

Models of Neutrino Masses and Mixings

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1 Introduction

At the School I gave three lectures on neutrino masses and mixings. Much of the material covered in my first two lectures is written down in a review on the subject that I published not long ago with F. Feruglio [1]. Moreover, there has been some (necessary and useful) overlap with other particularly related courses at this School, e.g. with [2], [3], [4] and with the experimental talks. Here, I make a relatively short summary (with updates) of the content of my first two lectures, referring to our review for a more detailed presentation, and then I expand on the content of the third lecture which was dedicated to recent work on A4 models of tri-bimaximal neutrino mixing which were not covered in the review.

By now there is convincing evidence for solar and atmospheric neutrino oscillations. The Δm^2 values and mixing angles are known with fair accuracy. A summary of the results, taken from Ref. [5] is shown in Table 1. For the Δm^2 we have: $\Delta m_{atm}^2 \sim 2.4 \cdot 10^{-3} eV^2$ and $\Delta m_{sol}^2 \sim 7.9 \cdot 10^{-5} eV^2$. As for the mixing angles, two are large and one is small. The atmospheric angle θ_{23} is large, actually compatible with maximal but not necessarily so: at 3σ : $0.29 \lesssim \sin^2 \theta_{23} \lesssim 0.71$ with central value around 0.44. The solar angle θ_{12} , the most precisely measured, is large, $\sin^2 \theta_{12} \sim 0.31$, but certainly not maximal (by about 6σ now). The third angle θ_{13} , strongly limited mainly by the CHOOZ experiment, has at present a 3σ upper limit given by about $\sin^2 \theta_{13} \lesssim 0.04$.

In spite of this experimental progress there are still many alternative routes in constructing models of neutrino masses. This variety is mostly due to the considerable ambiguities that remain. First of all, it is essential to know whether the LSND signal, which has not been confirmed by KARMEN and is currently being double-checked by MiniBoone, will be confirmed or will be excluded. If LSND is right we probably need at least four light neutrinos; if not we can do with only the three known ones, as we assume

Table 1. *Best fit values of squared mass differences and mixing angles[5]*

	lower limit (2σ)	best value	upper limit (2σ)
$(\Delta m_{sun}^2)_{LA}$ (10^{-5} eV ²)	7.2	7.9	8.6
Δm_{atm}^2 (10^{-3} eV ²)	1.8	2.4	2.9
$\sin^2 \theta_{12}$	0.27	0.31	0.37
$\sin^2 \theta_{23}$	0.34	0.44	0.62
$\sin^2 \theta_{13}$	0	0.009	0.032

here in the following. Then, as neutrino oscillations only determine mass squared differences a crucial missing input is the absolute scale of neutrino masses. Also the pattern of the neutrino mass spectrum is not known: it could be approximately degenerate with $m^2 \gg \Delta m_{ij}^2$, or of the inverse hierarchy type (with the 1,2 solar doublet on top) or of the normal hierarchy type (with the solar doublet below).

The following experimental information on the absolute scale of neutrino masses is available. From the endpoint of tritium beta decay spectrum we have an absolute upper limit of 2 eV (at 95% C.L.) on the mass of " $\bar{\nu}_e$ " [6], [7], which, combined with the observed oscillation frequencies under the assumption of three CPT-invariant light neutrinos, represents also an upper bound on the masses of all active neutrinos. A less direct information on the mass scale is obtained from neutrinoless double beta decay ($0\nu\beta\beta$) [8]. The discovery of $0\nu\beta\beta$ decay would be very important because it would directly establish lepton number violation and the Majorana nature of ν 's. The present limit from $0\nu\beta\beta$ is affected by a relatively large uncertainty due to ambiguities on nuclear matrix elements. We quote here two recent limits (90% c.l.): $|m_{ee}| < 0.33 - 1.35$ eV [IGEX(⁷⁶Ge) [9]] or $|m_{ee}| < (0.2 - 1.1)$ eV [Cuoricino(¹³⁰Te) [10]], where $m_{ee} = \sum U_{ei}^2 m_i$ in terms of the mixing matrix and the mass eigenvalues (see eq.(5)). Complementary information on the sum of neutrino masses is also provided by measurements in cosmology [3], where an extraordinary progress has been made in the last years, in particular data on the cosmic microwave background (CMB) anisotropies (WMAP), on the large scale structure of the mass distribution in the Universe (SDSS, 2dFGRS) and from the Lyman alpha forest [11]. WMAP by itself is not very restrictive: $\sum_i |m_i| < 2.11$ eV (at 95% C.L.). Combining CMB data with those on the large scale structure one obtains $\sum_i |m_i| < 0.68$ eV. Adding also the data from the Lyman alpha forest one has $\sum_i |m_i| < 0.17$ eV [12]. But this last combination is questionable because of some tension (at $\sim 2\sigma$'s) between the Lyman alpha forest data and those on the large scale structure. In any case, the cosmological bounds depend on a number of assumptions (or, in fashionable terms, priors) on the cosmological model. In summary, from cosmology for 3 degenerate neutrinos of mass m , depending on which data sets we include and on our degree of confidence in cosmological models, we can conclude that $m \lesssim 0.06 - 0.23 - 0.7$ eV.

Given that neutrino masses are certainly extremely small, it is really difficult from the theory point of view to avoid the conclusion that L conservation is probably violated.

In fact, in terms of lepton number violation the smallness of neutrino masses can be naturally explained as inversely proportional to the very large scale where L is violated, of order M_{GUT} or even M_{Pl} . If neutrinos are Majorana particles, their masses arise from the generic dimension-five non renormalizable operator of the form:

$$O_5 = \frac{(Hl)_i^T \lambda_{ij} (Hl)_j}{M} + h.c. \quad , \quad (1)$$

with H being the ordinary Higgs doublet, l_i the $SU(2)$ lepton doublets, λ a matrix in flavour space, M a large scale of mass and a charge conjugation matrix C between the lepton fields is understood. Neutrino masses generated by O_5 are of the order $m_\nu \approx v^2/M$ for $\lambda_{ij} \approx O(1)$, where $v \sim O(100 \text{ GeV})$ is the vacuum expectation value of the ordinary Higgs. A particular realization leading to comparable masses is the see-saw mechanism, where M derives from the exchange of heavy ν_R 's: the resulting neutrino mass matrix reads:

$$m_\nu = m_D^T M^{-1} m_D \quad . \quad (2)$$

that is, the light neutrino masses are quadratic in the Dirac masses and inversely proportional to the large Majorana mass. For $m_\nu \approx \sqrt{\Delta m_{atm}^2} \approx 0.05 \text{ eV}$ and $m_\nu \approx m_D^2/M$ with $m_D \approx v \approx 200 \text{ GeV}$ we find $M \approx 10^{15} \text{ GeV}$ which indeed is an impressive indication for M_{GUT} . Thus probably neutrino masses are a probe into the physics at M_{GUT} .

2 The ν -Mixing Matrix

If we take maximal s_{23} ($s_{ij} = \sin \theta_{ij}$) and keep only linear terms in $u = s_{13} e^{i\varphi}$, from experiment we find the following structure of the mixing matrix U_{fi} ($f = e, \mu, \tau, i = 1, 2, 3$), apart from sign convention redefinitions:

$$U_{fi} = \begin{pmatrix} c_{12} & s_{12} & u \\ -(s_{12} + c_{12}u^*)/\sqrt{2} & (c_{12} - s_{12}u^*)/\sqrt{2} & 1/\sqrt{2} \\ (s_{12} - c_{12}u^*)/\sqrt{2} & -(c_{12} + s_{12}u^*)/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \quad (3)$$

If s_{13} would be exactly zero there would be no CP violations in ν oscillations. A main target of the new planned oscillation experiments is to measure the actual size of s_{13} . In the next decade the upper limit on $\sin^2 2\theta_{13}$ will possibly go down by at least an order of magnitude (T2K, $\text{No}\nu\text{A}$, DoubleCHOOZ) [13]. Even for three neutrinos the pattern of the neutrino mass spectrum is still undetermined: it can be approximately degenerate, or of the inverse hierarchy type or normally hierarchical. Given the observed frequencies and the notation $\Delta m_{sun}^2 \equiv \Delta m_{12}^2$, $\Delta m_{atm}^2 \equiv |\Delta m_{23}^2|$ with $\Delta m_{12}^2 = |m_2|^2 - |m_1|^2 > 0$ and $\Delta m_{23}^2 = m_3^2 - |m_2|^2$, the three possible patterns of mass eigenvalues are:

$$\begin{aligned} \text{Degenerate} & : |m_1| \sim |m_2| \sim |m_3| \gg |m_i - m_j| \\ \text{Inverted hierarchy} & : |m_1| \sim |m_2| \gg |m_3| \\ \text{Normal hierarchy} & : |m_3| \gg |m_{2,1}| \end{aligned} \quad (4)$$

The sign of Δm_{23}^2 can be measured in the future through matter effects in long baseline experiments [13]. Models based on all these patterns have been proposed and studied and all are in fact viable at present.

The detection of neutrino-less double beta decay, besides its enormous intrinsic importance as a direct evidence of L non conservation, would also offer a way to possibly disentangle the 3 cases. The quantity which is bound by experiments is the 11 entry of the ν mass matrix, which in general, from $m_\nu = U^* m_{diag} U^\dagger$, is given by :

$$|m_{ee}| = |(1 - s_{13}^2) (m_1 c_{12}^2 + m_2 s_{12}^2) + m_3 e^{2i\phi} s_{13}^2| \quad (5)$$

Starting from this general formula it is simple to derive the following bounds for degenerate, inverse hierarchy or normal hierarchy mass patterns.

- a) Degenerate case. If $|m|$ is the common mass and we set $s_{13} = 0$, which is a safe approximation in this case, because $|m_3|$ cannot compensate for the smallness of s_{13} , we have $m_{ee} \sim |m|(c_{12}^2 \pm s_{12}^2)$. Here the phase ambiguity has been reduced to a sign ambiguity which is sufficient for deriving bounds. So, depending on the sign we have $m_{ee} = |m|$ or $m_{ee} = |m|\cos 2\theta_{12}$. We conclude that in this case m_{ee} could be as large as the present experimental limit but should be at least of order $O(\sqrt{\Delta m_{atm}^2}) \sim O(10^{-2} \text{ eV})$ given that the solar angle cannot be too close to maximal (in which case the minus sign option could be arbitrarily small). The experimental 2- σ range of the solar angle does not favour a cancellation by more than a factor of about 3.
- b) Inverse hierarchy case. In this case the same approximate formula $m_{ee} = |m|(c_{12}^2 \pm s_{12}^2)$ holds because m_3 is small and the s_{13} term in eq.(5) can be neglected. The difference is that here we know that $|m| \approx \sqrt{\Delta m_{atm}^2}$ so that $|m_{ee}| < \sqrt{\Delta m_{atm}^2} \sim 0.05 \text{ eV}$. At the same time, since a full cancellation between the two contributions cannot take place, we expect $|m_{ee}| > 0.01 \text{ eV}$.
- c) Normal hierarchy case. Here we cannot in general neglect the m_3 term. However in this case $|m_{ee}| \sim |\sqrt{\Delta m_{sun}^2} s_{12}^2 \pm \sqrt{\Delta m_{atm}^2} s_{13}^2|$ and we have the bound $|m_{ee}| < a \text{ few } 10^{-3} \text{ eV}$.

Recently some evidence for $0\nu\beta\beta$ was claimed [14] corresponding to $|m_{ee}| \sim (0.2 \div 0.6) \text{ eV}$ ($(0.1 \div 0.9) \text{ eV}$ in a more conservative estimate of the involved nuclear matrix elements). This result is not supported by the IGEX and Cuoricino measurements of a comparable sensitivity, but If confirmed it would rule out cases b) and c) and point to case a) or to models with more than 3 neutrinos. In the next few years a new generation of experiments will reach a larger sensitivity on $0\nu\beta\beta$ by about an order of magnitude [8]. If these experiments will observe a signal this would indicate that the inverse hierarchy is realized, if not, then the normal hierarchy case remains a possibility.

3 "Normal" versus "Exceptional" Models

After KamLAND, SNO and WMAP not too much hierarchy in neutrino masses is indicated by experiments:

$$r = \Delta m_{sol}^2 / \Delta m_{atm}^2 \sim 1/30. \quad (6)$$

Precisely at 2σ : $0.025 \lesssim r \lesssim 0.049$ [5]. Thus, for a hierarchical spectrum, $m_2/m_3 \sim \sqrt{r} \sim 0.2$, which is comparable to the Cabibbo angle $\lambda_C \sim 0.22$ or $\sqrt{m_\mu/m_\tau} \sim 0.24$. This suggests that the same hierarchy parameter (raised to powers with $o(1)$ exponents) applies for quark, charged lepton and neutrino mass matrices. This in turn indicates that, in absence of some special dynamical reason, we do not expect a quantity like θ_{13} to be too small. Indeed it would be very important to know how small the mixing angle θ_{13} is and how close to maximal is θ_{23} . Actually one can make a distinction between "normal" and "exceptional" models. For normal models θ_{23} is not too close to maximal and θ_{13} is not too small, typically a small power of the self-suggesting order parameter \sqrt{r} , with $r = \Delta m_{sol}^2/\Delta m_{atm}^2 \sim 1/30$. Exceptional models are those where some symmetry or dynamical feature assures in a natural way the near vanishing of θ_{13} and/or of $\theta_{23} - \pi/4$. Normal models are conceptually more economical and much simpler to construct. Typical categories of normal models are:

- a) Anarchy. These are models with approximately degenerate mass spectrum and no ordering principle, no approximate symmetry assumed in the neutrino mass sector [15] [1]. The small value of r is accidental, due to random fluctuations of matrix elements in the Dirac and Majorana neutrino mass matrices. Starting from a random input for each matrix element, the see-saw formula, being a product of 3 matrices, generates a broad distribution of r values. All mixing angles are generically large: so in this case one does not expect θ_{23} to be maximal and θ_{13} must probably be found near its upper bound.
- b) Semianarchy. We have seen that anarchy is the absence of structure in the neutrino sector. Here we consider an attenuation of anarchy where the absence of structure is limited to the 23 sector. The typical structure is in this case [16] [1]:

$$m_\nu \approx m \begin{pmatrix} \delta & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}, \quad (7)$$

where δ and ϵ are small and by 1 we mean entries of $o(1)$ and also the 23 determinant is of $o(1)$. This texture can be realized, for example, without see-saw from a suitable set of $U(1)_F$ charges for (l_1, l_2, l_3) , eg $(a, 0, 0)$ appearing in the dim. 5 operator of eq.(1). Clearly, in general we would expect two mass eigenvalues of order 1, in units of m , and one small, of order δ or ϵ^2 . This typical pattern would not fit the observed solar and atmospheric observed frequencies. However, given that \sqrt{r} is not too small, we can assume that its small value is generated accidentally, as for anarchy. We see that, if by chance the second eigenvalue $\eta \sim \sqrt{r} \sim \delta + \epsilon^2$, we can then obtain the correct value of r together with large but in general non maximal θ_{23} and θ_{12} and small $\theta_{13} \sim \epsilon$. The guaranteed smallness of θ_{13} is the main advantage over anarchy, and the relation with \sqrt{r} normally keeps θ_{13} not too small. For example, $\delta \sim \epsilon^2$ in typical $U(1)_F$ models that provide a very economical but effective realization of this scheme .

- c) Inverse hierarchy. One obtains inverted hierarchy, for example, in the limit of exact $L_e - L_\mu - L_\tau$ symmetry with $r = 0$ and bi-maximal mixing (both θ_{12} and θ_{23} are maximal) [1]. Simple forms of symmetry breaking cannot sufficiently displace θ_{12}

from the maximal value because typically $\tan^2 \theta_{12} \sim 1 + o(r)$. Viable normal models can be obtained by arranging large contributions to θ_{23} and θ_{12} from the charged lepton mass diagonalization. But then, it turns out that, in order to obtain the measured value of θ_{12} , the size of θ_{13} must be close to its present upper bound [17]. If indeed the shift from maximal θ_{12} is due to the charged lepton diagonalization, this could offer a possible track to explain the empirical Roldal relation $\theta_{12} + \theta_C = \pi/4$ [18] (with present data $\theta_{12} + \theta_C = (47.0 + 1.7 - 1.6)^\circ$). While it would not be difficult in this case to arrange that the shift from maximal is of the order of θ_C , it is not clear how to guarantee that it is precisely equal to θ_C [19]. Besides the effect of the charged lepton diagonalization, in a see-saw context, one can assume a strong additional breaking of $L_e - L_\mu - L_\tau$ from soft terms in the M_{RR} Majorana mass matrix [20]. Since ν_R 's are gauge singlets and thus essentially uncoupled, a large breaking in M_{RR} does not feedback in other sectors of the lagrangian. In this way one can obtain realistic values for θ_{12} and for all other masses and mixings, in particular also with a small θ_{13} .

- d) Normal hierarchy. Particularly interesting are models with 23 determinant suppressed by see-saw [1]: in the 23 sector one needs relatively large mass splittings to fit the small value of r but nearly maximal mixing. This can be obtained if the 23 sub-determinant is suppressed by some dynamical trick. Typical examples are lop-sided models with large off diagonal term in the Dirac matrices of charged leptons and/or neutrinos (in minimal SU(5) the d-quark and charged lepton mass matrices are one the transposed of the other, so that large left-handed mixings for charged leptons correspond to large unobservable right-handed mixings for d-quarks). Another class of typical examples is the dominance in the see-saw formula of a small eigenvalue in M_{RR} , the right-handed Majorana neutrino mass matrix. When the 23 determinant suppression is implemented in a 33 context, normally θ_{13} is not protected from contributions that vanish with the 23 determinant, hence with r .

The fact that some neutrino mixing angles are large and even nearly maximal, while surprising at the start, was eventually found to be well compatible with a unified picture of quark and lepton masses within GUTs. The symmetry group at M_{GUT} could be either (SUSY) SU(5) or SO(10) or a larger group. For example, normal models based on anarchy, semianarchy, inverted hierarchy or normal hierarchy can all be naturally implemented by simple assignments of $U(1)_F$ horizontal charges in a semiquantitative unified description of all quark and lepton masses in SUSY SU(5) \times $U(1)_F$. Actually, in this context, if one adopts a statistical criterium, hierarchical models appear to be preferred over anarchy and among them normal hierarchy with see-saw ends up as being the most likely [21].

In conclusion we expect that experiment will eventually find that θ_{13} is not too small and that θ_{23} is sizably not maximal. But if, on the contrary, either θ_{13} very small or θ_{23} very close to maximal will emerge from experiment or both, then theory will need to cope with this fact. Normal models have been extensively discussed in the literature [1], so we concentrate here on examples of exceptional models.

4 Tri-bimaximal Mixing

Here we want to discuss some particular exceptional models where both θ_{13} and $\theta_{23} - \pi/4$ exactly vanish (more precisely, they vanish in a suitable limit, with correction terms that can be made negligibly small) and, in addition, $s_{12} \sim 1/\sqrt{3}$, a value which is in very good agreement with present data. This is the so-called tri-bimaximal or Harrison-Perkins-Scott mixing pattern (HPS) [22], with the entries in the second column all equal to $1/\sqrt{3}$ in absolute value. Here we adopt the following phase convention:

$$U_{HPS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (8)$$

In the HPS scheme $\tan^2 \theta_{12} = 0.5$, to be compared with the latest experimental determination in Table 1: $\tan^2 \theta_{12} = 0.45_{-0.04}^{+0.07}$ (at 1σ). The challenge is to find natural and appealing schemes that lead to this matrix with good accuracy. Clearly, in a natural realization of this model, a very constraining and predictive dynamics must be underlying. It is interesting to explore particular structures giving rise to this very special set of models in a natural way. In this case we have a maximum of "order" implying special values for all mixing angles. Interesting ideas on how to obtain the HPS mixing matrix have been discussed in refs. [22], [23], [24]. Some attractive models are based on the discrete symmetry A4, which appears as particularly suitable for the purpose, and were presented in ref. [25],[26],[27], [28],[29], [30].

The HPS mixing matrix suggests that mixing angles are independent of mass ratios (while for quark mixings relations like $\lambda_C^2 \sim m_d/m_s$ are typical). In fact in the basis where charged lepton masses are diagonal, the effective neutrino mass matrix in the HPS case is given by $m_\nu = U_{HPS} \text{diag}(m_1, m_2, m_3) U_{HPS}^T$:

$$m_\nu = \left[\frac{m_3}{2} M_3 + \frac{m_2}{3} M_2 + \frac{m_1}{6} M_1 \right]. \quad (9)$$

where:

$$M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}. \quad (10)$$

The eigenvalues of m_ν are m_1, m_2, m_3 with eigenvectors $(-2, 1, 1)/\sqrt{6}$, $(1, 1, 1)/\sqrt{3}$ and $(0, 1, -1)/\sqrt{2}$, respectively. In general, disregarding possible Majorana phases, there are six parameters in a real symmetric matrix like m_ν : here only three are left after the values of the three mixing angles have been fixed à la HPS. For a hierarchical spectrum $m_3 \gg m_2 \gg m_1$, $m_3^2 \sim \Delta m_{atm}^2$, $m_2^2/m_3^2 \sim \Delta m_{sol}^2/\Delta m_{atm}^2$ and m_1 could be negligible. But also degenerate masses and inverse hierarchy can be reproduced: for example, by taking $m_3 = -m_2 = m_1$ we have a degenerate model, while for $m_1 = -m_2$ and $m_3 = 0$ an inverse hierarchy case is realized (stability under renormalization group running strongly prefers opposite signs for the first and the second eigenvalue which are related to solar

oscillations and have the smallest mass squared splitting). From the general expression of the eigenvectors one immediately sees that this mass matrix, independent of the values of m_i , leads to the HPS mixing matrix.

It is interesting to recall that the most general mass matrix, in the basis where charged leptons are diagonal, that corresponds to $\theta_{13} = 0$ and θ_{23} maximal is of the form [31]:

$$m = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \quad (11)$$

Note that this matrix is symmetric under 2-3 or $\mu - \tau$ exchange. It is however not easy to make a model where the $\mu - \tau$ applies to the whole lepton sector [32]. Imposing the symmetry on $l^T m_\nu l$ does not work because the Dirac mass term $l^c m_D l$ then produces a charged lepton mixing that completely spoils θ_{23} maximal. For example, in the model [33], the $\mu - \tau$ symmetry is badly broken in the charged lepton mass sector and, as a result, for parameter choices that fit the masses, θ_{23} is not necessarily close to maximal and θ_{13} is not too small: finally the model looks like a "normal" model! Similarly a symmetry $\nu_{\mu R} - \nu_{\tau R}$ in the RH neutrino sector does not lead to a $\mu - \tau$ symmetric neutrino mass after see-saw. A more elaborate broken symmetry is needed, like a set of discrete broken symmetries that make the charged lepton mass matrix diagonal and, at the same time, the Dirac neutrino mass matrix diagonal and $\mu - \tau$ symmetric and finally the permutational 2 - 3 symmetry is in the RR Majorana mass matrix. Thus the idea of a "simple" $\mu - \tau$ symmetry ends up with leading to complicated models.

For $\theta_{13} = 0$ there is no CP violation, so that, disregarding Majorana phases, we can restrict our consideration to real parameters. There are four of them in eq.(11) which correspond to three mass eigenvalues and one remaining mixing angle, θ_{12} . In particular, θ_{12} is given by:

$$\sin^2 2\theta_{12} = \frac{8y^2}{(x - w - z)^2 + 8y^2} \quad (12)$$

In the HPS case θ_{12} is also fixed and an additional parameter, for example x , can be eliminated, leading to:

$$m = \begin{pmatrix} z + w - y & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \quad (13)$$

It is easy to see that the HPS mass matrix in eqs.(9-10) is indeed of the form in eq.(13).

In the next sections we will present models of tri-bimaximal mixing based on the A4 group. We first introduce A4 and its representations and then we show that this group is particularly suited to the problem.

5 The A4 Group

A4 is the group of the even permutations of 4 objects. It has $4!/2=12$ elements. Geometrically, it can be seen as the invariance group of a tetrahedron (the odd permutations, for example the exchange of two vertices, cannot be obtained by moving a rigid solid). Let us denote a generic permutation $(1, 2, 3, 4) \rightarrow (n_1, n_2, n_3, n_4)$ simply by $(n_1 n_2 n_3 n_4)$. A4 can

Table 2. Characters of A_4

Class	χ^1	$\chi^{1'}$	$\chi^{1''}$	χ^3
C_1	1	1	1	3
C_2	1	ω	ω^2	0
C_3	1	ω^2	ω	0
C_4	1	1	1	-1

be generated by two basic permutations S and T given by $S = (4321)$ and $T = (2314)$. One checks immediately that:

$$S^2 = T^3 = (ST)^3 = 1 \quad (14)$$

This is called a "presentation" of the group. The 12 even permutations belong to 4 equivalence classes (h and k belong to the same class if there is a g in the group such that $ghg^{-1} = k$) and are generated from S and T as follows:

$$\begin{aligned}
 C1 & : I = (1234) & (15) \\
 C2 & : T = (2314), ST = (4132), TS = (3241), STS = (1423) \\
 C3 & : T^2 = (3124), ST^2 = (4213), T^2S = (2431), TST = (1342) \\
 C4 & : S = (4321), T^2ST = (3412), TST^2 = (2143)
 \end{aligned}$$

Note that, except for the identity I which always forms an equivalence class in itself, the other classes are according to the powers of T (in $C4$ S could as well be seen as ST^3).

In a finite group the squared dimensions of the inequivalent irreducible representations add up to N , the number of transformations in the group ($N = 12$ in A_4). A_4 has four inequivalent representations: three of dimension one, 1, 1' and 1'' and one of dimension 3. It is immediate to see that the one-dimensional unitary representations are obtained by:

$$\begin{aligned}
 1 & \quad S = 1 \quad T = 1 & (16) \\
 1' & \quad S = 1 \quad T = e^{i2\pi/3} \equiv \omega \\
 1'' & \quad S = 1 \quad T = e^{i4\pi/3} \equiv \omega^2
 \end{aligned}$$

Note that $\omega = -1/2 + \sqrt{3}/2$ is the cubic root of 1 and satisfies $\omega^2 = \omega^*$, $1 + \omega + \omega^2 = 0$.

The three-dimensional unitary representation, in a basis where the element S is diagonal, is built up from:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (17)$$

The characters of a group χ_g^R are defined, for each element g , as the trace of the matrix that maps the element in a given representation R . It is easy to see that equivalent

representations have the same characters and that characters have the same value for all elements in an equivalence class. Characters satisfy $\sum_g \chi_g^R \chi_g^{S*} = N\delta^{RS}$. Also, for each element h , the character of h in a direct product of representations is the product of the characters: $\chi_h^{R \otimes S} = \chi_h^R \chi_h^S$ and also is equal to the sum of the characters in each representation that appears in the decomposition of $R \otimes S$. The character table of A_4 is given in Table II [25]. From this Table one derives that indeed there are no more inequivalent irreducible representations other than $1, 1', 1''$ and 3 . Also, the multiplication rules are clear: the product of two 3 gives $3 \times 3 = 1 + 1' + 1'' + 3 + 3$ and $1' \times 1' = 1''$, $1' \times 1'' = 1$, $1'' \times 1'' = 1'$ etc. If $3 \sim (a_1, a_2, a_3)$ is a triplet transforming by the matrices in eq.(17) we have that under S : $S(a_1, a_2, a_3)^t = (a_1, -a_2, -a_3)^t$ (here the upper index t indicates transposition) and under T : $T(a_1, a_2, a_3)^t = (a_2, a_3, a_1)^t$. Then, from two such triplets $3_a \sim (a_1, a_2, a_3)$, $3_b \sim (b_1, b_2, b_3)$ the irreducible representations obtained from their product are:

$$1 = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (18)$$

$$1' = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \quad (19)$$

$$1'' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \quad (20)$$

$$3 \sim (a_2 b_3, a_3 b_1, a_1 b_2) \quad (21)$$

$$3 \sim (a_3 b_2, a_1 b_3, a_2 b_1) \quad (22)$$

In fact, take for example the expression for $1'' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3$. Under S it is invariant and under T it goes into $a_2 b_2 + \omega a_3 b_3 + \omega^2 a_1 b_1 = \omega^2 [a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3]$ which is exactly the transformation corresponding to $1''$.

In eq.(17) we have the representation 3 in a basis where S is diagonal. It is interesting to go to a basis where instead it is T which is diagonal. This is obtained through the unitary transformation:

$$T' = V T V^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (23)$$

$$S' = V S V^\dagger = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \quad (24)$$

where:

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \quad (25)$$

The matrix V is special in that it is a 3×3 unitary matrix with all entries of unit absolute value. It is interesting that this matrix was proposed long ago as a possible mixing matrix for neutrinos [34]. We shall see in the following that the matrix V appears in A_4 models as the unitary transformation that diagonalizes the charged lepton mass matrix.

An obvious representation of A_4 is obtained by considering the 4×4 matrices that directly realize each permutation. For $S = (4321)$ and $T = (2314)$ we have:

$$S_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (26)$$

The matrices S_4 and T_4 satisfy the relations (14), thus providing a representation of A_4 . Since the only irreducible representations of A_4 are a triplet and three singlets, the 4×4 representation described by S_4 and T_4 is not irreducible. It decomposes into the sum of the invariant singlet plus the triplet representation. This decomposition is realized by the unitary matrix [30] U given by:

$$U = \frac{1}{2} \begin{pmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \end{pmatrix} . \quad (27)$$

This matrix maps S_4 and T_4 into matrices that are block-diagonal:

$$US_4U^\dagger = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & S \end{array} \right) , \quad UT_4U^\dagger = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & T \end{array} \right) , \quad (28)$$

where S and T are the generators of the three-dimensional representation in eq.(17).

There is an interesting relation [29] between the A_4 model considered so far and the modular group. This relation could possibly be relevant to understand the origin of the A_4 symmetry from a more fundamental layer of the theory. The modular group Γ is the group of linear fractional transformations acting on a complex variable z :

$$z \rightarrow \frac{az + b}{cz + d} , \quad ad - bc = 1 , \quad (29)$$

where a, b, c, d are integers. There are infinite elements in Γ , but all of them can be generated by the two transformations:

$$s : z \rightarrow -\frac{1}{z} , \quad t : z \rightarrow z + 1 , \quad (30)$$

The transformations s and t in (30) satisfy the relations

$$s^2 = (st)^3 = 1 \quad (31)$$

and, conversely, these relations provide an abstract characterization of the modular group. Since the relations (14) are a particular case of the more general constraint (31), it is clear that A_4 is a very small subgroup of the modular group and that the A_4 representations discussed above are also representations of the modular group. In string theory the transformations (30) operate in many different contexts. For instance the role of the complex variable z can be played by a field, whose VEV can be related to a physical quantity like a compactification radius or a coupling constant. In that case s in eq. (30) represents a duality transformation and t in eq. (30) represent the transformation associated to an "axionic" symmetry.

A different way to understand the dynamical origin of A_4 was recently presented in ref. [30] where it is shown that the A_4 symmetry can be simply obtained by orbifolding starting from a model in 6 dimensions (6D). In this approach A_4 appears as the remnant

of the reduction from 6D to 4D space-time symmetry induced by the special orbifolding adopted. There are 4D branes at the four fixed points of the orbifolding and the tetrahedral symmetry of A_4 connects these branes. The standard model fields have components on the fixed point branes while the scalar fields necessary for the A_4 breaking are in the bulk. Each brane field, either a triplet or a singlet, has components on all of the four fixed points (in particular all components are equal for a singlet) but the interactions are local, i.e. all vertices involve products of field components at the same space-time point. This approach suggests a deep relation between flavour symmetry in 4D and space-time symmetry in extra dimensions.

The orbifolding is defined as follows. We consider a quantum field theory in 6 dimensions, with two extra dimensions compactified on an orbifold T^2/Z_2 . We denote by $z = x_5 + ix_6$ the complex coordinate describing the extra space. The torus T^2 is defined by identifying in the complex plane the points related by

$$\begin{aligned} z &\rightarrow z + 1 \\ z &\rightarrow z + \gamma \end{aligned} \quad \gamma = e^{i\frac{\pi}{3}} \quad , \quad (32)$$

where our length unit, $2\pi R$, has been set to 1 for the time being. The parity Z_2 is defined by

$$z \rightarrow -z \quad (33)$$

and the orbifold T^2/Z_2 can be represented by the fundamental region given by the triangle with vertices $0, 1, \gamma$, see Fig. 1. The orbifold has four fixed points, $(z_1, z_2, z_3, z_4) = (1/2, (1 + \gamma)/2, \gamma/2, 0)$. The fixed point z_4 is also represented by the vertices 1 and γ . In the orbifold, the segments labelled by a in Fig. 1, $(0, 1/2)$ and $(1, 1/2)$, are identified and similarly for those labelled by b , $(1, (1 + \gamma)/2)$ and $(\gamma, (1 + \gamma)/2)$, and those labelled by c , $(0, \gamma/2)$, $(\gamma, \gamma/2)$. Therefore the orbifold is a regular tetrahedron with vertices at the four fixed points.

The symmetry of the uncompactified 6D space time is broken by compactification. Here we assume that, before compactification, the space-time symmetry coincides with the product of 6D translations and 6D proper Lorentz transformations. The compactification breaks part of this symmetry. However, due to the special geometry of our orbifold, a discrete subgroup of rotations and translations in the extra space is left unbroken. This group can be generated by two transformations:

$$\begin{aligned} \mathcal{S} : z &\rightarrow z + \frac{1}{2} \\ \mathcal{T} : z &\rightarrow \omega z \end{aligned} \quad \omega \equiv \gamma^2 \quad . \quad (34)$$

Indeed \mathcal{S} and \mathcal{T} induce even permutations of the four fixed points:

$$\begin{aligned} \mathcal{S} : (z_1, z_2, z_3, z_4) &\rightarrow (z_4, z_3, z_2, z_1) \\ \mathcal{T} : (z_1, z_2, z_3, z_4) &\rightarrow (z_2, z_3, z_1, z_4) \end{aligned} \quad , \quad (35)$$

thus generating the group A_4 . From the previous equations we immediately verify that \mathcal{S} and \mathcal{T} satisfy the characteristic relations obeyed by the generators of A_4 : $\mathcal{S}^2 = \mathcal{T}^3 = (\mathcal{ST})^3 = 1$. These relations are actually satisfied not only at the fixed points, but on the whole orbifold, as can be easily checked from the general definitions of \mathcal{S} and \mathcal{T} in eq. (34), with the help of the orbifold defining rules in eqs. (32) and (33).

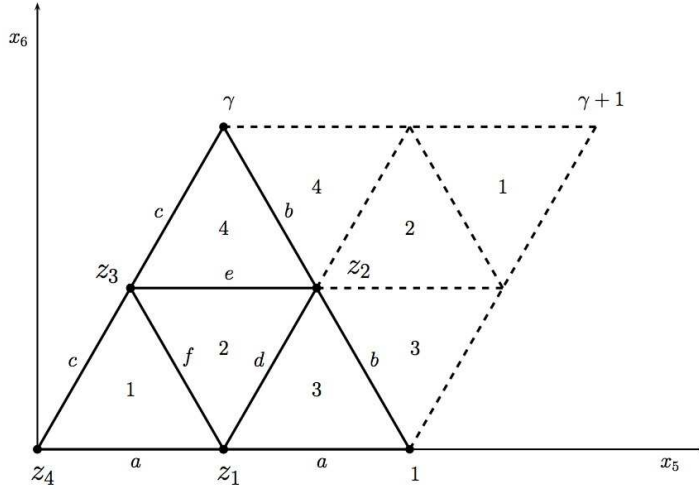


Figure 1. Orbifold T_2/Z_2 . The regions with the same numbers are identified with each other. The four triangles bounded by solid lines form the fundamental region, where also the edges with the same letters are identified. The orbifold T_2/Z_2 is exactly a regular tetrahedron with 6 edges a, b, c, d, e, f and four vertices z_1, z_2, z_3, z_4 , corresponding to the four fixed points of the orbifold.

6 Applying A4 to Lepton Masses and Mixings

A typical A4 model works as follows [28], [29]. One assigns leptons to the four inequivalent representations of A4: left-handed lepton doublets l transform as a triplet 3, while the right-handed charged leptons e^c , μ^c and τ^c transform as 1, $1'$ and $1''$, respectively. At this stage we do not introduce RH neutrinos, but later we will discuss a see-saw realization. The flavour symmetry is broken by two real triplets φ and φ' and by a real singlet ξ . These flavon fields are all gauge singlets. We also need one or two ordinary SM Higgs doublets $h_{u,d}$, which we take invariant under A4. The Yukawa interactions in the lepton sector read:

$$\begin{aligned} \mathcal{L}_Y &= y_e e^c(\varphi l) + y_\mu \mu^c(\varphi l)'' + y_\tau \tau^c(\varphi l)' \\ &+ x_a \xi(ll) + x_d(\varphi' ll) + h.c. + \dots \end{aligned} \quad (36)$$

In our notation, (33) transforms as 1, (33)' transforms as $1'$ and (33)'' transforms as $1''$. Also, to keep our notation compact, we use a two-component notation for the fermion fields and we set to 1 the Higgs fields $h_{u,d}$ and the cut-off scale Λ . For instance $y_e e^c(\varphi l)$ stands for $y_e e^c(\varphi l) h_d / \Lambda$, $x_a \xi(ll)$ stands for $x_a \xi(lh_u lh_u) / \Lambda^2$ and so on. The Lagrangian \mathcal{L}_Y contains the lowest order operators in an expansion in powers of $1/\Lambda$. Dots stand for higher dimensional operators that will be discussed later. Some terms allowed by the flavour symmetry, such as the terms obtained by the exchange $\varphi' \leftrightarrow \varphi$, or the term (ll) are missing in \mathcal{L}_Y . Their absence is crucial and, in each version of A4 models, is motivated by additional symmetries. For example (ll) , being of lower dimension with respect to $(\varphi' ll)$, would be the dominant component, proportional to the identity, of the neutrino mass matrix. In addition to that, the presence of the singlet flavon ξ plays an important

role in making the VEV directions of φ and φ' different.

For the model to work it is essential that the fields φ' , φ and ξ develop a VEV along the directions:

$$\begin{aligned}\langle\varphi'\rangle &= (v', 0, 0) \\ \langle\varphi\rangle &= (v, v, v) \\ \langle\xi\rangle &= u \quad .\end{aligned}\tag{37}$$

A crucial part of all serious A4 models is the dynamical generation of this alignment in a natural way. If the alignment is realized, at the leading order of the $1/\Lambda$ expansion, the mass matrices m_l and m_ν for charged leptons and neutrinos are given by:

$$m_l = v_d \frac{v}{\Lambda} \begin{pmatrix} y_e & y_e & y_e \\ y_\mu & y_\mu \omega^2 & y_\mu \omega \\ y_\tau & y_\tau \omega & y_\tau \omega^2 \end{pmatrix} ,\tag{38}$$

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a & 0 & 0 \\ 0 & a & d \\ 0 & d & a \end{pmatrix} ,\tag{39}$$

where

$$a \equiv x_a \frac{u}{\Lambda} \quad , \quad d \equiv x_d \frac{v'}{\Lambda} \quad .\tag{40}$$

Charged leptons are diagonalized by the matrix

$$l \rightarrow Vl = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} l \quad ,\tag{41}$$

This matrix was already introduced in eq.(25) as the unitary transformation between the S -diagonal to the T -diagonal 3x3 representation of A4. In fact, in this model, the S -diagonal basis is the Lagrangian basis and the T diagonal basis is that of diagonal charged leptons. The great virtue of A4 is to immediately produce the special unitary matrix V as the diagonalizing matrix of charged leptons and also to allow a singlet made up of three triplets, $(\phi' ll) = \phi'_1 l_2 l_3 + \phi'_2 l_3 l_1 + \phi'_3 l_1 l_2$ which leads, for the alignment in eq. (37), to the right neutrino mass matrix to finally obtain the HPS mixing matrix.

The charged fermion masses are given by:

$$m_e = \sqrt{3} y_e v_d \frac{v}{\Lambda} \quad , \quad m_\mu = \sqrt{3} y_\mu v_d \frac{v}{\Lambda} \quad , \quad m_\tau = \sqrt{3} y_\tau v_d \frac{v}{\Lambda} \quad .\tag{42}$$

We can easily obtain in a a natural way the observed hierarchy among m_e , m_μ and m_τ by introducing an additional $U(1)_F$ flavour symmetry under which only the right-handed lepton sector is charged. We assign F-charges 0, 2 and $3 \div 4$ to τ^c , μ^c and e^c , respectively. By assuming that a flavon θ , carrying a negative unit of F, acquires a VEV $\langle\theta\rangle/\Lambda \equiv \lambda < 1$, the Yukawa couplings become field dependent quantities $y_{e,\mu,\tau} = y_{e,\mu,\tau}(\theta)$ and we have

$$y_\tau \approx O(1) \quad , \quad y_\mu \approx O(\lambda^2) \quad , \quad y_e \approx O(\lambda^{3 \div 4}) \quad .\tag{43}$$

In the flavour basis the neutrino mass matrix reads [notice that the change of basis induced by V , because of the Majorana nature of neutrinos, will in general change the relative phases of the eigenvalues of m_ν (compare eq.(39) with eq.(44))]:

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a + 2d/3 & -d/3 & -d/3 \\ -d/3 & 2d/3 & a - d/3 \\ -d/3 & a - d/3 & 2d/3 \end{pmatrix}, \quad (44)$$

and is diagonalized by the transformation:

$$U^T m_\nu U = \frac{v_u^2}{\Lambda} \text{diag}(a + d, a, -a + d), \quad (45)$$

with

$$U = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix}. \quad (46)$$

The leading order predictions are $\tan^2 \theta_{23} = 1$, $\tan^2 \theta_{12} = 0.5$ and $\theta_{13} = 0$. The neutrino masses are $m_1 = a + d$, $m_2 = a$ and $m_3 = -a + d$, in units of v_u^2/Λ . We can express $|a|$, $|d|$ in terms of $r \equiv \Delta m_{sol}^2/\Delta m_{atm}^2 \equiv (|m_2|^2 - |m_1|^2)/|m_3|^2 - |m_1|^2$, $\Delta m_{atm}^2 \equiv |m_3|^2 - |m_1|^2$ and $\cos \Delta$, Δ being the phase difference between the complex numbers a and d :

$$\begin{aligned} \sqrt{2}|a| \frac{v_u^2}{\Lambda} &= \frac{-\sqrt{\Delta m_{atm}^2}}{2 \cos \Delta \sqrt{1 - 2r}} \\ \sqrt{2}|d| \frac{v_u^2}{\Lambda} &= \sqrt{1 - 2r} \sqrt{\Delta m_{atm}^2}. \end{aligned} \quad (47)$$

To satisfy these relations a moderate tuning is needed in this model. Due to the absence of (ll) in eq. (36) which we will motivate in the next section, a and d are of the same order in $1/\Lambda$, see eq. (40). Therefore we expect that $|a|$ and $|d|$ are close to each other and, to satisfy eqs. (47), $\cos \Delta$ should be negative and of order one. We obtain:

$$\begin{aligned} |m_1|^2 &= \left[-r + \frac{1}{8 \cos^2 \Delta (1 - 2r)} \right] \Delta m_{atm}^2 \\ |m_2|^2 &= \frac{1}{8 \cos^2 \Delta (1 - 2r)} \Delta m_{atm}^2 \\ |m_3|^2 &= \left[1 - r + \frac{1}{8 \cos^2 \Delta (1 - 2r)} \right] \Delta m_{atm}^2 \end{aligned} \quad (48)$$

If $\cos \Delta = -1$, we have a neutrino spectrum close to hierarchical:

$$|m_3| \approx 0.053 \text{ eV}, \quad |m_1| \approx |m_2| \approx 0.017 \text{ eV}. \quad (49)$$

In this case the sum of neutrino masses is about 0.087 eV. If $\cos \Delta$ is accidentally small, the neutrino spectrum becomes degenerate. The value of $|m_{ee}|$, the parameter characterizing the violation of total lepton number in neutrinoless double beta decay, is given by:

$$|m_{ee}|^2 = \left[-\frac{1 + 4r}{9} + \frac{1}{8 \cos^2 \Delta (1 - 2r)} \right] \Delta m_{atm}^2. \quad (50)$$

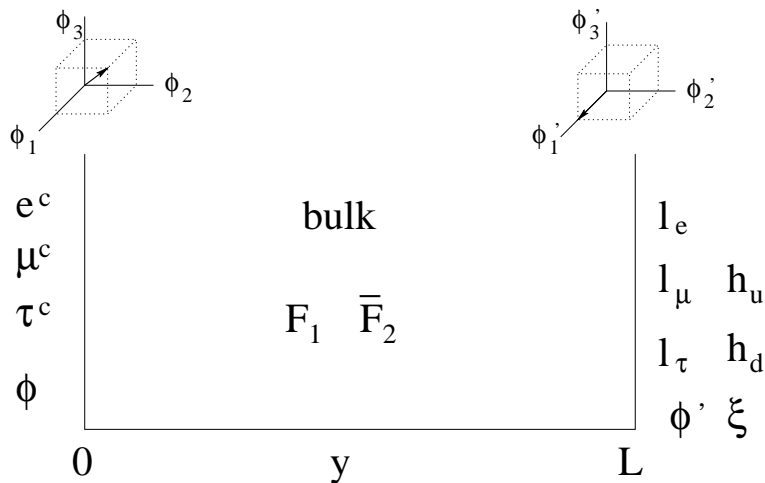


Figure 2. Fifth dimension and localization of scalar and fermion fields. The symmetry breaking sector includes the A_4 triplets φ and φ' , localized at the opposite ends of the interval. Their VEVs are dynamically aligned along the directions shown at the top of the figure.

For $\cos \Delta = -1$ we get $|m_{ee}| \approx 0.005$ eV, at the upper edge of the range allowed for normal hierarchy, but unfortunately too small to be detected in a near future. Independently from the value of the unknown phase Δ we get the relation:

$$|m_3|^2 = |m_{ee}|^2 + \frac{10}{9} \Delta m_{atm}^2 \left(1 - \frac{r}{2}\right) \quad , \quad (51)$$

which is a prediction of this model.

7 A4 model with an extra dimension

One of the problems we should solve in the quest for the correct alignment is that of keeping neutrino and charged lepton sectors separate, allowing φ and φ' to take different VEVs and also forbidding the exchange of one with the other in interaction terms. One possibility is that this separation is achieved by means of an extra spatial dimension. The space-time is assumed to be five-dimensional, the product of the four-dimensional Minkowski space-time times an interval going from $y = 0$ to $y = L$. At $y = 0$ and $y = L$ the space-time has two four-dimensional boundaries, called "branes". The idea is that matter SU(2) singlets such as e^c, μ^c, τ^c are localized at $y = 0$, while SU(2) doublets, such as l are localized at $y = L$ (see Fig.1). Neutrino masses arise from local operators at $y = L$. Charged lepton masses are produced by non-local effects involving both branes. The simplest possibility is to introduce a bulk fermion, depending on all space-time coordinates, that interacts with e^c, μ^c, τ^c at $y = 0$ and with l at $y = L$. The exchange of such a fermion can provide the desired non-local coupling between right-handed and left-handed ordinary fermions. Finally, assuming that φ and (φ', ξ) are localized respectively at $y = 0$ and $y = L$, one obtains a natural separation between the two sectors.

Such a separation also greatly simplifies the vacuum alignment problem. One can determine the minima of two scalar potentials V_0 and V_L , depending only, respectively, on φ and (φ', ξ) . Indeed, it is shown that there are whole regions of the parameter space where $V_0(\varphi)$ and $V_L(\varphi', \xi)$ have the minima given in eq. (37). Notice that in the present setup dealing with a discrete symmetry such as A4 provides a great advantage as far as the alignment problem is concerned. A continuous flavour symmetry such as, for instance, $\text{SO}(3)$ would need some extra structure to achieve the desired alignment. Indeed the potential energy $\int d^4x [V_0(\varphi) + V_L(\varphi', \xi)]$ would be invariant under a much bigger symmetry, $\text{SO}(3)_0 \times \text{SO}(3)_L$, with the $\text{SO}(3)_0$ acting on φ and leaving (φ', ξ) invariant and vice-versa for $\text{SO}(3)_L$. This symmetry would remove any alignment between the VEVs of φ and those of (φ', ξ) . If, for instance, (37) is minimum of the potential energy, then any other configuration obtained by acting on (37) with $\text{SO}(3)_0 \times \text{SO}(3)_L$ would also be a minimum and the relative orientation between the two sets of VEVs would be completely undetermined. A discrete symmetry such as A4 has not this problem, because applying separate A4 transformation on the minimum solutions on each brane a finite number of degenerate vacua is obtained which can be shown to correspond to the same physics apart from redefinitions of fields and parameters.

The Lagrangian in 5 dimensions includes a bulk fermion field $F(x, y) = (F_1, \overline{F_2})$, singlet under $\text{SU}(2)$ with hypercharge $Y = -1$ and transforming as a triplet of A4. One also imposes a discrete Z_4 symmetry under which $(f^c, l, F, \varphi, \varphi', \xi)$ transform into $(-if^c, il, iF, \varphi, -\varphi', -\xi)$. The complete action is

$$\begin{aligned}
 S = & \int d^4x dy \left\{ \left[iF_1 \sigma^\mu \partial_\mu \overline{F_1} + iF_2 \sigma^\mu \partial_\mu \overline{F_2} + \frac{1}{2} (F_2 \partial_y F_1 - \partial_y F_2 F_1 + h.c.) \right] \right. \\
 & - M(F_1 F_2 + \overline{F_1} \overline{F_2}) \\
 & + V_0(\varphi) \delta(y) + V_L(\varphi', \xi) \delta(y - L) \\
 & + [Y_e e^c (\varphi F_1) + Y_\mu \mu^c (\varphi F_1)'' + Y_\tau \tau^c (\varphi F_1)' + h.c.] \delta(y) \\
 & \left. + \left[\frac{x_a}{\Lambda^2} \xi (ll) h_u h_u + \frac{x_d}{\Lambda^2} (\varphi' ll) h_u h_u + Y_L (F_2 l) h_d + h.c. \right] \delta(y - L) \right\} + \dots \quad , \quad (52)
 \end{aligned}$$

where the constants Y have mass dimension $-1/2$. The first two lines represent the five-dimensional kinetic and mass terms of the bulk field F . The third line is the scalar potential and the remaining terms are the lowest order invariant operators localized at the two branes. Dots stand for the kinetic terms of $f^c, l, \varphi, \varphi', \xi$ and for higher-dimensional operators.

The potential energy is given, at lowest order by:

$$U = \int d^4x [V_0(\varphi) + V_L(\varphi', \xi)] \quad , \quad (53)$$

and, under the conditions discussed above, is minimized by eqs. (37) [28]. It is clear that at lowest order φ and (φ', ξ) are strictly separated.

We now discuss the effects of the tree-level exchange of F . To this purpose we consider the equations of motion for (F_1, F_2) :

$$\begin{aligned}
 i\sigma^\mu \partial_\mu \overline{F_2} + \partial_y F_1 - M F_1 &= 0 \\
 i\sigma^\mu \partial_\mu \overline{F_1} - \partial_y F_2 - M F_2 &= 0
 \end{aligned} \quad (54)$$

If M is large and positive, we can prove that all the modes contained in (F_1, F_2) become heavy, at a scale greater than or comparable to $1/L$, which we assume to be much higher than the electroweak scale. If we are only interested in energies much lower than $1/L$, we can solve the equations of motion in the static approximation, by neglecting the four-dimensional kinetic term:

$$\begin{aligned} F_1(y) &= F_1(L)e^{M(y-L)} \\ F_2(y) &= F_2(0)e^{-My} \quad . \end{aligned} \quad (55)$$

These equations must be supplemented with appropriate boundary conditions, which can be identified by varying the action S with respect the fields (F_1, F_2) . As a final result, as shown in detail in ref. [28], in lowest order approximation the Lagrangian \mathcal{L}_Y of eq. (36) is reproduced and the general discussion applies.

We also recall that, to account for the observed hierarchy of the charged lepton masses, one has to include an additional $U(1)$ flavour symmetry. Therefore, in the present picture, the quantities $Y_{e,\mu,\tau}$ stand for:

$$Y_e = \tilde{Y}_e \left(\frac{\theta}{\Lambda} \right)^4, \quad Y_\mu = \tilde{Y}_\mu \left(\frac{\theta}{\Lambda} \right)^2, \quad Y_\tau = \tilde{Y}_\tau, \quad (56)$$

where $\tilde{Y}_{e,\mu,\tau}$ are field-independent constants having similar values. After spontaneous breaking of $U(1)$, the Yukawa couplings y_f possess the desired hierarchy.

8 A4 model with SUSY in 4 Dimensions

We now discuss an alternative supersymmetric solution to the vacuum alignment problem [29]. In a SUSY context, the right-hand side of eq. (36) should be interpreted as the superpotential w_l of the theory, in the lepton sector:

$$\begin{aligned} w_l &= y_e e^c(\varphi l) + y_\mu \mu^c(\varphi l)'' + y_\tau \tau^c(\varphi l)' + \\ &+ (x_a \xi + \tilde{x}_a \tilde{\xi})(ll) + x_b(\varphi' ll) + h.c. + \dots \end{aligned} \quad (57)$$

where dots stand for higher dimensional operators and where we have also added an additional A4-invariant singlet $\tilde{\xi}$. Such a singlet does not modify the structure of the mass matrices discussed previously, but plays an important role in the vacuum alignment mechanism. A key observation is that the superpotential w_l is invariant not only with respect to the gauge symmetry $SU(2) \times U(1)$ and the flavour symmetry $U(1)_F \times A_4$, but also under a discrete Z_3 symmetry and a continuous $U(1)_R$ symmetry under which the fields transform as shown in the following table.

Field	1	e^c	μ^c	τ^c	$h_{u,d}$	φ	φ'	ξ	$\tilde{\xi}$	φ_0	φ'_0	ξ_0
A4	3	1	1'	1''	1	3	3	1	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	ω	1	ω	ω
$U(1)_R$	1	1	1	1	0	0	0	0	0	2	2	2

We see that the Z_3 symmetry explains the absence of the term (ll) in w_l : such a term transforms as ω^2 under Z_3 and need to be compensated by the field ξ in our construction. At the same time Z_3 does not allow the interchange between φ and φ' , which transform differently under Z_3 . The singlets ξ and $\tilde{\xi}$ have the same transformation properties under all symmetries and, as we shall see, in a finite range of parameters, the VEV of $\tilde{\xi}$ vanishes and does not contribute to neutrino masses. Charged leptons and neutrinos acquire masses from two independent sets of fields. If the two sets of fields develop VEVs according to the alignment described in eq. (37), then the desired mass matrices follow.

Finally, there is a continuous $U(1)_R$ symmetry that contains the usual R -parity as a subgroup. Suitably extended to the quark sector, this symmetry forbids the unwanted dimension two and three terms in the superpotential that violate baryon and lepton number at the renormalizable level. The $U(1)_R$ symmetry allows us to classify fields into three sectors. There are ‘‘matter fields’’ such as the leptons l , e^c , μ^c and τ^c , which occur in the superpotential through bilinear combinations. There is a ‘‘symmetry breaking sector’’ including the higgs doublets $h_{u,d}$ and the flavons φ , φ' , $(\xi, \tilde{\xi})$. Finally, there are ‘‘driving fields’’ such as φ_0 , φ'_0 and ξ_0 that allows to build a non-trivial scalar potential in the symmetry breaking sector. Since driving fields have R -charge equal to two, the superpotential is linear in these fields.

The full superpotential of the model is

$$w = w_l + w_d \quad (58)$$

where, at leading order in a $1/\Lambda$ expansion, w_l is given by eq. (57) and the ‘‘driving’’ term w_d reads:

$$\begin{aligned} w_d = & M(\varphi_0\varphi) + g(\varphi_0\varphi\varphi) + g_1(\varphi'_0\varphi'\varphi') + g_2\tilde{\xi}(\varphi'_0\varphi') + g_3\xi_0(\varphi'\varphi') \\ & + g_4\xi_0\xi^2 + g_5\xi_0\xi\tilde{\xi} + g_6\xi_0\tilde{\xi}^2 \quad . \end{aligned} \quad (59)$$

At this level there is no fundamental distinction between the singlets ξ and $\tilde{\xi}$. Thus we are free to define $\tilde{\xi}$ as the combination that couples to $(\varphi'_0\varphi')$ in the superpotential w_d . We notice that at the leading order there are no terms involving the Higgs fields $h_{u,d}$. We assume that the electroweak symmetry is broken by some mechanism, such as radiative effects when SUSY is broken. It is interesting that at the leading order the electroweak scale does not mix with the potentially large scales u , v and v' . The scalar potential is given by:

$$V = \sum_i \left| \frac{\partial w}{\partial \phi_i} \right|^2 + m_i^2 |\phi_i|^2 + \dots \quad (60)$$

where ϕ_i denote collectively all the scalar fields of the theory, m_i^2 are soft masses and dots stand for D-terms for the fields charged under the gauge group and possible additional soft breaking terms. Since m_i are expected to be much smaller than the mass scales involved in w_d , it makes sense to minimize V in the supersymmetric limit and to account for soft breaking effects subsequently. A detailed minimization analysis, presented in ref.[29], shows the the desired alignment solution is indeed realized. In ref.[30] we have shown that it is straightforward to reformulate this SUSY model in the approach where the A4 symmetry is derived from orbifolding.

9 Corrections to the Lowest Approximation

The results of the previous sections hold to first approximation. Higher-dimensional operators, suppressed by additional powers of the cut-off Λ , can be added to the leading terms in the lagrangian. These corrections have been classified and discussed in detail in refs.[28], [29]. They are completely under control in our models and can be made negligibly small without any fine-tuning: one only needs to assume that the VEV's are sufficiently smaller than the cutoff Λ . Higher-order operators contribute corrections to the charged lepton masses, to the neutrino mass matrix and to the vacuum alignment. These corrections, suppressed by powers of VEVs/Λ , with different exponents in different versions of A4 models, affect all the relevant observable with terms of the same order: s_{13} , s_{12} , s_{23} , r . If we require that the subleading terms do not spoil the leading order picture, these deviations should not be larger than about 0.05. This can be inferred by the agreement of the HPS value of $\tan^2 \theta_{12}$ with the experimental value, from the present bound on θ_{13} or from requiring that the corrections do not exceed the measured value of r . In the SUSY model, where the largest corrections are linear in VEVs/Λ [29], this implies the bound

$$\frac{v_S}{\Lambda} \approx \frac{v_T}{\Lambda} \approx \frac{u}{\Lambda} < 0.05 \quad (61)$$

which does not look unreasonable, for example if $\text{VEVs} \sim M_{GUT}$ and $\Lambda \sim M_{Planck}$.

10 See-saw Realization

We can easily modify the previous model to implement the see-saw mechanism. We introduce conjugate right-handed neutrino fields ν^c transforming as a triplet of A4 and we modify the transformation law of the other fields according to the following table:

Field	ν^c	φ'	ξ	$\tilde{\xi}$	φ'_0	ξ_0
A4	3	3	1	1	3	1
Z_3	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
$U(1)_R$	1	0	0	0	2	2

The superpotential becomes

$$w = w_l + w_d \quad (62)$$

where the 'driving' part is unchanged, whereas w_l is now given by:

$$\begin{aligned} w_l = & y_e e^c(\varphi l) + y_\mu \mu^c(\varphi l) + y_\tau \tau^c(\varphi l) + y(\nu^c l) + (x_A \xi + \tilde{x}_A \tilde{\xi})(\nu^c \nu^c) \\ & + x_B(\varphi' \nu^c \nu^c) + h.c. + \dots \end{aligned} \quad (63)$$

dots denoting higher-order contributions. The vacuum alignment proceeds exactly as discussed in section 8 and also the charged lepton sector is unaffected by the modifications. In the neutrino sector, after electroweak and A4 symmetry breaking we have Dirac and

Majorana masses:

$$m_\nu^D = yv_u \mathbf{1}, \quad M = \begin{pmatrix} A & 0 & 0 \\ 0 & A & B \\ 0 & B & A \end{pmatrix} u \quad , \quad (64)$$

where $\mathbf{1}$ is the unit 3×3 matrix and

$$A \equiv 2x_A \quad , \quad B \equiv 2x_B \frac{v_S}{u} \quad . \quad (65)$$

The mass matrix for light neutrinos is $m_\nu = (m_\nu^D)^T M^{-1} m_\nu^D$ with eigenvalues

$$m_1 = \frac{y^2 v_u^2}{A+B} \quad , \quad m_2 = \frac{y^2 v_u^2}{A} \quad , \quad m_3 = \frac{y^2 v_u^2}{-A+B} \quad . \quad (66)$$

The mixing matrix is the HPS one, eq. (8). In the presence of a see-saw mechanism both normal and inverted hierarchies in the neutrino mass spectrum can be realized. If we call Φ the relative phase between the complex number A and B , then $\cos \Phi > -|B|/2|A|$ is required to have $|m_2| > |m_1|$. In the interval $-|B|/2|A| < \cos \Phi \leq 0$, the spectrum is of inverted hierarchy type, whereas in $|B|/2|A| \leq \cos \Phi \leq 1$ the neutrino hierarchy is of normal type. It is interesting that this model is an example of model with inverse hierarchy, realistic θ_{12} and θ_{23} and, at least in a first approximation, $\theta_{13} = 0$. The quantity $|B|/2|A|$ cannot be too large, otherwise the ratio r cannot be reproduced. When $|B| \ll |A|$ the spectrum is quasi degenerate. When $|B| \approx |A|$ we obtain the strongest hierarchy. For instance, if $B = -2A + z$ ($|z| \ll |A|, |B|$), we find the following spectrum:

$$\begin{aligned} |m_1|^2 &\approx \Delta m_{atm}^2 \left(\frac{9}{8} + \frac{1}{12} r \right), \\ |m_2|^2 &\approx \Delta m_{atm}^2 \left(\frac{9}{8} + \frac{13}{12} r \right), \\ |m_3|^2 &\approx \Delta m_{atm}^2 \left(\frac{1}{8} + \frac{1}{12} r \right). \end{aligned} \quad (67)$$

When $B = A + z$ ($|z| \ll |A|, |B|$), we obtain:

$$\begin{aligned} |m_1|^2 &\approx \Delta m_{atm}^2 \left(\frac{1}{3} r \right), \\ |m_2|^2 &\approx \Delta m_{atm}^2 \left(\frac{4}{3} r \right), \\ |m_3|^2 &\approx \Delta m_{atm}^2 \left(1 - \frac{1}{3} r \right). \end{aligned} \quad (68)$$

These results are affected by higher-order corrections induced by non renormalizable operators with similar results as in the version with no see-saw. In conclusion, the symmetry structure of the model is fully compatible with the see-saw mechanism.

11 Quarks

There are several possibilities to include quarks. At first sight the most appealing one is to adopt for quarks the same classification scheme under A4 that we have used for

leptons. Thus we tentatively assume that left-handed quark doublets q transform as a triplet 3, while the right-handed quarks (u^c, d^c) , (c^c, s^c) and (t^c, b^c) transform as 1, 1' and 1'', respectively. We can similarly extend to quarks the transformations of Z_3 and $U(1)_R$ given for leptons in the table of section 8. The superpotential for quarks reads:

$$w_q = y_d d^c(\varphi q) + y_s s^c(\varphi q) + y_b b^c(\varphi q)' \quad (69)$$

$$+ y_u u^c(\varphi q) + y_c c^c(\varphi q) + y_t t^c(\varphi q)' + h.c. + \dots$$

It is interesting to note that such an extrapolation to quarks leads to a diagonal CKM mixing matrix in first approximation [25, 26, 29, 35]. In fact, starting from eq. (69) and proceeding as described in detail for the lepton sector, we see that the up quark and down quark mass matrices are separately diagonal with mass eigenvalues which are left unspecified by A4 and with a hierarchy that could be accommodated by a suitable $U(1)_F$ set of charge assignments for quarks. Thus the V_{CKM} matrix is the identity in leading order, providing a good first order approximation.

The problems come when we discuss non-leading corrections. As seen in section 9, first-order corrections to the lepton sector should be typically below 0.05, approximately the square of the Cabibbo angle. Also, by inspecting these corrections more closely, we see that, up to very small terms of order $y_{u(d)}^2/y_{t(b)}^2$ and $y_{c(s)}^2/y_{t(b)}^2$, all corrections are the same in the up and down sectors and therefore they almost exactly cancel in the mixing matrix V_{CKM} . We conclude that, if one insists in adopting for quarks the same flavour properties as for leptons, than new sources of A4 breaking are needed in order to produce an acceptable V_{CKM} .

The A4 classification for quarks and leptons discussed in this section, which leads to an appealing first approximation with $V_{CKM} \sim 1$ for quark mixing and to U_{HPS} for neutrino mixings, is not compatible with A4 commuting with $SU(5)$ or $SO(10)$. In fact for this to be true all particles in a representation of $SU(5)$ should have the same A4 classification. But, for example, both the $Q = (u, d)_L$ LH quark doublet and the RH charged leptons l^c belong to the 10 of $SU(5)$, yet they have different A4 transformation properties. In a recent paper [36] the possibility of classifying all fermion multiplets as triplets was advanced. But the crucial issues of the correct alignment and of reproducing in a natural way the observed hierarchy of, for example, the charged leptons were not addressed and are difficult to realize in this case.

12 Conclusion

In the last decade we have learnt a lot about neutrino masses and mixings. A list of important conclusions have been reached. Neutrinos are not all massless but their masses are very small. Probably masses are small because neutrinos are Majorana particles with masses inversely proportional to the large scale M of lepton number violation. It is quite remarkable that M is empirically close to $10^{14-15} GeV$ not far from M_{GUT} , so that neutrino masses fit well in the SUSY GUT picture. Also out of equilibrium decays with CP and L violation of heavy RH neutrinos can produce a B-L asymmetry, then converted near the weak scale by instantons into an amount of B asymmetry compatible with observations (baryogenesis via leptogenesis) [4], [37]. It has been established that neutrinos are not a significant component of dark matter in the Universe. We have also understood there

there is no contradiction between large neutrino mixings and small quark mixings, even in the context of GUTs.

This is a very impressive list of achievements. Coming to a detailed analysis of neutrino masses and mixings a very long collection of models have been formulated over the years. With a continuous improvement of the data and a progressive narrowing of the values of the mixing angles most of the models have been discarded by experiment. Still the missing elements in the picture like, for example, the scale of the average neutrino m^2 , the pattern of the spectrum (degenerate or inverse or normal hierarchy) and the value of θ_{13} have left many different viable alternatives for models. It certainly is a reason of satisfaction that so much has been learnt recently from experiments on neutrino mixings. By now, besides the detailed knowledge of the entries of the V_{CKM} matrix we also have a reasonable determination of the neutrino mixing matrix. It is remarkable that neutrino and quark mixings have such a different qualitative pattern. One could have imagined that neutrinos would bring a decisive boost towards the formulation of a comprehensive understanding of fermion masses and mixings. In reality it is frustrating that no real illumination was sparked on the problem of flavour. We can reproduce in many different ways the observations but we have not yet been able to single out a unique and convincing baseline for the understanding of fermion masses and mixings. In spite of many interesting ideas and the formulation of many elegant models, some of them presented in these lectures, the mysteries of the flavour structure of the three generations of fermions have not been much unveiled.

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