with respect to $\frac{\partial u}{\partial x_{i-1}}$ for $2 \leq i \leq n+1$, evaluated at $(u, \nabla u)$. The boundary conditions can be written in the following weak form:

$$-\int_{\partial\Omega} \psi q = \int_{\partial\Omega} J_{B,2}'[\nu \nabla u \cdot \vec{n}](q), \quad \forall q \in H^{1/2}(\partial\Omega).$$

Consider the following DG discretization: find $u_h \in \mathcal{V}_h^p$ such that

$$R_h(u_h, v_h) \equiv B_h(u_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f(u_h, \nabla u_h) = 0, \quad \forall v_h \in \mathcal{V}_h^p, \tag{16}$$

where B_h is a consistent and dual consistent bilinear form for the diffusion operator (e.g. $R_{h,DC}$ from Section 4, BR2, or LDG).

Proposition 4. *The discretization defined by the semi-linear form R_h, defined in* (16), *together with the discrete functional $\mathcal{J}_h$, defined in* (6), *is dual inconsistent.*

Proof. Linearizing R_h about the exact solution and integrating by parts gives

$$\begin{aligned} R'_h[u](w_h, v_h) \;\; &= \;\; B_h(w_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa w_h(D_1 f(u, \nabla u) v_h - \nabla \cdot (D_{\nabla u} f(u, \nabla u) v_h)) \\ &\quad - \int_{\Gamma_i} (\llbracket w_h \rrbracket \cdot \{D_{\nabla u} f(u, \nabla u) v_h\} + \{w_h\} \llbracket D_{\nabla u} f(u, \nabla u) v_h \rrbracket) \\ &\quad - \int_{\partial\Omega} w_h^+ (D_{\nabla u} f(u, \nabla u) v_h^+) \cdot \vec{n}^+. \end{aligned}$$

Since $\psi \in H^2(\Omega)$, $u \in H^2(\Omega)$, and $f \in \mathcal{C}^1(\mathbb{R}^{n+1})$, it is clear that $\{D_{\nabla u} f(u, \nabla u)\psi\} = D_{\nabla u} f(u, \nabla u)\psi$ and $\llbracket D_{\nabla u} f(u, \nabla u)\psi \rrbracket = 0$. Thus,

$$\begin{aligned} R'_h[u](v_h, \psi) \;\; &= \;\; B_h(v_h, \psi) - \sum_{\kappa \in T_h} \int_\kappa v_h(D_1 f(u, \nabla u)\psi - \nabla \cdot (D_{\nabla u} f(u, \nabla u)\psi)) \\ &\quad - \int_{\Gamma_i} \llbracket v_h \rrbracket \cdot (D_{\nabla u} f(u, \nabla u)\psi) - \int_{\partial\Omega} v_h^+ (D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{n}^+. \end{aligned}$$

Evaluating the dual consistency using the discrete functional as defined in (6) gives

$$R'_h[u](v_h, \psi) - \mathcal{J}'_h[u](v_h) = (\mathcal{L}_{h,I}(u, \psi))(v_h) + (\mathcal{L}_{h,B}(u, \psi))(v_h),$$

where

$$(\mathcal{L}_{h,I}(u, \psi))(v_h) \;\; \equiv \;\; - \int_{\Gamma_i} \llbracket v_h \rrbracket \cdot (D_{\nabla u} f(u, \nabla u)\psi), \tag{17}$$

$$\begin{aligned} (\mathcal{L}_{h,B}(u, \psi))(v_h) \;\; &\equiv \;\; - \int_{\partial\Omega} \psi \vec{q}_b(v_h) \cdot \vec{n}^+ - \int_{\partial\Omega} v_h^+ (D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{n}^+ \\ &\quad - \int_{\partial\Omega} \left(J'_{B,1}[u](v_h) + J'_{B,2}[\nu \nabla u \cdot \vec{n}](\nu \nabla v_h^+ \cdot \vec{n}) \right). \end{aligned} \tag{18}$$

In general, there exists $v_h \in \mathcal{V}_h^p$ such that $(\mathcal{L}_{h,I}(u, \psi))(v_h)$ and $(\mathcal{L}_{h,B}(u, \psi))(v_h)$ do not vanish. Thus, the scheme is dual inconsistent. $\square$

Remark 5. As in Section 4, a dual consistent scheme is derived by subtracting terms from the semi-linear form R_h and modifying the discrete functional $\mathcal{J}_h$. The interior face and boundary face contributions to the dual inconsistency are examined separately.

To eliminate the dual inconsistency from the interior faces, the following term is subtracted from the semi-linear form R_h:

$$A_{h,I}(w_h, v_h) = -\int_{\Gamma_i} [\![w_h]\!] \cdot \{\vec{\beta}_i(w_h, v_h)\},$$

where dual consistency requires that $\{\vec{\beta}_i(u, \psi)\} = D_{\nabla u} f(u, \nabla u)\psi$.

Using the functional $\mathcal{J}_{h,DC}$ defined in (10), and the resulting derivative, shown in (11), the dual inconsistency contribution from the boundary faces becomes

$$-\int_{\partial\Omega} v_h^+ (D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{n}^+.$$

To eliminate this dual inconsistency, the following term is subtracted from R_h:

$$A_{h,B}(w_h, v_h) = -\int_{\partial\Omega} w_h^+ \vec{\beta}_b(w_h, v_h) \cdot \vec{n},$$

where dual consistency requires $\vec{\beta}_b(u, \psi) = D_{\nabla u} f(u, \nabla u)\psi$.

Proposition 5. *Let B_h be a dual consistent bilinear form corresponding to the diffusion operator. Furthermore, let $\vec{\beta}_i$ be a function such that, for $u \in H^2(\Omega)$ satisfying* (14) *and $\psi \in H^2(\Omega)$ satisfying* (15)*, $\vec{\beta}_i(u, \psi) = D_{\nabla u} f(u, \nabla u)\psi$, and let $\vec{\beta}_b$ be a function such that $\vec{\beta}_b(u, \psi) = D_{\nabla u} f(u, \nabla u)\psi$. Then, the semi-linear form given by*

$$\begin{aligned} R_{h,DC}(w_h, v_h) \quad = \quad & B_h(w_h, v_h) - \sum_{\kappa \in T_h} \int_{\kappa} v_h f(w_h, \nabla w_h) \\ & + \int_{\Gamma_i} [\![w_h]\!] \cdot \{\vec{\beta}_i(w_h, v_h)\} + \int_{\partial\Omega} w_h^+ \vec{\beta}_b(w_h, v_h) \cdot \vec{n}, \end{aligned}$$

together with the discrete functional $J_{h,DC}$, defined in (10)*, is dual consistent.*

Proof. The proof is clear from Remark 5. ☐

Remark 6. The choices of $\vec{\beta}_i$ and $\vec{\beta}_b$ are not fully determined by requiring dual consistency. One valid choice is given by

$$\vec{\beta}_i(w_h, v_h) = \{D_{\nabla u} f(w_h, \nabla w_h) v_h\}; \;\; \vec{\beta}_b(w_h, v_h) = D_{\nabla u} f(w_h^+, \nabla w_h^+) v_h^+.$$

Then,

$$R_{h,DC}(w_h, v_h) = B_h(w_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f + \int_{\Gamma_i} [\![w_h]\!] \cdot \{D_{\nabla u} f(w_h, \nabla w_h) v_h\} + \int_{\partial\Omega} w_h^+ v_h^+ D_{\nabla u} f(w_h^+, \nabla w_h^+) \cdot \vec{n}. \tag{19}$$

However, if necessary, one could construct more complex functions that satisfy the dual consistency requirement as well as add stability to the discretization.

Remark 7. In addition to being dual consistent, if B_h is a consistent bilinear form for Poisson's equation, the discretization of Proposition 5 is consistent for any choice of $\vec{\beta}_i$ and $\vec{\beta}_b$ because, for an exact solution $u \in H^2(\Omega)$, $[\![u]\!] = 0$ and $u|_{\partial\Omega} = 0$. Thus,

$$R_{h,DC}(u, v_h) = B_h(u, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f(u, \nabla u) = 0, \quad \forall v_h \in \mathcal{V}_h^p.$$

6 The Mixed Formulation for Source Terms

In addition to the standard weighting source term treatment discussed in Section 5, another source term treatment of interest has appeared in the DG literature. In this method, known as the mixed formulation, the gradient of the state is replaced by a variable that is solved for simultaneously with the primal state [5]. Variants of this technique are widely used in DG discretizations of second-order operators. See [4] for a full analysis of those discretizations. This section provides a brief derivation of the mixed method as applied to source terms involving the gradient of the state. Furthermore, it shows that discretizations derived in this manner are, in general, asymptotically dual consistent.

6.1 Discretization Derivation

Consider (14)—i.e. the model problem considered in Section 5—and consider the following discretization: find $u_h \in \mathcal{V}_h^p$ and $\vec{g}_h \in \left[\mathcal{V}_h^p\right]^n$ such that

$$R_h(u_h, v_h) \equiv B_h(u_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f(u_h, \vec{g}_h) = 0, \quad \forall v_h \in \mathcal{V}_h^p, \tag{20}$$

$$\sum_{\kappa \in T_h} \int_\kappa \vec{\tau}_h \cdot \vec{g}_h = -\sum_{\kappa \in T_h} \int_\kappa u_h \nabla \cdot \vec{\tau}_h + \int_{\Gamma_i} \left([\![\hat{u}]\!] \cdot \{\vec{\tau}_h\} + \{\hat{u}\}[\![\vec{\tau}_h]\!]\right) + \int_{\partial\Omega} u^b \vec{\tau}_h^+ \cdot \vec{n}, \quad \forall \vec{\tau}_h \in \left[\mathcal{V}_h^p\right]^n \tag{21}$$

where $\hat{u}$ and u^b are numerical flux functions approximating u on interior and boundary faces respectively. Integrating by parts on (21) gives

$$\begin{aligned}\sum_{\kappa\in T_h}\int_\kappa \vec{\tau}_h\cdot\vec{g}_h &= \sum_{\kappa\in T_h}\int_\kappa \vec{\tau}_h\cdot\nabla u_h + \int_{\Gamma_i}\left([\![\hat{u}-u_h]\!]\cdot\{\vec{\tau}_h\} + \{\hat{u}-u_h\}[\![\vec{\tau}_h]\!]\right) \\ &\quad + \int_{\partial\Omega}(u^b-u_h^+)\vec{\tau}_h^+\cdot\vec{n}, \quad \forall\vec{\tau}_h\in\left[\mathcal{V}_h^p\right]^n.\end{aligned} \tag{22}$$

Defining the lifting operators $\vec{r}_h$ and $\vec{\ell}_h$ (see e.g. [4]) by

$$\begin{aligned}\sum_{\kappa\in T_h}\int_\kappa \vec{\tau}_h\cdot\vec{r}_h(u_h) &= -\int_{\Gamma_i}[\![\hat{u}-u_h]\!]\cdot\{\vec{\tau}_h\} \\ &\quad -\int_{\partial\Omega}(u^b-u_h^+)\vec{\tau}_h^+\cdot\vec{n}, \quad \forall\vec{\tau}_h\in\left[\mathcal{V}_h^p\right]^n,\end{aligned} \tag{23}$$

$$\sum_{\kappa\in T_h}\int_\kappa \vec{\tau}_h\cdot\vec{\ell}_h(u_h) = -\int_{\Gamma_i}\{\hat{u}-u_h\}[\![\vec{\tau}_h]\!], \quad \forall\vec{\tau}_h\in\left[\mathcal{V}_h^p\right]^n, \tag{24}$$

and using (22) gives

$$\vec{g}_h = \nabla u_h - \vec{r}_h(u_h) - \vec{\ell}_h(u_h). \tag{25}$$

Substituting (25) into (20) gives the following discretization: find $u_h\in\mathcal{V}_h^p$ such that

$$B_h(u_h,v_h) - \sum_{\kappa\in T_h}\int_\kappa v_h f(u_h,\nabla u_h - \vec{r}_h(u_h)-\vec{\ell}_h(u_h)) = 0, \quad \forall v_h\in\mathcal{V}_h^p, \tag{26}$$

where $\vec{r}_h$ and $\vec{\ell}_h$ are defined in (23) and (24), respectively.

6.2 Dual Consistency

Proposition 6. *If the interior face numerical flux function, $\hat{u}$, is linear, consistent, and conservative and the boundary face numerical flux function is $u^b = 0$, then the semi-linear form R_h defined in* (26) *together with the discrete function $\mathcal{J}_{h,DC}$ defined in* (10) *forms an asymptotically dual consistent discretization.*

Proof. Noting that the lifting operators $\vec{r}_h$ and $\vec{\ell}_h$ are linear functionals and that $\vec{r}_h(u) = \vec{\ell}_h(u) = 0$, linearizing R_h about the exact solution gives

$$\begin{aligned} R'_h[u](w_h, v_h) &= B_h(w_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h D_1 f(u, \nabla u) w_h \\ &\quad - \sum_{\kappa \in T_h} \int_\kappa v_h D_{\nabla u} f(u, \nabla u) \cdot (\nabla w_h - \vec{r}_h(w_h) - \vec{\ell}_h(w_h)). \end{aligned}$$

Thus, integrating by parts gives

$$\begin{aligned} R'_h[u](w_h, v_h) &= B_h(w_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa (w_h D_1 f(u, \nabla u) v_h - w_h \nabla \cdot (D_{\nabla u} f(u, \nabla u) v_h)) \\ &\quad - \int_{\Gamma_i} ([\![w_h]\!] \cdot \{D_{\nabla u} f(u, \nabla u) v_h\} + \{w\}[\![D_{\nabla u} f(u, \nabla u) v_h]\!]) \\ &\quad - \int_{\partial\Omega} w_h^+ (D_{\nabla u} f(u, \nabla u) v_h^+) \cdot \vec{n} \\ &\quad + \sum_{\kappa \in T_h} \int_\kappa v_h D_{\nabla u} f(u, \nabla u) \cdot (\vec{r}_h(w_h) + \vec{\ell}_h(w_h)). \end{aligned}$$

Now, *assume* that $D_{\nabla u} f(u, \nabla u)\psi \in [\mathcal{V}_h^p]^n$. While, for the general case, there is no reason to expect this to hold for finite values of h and p, if it is true, substituting for $\vec{r}_h$ and $\vec{\ell}_h$ and using the assumptions about $\hat{u}$ and u^b gives

$$R'_h[u](v_h, \psi) = B_h(v_h, \psi) - \sum_{\kappa \in T_h} \int_\kappa (v_h D_1 f(u, \nabla u)\psi - v_h \nabla \cdot (D_{\nabla u} f(u, \nabla u)\psi)).$$

Thus,

$$R'_h[u](v_h, \psi) - \mathcal{J}'_{h,DC}[u](v_h) = 0, \quad \forall v_h \in \mathcal{W}_h^p.$$

Returning to the assumption that $D_{\nabla u} f(u, \nabla u)\psi \in [\mathcal{V}_h^p]^n$, if the assumption is satisfied for all h and p, then the discretization is dual consistent. In general, the assumption will not be satisfied for finite h and p. However, in the limit as $h \to 0$ and the discrete solution space approaches an infinite dimensional space, it will be satisfied. Thus, the scheme is asymptotically dual consistent. □

7 Dual Consistency Error Bounds

As shown in Section 3, the dual consistency error contributes to the error in computed functional outputs. This error also appears in the analysis of the L^2 norm of the primal error [4]. Of course, for dual consistent schemes, the dual consistency error is exactly zero. However, for asymptotically dual consistent or dual inconsistent schemes, it is not zero for finite h and p. Furthermore, as noted in Remark 2, a simple application of boundedness does not typically give a tight bound on the dual consistency error. In this section, tighter bounds on the dual consistency error for the schemes shown in Sections 5 and 6 are derived.

7.1 Preliminaries

To begin, it is necessary to establish four lemmas that will be used in Sections 7.2 and 7.3.

For the remainder of the section, let $\mathcal{E}_h$ denote the set of all faces in the triangulation T_h. Define the jump operator, $[\![\cdot]\!]$, on boundary faces by $[\![s]\!] = s^+\vec{n}^+$ for scalar quantities and $[\![\vec{v}]\!] = \vec{v}^+ \cdot \vec{n}^+$ for vector quantities. Define the average operator, $\{\cdot\}$, on boundary faces by $\{\vec{v}\} = \vec{v}^+$. Let $\mathcal{W}_h^p \equiv \mathcal{V} + \mathcal{V}_h^p$, and $\mathcal{V} \equiv H^2(\Omega) \cap H_0^1(\Omega)$. Finally, assume that the set $[T_h]_{h>0}$ of triangulations of $\Omega \subset \mathbb{R}^n$ is quasi-uniform (see [16, 24] for definition).

Lemma 1. *There exists a norm, $|||\cdot|||_* : \mathcal{W}_h^p \to \mathbb{R}$, and a constant, c, such that*

$$h^{-1/2} \sum_{e \in \mathcal{E}_h} \|[\![v]\!]\|_{0,e} \leq \sum_{e \in \mathcal{E}_h} h_{\kappa_e}^{-1/2} \|[\![v]\!]\|_{0,e} \leq c|||v|||_*, \quad \forall v \in \mathcal{W}_h^p,$$

where κ_e is such that $e \subset \partial\kappa_e$, $h = \max_{\kappa \in T_h} h_\kappa$, and $h_\kappa = \sup_{x,y \in \kappa} |x - y|$.

Proof. See [4], Section 4.1, and [11], Lemma 2. □

Lemma 2. *For a face, $e \in \mathcal{E}_h$, such that $e \subset \partial\kappa$, there exists a constant, c, such that the following inequality holds for all $v \in H^1(\kappa)$:*

$$\|v\|_{0,e} \leq c h_\kappa^{-1/2} (\|v\|_{0,\kappa} + h_\kappa |v|_{1,\kappa}).$$

Proof. This statement is a standard trace theorem. See [3, 1]. □

Lemma 3. *For a face, $e \in \mathcal{E}_h$, such that $e \subset \partial\kappa$, there exists a constant, c, such that, for all $v \in H^1(\kappa)$ and $w \in L^2(e)$,*

$$\int_e |vw| \leq c h_\kappa^{-1/2} (\|v\|_{0,\kappa} + h_\kappa |v|_{1,\kappa}) \|w\|_{0,e}.$$

Proof. Apply the Cauchy-Schwarz inequality and Lemma 2. □

Lemma 4. *For all $v \in H^{p+1}(\Omega)$ there exists a constant, c, such that*

$$\sum_{\kappa \in T_h} \|v - \Pi_h^p(v)\|_{1,\kappa} \leq ch^p |v|_{p+1,\Omega},$$

where $\Pi_h^p : L^2(\Omega) \to \mathcal{V}_h^p$ is the $L^2(\Omega)$-orthogonal projection onto $\mathcal{V}_h^p$.

Proof. If $\Pi_\kappa^p : L^2(\kappa) \to P^p$ is the $L^2(\kappa)$-projection onto P^p, then

$$\Pi_h^p(v)|_\kappa = \Pi_\kappa^p(v|_\kappa).$$

To complete the proof, apply Proposition 1.134(iii) from [16] to each element κ and sum over the elements. $\square$

7.2 Standard Weighting Technique

This section shows a bound on the dual consistency error resulting from the standard weighting discretization.

Proposition 7. *If R_h is the semi-linear form defined in (16) and $\mathcal{J}_{h,DC}$ is the discrete functional defined in (10), then there exists a constant, c, such that*

$$|R_h'[u](u_h, \psi - \psi_h)| = |R_h'[u](u_h, \psi) - \mathcal{J}_{h,DC}'[u](u_h)| \leq c|||u - u_h|||_*.$$

Proof. From (17), (18), and Remark 5,

$$E_h \equiv R_h'[u](u_h, \psi) - \mathcal{J}_{h,DC}'[u](u_h) = -\int_\Gamma [\![u_h]\!] \cdot (D_{\nabla u} f(u, \nabla u)\psi),$$

where $\Gamma = \Gamma_i \cup \partial\Omega$. Since $u \in H^2(\Omega)$ and $u|_{\partial\Omega} = 0$,

$$E_h = -\int_\Gamma [\![u_h - u]\!] \cdot (D_{\nabla u} f(u, \nabla u)\psi).$$

Thus,

$$\begin{aligned} |E_h| &\leq \sum_{e \in \mathcal{E}_h} \int_e |[\![u_h - u]\!]| \, |(D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{n}^+| \\ &\leq \sum_{e \in \mathcal{E}_h} \int_e |[\![u_h - u]\!]| \, M, \end{aligned}$$

where $M = \sum_{i=1}^{n} |(D_{\nabla u} f(u, \nabla u)\psi)_i|$. Applying Lemma 3 gives

$$|E_h| \leq c_1 \sum_{e \in \mathcal{E}_h} h_{\kappa_e}^{-1/2} (\|M\|_{0,\kappa_e} + h_{\kappa_e} |M|_{1,\kappa_e}) \|[\![u_h - u]\!]\|_{0,e},$$

where $e \subset \partial\kappa_e$. Thus, by Lemma 1,

$$\begin{aligned} |E_h| &\leq c_2 \left(\sum_{\kappa \in T_h} (\|M\|_{0,\kappa} + h_\kappa |M|_{1,\kappa}) \right) \left(\sum_{e \in \mathcal{E}_h} h_{\kappa_e}^{-1/2} \|[\![u_h - u]\!]\|_{0,e} \right) \\ &\leq c_3 |\!|\!| u_h - u |\!|\!|_*, \end{aligned}$$

for some constant c_3. $\square$

Remark 8. If the discretization is optimal with respect to the $|\!|\!| \cdot |\!|\!|_*$ norm and if $u \in H^{p+1}(\Omega)$, then, by Proposition 7, there exists a constant, c, such that

$$|E_h| \leq c h^p |u|_{p+1,\Omega}.$$

Thus, assuming that the scheme is otherwise optimal, the standard weighting discretization dual consistency error is $O(h^p)$.

7.3 Mixed Formulation

This section shows a bound on the dual consistency error for two variants of the mixed formulation. Specifically, consider R_h as defined in (26) for $\hat{u} = \{u_h\}$ or $\hat{u} = \{u_h\} + \vec{d}_e \cdot [\![u_h]\!]$ and $u^b = 0$. These two $\hat{u}$ options are analyzed because they are used often in DG discretizations of Poisson's equation, in particular in the BR2 and LDG discretizations (see [4]).

Proposition 8. *Define $\vec{\pi} \in [\mathcal{V}_h^p]^n$ by*

$$\pi_j = \Pi_h^p((D_{\nabla u} f(u, \nabla u)\psi)_j), \quad \text{for } j = 1, \ldots, n$$

and define $\vec{\epsilon} \in [L^2(\Omega)]^n$ by

$$\epsilon_j = (D_{\nabla u} f(u, \nabla u)\psi)_j - \pi_j, \quad \text{for } j = 1, \ldots, n.$$

Then, if $D_{\nabla u} f(u, \nabla u)\psi \in [H^1(\Omega)]^n$, $\vec{\epsilon}|_\kappa \in [H^1(\kappa)]^n$, for all $\kappa \in T_h$.

Furthermore, if R_h is the semi-linear form defined in (26) and $\mathcal{J}_{h,DC}$ is the discrete functional *defined in* (10)*, then, for* $h < 1$*, there exists a constant,* c*, such that*

$$|R_h'[u](u_h, \psi - \psi_h)| = |R_h'[u](u_h, \psi) - \mathcal{J}_{h,DC}'[u](u_h)| \leq c|||u - u_h|||_* \sum_{j=1}^{n} \sum_{\kappa \in T_h} \|\epsilon_j\|_{1,\kappa}.$$

Proof. The fact that $\vec{\epsilon}|_\kappa \in [H^1(\kappa)]^n$ follows trivially from $D_{\nabla u} f(u, \nabla u)\psi \in [H^1(\Omega)]^n$ and the definition of the projection Π_h^p.

Furthermore, Section 6 shows that

$$\begin{aligned} E_h &\equiv R_h'[u](u_h, \psi) - \mathcal{J}_{h,DC}'[u](u_h) \\ &= -\int_\Gamma [\![u_h]\!] \cdot (D_{\nabla u} f(u, \nabla u)\psi) + \sum_{\kappa \in T_h} \int_\kappa (D_{\nabla u} f(u, \nabla u)\psi) \cdot (\vec{r}_h(u_h) + \vec{\ell}_h(u_h)). \end{aligned}$$

Substituting $D_{\nabla u} f(u, \nabla u)\psi = \vec{\pi} + \vec{\epsilon}$ gives

$$E_h = -\int_\Gamma [\![u_h]\!] \cdot \{\vec{\pi} + \vec{\epsilon}\} + \sum_{\kappa \in T_h} \int_\kappa (\vec{\pi} + \vec{\epsilon}) \cdot (\vec{r}_h(u_h) + \vec{\ell}_h(u_h)).$$

Requiring that $[\![\hat{u}]\!] = 0$ and $u^b = 0$ and using (23) and (24) gives

$$E_h = -\int_\Gamma [\![u_h]\!] \cdot \{\vec{\epsilon}\} - \int_{\Gamma_i} (\hat{u} - \{u_h\})[\![\vec{\pi}]\!] + \sum_{\kappa \in T_h} \int_\kappa \vec{\epsilon} \cdot (\vec{r}_h(u_h) + \vec{\ell}_h(u_h)).$$

By definition, $\vec{\epsilon}$ is $L^2(\Omega)$-orthogonal to $[\mathcal{V}_h^p]^n$. Furthermore, it is straightforward to show that $[\![\vec{\pi}]\!] = -[\![\vec{\epsilon}]\!]$. Thus,

$$E_h = -\int_\Gamma [\![u_h]\!] \cdot \{\vec{\epsilon}\} + \int_{\Gamma_i} (\hat{u} - \{u_h\})[\![\vec{\epsilon}]\!].$$

Noting that $u \in H^2(\Omega)$ and $u|_{\partial\Omega} = 0$ gives

$$E_h = -\int_\Gamma [\![u_h - u]\!] \cdot \{\vec{\epsilon}\} + \int_{\Gamma_i} (\hat{u} - \{u_h\})[\![\vec{\epsilon}]\!] \equiv T_1 + T_2.$$

Thus,

$$|E_h| \leq |T_1| + |T_2|.$$

The terms T_1 and T_2 are considered separately. Examining T_1,

$$\begin{aligned} |T_1| &\leq \int_\Gamma |[\![u_h - u]\!]| \, |\{\vec{\epsilon}\} \cdot \vec{n}^+| \\ &\leq \sum_{j=1}^{n} \left(\int_\Gamma |[\![u_h - u]\!]| \, |\epsilon_j^+| + \int_{\Gamma_i} |[\![u_h - u]\!]| \, |\epsilon_j^-| \right). \end{aligned} \tag{27}$$

By Lemmas 3 and 1,

$$\begin{aligned}|T_1| &\leq c_1 \sum_{j=1}^{n} \left(\left(\sum_{\kappa \in T_h} (\|\epsilon_j\|_{0,\kappa} + h_\kappa |\epsilon_j|_{1,\kappa}) \right) \times \left(\sum_{e \in \mathcal{E}_h} h^{-1/2} \|[\![u_h - u]\!]\|_{0,e} \right) \right) \\ &\leq c_2 |\!|\!| u_h - u |\!|\!|_* \sum_{j=1}^{n} \sum_{\kappa \in T_h} (\|\epsilon_j\|_{0,\kappa} + h_\kappa |\epsilon_j|_{1,\kappa}).\end{aligned}$$

For $h < 1$, $(\|\epsilon_i\|_{0,\kappa} + h_\kappa |\epsilon_i|_{1,\kappa}) \leq \|\epsilon_i\|_{1,\kappa}$. Thus, when $h < 1$,

$$|T_1| \leq c_2 \, |\!|\!| u_h - u |\!|\!|_* \sum_{j=1}^{n} \sum_{\kappa \in T_h} \|\epsilon_j\|_{1,\kappa}.$$

The term T_2 is examined for two common choices of $\hat{u}$: $\hat{u} = \{u\}$ and $\hat{u} = \{u\} + \vec{d_e} \cdot [\![u_h]\!]$, where $\vec{d_e}$ is a constant vector defined for each face. In the former case, it is clear that $T_2 = 0$. In the latter case,

$$\begin{aligned}|T_2| &\leq \int_{\Gamma_i} |\vec{d_e} \cdot [\![u_h - u]\!]| \, |[\![\vec{\epsilon}]\!]| \\ &\leq c_3 \sum_{j=1}^{n} \int_{\Gamma_i} |[\![u_h - u]\!]| \, (|\epsilon_j^+| + |\epsilon_j^-|).\end{aligned}$$

The right-hand side above also appears in (27) and thus,

$$|T_2| \leq c_4 |\!|\!| u_h - u |\!|\!|_* \sum_{j=1}^{n} \sum_{\kappa \in T_h} \|\epsilon_j\|_{1,\kappa},$$

which completes the proof. □

Remark 9. If the scheme achieves the optimal rate in the $|\!|\!| \cdot |\!|\!|_*$ norm and $D_{\nabla u} f(u, \nabla u) \psi \in [H^{p+1}(\Omega)]^n$, by Proposition 8 and Lemma 4,

$$|E_h| \leq c h^{2p} |u|_{p+1,\Omega} \sum_{j=1}^{n} |(D_{\nabla u} f(u, \nabla u) \psi)_j|_{p+1,\Omega}.$$

8 Numerical Results

As a demonstration of the effects of dual consistency, a simple test problem based on a nonlinear ODE is considered. The effect of dual consistency on the convergence rates of the solution and adjoint solution errors as well as a simple functional output is demonstrated.

Consider the following ODE:

$$
\begin{aligned}
-((\nu+u)u_x)_x - cu_xu_x &= g \quad \text{for } x \in (0,1), \\
u(0) = u(1) &= 0,
\end{aligned}
$$

where $\nu = 1$ and $c = \frac{1}{2}$. Setting the source term as

$$g(x) = \pi^2((\nu + \sin(\pi x))\sin(\pi x) - (1+c)\cos^2(\pi x)),$$

it is easy to show that the exact solution is given by

$$u_e(x) = \sin(\pi x).$$

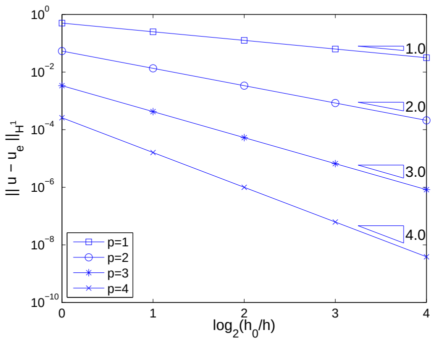
This nonlinear problem has been solved using three discretizations: the standard weighting method as shown in (16), a dual consistent method with a penalty term like that shown in (19), and an asymptotically dual consistent mixed method with $\hat{u} = \{u\}$. In all cases, BR2 is used to discretize the nonlinear diffusion operator.

Figure 1 shows the error in the primal solution versus grid refinement. The error is measured in a "broken" H^1 norm defined by

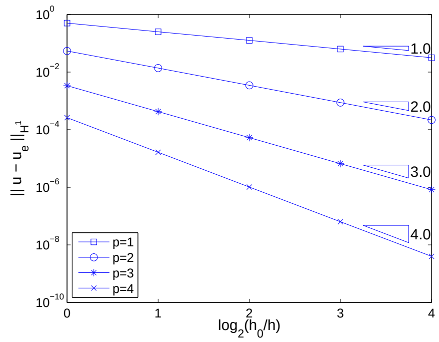
$$\|v\|_{H^1} = \sum_{\kappa \in T_h} \int_\kappa (v^2 + v_x^2).$$

In this norm, all three schemes produce essentially the same error in the primal solution. However, as shown in Figure 2, the dual consistent and asymptotically dual consistent schemes produce superior results when the error is measured in the L^2 norm. In particular, for the dual inconsistent discretization, the L^2 norm of the error is proportional to $O(h^p)$ for even p and proportional to $O(h^{p+1})$ for odd p. While the even p result agrees with the bound established in Section 7, the error converges faster than expected for odd p. Similar results have been obtained for other dual inconsistent discretizations [20]. Although this phenomenon is not well understood, it appears that, at least in some cases, the bound on the dual consistency error in Section 7 may not be tight for odd p. Alternatively, the dual consistent and asymptotically dual consistent discretizations give $O(h^{p+1})$ for all p tested. Furthermore, it is interesting to note that the asymptotically dual consistent method produces essentially exactly the same results as the dual consistent discretization. This result makes sense given that the dual consistency error for the asymptotically dual consistent scheme is $O(h^{2p})$.

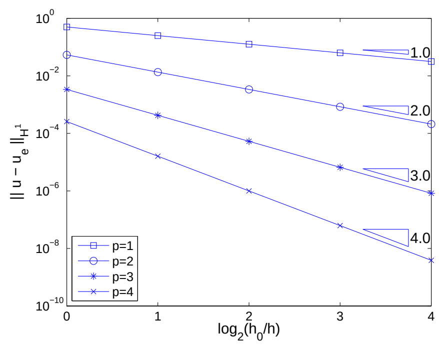
Examining the adjoint solution error, one can see that the dual consistent and asymptotically



(a) Dual consistent

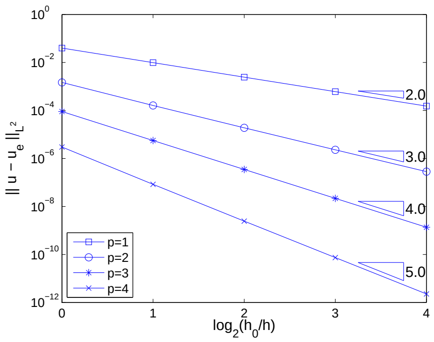


(b) Dual inconsistent

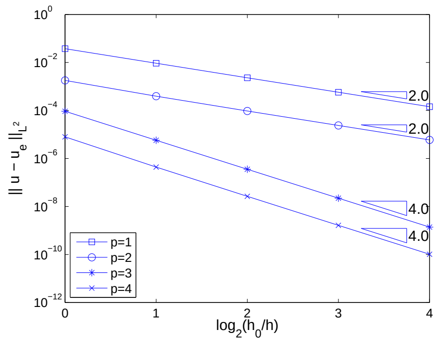


(c) Asymptotically dual consistent

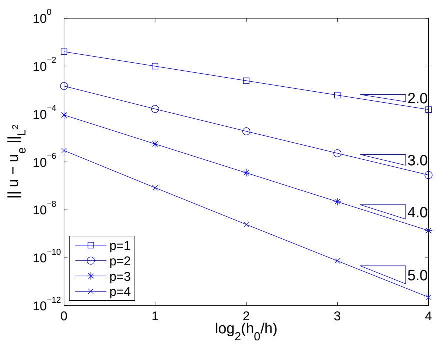
Figure 1: Primal error in the broken H^1 norm



(a) Dual consistent



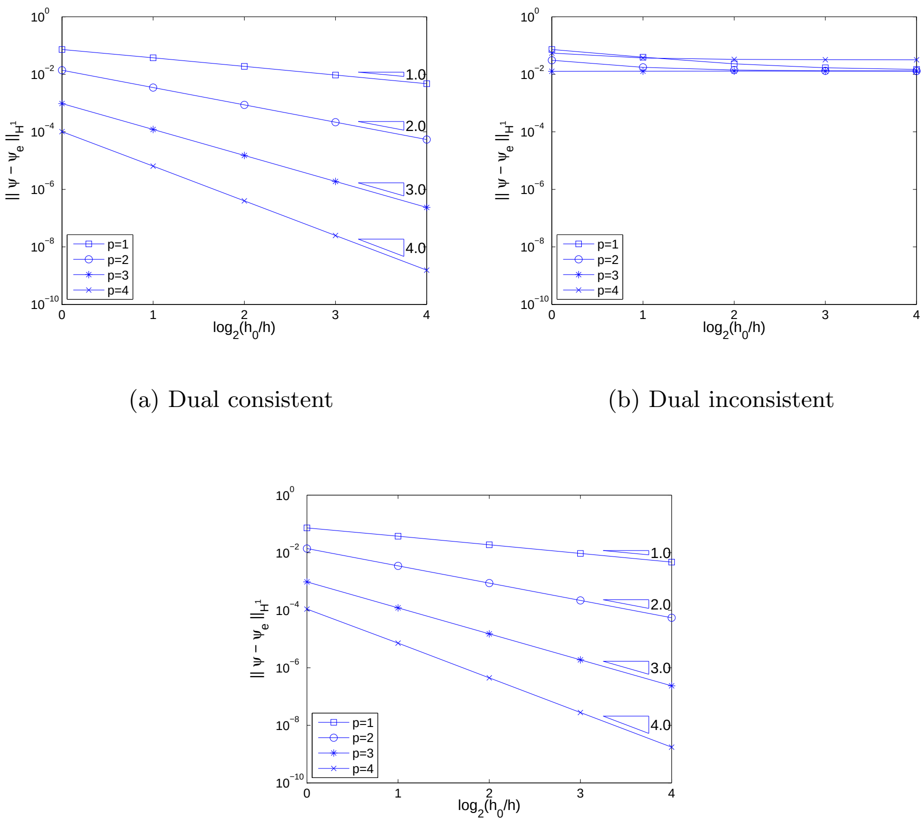
(b) Dual inconsistent



(c) Asymptotically dual consistent

Figure 2: Primal error in the L^2 norm

dual consistent schemes are superior for computing the adjoint. Figure 3 shows the adjoint error in the broken H^1 norm. The adjoint error is computed relative to a 40th order solution of a Galerkin



(a) Dual consistent

(b) Dual inconsistent

(c) Asymptotically dual consistent

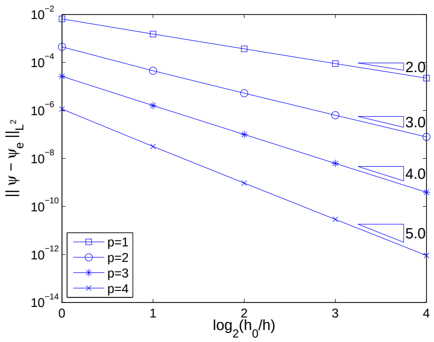
Figure 3: Adjoint error in the broken H^1 norm

spectral discretization of the dual problem. When using the dual inconsistent discretization, the broken H^1 norm of the adjoint error does not converge to zero with grid refinement. For the dual consistent and asymptotically dual consistent schemes, this error converges at $O(h^p)$. Similarly, Figure 4 shows that the L^2 norm of the adjoint error converges at $O(h)$ when using the dual inconsistent scheme, regardless of p, while, for the dual consistent and asymptotically dual consistent schemes, this error converges at $O(h^{p+1})$.

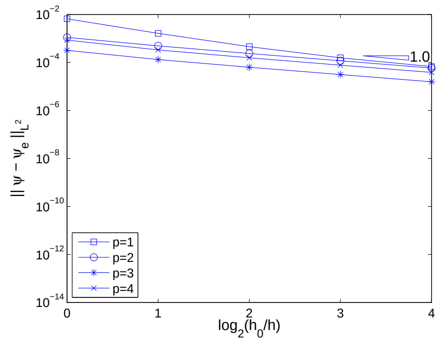
Finally, let

$$\mathcal{J}(u) = \frac{1}{2}\int_0^1 (w-u)^2,$$

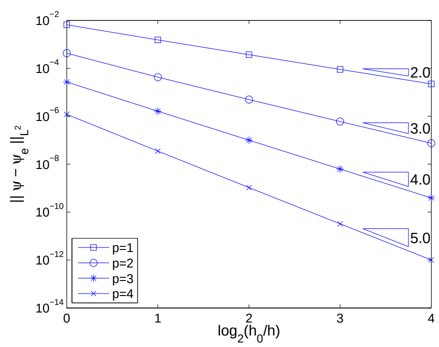
where $w(x) = 2\sin(\pi x)$ be the output of interest. Then, computing the exact functional output is



(a) Dual consistent



(b) Dual inconsistent

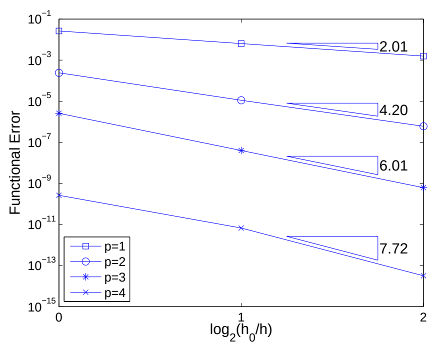


(c) Asymptotically dual consistent

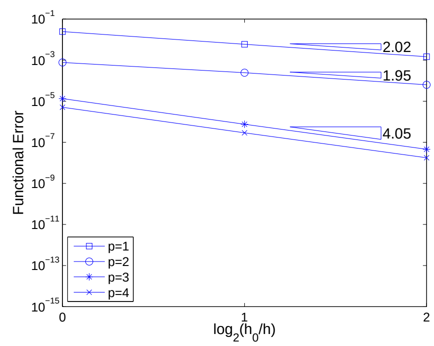
Figure 4: Adjoint error in the L^2 norm

trivial, enabling comparison of the computed result with the exact value. In particular, $\mathcal{J}(u_e) = 1/4$.

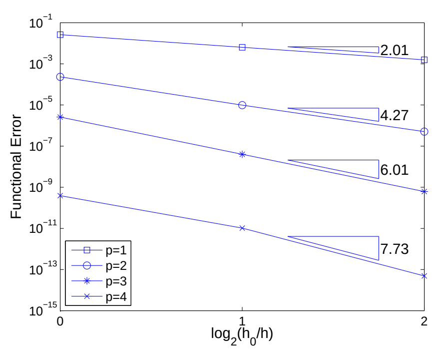
Figure 5 shows the error in the computed functional for the three discretizations considered. The figure shows that, as in the state and adjoint results, the performance the dual consistent and



(a) Dual consistent



(b) Dual inconsistent



(c) Asymptotically dual consistent

Figure 5: Functional output error

asymptotically dual consistent schemes is very similar. Both schemes achieve $O(h^{2p})$ for $1 \leq p \leq 4$. However, for the dual inconsistent scheme, the convergence rate of the functional is $O(h^p)$ for even p and $O(h^{p+1})$ for odd p. Thus, the dual consistent and asymptotically dual consistent discretizations predict the functional with greater accuracy than the dual inconsistent discretization for similar numbers of degrees of freedom.

9 Conclusions

The effect of dual consistency on DG discretizations of solution and solution dependent source terms has been examined. In particular, the standard weighting DG discretization of source terms depending on the gradient of the solution has been analyzed and shown to be dual inconsistent. Starting from this dual inconsistent scheme a dual consistent discretization has been developed. Furthermore, discretizations derived using the mixed formulation have been shown to be asymptotically dual consistent.

A priori error estimates have been derived to show the effect of dual inconsistency on the accuracy of computed functional outputs. In particular, for the dual consistent and asymptotically dual consistent schemes and type of output considered here, $O(h^{2p})$ convergence can be shown, while for similar but dual inconsistent schemes, only $O(h^p)$ can be proved. Numerical results from a simple test problem demonstrate that indeed the dual consistent and asymptotically dual consistent schemes are superior both in terms of solution accuracy and output accuracy.

Further work is required in many areas. For example, only simple outputs depending on the solution in the interior of the domain have been considered in the *a priori* analysis. This analysis should be extended to more general outputs—in particular, to outputs depending on both the state and flux at the boundary of the domain. Futhermore, this work has considered only the effect of dual consistency. Using the methods presented in Section 5 and Section 6, one could construct a consistent and dual consistent but unstable scheme. Such schemes may not be viable discretizations. Thus, while techniques for constructing a consistent and dual consistent (or asymptotically dual consistent) discretization have been shown, a method for ensuring that the resulting scheme is stable is left for future research. Finally, given the extremely similar results shown for the dual consistent and asymptotically dual consistent schemes considered, it remains to be determined which of these schemes is most effective for practical problems.

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