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Noise-Induced Phenomena in Slow-Fast Dynamical Systems

A Sample-Paths Approach

With 57 Figures

 Springer

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