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New baryon matter in the lattice Gross-Neveu model

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We investigate the Gross-Neveu model on the lattice at finite temperature and chemical potential in the limit of an infinite number of fermion flavours. We check the universality of the continuum limit of staggered and overlap fermions at finite temperature and chemical potential. We show that at finite density a recently discovered phase of cold baryonic matter emerges as a baryon crystal from a spatially inhomogeneous fermion condensate. However, we also demonstrate that on the lattice, this new phase disappears at large coupling or in small volumes. Furthermore, we investigate unusual finite size effects that appear at finite chemical potential. Finally, we speculate on the implications of our findings for QCD.

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1. Introduction

In order to understand the phases of matter at finite temperature and density it is necessary to understand the properties of the non-perturbative vacuum of QCD or related models. The Gross-Neveu model [1] resembles QCD in many respects and can be solved analytically in the limit of an infinite number of fermion flavours N. Here we report on our investigation of this model on a discrete space-time lattice at finite temperature and density in the large-N limit.

2. The Gross-Neveu model

Let us start with writing the Euclidean lagrangian density of the Gross-Neveu (GN) model [1] in $1+1$ dimensions,

$$\mathscr{L} = \sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x) \partial\!\!\!/ \, \psi^{\alpha}(x) - \frac{g^2}{2} \left(\sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x) \psi^{\alpha}(x) \right)^2, \tag{2.1}$$

where $\psi^{\alpha}(x)$ are 2-component Dirac spinors and α is a flavour index. Usually, one introduces a scalar field $\sigma(x)$ conjugate to $\sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x)\psi^{\alpha}(x)$ in order to transform away the 4-fermion coupling,

$$\mathscr{L} = \sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x) \partial\!\!\!/ \, \psi^{\alpha}(x) + \frac{1}{2g^2}\sigma(x)^2 + \sigma(x) \sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x)\psi^{\alpha}(x). \tag{2.2}$$

The GN model shares many interesting properties with QCD. In particular, it is renormalisable and asymptotically free, with the β-function given to lowest order by $\beta(g) = -\frac{N-1}{2\pi} g^3 + O(g^5)$. Moreover it enjoys a $O(2N) \times \Gamma$ global symmetry, where Γ is the discrete chiral symmetry

$$\Gamma: \qquad \psi \to \gamma_5 \psi, \quad \bar{\psi} \to -\bar{\psi}\gamma_5, \quad \sigma \to -\sigma, \tag{2.3}$$

and it exhibits spontaneous breaking of this discrete chiral symmetry which in turn leads to the fermions acquiring a non-vanishing mass $\sigma_0 = \langle \sigma \rangle$ (dimensional transmutation)[1].

In the large-N limit where the number of fermion flavours N is taken to infinity while keeping $\lambda = g^2 N$ fixed, the model can be solved analytically. One can integrate out the fermions to obtain $Z = \int [d\sigma] \exp\{-S_{\text{eff}}\}$ with

$$S_{\text{eff}} = N \left\{ \int dx \frac{\sigma(x)^2}{2\lambda} - \text{Tr} \log \left[\partial\!\!\!/ + \sigma \right] \right\}. \tag{2.4}$$

The minimum of the effective potential is given by a set of equations,

$$\partial_{\sigma(x)} S_{\text{eff}}/N = \frac{\sigma(x)}{\lambda} - \partial_{\sigma(x)} \text{Tr} \log \left[\partial\!\!\!/ + \sigma \right] = 0, \quad \forall x, \tag{2.5}$$

and for a homogeneous condensate $\sigma(x) = \sigma$ this set reduces to a single equation

$$\frac{\sigma}{\lambda} = \partial_{\sigma} \text{Tr} \log \left[\partial\!\!\!/ + \sigma \right], \tag{2.6}$$

[1] Note that there is no Goldstone boson since Γ is a discrete symmetry.

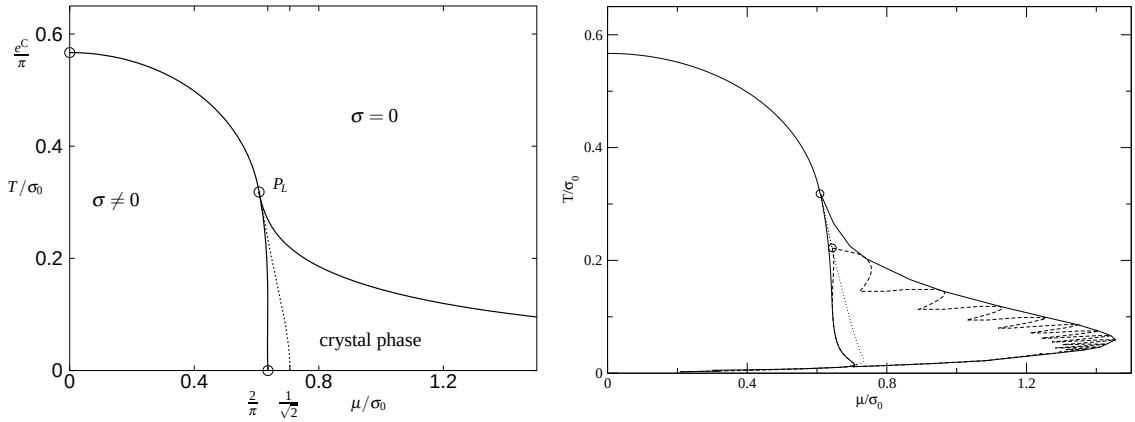

Figure 1: Revised phase diagram of the GN model in the continuum from [4] (left) and on the lattice (right). Full lines correspond to second order phase boundaries, dotted lines denote the (incorrect) first order phase boundaries from the translationally invariant calculation. P_L is a multicritical Lifshitz point and C is the Euler constant. The dashed lines in the right figure illustrate the finite size artefacts, in particular the incommensurability effects at the right phase boundary of the crystal phase.

or in momentum space

$$\sigma = 0 \quad \text{or} \quad \frac{1}{\lambda} = \int \frac{d^2k}{(2\pi)^2} \frac{2}{k^2+\sigma^2} \tag{2.7}$$

which is a self consistency equation for the fermion condensate or simply the *gap equation*. Equivalent equations can be derived via Hartree-Fock, Schwinger-Dyson or Bethe-Salpeter approaches. With the help of these equations one can derive the spectrum of the GN model [2] – it contains fermions with mass $m = \sigma_0$, n-fermion bound states, and baryons with mass $m_B = \sigma_0 \cdot \frac{2N}{\pi}$.

What makes the GN model particularly interesting for our purpose is its rich phase structure in the (μ, T)–plane where μ represents the fermion chemical potential and T the temperature[2]. In the homogeneous mean field approximation this was studied by Wolff [3] who found two phases, a massive and a massless Fermi gas, separated by a line of first and second order transitions meeting in a tri-critical point, cf. Figure 1. Recently the phase structure has been further clarified by Thies and Urlichs [4, 5] who relaxed the tacit assumption of translational invariance of the condensate. Motivated by the fact that matter at low density forms isolated baryons they analytically found an inhomogeneous solution $\sigma(x)$ to eq.(2.5). Indeed, they discovered that there exists a new baryonic matter phase at low temperature where baryons form a crystal structure. The transitions from the massive to the crystal phase and further to the massless phase are all second order.

3. Lattice formulations of the Gross-Neveu model

Let us now consider the GN model on a two-dimensional lattice,

$$S = N \sum_x \frac{\sigma(x)^2}{2\lambda} + \sum_{x,y} \sum_{\alpha=1}^{N} \bar{\chi}^\alpha(x) \left[D_{xy} + \Sigma_{xy} \right] \chi^\alpha(y) \tag{3.1}$$

where the staggered Dirac operator

$$D_{xy} = \frac{1}{2} \left[\delta_{x,y+\hat{1}} - \delta_{x,y-\hat{1}} \right] + \frac{1}{2} (-1)^{x_1} \left[e^{+\mu} \delta_{x,y+\hat{2}} - e^{-\mu} \delta_{x,y-\hat{2}} \right] \tag{3.2}$$

[2] In the large-N limit, a discrete symmetry can break spontaneously in $(1+1)d$ even at finite temperature.

describes 2 flavours and

$$\Sigma_{xy} = \frac{1}{4}\delta_{xy}\left(\sigma(x)+\sigma(x-\hat{1})+\sigma(x-\hat{2})+\sigma(x-\hat{1}-\hat{2})\right). \tag{3.3}$$

The modification of the naive discretisation $\sigma(x)\delta_{xy} \to \Sigma_{xy}$ is necessary to ensure the correct continuum limit [6]. With the staggered discretisation the following discrete chiral symmetry is preserved,

$$\chi(x) \to (-1)^{x_1+x_2}\chi(x), \quad \bar{\chi}(x) \to -(-1)^{x_1+x_2}\bar{\chi}(x), \quad \sigma(x) \to -\sigma(x). \tag{3.4}$$

Alternatively we consider a discretisation respecting exact chiral symmetry by employing the overlap Dirac operator

$$D = m\left\{1 + D_W(-m)\left[D_W^\dagger(-m)D_W(-m)\right]^{-1/2}\right\} \tag{3.5}$$

satisfying the Ginsparg-Wilson relation $D^\dagger + D = \frac{1}{m}D^\dagger D$. The coupling of the fermionic fields to the scalar field is introduced according to

$$\mathscr{L} = \bar{\psi}(x)\left[D_{x,y} - \frac{\sigma(x)}{4m}D_{x,y} - D_{x,y}\frac{\sigma(y)}{4m} + \sigma(x)\delta_{x,y}\right]\psi(y) \tag{3.6}$$

which is consistent with a scalar density transforming covariantly under a lattice chiral symmetry transformation. For a homogeneous condensate $\sigma \to$ const. it is just the usual mass term of the overlap operator.

The large-N limit of the lattice theory is obtained by minimising the free energy. Using a homogeneous mean field, we obtain σ as a function of λ from the the gap equation. In Figure 2 we show these scaling functions for various lattice sizes. The dashed red curve describes the asymptotic scaling curve $2^{3/2}e^{-\pi/2\lambda}$ for staggered and $1.5539...e^{-\pi/\lambda}$ for overlap fermions. As expected, the staggered fermion formulation exhibits an additional factor of 2 in the exponent of the scaling curve due to the doubling of the number of flavours.

To start our investigation of the (μ,T)–phase diagram we first determine the chiral transition temperature T_c at $\mu = 0$ where the chiral condensate $\sigma(T)$ vanishes. The results are shown in Figure 3 where we plot the scaling of T_c/σ_0 versus $(a\sigma_0)^2$ on various lattice sizes for the staggered

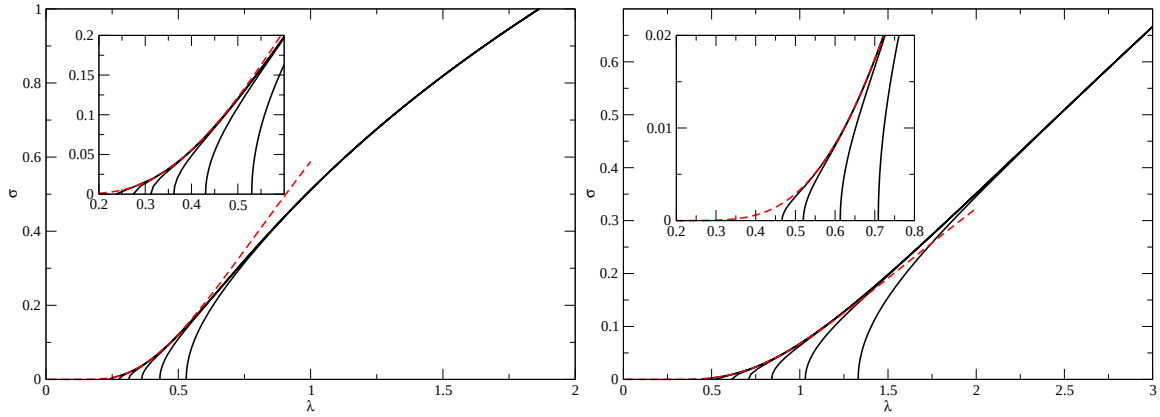

Figure 2: Scaling of the condensate σ as a function of the coupling λ for staggered (left) and overlap fermions (right) for various lattice sizes. The dashed red lines are the asymptotic scaling curves.

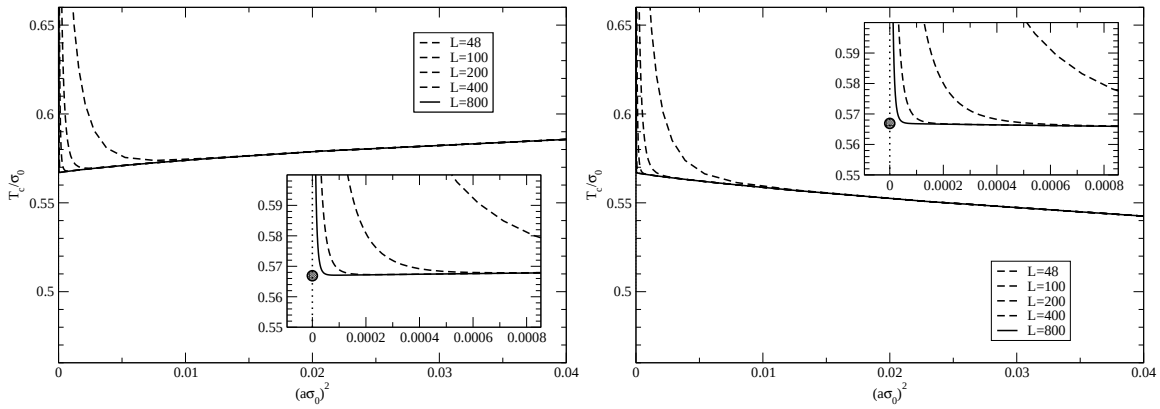

Figure 3: Scaling of T_c/σ_0 vs $(a\sigma_0)^2$ on various lattice sizes for the staggered (left) and the overlap operator (right).

operator on the left and the overlap operator on the right. Both formulations exhibit lattice artefacts of similar size, but with a different sign, and nicely approach the analytically known value $T_c/\sigma_0 = 0.5669...$ in the continuum [3], thereby confirming the universality of the continuum limit.

4. Baryonic matter in the Gross-Neveu model

In both formulations a finite chemical potential can be introduced by weighting the temporal derivatives with factors $\exp(\pm\mu)$ [7]. We can then check the universality of the continuum limit of μ_c at $T \simeq 0$ and look at the discretisation artefacts for the two lattice formulations. In Figure 4 we show the scaling of μ_c/σ_0 versus $(a\sigma_0)^2$ for the staggered Dirac operator on the left and the overlap on the right, still using only the homogeneous ansatz for the solution of the gap equation. Again, the two formulations show discretisation artefacts of similar size, but with a different sign,

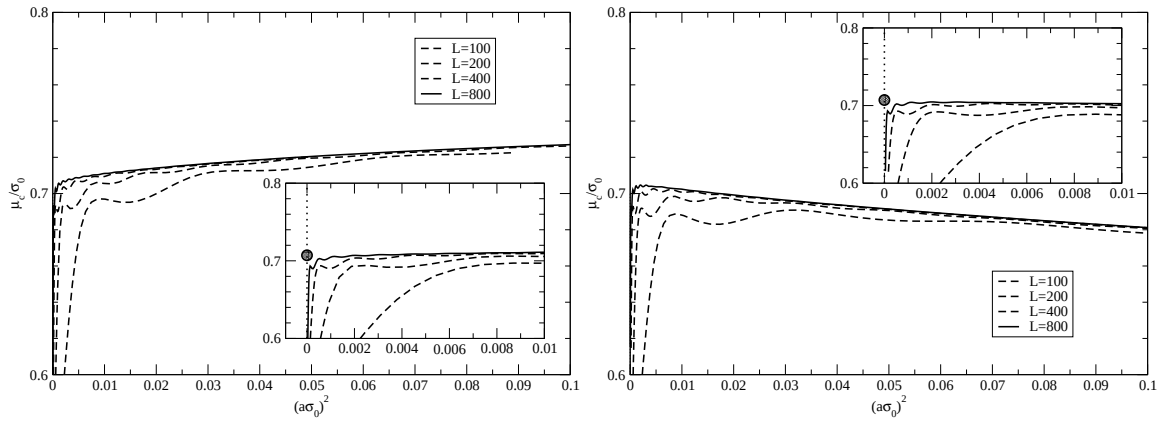

Figure 4: Scaling of μ_c/σ_0 vs $(a\sigma_0)^2$ on various lattice sizes for the staggered (left) and the overlap operator (right) using the homogeneous ansatz for the solution of the gap equation.

and nicely scale to the analytically known value $\mu_c = 1/\sqrt{2}$ from the homogeneous ansatz of the condensate [3]. However, in both cases the continuum value is approached non-monotonically – this is caused by the fact that the Fermi momentum changes continuously with μ while the lattice

momentum is quantised by the finite lattice size. This finite volume effect would be most striking at exactly $T = 0$ where one would expect a sawtooth behaviour; at $T > 0$, however, it is smoothened due to the softening of the Fermi surface.

In Figure 5 we show the full phase diagram for the staggered operator at weak ($L_t = 80$) and strong coupling ($L_t = 4$), still using the homogeneous ansatz for the condensate. We note that both the region of metastability and that of the chirally broken phase shrink considerably towards strong coupling.

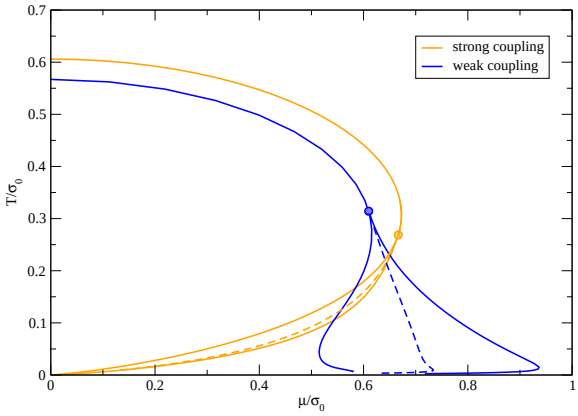

Figure 5: Phase diagram at weak ($L_t = 80$) and strong ($L_t = 4$) coupling using the homogeneous ansatz for the condensate. The full line above the tricritical point marked by the dot denotes the second order transition, the dashed line below denotes the first order transitions while the two full lines to the left and right of it bound the region of metastability associated with the first order transition.

Let us now relax the assumption of translation invariance. In order to determine the phase boundaries related to the transitions into the new baryonic matter phase, we perform on the one hand a brute force minimisation of the free energy and on the other hand check for instabilities of the homogeneous vacuum. The latter is achieved by monitoring the eigenvalues of the Hessian matrix in the homogeneous vacuum – a negative eigenvalue indicates that the free energy can be lowered further by an inhomogeneous perturbation. This is illustrated in Figure 6 where we show the lowest eigenvalue of the Hessian matrix associated with spatial variations of the condensate as a function of μ for a fixed temperature. It is clearly seen that for $\mu \gtrsim 0.075$, where the preferred homogeneous condensate is $\sigma = 0$, an inhomogeneous one is favoured. In fact, a brute force free energy minimisation shows that this is also true for some range $\mu \lesssim 0.075$, however finite-size effects cause a small free energy barrier between the two vacua. We also note that the non-monotonic behaviour of the lowest eigenvalue is an artefact of the non-commensurability of the spatial lattice size with the intrinsic length scale of the inhomogeneous condensate (compare the two unit cell volumes in Figure 6). In Figure 1 right we show the new phase diagram in the thermodynamic limit, at fixed $L_t = 80$, to be compared with the continuum phase diagram on the left. The dashed lines show the effects from finite volume. In particular we point to the fringes at the right phase boundary of the crystal phase which are the reflection of the incommensurability effects discussed above. In the thermodynamic limit these effects disappear.

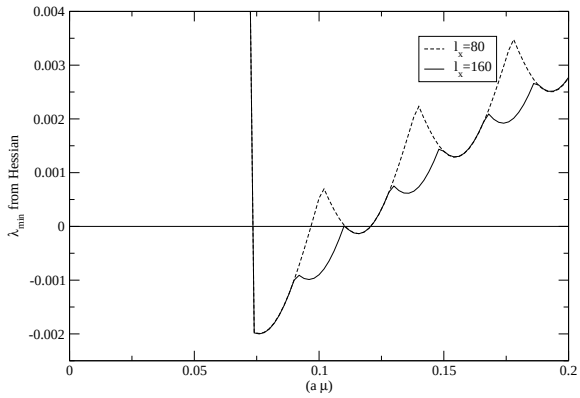

Figure 6: The lowest eigenvalue of the Hessian matrix associated with spatial variations of the condensate as a function of μ for a fixed temperature and two unit cell volumes.

5. Summary and Outlook

We have investigated the phase structure of the GN model in the (μ, T)–plane. The breakdown of translational invariance of the chiral condensate requires a revision of the GN model phase diagram. Besides the massive and massless Fermi gas phase, a new phase of baryonic matter emerges which forms a baryon crystal. The transition to the new phase is always second order for any temperature. Our investigation of the phase structure on the lattice indicates that the new crystal phase disappears at strong coupling, because the topological excitations associated with the forming of the crystal fall through the lattice. Furthermore, large volumes are needed to detect the baryon crystal phase and to avoid or reduce complicated artefacts due to the incommensurability of the intrinsic length scale of the inhomogeneous condensate and the lattice volume, which distort both the phase boundary and the order of the phase transition.

In this exactly solvable model, the crystal is formed by topological defects, the kinks and antikinks, which carry the baryon number. This crystal becomes stable for a sufficiently large chemical potential. One may wonder how general this phenomenon is. A similar topological crystal may occur in the $(2+1)d$ Nambu-Jona-Lasinio model. If it does, perhaps it also occurs in QCD.

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