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Oscillatory Behavior of Higher Order Nonlinear Difference Equations

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Abstract. The authors present some new oscillation criteria for higher order nonlinear difference equations with nonnegative real coefficients of the form

$$\Delta\left(\left(a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}\right)+q(t)\,x^{\beta}(t-m+1)=0.$$

Both of the cases n even and n odd are considered. They give examples to illustrate their results.

Keywords: oscillation, higher order, difference equations.

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1 Introduction

In this paper, we study the oscillatory behavior of all solutions of the higher order nonlinear difference equation

$$\Delta\left(\left(a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}\right)+q(t)x^{\beta}(t-m+1)=0,\quad t\geq t_0,\tag{1.1}$$

where $n\geq 3$. We assume throughout that:

(i) α and β are the ratios of odd positive integers with $\alpha\geq\beta$;

(ii) $\{a(t)\}$ and $\{q(t)\}$ are sequences of positive real numbers for $t \geq t_0$;

(iii) $m > 1$.

We let

$$A(v,u) = \sum_{s=u}^{v} \frac{1}{a^{1/\alpha}(s)}, \quad v \geq u \geq t_0,$$

and assume that

$$\lim_{t\to\infty} A(t,t_0) < \infty. \tag{1.2}$$

By a *solution* of Equation (1.1), we mean a real sequence $\{x(t)\}$ defined for all $t \geq t_0 - m + 1$ that satisfies Equation (1.1) for all $t \geq t_0$. A solution of Equation (1.1) is called *oscillatory* if its terms are neither eventually positive nor eventually negative, and it is called *nonoscillatory* otherwise. If all solutions of the equation are oscillatory, then the equation itself is called oscillatory.

In recent years there has been much research activity concerning the oscillation and asymptotic behavior of solutions of various classes of difference equations and we mention [1, 2, 3, 4, 5, 6, 7] and the references cited therein as examples of some recent contributions in this area. There have been numerous studies on second order difference equations due to their use in the natural sciences and as well as for theoretical interests. Recent results on the oscillatory and asymptotic behavior of solutions of second order difference equations can be found, for example, in [8,9,10,11,12,13,17,18,19]. However, it appears that there are no known results regarding the oscillation of solutions of higher order difference equations of the form of Equation (1.1). In view of this, our aim in this paper is to present some new sufficient conditions that ensure that all solutions of Equation (1.1) are oscillatory.

2 Main results

We begin with some useful lemmas.

Lemma 1. ([1, Theorem 1.8.11], [3, Lemma 1]) *Let $\{x(t)\}$ be defined for $t \geq t_0$ and $x(t) > 0$ with $\Delta^n x(t)$ of constant sign and not identically zero for $t \geq t_0$. Then there exists an integer p, $0 \leq p \leq n$, with $p + n$ even if $\Delta^n x(t) \geq 0$ and $p + n$ odd for $\Delta^n x(t) \leq 0$, such that:*

i) $p \leq n-1$ implies $(-1)^{p+i}\Delta^i x(t) > 0$ for $t \geq t_0$ and $p \leq i \leq n-1$;

ii) $p \geq 1$ implies $\Delta^i x(t) > 0$ for all large $t \geq t_0$ and $1 \leq i \leq p-1$.

Lemma 2. ([1, Corollary 1.8.12], [3, Lemma 2]) *Let $\{x(t)\}$ be defined for $t \geq t_0$ with $x(t) > 0$ and $\Delta^n x(t) \leq 0$ and not identically zero for $t \geq t_0$. Then there exists $t_1 \geq t_0$ such that*

$$x(t) \geq \frac{1}{(n-1)!}(t-t_1)^{n-1}\Delta^{n-1}x(2^{n-p-1}t) \quad \text{for } t \geq t_1,$$

where p is defined as in Lemma 1. Furthermore, if $\{x(t)\}$ is increasing, then

$$x(t) \geq \frac{1}{(n-1)!}\left(\frac{t}{2^{n-1}}\right)^{n-1}\Delta^{n-1}x(t) \quad \text{for } t \geq t_1.$$

The following lemma is an extension of the discrete analogue of known results in [15] and [16, Corollary 1]; it can also be found in [2, Lemma 6.2.2] and [14, Corollary 7.4.1]. The proof is immediate.

Lemma 3. *Let $\{q(t)\}$ be a sequence of positive real numbers, m and p be positive numbers, and $f : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function with $xf(x) > 0$ for $x \neq 0$. If the first order delay inequality*

$$\Delta y(t) + q(t)f(y(t-m+1)) \leq 0$$

has an eventually positive solution, then so does the delay equation

$$\Delta y(t) + q(t)f(y(t-m+1)) = 0.$$

We are now ready for our first oscillation result; it is for the case where n is even.

Theorem 1. *Let n be even and assume that there is a number k such that $m > (n-2)k+1$. If the first order equations*

$$\Delta Y(t) + q(t)\left(\frac{1}{2^{n-1}(n-1)!}\right)^{\beta}\left((t-m+1)^{(n-1)\beta}\, a^{-\beta/\alpha}(t-m+1)\right) \times Y^{\beta/\alpha}(t-m+1) = 0, \tag{2.1}$$

$$\Delta Z(t) + \left(\frac{1}{a(t)}\sum_{s=t_0}^{t-1}\left(\frac{1}{2^{n-2}(n-2)!}\right)^{\beta} q(s)\,(s-m+1)^{(n-2)\beta}\right)^{1/\alpha} \times Z^{\beta/\alpha}(t-m+1) = 0, \tag{2.2}$$

and

$$\Delta W(t) + \frac{\left(k^{n-2}\right)^{\beta/\alpha}}{a^{1/\alpha}(t)}\left(\sum_{s=t_0}^{t-1} q(s)\right)^{1/\alpha} W^{\beta/\alpha}(t-m+(n-2)k+1) = 0,$$

are oscillatory, then Equation (1.1) *is oscillatory.*

Proof. Let $\{x(t)\}$ be a nonoscillatory solution of Equation (1.1), say $x(t) > 0$ and $x(t-m+1) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from Equation (1.1) that

$$\Delta\left(a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}\right) \leq -q(t)x^{\beta}(t-m+1) \leq 0 \tag{2.3}$$

for $t \geq t_1$. Hence, $a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}$ is nonincreasing and eventually of one sign. That is, there exists $t_2 \geq t_1$ such that

(I) $\Delta^{n-1}x(t) > 0$ or (II) $\Delta^{n-1}x(t) < 0$ and $\Delta^{n-2}x(t) > 0$ for $t \geq t_2$.

First, we consider Case (I). By Lemma 1, we have $\Delta x(t) > 0$ for $t \geq t_2$. From Lemma 2, we see that

$$x(t) \geq \frac{1}{(n-1)!}\left(\frac{t}{2^{n-1}}\right)^{n-1} \Delta^{n-1}x(t) \quad \text{for} \quad t \geq t_2,$$

and so

$$x(t-m+1) \geq \frac{1}{(n-1)!}\left(\frac{t-m+1}{2^{n-1}}\right)^{n-1} \Delta^{n-1}x(t-m+1) \quad \text{for} \quad t \geq t_3, \tag{2.4}$$

for some $t_3 \geq t_2$. Using (2.4) in (2.3), we obtain

$$\begin{aligned}&\Delta\left(a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}\right)\\ &\quad \leq -q(t)\left(\frac{1}{2^{n-1}(n-1)!}\right)^{\beta}(t-m+1)^{(n-1)\beta}\left(\Delta^{n-1}x(t-m+1)\right)^{\beta}\end{aligned}$$

or

$$\begin{aligned}\Delta Y(t) + q(t)\left(\frac{1}{2^{n-1}(n-1)!}\right)^{\beta}&\left((t-m+1)^{(n-1)\beta}\, a^{-\beta/\alpha}(t-m+1)\right)\\ \times Y^{\beta/\alpha}(t-m+1) &\leq 0 \end{aligned} \tag{2.5}$$

for $t \geq t_3$, where Y(t) $= a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}$. It follows from Lemma 3 that the corresponding Equation (2.1) also has a positive solution, which is a contradiction.

Next, we consider Case (II). Since n is even, we distinguish the following two cases (recall that (2.3) holds):

(III) $x(t) > 0,\ \Delta x(t) > 0,\ \ldots,\ \Delta^{n-2}x(t) > 0,\ \Delta^{n-1}x(t) < 0 \quad \text{for} \quad t \geq t_2,$

or

(IV) $x(t) > 0,\ \Delta x(t) < 0,\ \ldots,\ \Delta^{n-2}x(t) > 0,\ \Delta^{n-1}x(t) < 0 \quad \text{for} \quad t \geq t_2.$

If (III) holds, then by Lemma 2, we see that

$$x(t-m+1) \geq \frac{1}{(n-2)!}\left(\frac{t-m+1}{2^{n-2}}\right)^{n-2} \Delta^{n-2}x(t-m+1) \text{ for } t \geq t_3 \tag{2.6}$$

for some $t_3 \geq t_2$. Using (2.6) in (2.3), we have

$$\begin{aligned}&\Delta\left(a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}\right)\\ &\quad \leq -q(t)\left(\frac{1}{2^{n-2}(n-2)!}\right)^{\beta}(t-m+1)^{(n-2)\beta}\left(\Delta^{n-2}x(t-m+1)\right)^{\beta}\end{aligned}$$

for $t \geq t_3$. Summing this inequality from t_3 to $t-1$ gives

$$\begin{aligned}&-\left(a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}\right)\\ &\quad \geq \left(\sum_{s=t_3}^{t-1}\left(\frac{1}{2^{n-2}(n-2)!}\right)^{\beta} q(s)\,(s-m+1)^{(n-2)\beta}\right)\left(\Delta^{n-2}x(t-m+1)\right)^{\beta}\end{aligned}$$

or

$$-\Delta^{n-1}x(t) \geq \left(\frac{1}{a(t)} \sum_{s=t_3}^{t-1} \left(\frac{1}{2^{n-2}(n-2)!} \right)^{\beta} q(s)\,(s-m+1)^{(n-2)\beta} \right)^{1/\alpha} \times \left(\Delta^{n-2}x(t-m+1) \right)^{\beta/\alpha},$$

so

$$\Delta Z(t) + \left(\frac{1}{a(t)} \sum_{s=t_3}^{t-1} \left(\frac{1}{2^{n-2}(n-2)!} \right)^{\beta} q(s)\,(s-m+1)^{(n-2)\beta} \right)^{1/\alpha} \times Z^{\beta/\alpha}(t-m+1) \leq 0$$

for $t \geq t_3$, where $Z(t) = \Delta^{n-2}x(t) > 0$. The remainder of the proof in this case is similar to that of Case (I) and is omitted.

Finally, we consider Case (IV). We have

$$(-1)^i \Delta^i x(t) > 0 \quad \text{for} \quad i = 0, 1, \ldots, n-1, \quad \text{and } t \geq t_2. \tag{2.7}$$

Notice that

$$-\Delta^{n-3}x(t) \geq \Delta^{n-3}x(t+k) - \Delta^{n-3}x(t) = \sum_{s=t}^{t+k-1} \Delta^{n-2}x(s) \geq k(\Delta^{n-2}x(t+k)).$$

Repeating this $n-2$ times, we arrive at

$$x(t) \geq k^{n-2}\Delta^{n-2}(t+(n-2)k),$$

so

$$x(t-m+1) \geq k^{n-2}\Delta^{n-2}(t-m+(n-2)k+1).$$

Using this inequality in (2.3), we have

$$\Delta \left(a(t)\left(-\Delta^{n-1}x(t)\right)^{\alpha} \right) \geq q(t)\left(k^{n-2}\Delta^{n-2}(t-m+(n-2)k+1)\right)^{\beta}.$$

Summing from t_2 to $t-1$ yields

$$a(t)\left(-\Delta^{n-1}x(t)\right)^{\alpha} - a(t_2)\left(-\Delta^{n-1}x(t_2)\right)^{\alpha} \geq \sum_{s=t_2}^{t-1} q(s)\left(k^{n-2}\Delta^{n-2}x(s-m+(n-2)k+1)\right)^{\beta}$$

or

$$-\Delta^{n-1}x(t) \geq \frac{\left(k^{n-2}\right)^{\beta/\alpha}}{a^{1/\alpha}(t)} \left(\sum_{s=t_2}^{t-1} q(s) \right)^{1/\alpha} \left(\Delta^{n-2}x(t-m+(n-2)k+1)\right)^{\beta/\alpha},$$

and hence we obtain

$$\Delta W(t) + \frac{\left(k^{n-2}\right)^{\beta/\alpha}}{a^{1/\alpha}(t)} \left(\sum_{s=t_1}^{t-1} q(s) \right)^{1/\alpha} W^{\beta/\alpha}(t-m+(n-2)k+1) \leq 0,$$

where $W(t) = \Delta^{n-2}x(t) > 0$. The rest of the proof is similar to that of Case (I) and is left to the reader. This completes the proof of the theorem. □

To obtain some consequences of the above theorem, we let

$$P(t) \le \min\left\{ q(t)\left(\frac{1}{2^{n-1}(n-1)!}\right)^{\beta}(t-m+1)^{(n-1)\beta}a^{-\beta/\alpha}(t-m+1), \left(\frac{1}{a(t)}\right.\right.$$
$$\left.\left.\times\sum_{s=t_0}^{t-1}\left(\frac{1}{2^{n-2}(n-2)!}\right)^{\beta}q(s)\,(s-m+1)^{(n-2)\beta}\right)^{\frac{1}{\alpha}}, \frac{\left(k^{n-2}\right)^{\frac{\beta}{\alpha}}}{a^{1/\alpha}(t)}\left(\sum_{s=t_0}^{t-1}q(s)\right)^{\frac{1}{\alpha}}\right\}.$$

and $\tau = \max\{m-(n-2)k-1,\, m-1\}$.

The following corollaries are immediate consequences of known results.

Corollary 1. Let n be even and k be a number such that $m > (n-2)k+1$. If the first order delay equation

$$\Delta y(t) + P(t)y^{\beta/\alpha}(t-\tau) = 0$$

is oscillatory, then Equation (1.1) is oscillatory.

Corollary 2. Let n be even and k be a number such that $m > (n-2)k+1$. If

$$\liminf_{t\to\infty}\sum_{s=t-m+1}^{t-1}P(s)\begin{cases} > \frac{(\tau-1)^{\tau-1}}{\tau^{\tau}}, & \text{if } \beta=\alpha,\\ = \infty, & \text{if } \beta<\alpha,\end{cases}$$

then Equation (1.1) is oscillatory.

We now turn our attention to the case where n is odd.

Theorem 2. *Let n be odd and k be a number such that $m > (n-1)k+1$. If the first order Equations* (2.1)–(2.2)*, and*

$$\Delta W(t) + q(t)\left(\frac{k^{n-1}}{a^{1/\alpha}(t)}\right)^{\beta}W^{\beta/\alpha}(t-m+(n-1)k+1) = 0 \qquad (2.8)$$

are oscillatory, then Equation (1.1) *is oscillatory.*

Proof. Let $\{x(t)\}$ be a nonoscillatory solution of Equation (1.1), say $x(t) > 0$ and $x(t-m+1) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. As in the proof of Theorem 1, we see that $a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}$ is nonincreasing and eventually of one sign. That is, there exists $t_2 \ge t_1$ such that

(I) $\Delta^{n-1}x(t) > 0$ or (II) $\Delta^{n-1}x(t) < 0$ and $\Delta^{n-2}x(t) > 0$ for $t \ge t_2$.

If Case (I) holds, either we have (A) $\Delta x(t) > 0$, or (B) $\Delta x(t) < 0$ for $t \ge t_2$. By Lemma 2 and (A), we obtain (2.4) and (2.5) which leads to a contradiction. If (B) holds, we see that (2.7) holds, and as in the proof of Theorem 1, we have

$$x(t-m+1) \ge k^{n-1}\Delta^{n-1}(t-m+(n-1)k+1).$$

Thus,

$$\Delta\left(a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha}\right) \le -q(t)\left(k^{n-1}\Delta^{n-1}x(t-m+(n-1)k+1)\right)^{\beta}$$

or

$$\Delta W(t) \le -q(t)\left(\frac{k^{n-1}}{a^{1/\alpha}(t-m+(n-1)k+1)}\right)^{\beta} W^{\beta/\alpha}(t-m+(n-1)k+1),$$

where $W(t) = a(t)\left(\Delta^{n-1}x(t)\right)^{\alpha} > 0$. The rest of the proof is similar to that of the proof of Case I in Theorem 1 and hence is omitted. □

Next, we consider Case (II). By Lemma 2 and as in the proof of Theorem 1, we obtain (2.6). The rest of the proof is similar to that of Case (III) in Theorem 1 and so we omit the details. This proves the theorem.

Now let

$$P^*(t) \le \min\left\{q(t)\left(\frac{1}{2^{n-1}(n-1)!}\right)^{\beta}\left((t-m+1)^{(n-1)\beta}\, a^{-\beta/\alpha}(t-m+1)\right),\right.$$
$$\left.\left(\frac{1}{a(t)}\sum_{s=t_0}^{t-1}\left(\frac{1}{2^{n-2}(n-2)!}\right)^{\beta} q(s)\,(s-m+1)^{(n-2)\beta}\right)^{\frac{1}{\alpha}}, \frac{\left(k^{n-1}\right)^{\frac{\beta}{\alpha}}}{a^{1/\alpha}(t)}\left(\sum_{s=t_0}^{t-1} q(s)\right)^{\frac{1}{\alpha}}\right\}$$

and $\tau^* = \max\{m-(n-1)k-1, m-1\}$.

The following corollaries are analogous to those for the case where n is even.

Corollary 3. Let n be odd and k be a number such that $m > (n-1)k+1$. If the first order delay equation

$$\Delta y(t) + P^*(t)y^{\beta/\alpha}(t-\tau^*) = 0$$

is oscillatory, then Equation (1.1) is oscillatory.

Corollary 4. Let n be odd and k be a number with $m > (n-1)k+1$. If

$$\liminf_{t\to\infty}\sum_{s=t-m+1}^{t-1} P^*(s)\begin{cases} > \dfrac{(\tau^*-1)^{\tau^*-1}}{\tau^{*\tau^*}}, & \text{if } \beta=\alpha,\\ = \infty, & \text{if } \beta<\alpha,\end{cases}$$

then Equation (1.1) is oscillatory.

Remark 1. Clearly our results hold if $n = 2$, and they appear to be new in this case as well.

If we replace condition (1.2) by

$$\lim_{t\to\infty} A(t,t_0) = \infty, \tag{2.9}$$

it is easy to obtain the following results.

Theorem 3. *Let n be even, condition* (2.9) *hold, and there exists a number k with $m > (n-2)k+1$. If the first order Equation* (2.1) *is oscillatory, then Equation* (1.1) *is oscillatory.*

Theorem 4. *Let n be odd, condition* (2.9) *hold, and k be a number with $m > (n-1)k+1$. If the first order Equations* (2.1) *and* (2.8) *are oscillatory, then Equation* (1.1) *is oscillatory.*

As an example to illustrate our results, consider the difference equation

$$\Delta\left(\frac{1}{t^6}\left(\Delta^{n-1}x(t)\right)^3\right) + x^3(t-m+1) = 0. \tag{2.10}$$

It is easy to see that if n is even and $m > (n-2)k+1$ (n is odd and $m > (n-1)k+1$) the conditions of Corollary 2.2 (respectively Corollary 2.4) are satisfied and hence we conclude that Equation (2.10) is oscillatory.

We conclude this paper with a suggestion for future research, namely, to study the oscillatory behavior of solutions of Equation (1.1) in case $\beta > \alpha$.

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