

**Expandability, Reversibility, and Optimal Capacity  
Choice**

by

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# EXPANDABILITY, REVERSIBILITY, AND OPTIMAL CAPACITY CHOICE

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**Abstract:** We develop continuous-time models of capacity choice when demand fluctuates stochastically, and the firm's opportunities to expand or contract are limited. Specifically, we consider costs of investing or disinvesting that vary with time, or with the amount of capacity already installed. The firm's limited opportunities to expand or contract create call and put options on incremental units of capital; we show how the values of these options affect the firm's investment decisions.

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# 1 Introduction.

Our recent book and survey articles on the real options approach to investment identify three characteristics of most investment decisions: (1) uncertainty over future profit streams, (2) irreversibility, i.e., the existence of some sunk costs that cannot be recouped if the firm changes its mind later, and (3) the choice of timing, i.e., the opportunity to delay the investment decision.<sup>1</sup> We argued that because of the *interaction* of these three forces, optimal investment decisions have to satisfy more stringent hurdles for their expected rates of return than the naive NPV criterion would indicate. The uncertainty implies that there may be future eventualities where the firm would regret having invested. The irreversibility implies that if the firm invests now, it cannot costlessly disinvest should such an eventuality materialize. And the opportunity to wait allows it to learn more about the uncertain future and reduce the likelihood of such regret.

By analogy with financial economics, the opportunity to invest is a call option — a right but not an obligation to make the investment. To invest is to exercise the option. Because of the uncertainty, the option has a time premium or holding value. It should not be exercised as soon as it is “in the money,” even though doing so has a positive NPV. The optimal exercise point comes only when the option is sufficiently “deep in the money” — the NPV of exercise is large enough to offset the value of waiting for more information. This conclusion is probably the most widely known “result” of the real options literature.

Of the triad of conditions mentioned above, the literature has focused on irreversibility. But most formal models assume simultaneously total irreversibility and a completely costless ability to wait, so they cannot separately identify the contributions of these two conditions. Exceptions to this include the seminal article by Brennan and Schwartz (1985), which examined an investment in a mining project and allowed for both an option to invest and an abandonment option, the models developed by Trigeorgis (1993, 1996) that allow for a variety of different options interacting within a single project, including options to expand and contract, and the work of Kulatilaka (1995) on substitutability and complementarity

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<sup>1</sup>See Dixit (1992), Pindyck (1991), and Dixit and Pindyck (1996).

in real options. Another exception is our recent article co-authored with Abel and Eberly (1996), henceforth referred to as ADEP, which developed a two-period model that allowed for arbitrary degrees of irreversibility and future expandability.

ADEP showed that a firm that makes an investment that is reversible (partially or totally) acquires a put option, namely the ability to pull out should future conditions be sufficiently adverse. This option has value if future uncertainty has a sufficiently large downside that the probability is positive that the firm will want to exercise the option. Therefore, recognition of this put option will make the firm *more* willing to invest than it would be under a naive NPV calculation that assumes that the project continues for its physical lifetime and omits the possibility of future disinvestment.<sup>2</sup> Likewise, a firm that can expand by making an investment now or in the future (at a finite cost) is exercising a call option, namely it is acting now when it might have waited. This option has value if future uncertainty has a sufficiently large downside that waiting would have been preferable. Therefore recognition of this call option will make the firm *less* willing to invest than it would be under a naive NPV calculation that assumes that the project must be started now or never, and ignores the possibility of a future optimal startup decision.

For many real-world investments, both of these options exist to some degree. Firms typically have at least some ability to expand their capacity at a time of their choosing, and sometimes can partially reverse their decisions by selling off capital and recovering part of their investment. The net effect of these two options will in general be ambiguous, depending on the degrees of reversibility and expandability, and the extent and nature of the uncertainty.

If the investment is totally irreversible, there is only a call option and no put option, and hence the investment must necessarily satisfy a stiffer hurdle than a positive NPV naively calculated. But as ADEP showed, it is not the irreversibility that gives rise to the call option; it is the *expandability* that does so. What irreversibility does is to eliminate the put option that acts in the opposite direction.

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<sup>2</sup>The option to abandon a project midstream is an example of this. Myers and Majd (1984) showed how this option can be valued and its implications for the investment decision.

One might argue that in practice irreversibility is more important than limited expandability, in part because of “lemons effects,” but mostly because many unpredictable shocks are industry-specific.<sup>3</sup> However, expandability can also be limited, e.g., because of limited land, natural resource reserves, or because of the need for a permit or license of which only a limited number are being issued. Thus it is important to recognize and clarify the effects of these different underlying economic conditions.

In this paper, we move beyond the two-period analysis in ADEP by examining a set of continuous-time models. These models allow for incremental capacity expansion and/or contraction over time, and thereby provide further insight into the effects of irreversibility, expandability, and the ability to wait. In this continuous-time setting, limited reversibility and expandability lead to clearly identifiable (and measurable) put and call options, which have opposite effects on the firm’s incentive to invest.

Most of our analysis will deal with exogenous and time-dependent limitations on the firm’s ability to expand or contract. Specifically, we will consider models in which the cost of investing increases over time (limited expandability), and the price that the firm can get by selling previously installed capital falls over time (limited reversibility). Our general framework is described in the next section. In Section 3, models with time-varying costs are presented in detail, and their implications for investment and capacity choice are explored. In Section 4 we briefly discuss capacity choice decisions when the cost of investing or disinvesting varies with the amount of capacity already installed. In the concluding Section 5, we suggest some extensions of our model for future work.

## 2 Continuous-Time Models of Capacity Choice.

The two-period model developed in ADEP showed the effects of the call and put options associated with investment in the simplest possible way. For a more realistic picture, however, we need a longer horizon, with ongoing uncertainty and repeated opportunities for the firm

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<sup>3</sup>A steel manufacturer will want to sell a steel plant when the steel market is depressed, but that is precisely the time when no one else will want to pay a price for it anywhere near its replacement cost. Therefore investment in a steel plant is largely irreversible.

to expand or contract in response to the changing circumstances. In such a setting, partial reversibility and expandability will arise when the costs of capacity contraction or expansion vary in response to changes in one or more exogenous or endogenous variables. We will consider two such variables in this paper.

First, we will examine what happens when the cost of investing or disinvesting varies exogenously with time. This would be the case, for example, if the cost of capacity expansion rises over time as the resources needed for such expansion (such as land or mineral reserves) are used up by other firms, or dwindle for physical reasons such as land erosion, or the depletion of a potentially discoverable resource base. Likewise, the sale price of used capital could fall over time, perhaps as a result of the increasing obsolescence of the capital.

Second, we will examine investment decisions when the cost of investing varies endogenously with the amount of capacity already installed by the firm in question. This kind of limited expandability would arise when the firm itself (which presumably has some monopoly power) uses up limited resources as it expands.

In all cases we will assume that the firm faces an isoelastic demand curve:

$$P = \theta(t)Q^{-1/\eta}, \quad (1)$$

where  $\eta$  is the elasticity of demand. We let the shift variable  $\theta$  vary stochastically according to a geometric Brownian motion:

$$d\theta = \alpha\theta dt + \sigma\theta dz. \quad (2)$$

Although this is certainly not critical, we will assume for convenience that the uncertainty over future values of  $\theta$  is spanned by capital markets. Hence there is some risk-adjusted rate of return for  $\theta$ , which we will denote by  $\mu$ . We will let  $\delta = \mu - \alpha$  denote the rate-of-return shortfall.

To simplify matters, we will assume that the firm has zero operating costs, and hence will always produce at its capacity, denoted by  $K$ . This eliminates any “operating options” that can affect the value of a unit of installed capital, and lets us focus on options associated purely with the investment decision.<sup>4</sup>

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<sup>4</sup>The most important operating option is the ability of the firm to reduce output or even shut down and



As in Pindyck (1988), we will examine the firm's incremental investment decisions. Let  $\Delta V(K; \theta, t)$  denote the value of the last incremental unit of installed capital, and let  $\Delta F(K; \theta, t)$  denote the value of the firm's option to install one more incremental unit. In the standard neoclassical model of investment,  $\Delta V$  would simply be the present value of the expected flow of marginal revenue from the unit, i.e.,

$$\Delta V_0(K; \theta, t) = \omega(K)\theta, \quad (3)$$

where

$$\omega(K) \equiv \left( \frac{\eta - 1}{\eta \delta} \right) K^{-1/\eta}. \quad (4)$$

Likewise,  $\Delta F$  would be the greater of zero or the NPV of immediate investment in this incremental unit. If the cost of an incremental unit of capital were fixed at  $k_0$ , then  $\Delta F$  in the neoclassical model would be given by:

$$\Delta F_0(K; \theta, t) = \max[0, \omega(K)\theta - k_0]. \quad (5)$$

The neoclassical model, however, ignores the values of the options that the firm has to buy or sell capacity in the future. These option values depend on the ability of the firm to make such purchases or sales, and the prices it will pay or receive for capital. In the next section, we let those prices vary with time.

### 3 Time-Dependent Costs.

Suppose that additional capacity can be added at a cost  $k(t) = k_0 e^{\rho t}$  per unit, with  $\rho \geq 0$ . In this setting, if  $\rho > 0$  so that the cost of adding capacity is rising over time, there is *partial expandability*; with  $\rho = 0$  there is complete expandability, and for  $\rho \rightarrow \infty$  there is no expandability. Note that in this model, limits to expandability are exogenous to the firm's actions;  $k(t)$  might rise, for example, because of continual entry or expansion by other firms that pushes up capital costs. Also, while we only consider values of  $\rho \geq 0$ , in practice  $\rho$

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thereby avoid variable costs. As one would expect, and as demonstrated by McDonald and Siegel (1985), this operating option raises the value of a unit of capital. For a discussion of this and other operating options, see Chapter 6 of Dixit and Pindyck (1996).

might be negative. This could occur if continual technological improvements or learning by doing cause per unit capital costs to fall over time.

Similarly, we will assume that installed capital can be sold, but only for a price  $S(t) = k_1 e^{-st}$  per unit, with  $s \geq 0$ . Hence there is *partial reversibility* that is completely time-dependent, reflecting, for example, the increasing obsolescence of capital (as opposed to its physical depreciation). If  $k_1 = k_0$ , then at  $s = 0$ , investment is completely reversible. If  $k_1 < k_0$ , then there is some irreversibility even at  $s = 0$ ; this can arise because of “lemons” effect. In either case, if  $s \rightarrow \infty$ , investment is completely irreversible. We call the effect of an initial gap between the purchase and sale prices of capital the “static” aspect of irreversibility, and the widening of the gap over time because of  $\rho > 0$  and  $s > 0$  the dynamic effects. For a while we focus on the dynamic effects by assuming  $k_1 = k_0$ . In Section 3.4 we bring in the dynamic effect, and compare the two.

We pause to explain our modeling choices. In making the cost of purchasing capital purely a function of time, we have in mind that cost increases are largely the result of the activities of other firms. For example, in an extractive industry such as oil or copper, other firms will deplete the potentially discoverable resource base over time; then expansion by a given firm becomes more expensive over time as the new deposits it seeks are harder to find and costlier to develop. Or, in the residential and commercial construction industries, other firms will buy and develop choice parcels of land over time, again making expansion by a given firm more expensive. Ideally, this process should be modelled in an equilibrium setting, so that each firm in the industry (including possible new entrants) makes its decisions consistent with rational expectations of the optimal behavior of all other firms. Although equilibrium models of entry and exit with sunk costs have appeared in the literature (see Chapters 8 and 9 of Dixit and Pindyck (1996) for an overview), here we focus on the optimal decisions of the manager of one firm. Most managers base their decisions on expectations of changes in market parameters, including capital purchase and resale prices. Managers may or may not think in terms of an overall industry equilibrium when they form these expectations, but they tend to treat these price movements as exogenous functions of time, much as we have treated them here.

We offer the same sort of justification for our assumption that capital can be sold for a price that declines with time, *irrespective of when the capital was purchased*. (Hence we are not considering physical depreciation, as in Chapter 6 or Dixit and Pindyck (1996), in which case the sale price would begin declining only after the capital has been purchased.) Again, we have in mind a pattern of obsolescence that is largely caused by other firms that are continually developing superior processes and/or products. Again, a fuller theory of such a pattern of technological “leapfrogging” might best be described in an equilibrium framework that would go beyond what we have tried to do here.

Finally, one might argue with our assumption that the purchase and sale prices of capital evolve *exponentially* with time. Obviously, this functional form was chosen for analytical convenience. One might imagine other forms of time dependence that might be more realistic for particular industries, but that may complicate the arithmetic that follows.

With these caveats in mind, we can note that limited expandability and reversibility creates options, the values of which must be taken into account when determining the firm’s optimal investment rules. In contrast to the neoclassical model,  $\Delta V$  in fact has two components: the value of the expected profit flow from the use of the unit, and the value of the (put) option to sell the unit and receive  $k_0 e^{-st}$ . Likewise,  $\Delta F$  must account for the full option value of the investment, i.e., the fact that the option has a time value and need not be exercised immediately.

Using standard methods, it is easy to show that  $\Delta V$  must satisfy the following differential equation:

$$\frac{1}{2}\sigma^2\theta^2\Delta V_{\theta\theta} + (r - \delta)\theta\Delta V_{\theta} + \Delta V_t - r\Delta V + \delta\omega(K)\theta = 0, \quad (6)$$

where  $\omega(K)$  is given by eqn. (4). The solution must also satisfy the following boundary conditions:

$$\lim_{\theta \rightarrow \infty} (\Delta V/\theta) = \omega(K) \quad (7)$$

$$\Delta V(K; \theta^{**}, t) = \Delta F(K; \theta^{**}, t) + k_0 e^{-st} \quad (8)$$

$$\Delta V_{\theta}(K; \theta^{**}, t) = \Delta F_{\theta}(K; \theta^{**}, t) \quad (9)$$

Here  $\theta^{**} = \theta^{**}(K, t)$  is the critical value of  $\theta$  at which it is optimal to exercise the put option and sell the unit of capital. Boundary condition (7) simply says that if  $\theta$  is very large, the firm will never want to sell off the unit of capital, so that its value is just the present value of the expected profit flow that it generates. Conditions (8) and (9) are the standard value matching and smooth pasting conditions that apply at the critical exercise point  $\theta^{**}$ .

Likewise,  $\Delta F$  must satisfy:

$$\frac{1}{2}\sigma^2\theta^2\Delta F_{\theta\theta} + (r - \delta)\theta\Delta F_{\theta} + \Delta F_t - r\Delta F = 0, \quad (10)$$

subject to boundary conditions

$$\Delta F(K; 0, t) = 0 \quad (11)$$

$$\Delta F(K; \theta^*, t) = \Delta V(K; \theta^*, t) - k_0 e^{\rho t} \quad (12)$$

$$\Delta F_{\theta}(K; \theta^*, t) = \Delta V_{\theta}(K; \theta^*, t) \quad (13)$$

$$\lim_{t \rightarrow \infty} \Delta F(K; \theta, t) = 0 \quad (14)$$

The first three of these conditions are standard; the last one says that (with  $\rho > 0$ ) the value of the call option to install an incremental unit approaches zero as time passes, because the cost of exercising is rising exponentially.

To clarify the nature of the optimal investment decision, it is best to proceed in steps. As we noted in the Introduction, most of the literature assumes that investment is completely irreversible and completely expandable. We will begin by considering the case in which investment is completely irreversible, but only partially expandable. Hence there is only a single boundary,  $\theta^*(K, t)$ , which triggers investment. This special case will help to elucidate the nature of the call option and its dependence on the extent of expandability. Next we will examine the case in which investment is partially irreversible but completely non-expandable, so that investment entails a put option (the value of which depends on the extent of reversibility), but no call option. In this case there is again only a single boundary, now  $\theta^{**}(K, t)$ , which triggers disinvestment. Then we will return to the general case set forth above.

### 3.1 Complete Irreversibility, Partial Expandability.

In this special case,  $s = \infty$  so the firm cannot disinvest. Then  $\Delta V$  is simply the present value of the flow of marginal revenue product from an incremental unit of capital:

$$\Delta V(K; \theta, t) = \left( \frac{\eta - 1}{\eta \delta} \right) K^{-1/\eta} \theta = \omega(K) \theta. \quad (15)$$

We can find the solution to eqn. (10) for the value of the option to install an additional unit of capital by guessing a functional form and choosing its parameters to satisfy all of the boundary conditions.

We will guess (and then verify) that the solution to eqn. (10) for  $\Delta F$  has the form:

$$\Delta F = a(K) \theta^{\beta_1} e^{-gt}. \quad (16)$$

The parameters  $\beta_1$ ,  $g$ , and  $a(K)$ , along with the critical value  $\theta^*$  are found from the boundary conditions. By substituting eqn. (16) for  $\Delta F$  into eqn. (10), we know that  $\beta_1$  must be a solution to the fundamental quadratic equation

$$\frac{1}{2} \sigma^2 \beta_1 (\beta_1 - 1) + (r - \delta) \beta_1 - r - g = 0. \quad (17)$$

From condition (11), we know that  $\beta_1$  must be the positive solution to this equation, i.e.,

$$\beta_1 = \frac{1}{2} - \frac{(r - \delta)}{\sigma^2} + \sqrt{[(r - \delta)/\sigma^2 - \frac{1}{2}]^2 + 2(r + g)/\sigma^2} > 1. \quad (18)$$

From conditions (12) and (13), the critical value  $\theta^*$  is given by:

$$\theta^*(K, t) = \left( \frac{\beta_1}{\beta_1 - 1} \right) \left( \frac{\eta \delta}{\eta - 1} \right) K^{1/\eta} k_0 e^{\rho t}. \quad (19)$$

Substituting this into boundary condition (12) gives the following expression for  $a(K)$ :

$$a(K) = (\beta_1 - 1)^{\beta_1 - 1} \left( \frac{\eta - 1}{\eta \delta \beta_1} \right)^{\beta_1} K^{-\beta_1/\eta} k_0^{1 - \beta_1} e^{[g - \rho(\beta_1 - 1)]t} \quad (20)$$

Since  $a(K)$  cannot depend on  $t$ , we know that  $g = \rho[\beta_1(g) - 1]$ . Substituting eqn. (18) for  $\beta_1(g)$  and solving the resulting equation gives the following expression for  $g$ :

$$g = \rho \left[ -\frac{1}{2} - \frac{(r - \delta - \rho)}{\sigma^2} + \sqrt{[(r - \delta - \rho)/\sigma^2 + \frac{1}{2}]^2 + 2\delta/\sigma^2} \right] > 0 \quad (21)$$

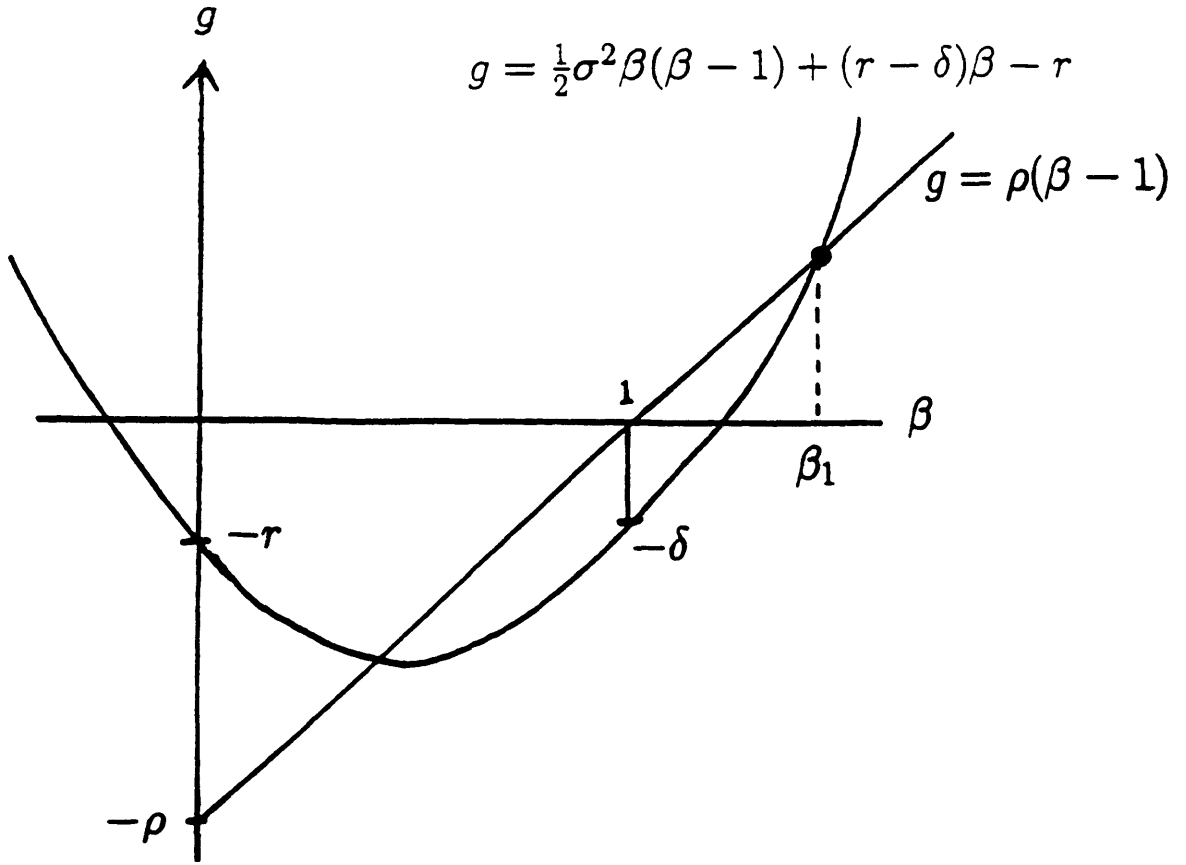


Figure 1: Complete Irreversibility, Partial Expandability — Graph of Eqn. (17) and the Line  $g = \rho(\beta - 1)$ , Showing Solution for  $g$  and  $\beta_1$ .

This solution for  $g$  and the relationship between  $g$  and  $\beta_1$  can be depicted more intuitively by rewriting (17) as

$$g = \frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r,$$

and plotting this along with the line  $g = \rho[\beta - 1]$ , as we have done in Figure 1. Note that the solution for  $g$  and  $\beta_1$  is found at the intersection of these two curves at which  $\beta > 1$ .

Here,  $\beta_1/(\beta_1 - 1) > 1$  is the standard “wedge” that arises in irreversible investment problems. But as  $\rho \rightarrow \infty$ ,  $\beta_1 \rightarrow \infty$  and  $g \rightarrow \infty$ . This can be seen either algebraically from eqns. (17) and (21), or from Figure 1) by observing that as  $\rho$  increases, the line  $g = \rho(\beta - 1)$  twists counterclockwise around the point  $(1, 0)$ . Then  $\beta_1/(\beta_1 - 1) \rightarrow 1$ , so that for  $t = 0$ ,  $\theta^* \rightarrow k_0/\omega(K)$ , i.e., the value it would have in the absence of uncertainty. One can also see from Figure 1 that if  $\rho \rightarrow \infty$ ,  $\beta_1 \rightarrow \infty$  and  $g \rightarrow \infty$ , so that  $\Delta F(K; \theta, t) = 0$  for  $t > 0$ , and

$$\Delta F = \max[0, \omega(K)\theta - k_0]$$

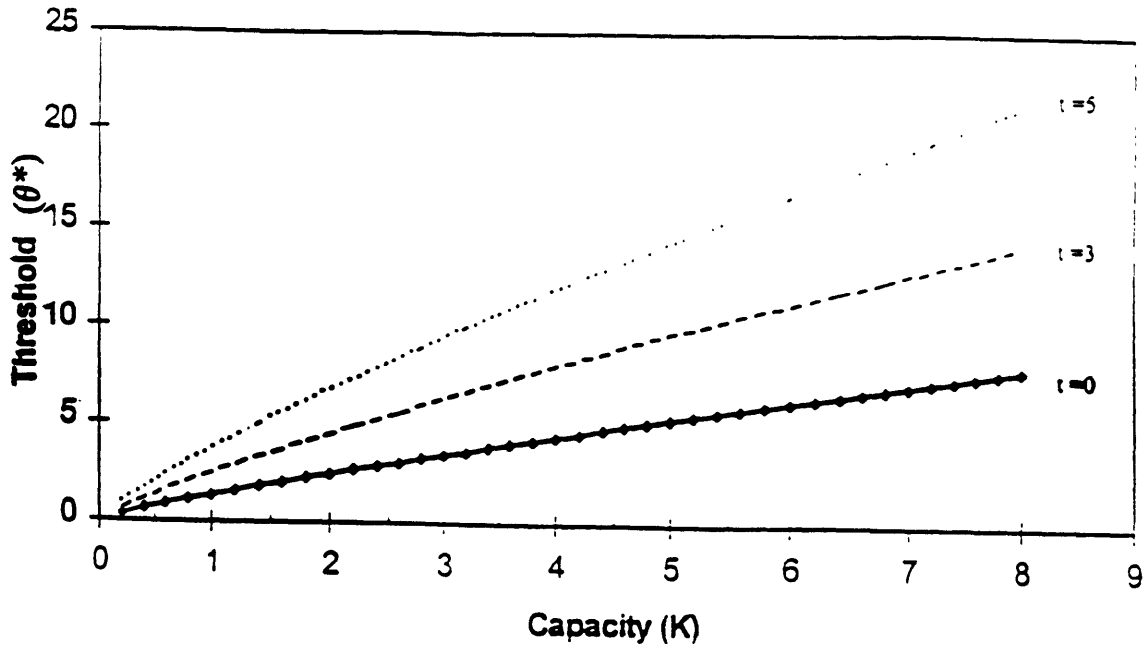


Figure 2: Complete Irreversibility, Partial Expandability — Solution for  $\theta^*(K, t)$  Plotted as a Function of  $K$ . (Parameter Values:  $r = .05$ ,  $\delta = .05$ ,  $\sigma = .40$ ,  $\eta = 1.20$ ,  $\rho = .20$ , and  $k_0 = 3.0$ .)

for  $t = 0$ . Note that then  $\Delta F$  is either zero or the net present value of the incremental investment — there is no option to invest after  $t = 0$ .

Figure 2 shows the solution for  $\theta^*(K, t)$  plotted as a function of  $K$  for three values of  $t$ . (The parameter values are  $r = .05$ ,  $\delta = .05$ ,  $\sigma = .40$ ,  $\eta = 1.20$ ,  $\rho = .20$ , and  $k_0 = 3.0$ .) Observe that the boundary moves up over time as the cost of investing increases. Figure 3 shows  $\theta^*(K, t)$  as a function of  $K$ , at  $t = 3$ , for three different values of  $\sigma$ . As is typical in investment problems of this kind, the value of the call option increases as  $\sigma$  increases, and thus so does the threshold that triggers investment.

Finally, note that if in addition to  $s = \infty$  we let  $\rho = 0$ , we have the case that has received the most attention in the literature, namely, complete irreversibility and complete expandability. Then  $g = 0$ ,  $\beta_1$  is the solution to the standard quadratic equation, and  $\theta^*$  is independent of time. See, e.g., Dixit and Pindyck (1996).

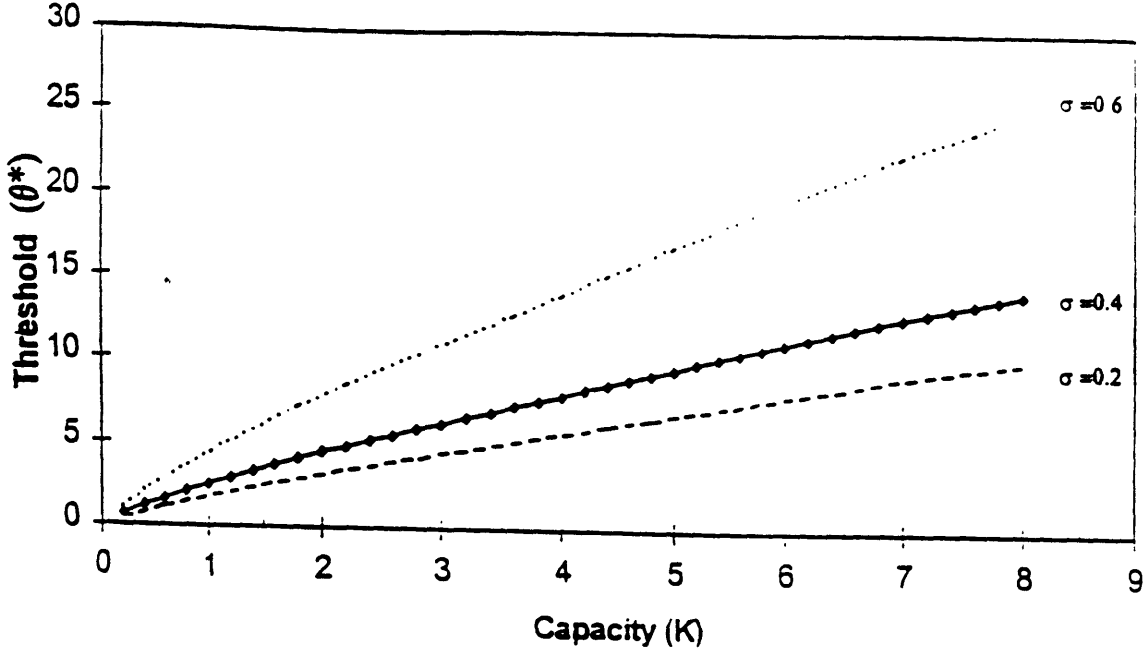


Figure 3: Complete Irreversibility, Partial Expandability — Dependence of  $\theta^*(K, t)$  on  $\sigma$ .

### 3.2 Partial Reversibility, No Expandability.

This is the case for which  $\rho = \infty$  and  $s > 0$ , so the firm can disinvest but cannot expand. Now the solution to eqn. (6) for  $\Delta V$  has the form:

$$\Delta V(K; \theta, t) = b(K)\theta^{\beta_2} e^{-ht} + \left( \frac{\eta - 1}{\eta\delta} \right) K^{-1/\eta} \theta, \quad (22)$$

where the first term on the right-hand side is the value of the put option to sell the unit of capital. This solution can be verified by direct substitution, and expressions for  $\beta_2$ ,  $h$ ,  $b(K)$ , and the critical value  $\theta^{**}$  can be found from boundary conditions (7) to (9).

Investment can either occur immediately, at  $t = 0$ , or never, so  $\Delta F$  is given by the standard NPV rule:

$$\Delta F(K; \theta, t) = \max[0, \Delta V(K; \theta, t) - k_0] \quad (23)$$

for  $t = 0$ , and  $\Delta F = 0$  for  $t > 0$ . In this case the boundary conditions that apply to  $\Delta V$  are not linked to those for  $\Delta F$ , and we can determine  $\Delta V$  independently from  $\Delta F$ . Since the firm has no call option to invest in the future, it will set its initial capacity  $K$  at the point where  $\Delta V(K; \theta, 0) = k_0$ . Hence the only issue is to determine  $\Delta V$ .



Substituting eqn. (22) into (6) and using boundary condition (7), we find that  $\beta_2$  is the negative solution to the quadratic equation (17), with  $g$  replaced by  $h$ :

$$\beta_2 = \frac{1}{2} - \frac{(r - \delta)}{\sigma^2} - \sqrt{[(r - \delta)/\sigma^2 - \frac{1}{2}]^2 + 2(r + h)/\sigma^2} < 0 \quad (24)$$

To obtain solutions for  $h$ ,  $b(K)$ , and  $\theta^{**}$ , we proceed as in the previous case, making use of boundary conditions (8) and (9) and the fact that  $b(K)$  must be independent of  $t$ :

$$\theta^{**}(K, t) = \left( \frac{\beta_2}{\beta_2 - 1} \right) \left( \frac{\eta \delta}{\eta - 1} \right) K^{1/\eta} k_0 e^{-st}, \quad (25)$$

$$b(K) = -\frac{1}{\beta_2} \left( \frac{\beta_2 - 1}{\beta_2} \right)^{\beta_2 - 1} \left( \frac{\eta - 1}{\eta \delta} \right)^{\beta_2} K^{-\beta_2/\eta} k_0^{1 - \beta_2}, \quad (26)$$

and

$$h = s \left[ \frac{1}{2} + \frac{(r - \delta + s)}{\sigma^2} + \sqrt{[(r - \delta + s)/\sigma^2 + \frac{1}{2}]^2 + 2\delta/\sigma^2} \right] > s. \quad (27)$$

Note that  $0 < \beta_2/(\beta_2 - 1) < 1$ . As  $\sigma$  increases,  $\beta_2$  increases toward 0, so that this multiple becomes smaller in magnitude. Thus the more uncertainty there is, the lower is the critical value of  $\theta$  that will trigger disinvestment. This is a standard result; see, e.g., the model of entry and exit in Dixit (1989) and Chapter 7 of Dixit and Pindyck (1994). But note that now this multiple depends on  $s$ , the rate at which the resale value of capital is falling. The larger is  $s$ , the closer this multiple is to one, and the smaller is  $b(K)$  and hence the value of the put option. As  $s \rightarrow \infty$ ,  $\beta_2 \rightarrow -\infty$ , and  $\beta_2/(\beta_2 - 1) \rightarrow 1$ . Then there is no put option, so that for  $t = 0$ ,  $\theta^{**}(K) \rightarrow k_0/\omega(K)$ , which is the value it would have in the absence of uncertainty.

Figure 4 shows the solution for  $\theta^{**}(K, t)$ , again plotted as a function of  $K$  for three values of  $t$ . (As before, the parameter values are  $r = .05$ ,  $\delta = .05$ ,  $\sigma = .40$ ,  $\eta = 1.20$ , and  $k_0 = 3.0$ , and now  $s = .20$ .) Observe that the boundary moves down over time as the price that the firm can receive for installed capital decreases. Figure 5 shows  $\theta^{**}(K, t)$  as a function of  $K$ , at  $t = 3$ , for three different values of  $\sigma$ . Now the value of the firm's put option increases as  $\sigma$  increases, and thus the threshold that triggers disinvestment moves down.

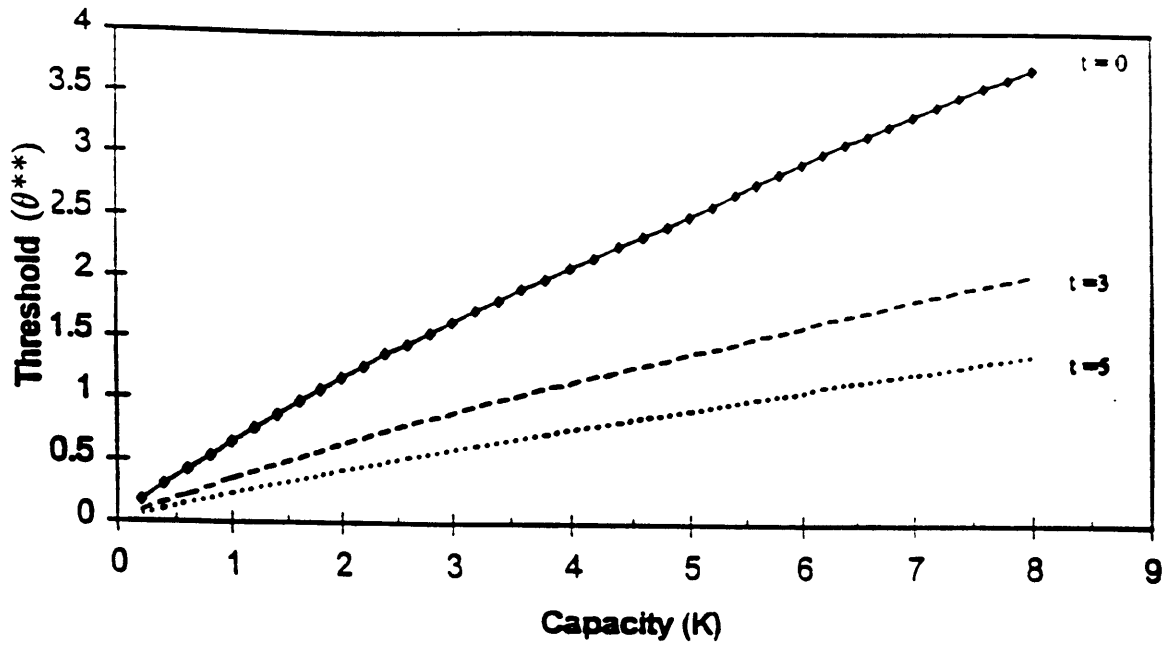


Figure 4: Partial Reversibility, No Expandability — Solution for  $\theta^{**}(K, t)$  Plotted as a Function of  $K$ . (Parameter Values:  $r = .05$ ,  $\delta = .05$ ,  $\sigma = .40$ ,  $\eta = 1.20$ ,  $k_0 = 3.0$ , and  $s = .20$ .)

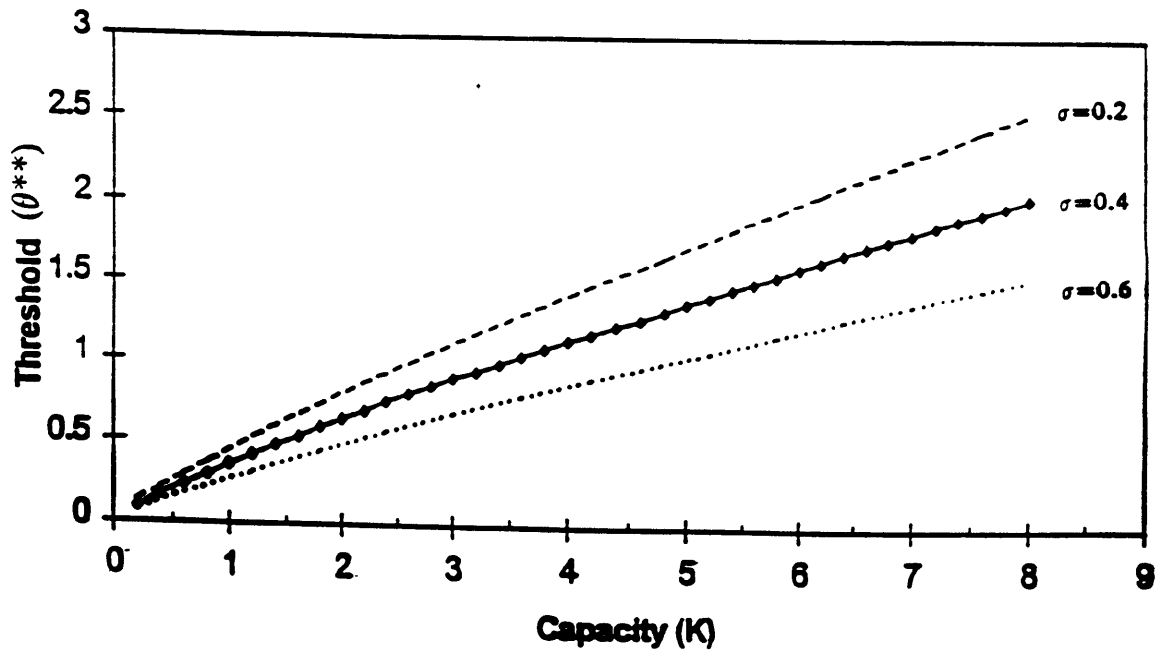


Figure 5: Partial Reversibility, No Expandability — Dependence of  $\theta^{**}(K, t)$  on  $\sigma$ .

### 3.3 The General Case.

In the general case,  $\rho$  and  $s$  are both positive and finite. Then  $\Delta V$  satisfies eqn. (6) subject to boundary conditions (7) – (9), and  $\Delta F$  satisfies eqn. (10), subject to boundary conditions (11) – (14). Equations (6) and (10) cannot be solved analytically in this case. Furthermore, it is difficult to obtain numerical solutions — although these are parabolic partial differential equations, they are linked to each other through the two sets of boundary conditions. Fortunately, however, we can obtain approximate solutions as long as  $gt$  and  $ht$  are not too small.

We make use of the fact that if  $gt$  and  $ht$  are large, the investment and disinvestment boundaries will be far apart, and thus the two sets of boundary conditions will be relatively independent of each other. (Intuitively, if the investment boundary is hit, it is likely to take a long period of time before the disinvestment boundary is hit, and vice versa.) It can be verified numerically that with  $gt$  and  $ht$  are large, the solutions to (6) and (10) will be of the time-separable form:

$$\Delta F = A(K)\theta^{\beta_1}e^{-gt}, \quad (28)$$

and

$$\Delta V(K; \theta, t) = B(K)\theta^{\beta_2}e^{-ht} + \left(\frac{\eta - 1}{\eta\delta}\right) K^{-1/\eta}\theta, \quad (29)$$

with  $\beta_1$ ,  $\beta_2$ ,  $g$ , and  $h$  again given by eqns. (18), (24), (21), and (27). The functions  $A(K)$  and  $B(K)$  and the critical values  $\theta^*(K, t)$  and  $\theta^{**}(K, t)$  can then be found from boundary conditions (8), (9), (12), and (13). Making the substitutions, these conditions become:

$$B(K)(\theta^{**})^{\beta_2}e^{-ht} + \omega(K)\theta^{**} = A(K)(\theta^{**})^{\beta_1}e^{-gt} + k_0e^{-st}, \quad (30)$$

$$\beta_2 B(K)(\theta^{**})^{\beta_2-1}e^{-ht} + \omega(K) = \beta_1 A(K)(\theta^{**})^{\beta_1-1}e^{-gt}, \quad (31)$$

$$A(K)(\theta^*)^{\beta_1}e^{-gt} = B(K)(\theta^*)^{\beta_2}e^{-ht} + \omega(K)\theta^* - k_0e^{\rho t}, \quad (32)$$

and

$$\beta_1 A(K)(\theta^*)^{\beta_1-1}e^{-gt} = \beta_2 B(K)(\theta^*)^{\beta_2-1}e^{-ht} + \omega(K). \quad (33)$$

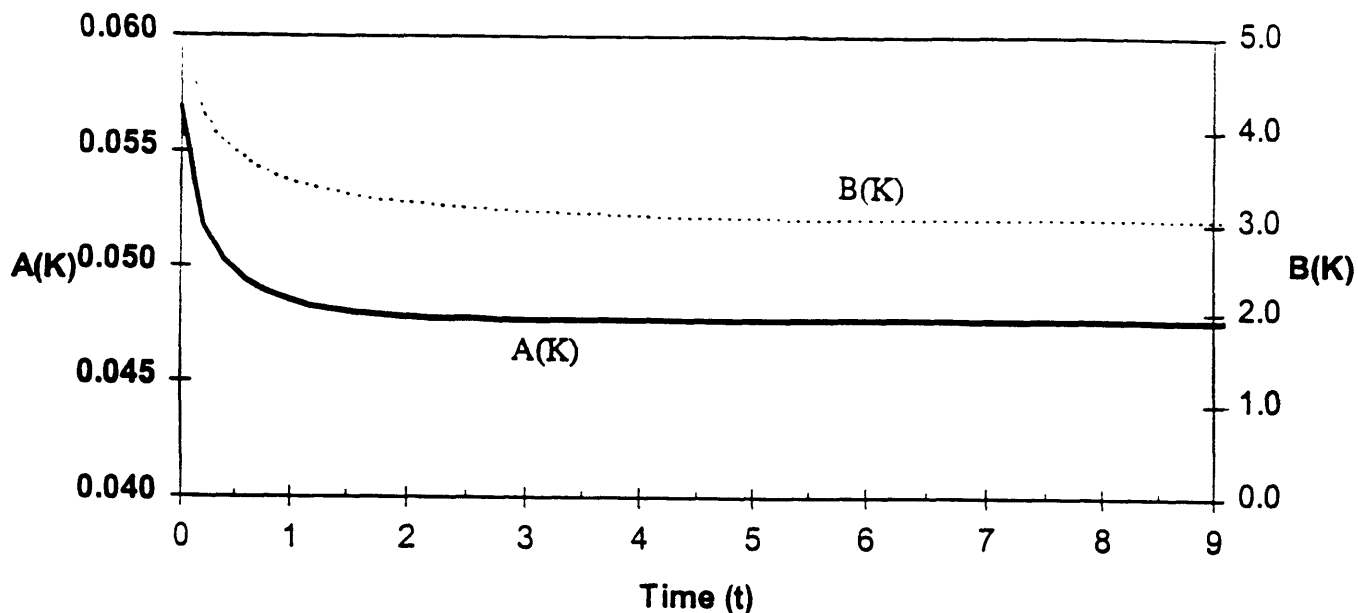


Figure 6: General Case — Numerical Solutions for  $A(K)$  and  $B(K)$ . (Parameter values:  $r = \delta = .05$ ,  $\sigma = .40$ ,  $\eta = 1.20$ ,  $k_0 = 3.0$ , and  $\rho = s = .20$ .)

Given values for  $K$  and  $t$ , these four equations can be solved numerically for  $A(K)$ ,  $B(K)$ ,  $\theta^*(K, t)$ , and  $\theta^{**}(K, t)$ . We can also check the accuracy of these approximate solutions by determining whether  $A(K)$  and  $B(K)$  remain constant as we vary  $t$ .

This is illustrated in Figure 6, which shows numerical solutions of eqns. (30) – (33) for  $A(K)$  and  $B(K)$  for  $K = 3$ , as  $t$  varies from 0 to 9. (The parameter values are  $r = \delta = .05$ ,  $\sigma = .40$ ,  $\eta = 1.20$ ,  $k_0 = 3.0$ , and  $\rho = s = .20$ .) Observe that  $A(K)$  and  $B(K)$  become roughly constant once  $t$  is greater than about 2.

Figure 7 shows solutions for the investment and disinvestment thresholds,  $\theta^*(K)$  and  $\theta^{**}(K)$ , as functions of  $K$ , for  $t = 2$  and 5. There are now three regions: If  $\theta > \theta^*(K)$ , the firm should immediately invest, increasing  $K$  (and thus increasing  $\theta^*$ ) until  $\theta = \theta^*$ . If  $\theta < \theta^{**}(K)$ , the firm should disinvest until  $\theta = \theta^{**}(K)$ . If  $\theta^{**} \leq \theta \leq \theta^*$ , the firm should take no action. Note that the thresholds  $\theta^*(K)$  and  $\theta^{**}(K)$  move apart over time, increasing the zone of inaction. This is also illustrated in Figure 8, which shows  $\theta^*$  and  $\theta^{**}$  as functions of time, for  $K = 3$ .

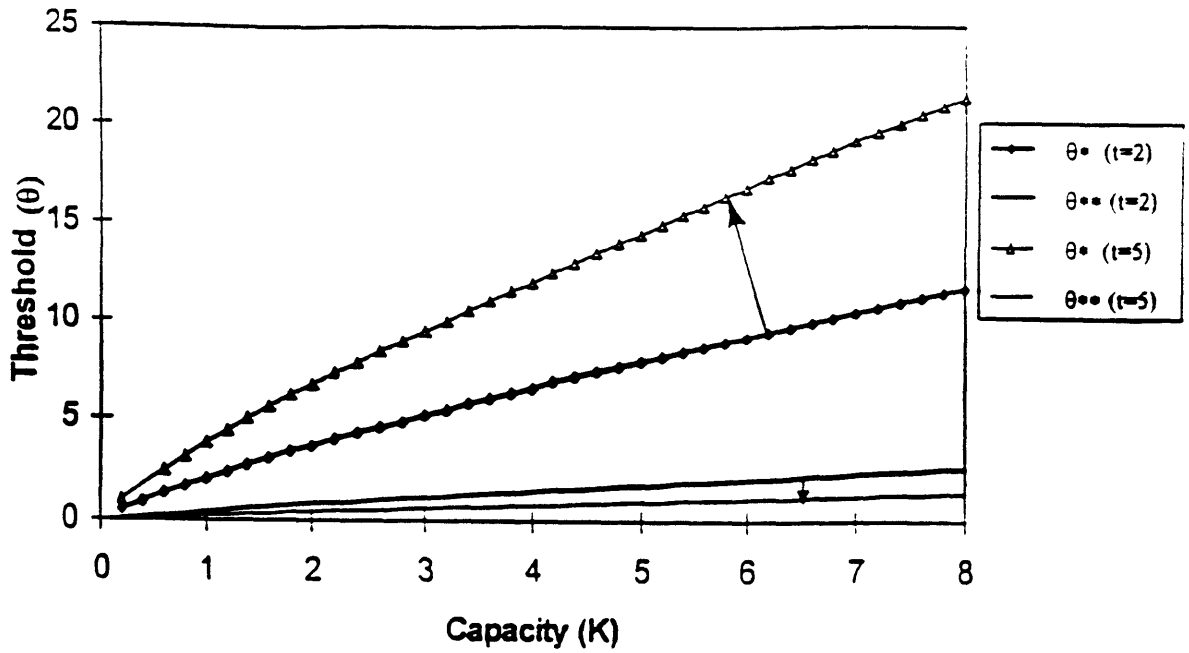


Figure 7: General Case — Investment and Disinvestment Thresholds,  $\theta^*$  and  $\theta^{**}$ , as Functions of  $K$ , for  $t = 2$  and  $t = 5$ .

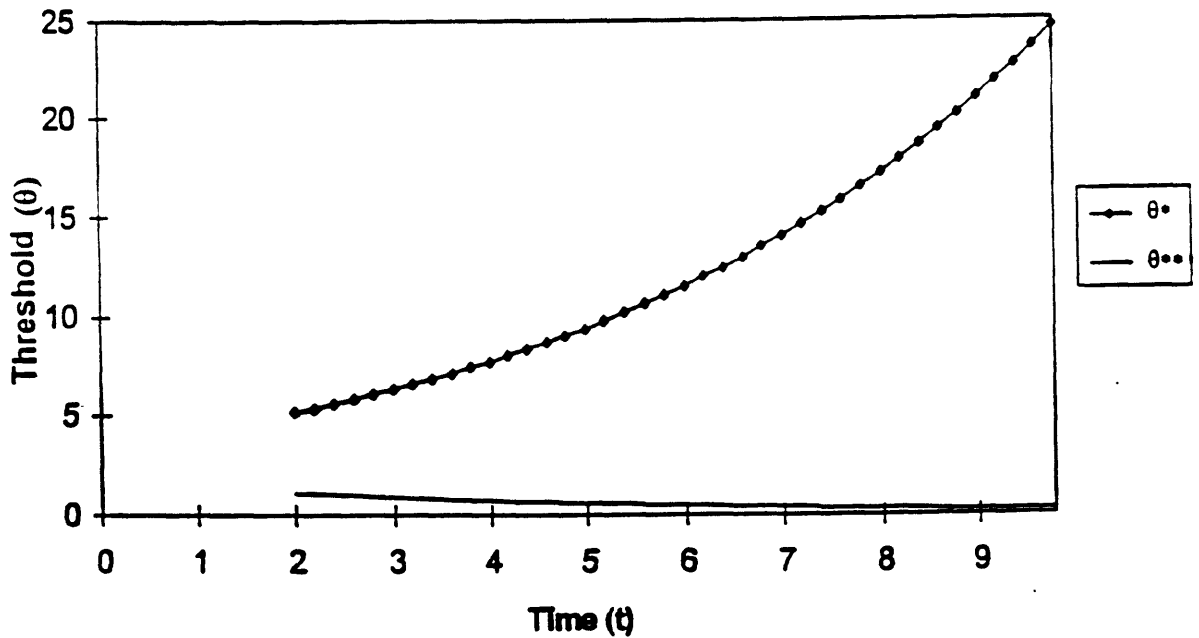


Figure 8: General Case — Movement of Investment and Disinvestment Thresholds over Time, for  $K = 3$ .

Finally, Figure 9 shows sample paths for  $\theta(t)$  and for capacity  $K(t)$ . Starting with no capital, at time  $t_0$  the firm immediately invests, bringing its capacity to  $K_0$ , such that  $\theta_0 = \theta^*(K_0, t_0)$ . From  $t_0$  until  $t_1$ ,  $\theta^{**}(K_0, t) < \theta(t) < \theta^*(K_0, t)$ , so the firm neither invests nor disinvests. Note that over this interval of time, the threshold  $\theta^*$  gradually increases as the cost of adding capacity increases, and the threshold  $\theta^{**}$  gradually decreases as the selling price of capacity decreases. At time  $t_1$ ,  $\theta(t)$  hits the upper threshold  $\theta^*$ , so the firm adds capacity. Over the interval  $t_1$  to  $t_2$ ,  $\theta(t)$  is increasing, and capacity is increased from  $K_0$  to  $K_1$ , so that  $\theta^*(K, t) = \theta(t)$ . (Note that the lower threshold  $\theta^{**}(K, t)$  also increases as  $K$  increases.) From  $t_2$  to  $t_3$ ,  $\theta^{**}(K_1, t) < \theta(t) < \theta^*(K_1, t)$ , so the firm is again inactive. At time  $t_3$ ,  $\theta(t)$  hits the lower threshold  $\theta^{**}$ , so the firm disinvests. From  $t_3$  to  $t_4$ ,  $\theta(t)$  continues to fall, and the firm's capacity is gradually reduced from  $K_1$  to  $K_2$ . After time  $t_4$  the firm is again inactive. Observe that as time goes on,  $\theta^*(K_2, t)$  gradually increases and  $\theta^{**}(K_2, t)$  decreases, so that the periods of investment or disinvestment become less and less frequent.

### 3.4 Static versus Dynamic Effects of Sunk Costs.

A difference between the current prices at which capital can be bought or sold will by itself create a zone of inaction in which the firm neither increases nor decreases capacity. This “static effect” of sunk costs of entry and exit is a standard result; see, e.g., Chapter 7 of Dixit and Pindyck (1996). However, the expectation that the purchase and sales prices will, respectively, increase and decrease *in the future* also affects the *current* investment thresholds. It is useful to separate these “static” and “dynamic” effects of limited expandability and reversibility.

Suppose we begin at some time  $t_1$  at which the purchase price of a unit of capital,  $k_p$ , exceeds the resale price,  $k_r$ . To determine the static effect of this differential, we calculate the investment and disinvestment thresholds,  $\theta^*(K)$  and  $\theta^{**}(K)$ , under the assumption that these prices will remain fixed over time from  $t_1$  onward. (These thresholds will, of course, also be fixed through time.) Next, we calculate  $\theta^*(K, t)$  and  $\theta^{**}(K, t)$  under the assumption that at any future time  $t > t_1$ , the purchase price will be  $k_p e^{\rho(t-t_1)}$  and the resale price will be  $k_r e^{-s(t-t_1)}$ . Of course this “dynamic”  $\theta^*(K, t)$  will rise over time, and this “dynamic”

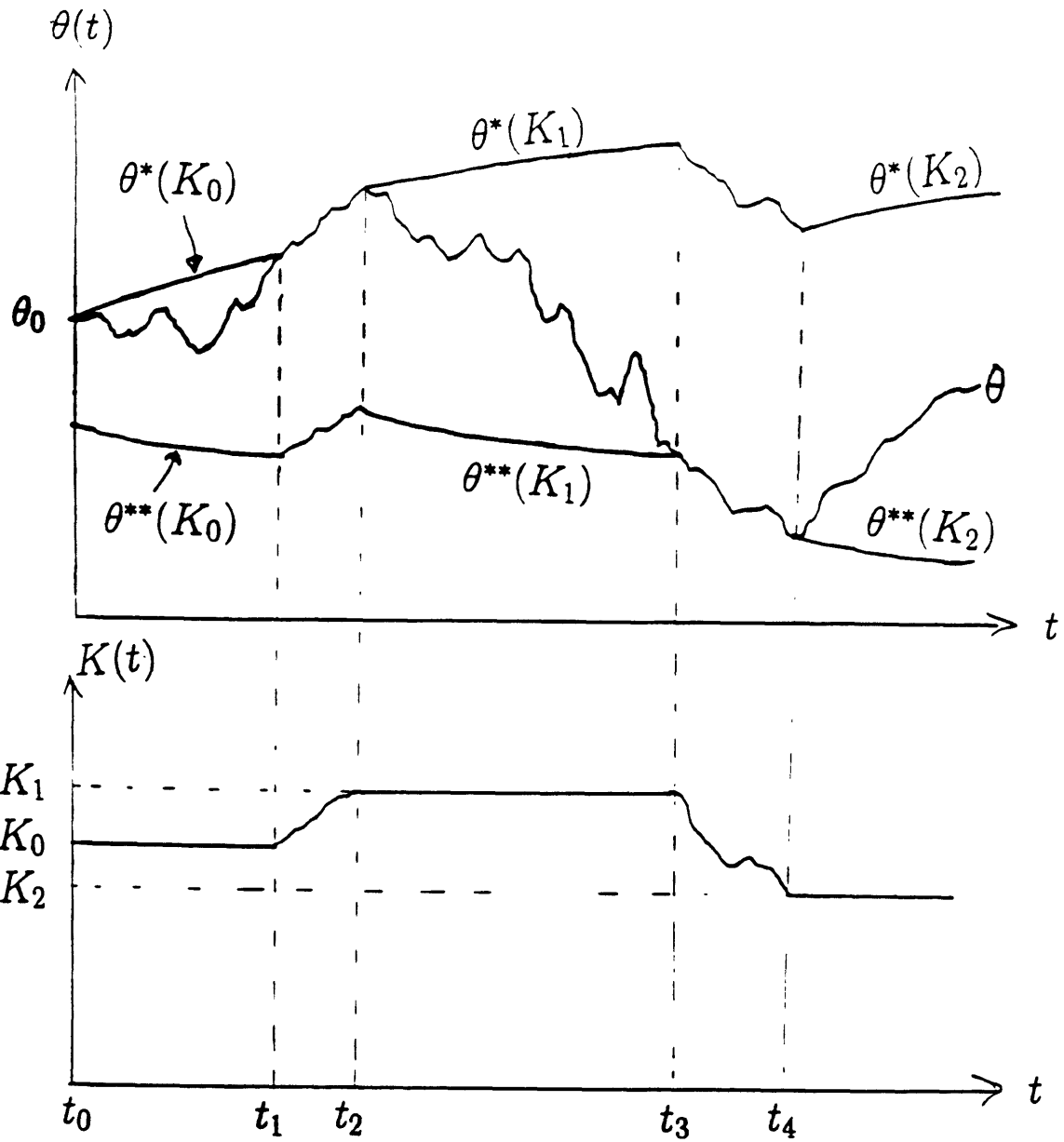


Figure 9: Optimal Investment and Disinvestment — Sample Paths of  $\theta(t)$  and Capacity  $K(t)$ .

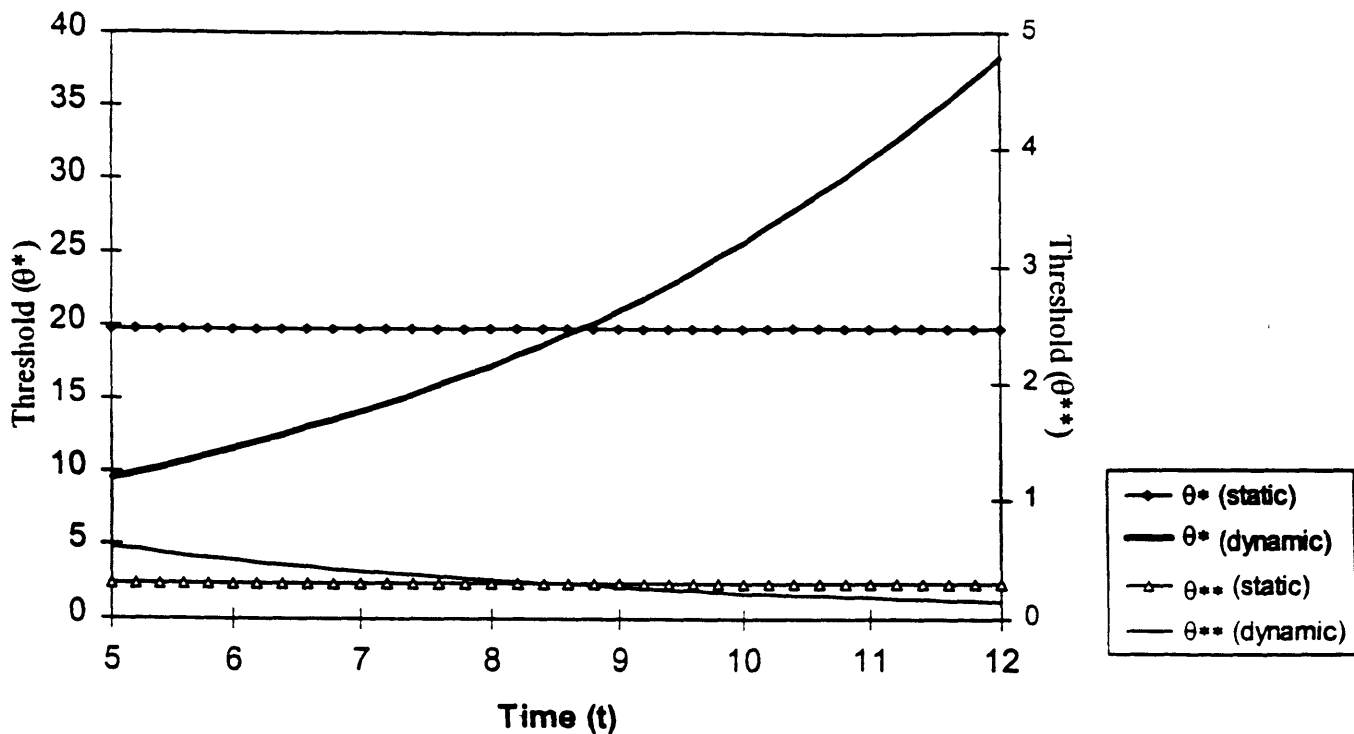


Figure 10: Investment and Disinvestment Thresholds for Static versus Dynamic Capital Costs. (Starting at  $t = 5$ , Purchase Price is  $k_p = k_0 e$  and Resale Price is  $k_r = k_0 e^{-1}$ . Other Parameters:  $r = \delta = .05$ ,  $\sigma = .40$ ,  $\eta = 1.20$ ,  $k_0 = 3.0$ ,  $K = 3$ , and  $s = \rho = 0.2$ .)

$\theta^{**}(K, t)$  will fall over time, but it is of interest to compare the static and dynamic thresholds at the initial time,  $t_1$ .

We have done this, and illustrated the results in Figure 10. In that figure, we start out at  $t = 5$  with  $k_p = k_0 e$  and  $k_r = k_0 e^{-1}$ . (The other parameter values are  $r = \delta = .05$ ,  $\sigma = .40$ ,  $\eta = 1.20$ ,  $k_0 = 3.0$ , and  $K = 3$ .) Static thresholds are calculated assuming  $k_p$  and  $k_r$  remain fixed at these levels, and dynamic thresholds are calculated assuming that  $k_p(t) = k_0 e^{\rho t}$  and  $k_r(t) = k_0 e^{-st}$ . As can be seen in the figure, initially the zone of inaction is smaller in the dynamic case than in the static one. However, this zone of inaction grows in the dynamic case as  $\theta^*$  rises and  $\theta^{**}$  falls, so that it eventually exceeds the zone in the static case.

Why is  $\theta^*(K)$  initially lower in the dynamic case? There are two forces at work, and they have opposite effects. First, the fact that the purchase price of capital is expected to rise in the future reduces the value of the firm's call option on an incremental unit of capital, which reduces the value of waiting and pushes  $\theta^*(K)$  down. Second, the fact that the resale price



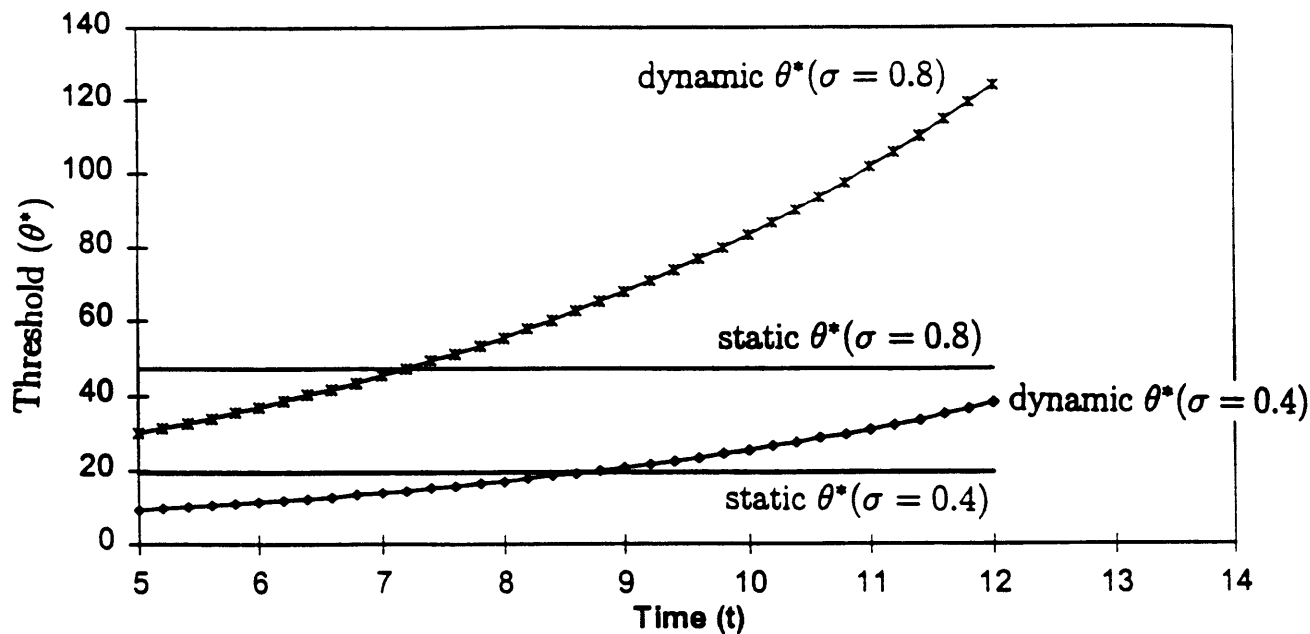


Figure 11: Static and Dynamic Investment Thresholds:  $\sigma = .4, .8$ .

of capital is expected to fall in the future reduces the value of the firm's put option on an incremental unit of installed capital, which reduces the total value of the unit of capital, and pushes  $\theta^*(K)$  up. In the example shown in Figure 10 the first effect outweighs the second, so  $\theta^*(K)$  falls.

The situation is similar with respect to  $\theta^{**}(K)$ . Again, the fact that the resale price  $k_r$  is expected to fall reduces the value of the firm's put option on an incremental unit of capital, which reduces the value of waiting to disinvest, and pushes  $\theta^{**}(K)$  up. And the fact that the purchase price  $k_p$  is expected to rise reduces the value of the call option, which raises the cost of disinvesting now, and pushes  $\theta^{**}(K)$  down. Once again, in this example the first effect outweighs the second, so  $\theta^{**}(K)$  rises.

Of course, the magnitudes of these effects will depend on various parameter values besides those of  $\rho$  and  $s$ . For example, Figures 11 and 12 show these static and dynamic thresholds, and their movements over time, for two different values of  $\sigma$ , the volatility of demand fluctuations. Note that when  $\sigma$  is larger, both the static and dynamic investment thresholds are higher, and the static and dynamic disinvestment thresholds are lower.

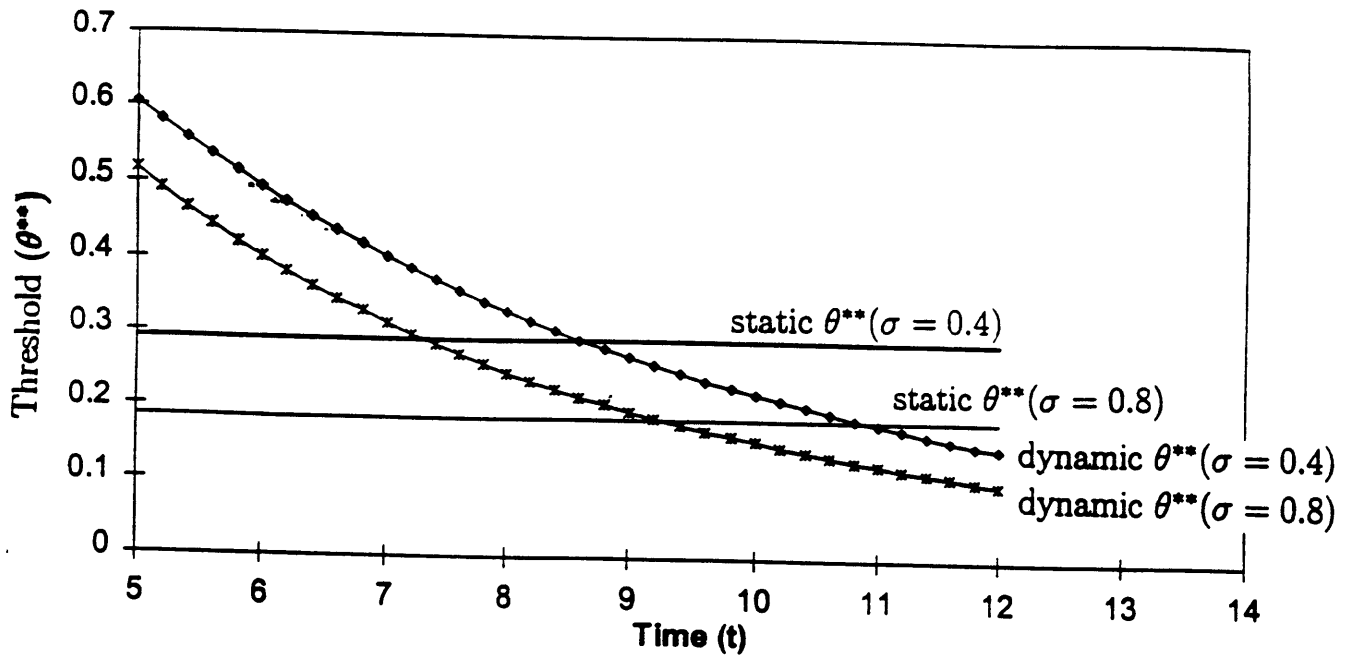


Figure 12: Static and Dynamic Disinvestment Thresholds:  $\sigma = .4, .8$ .

Now that we have a better understanding of the source and nature of the dynamic effects, we can show their dependence on the rates of change of the purchase and resale prices of capital,  $\rho$  and  $s$  respectively. We do this in Table 1 for a representative case. The initial investment cost is 3 and the initial resale value is 1. This gap, and the uncertainty ( $\sigma = 0.4$ ), are so large that under static conditions ( $\rho = s = 0$  from here on), the initial investment threshold is more than 6.5, and the initial disinvestment threshold less than 0.1. The table shows what happens to the corresponding initial values of the dynamic thresholds as we vary  $\rho$  and  $s$ . The upper panel shows the investment threshold  $\theta^*$ . It decreases as  $\rho$  increases, because the call option becomes less valuable. In fact, for very large values of  $\rho$ , it may become optimal to invest even though  $\theta$  is less than the purchase price of capital. The latter is expected to grow so fast that it pays the firm to acquire the capital right away while it is cheap. Also,  $\theta^*$  increases as  $s$  increases, because the put option of disinvestment is less valuable. These results reconfirm the intuition we stated above. But the numerical calculations reveal an interesting insensitivity: the effect of  $s$  on  $\theta^*$  is very small in magnitude. The gap between the two thresholds is sufficiently large that when

Table 1: Effects of  $\rho$  and  $s$  on Thresholds

(Parameters:  $r = 0.05$ ,  $\delta = 0.05$ ,  $\sigma = 0.4$ ,  $\eta = 1.2$ ,  $k_p = 3$ ,  $k_r = 1$ )  
 Static Thresholds  $\theta^* = 6.58$ ,  $\theta^{**} = 0.098$

Initial Dynamic Investment Threshold $\theta^*(t_1)$				
$\rho$	$s$			
	0.0	0.2	0.4	0.6
0.0	6.586	6.716	6.716	6.716
0.2	3.073	3.144	3.144	3.144
0.4	2.469	2.528	2.528	2.528

Initial Dynamic Disinvestment Threshold $\theta^{**}(t_1)$				
$\rho$	$s$			
	0.0	0.2	0.4	0.6
0.0	0.098	0.246	0.283	0.300
0.2	0.084	0.201	0.231	0.244
0.4	0.084	0.200	0.230	0.244

$\theta$  is at the upper threshold, it is unlikely to fall to the lower threshold in any reasonable future. Therefore options that get exercised in that unlikely and remote eventuality do not have a significant effect on today's decision. The investment and disinvestment thresholds get effectively separated, as studied in our earlier separate discussions of the two choices. This seems quite a robust feature of our numerical calculations, and it may provide a useful simplification for solution of combined investment and disinvestment problems when one or both of the degree of irreversibility and the uncertainty are large. Bonomo (1994) discusses a similar issue in the context of models of impulse control, namely when a one-sided s-S rule is a sufficiently good approximation to a two-sided s-S policy.

The lower panel of the table shows the dynamic disinvestment threshold  $\theta^{**}$  in relation to

$\rho$  and  $s$ . The results reconfirm the intuition stated above: the threshold rises as  $s$  increases, because the put option exercised by disinvesting is less valuable, and it falls as  $\rho$  increases, because the call option that would be acquired upon investment is less valuable. And again we find an effective separation of the two decisions:  $\theta^{**}$  is relatively insensitive to changes in  $\rho$ , particularly for higher values.

## 4 Capacity-Dependent Costs.

We turn now to endogenous variations in the costs of investing and disinvesting. We will briefly consider situations in which the ability of the firm to add to or reduce its capacity in the future is dependent on its own past actions, and in particular on the amount of capacity that it already has in place, rather than the amount of time that has elapsed.

Specifically, we will assume that the firm can add capacity at any time in the future at a cost  $k_0 + \rho(K)$  per unit of capital, with  $d\rho/dK > 0$ . In effect, it becomes more expensive to add capacity the more capacity the firm already has. Thus the limits to expandability are driven by market parameters, such as population, available land, etc., but depend on the firm's own actions rather than the actions of its competitors. In the simplest case, one could make the incremental cost of capacity expansion linear, i.e.,  $\rho(K) = \rho K$ . Then, if  $\rho = \infty$  there is no expandability (even from zero), and if  $\rho = 0$  there is complete expandability.

Likewise, we could assume that the firm can sell off capacity at any time in the future, and if its current capacity is  $K$ , it will receive  $s_0 k_0 + s_1 \rho(K)$  for an incremental unit, with  $0 \leq s_0 \leq 1$ , and  $s_1 < 1$ . In the simplest linear case, the firm receives  $s_0 k_0 + s_1 \rho K$ . Thus the degree of irreversibility is different for each marginal unit. We would expect that irreversibility would be greater the greater is  $K$ , since the demand for used industry-specific capital is likely to be smaller the greater is the amount of installed capacity already in place. In this case,  $s_1 < s_0$ . Also, it might be the case that  $s_1 < 0$ , so that if  $K$  is large enough, the firm receives a negative amount on its sale of an incremental unit. (This could occur, for example, if the firm faces land reclamation costs.) Finally, if  $s_0 = s_1 = 0$  there is complete irreversibility, and if  $s_0 = s_1 = 1$  there is complete reversibility.

Eqns. (6) and (10) will again apply for  $\Delta V$  and  $\Delta F$ , but now without the  $\Delta V_t$  and  $\Delta F_t$  terms. Hence the investment problem is now much simpler;  $\theta^{**}$  and  $\theta^*$  each depend only on  $K$ , and not on  $t$ . In the general case, the boundary conditions again result in four nonlinear equations for  $A(K)$ ,  $B(K)$ ,  $\theta^{**}(K)$  and  $\theta^*(K)$ , and these can easily be solved numerically for each value of  $K$ .

We do not present numerical solutions here, because the basic effects are similar to those in the entry/exit models discussed in Dixit and Pindyck (1996). What is different here is that the investment and disinvestment thresholds,  $\theta^*(K)$  and  $\theta^{**}(K)$ , depend on  $K$ . As  $K$  increases, the direct value (i.e., present value of the marginal revenue product stream) of an incremental unit of capital falls, as does the value of the call option on the unit. The drop in the direct value raises the investment threshold  $\theta^*(K)$ , and the drop in the option value reduces the threshold. The first effect dominates, however, so that  $\theta^*(K)$  increases with  $K$ . The opposite may be true for the disinvestment threshold  $\theta^{**}(K)$ . As  $K$  increases, the direct value of the incremental unit of capital falls, but the value of the put option on the unit increases. The degree of reversibility (i.e., the resale value of capital) determines whether value of the put option exceeds the direct value of the incremental unit. If it does,  $\theta^{**}(K)$  will also increase with  $K$ .

## 5 Concluding Remarks.

We have illustrated how the call and put options associated with expandability and reversibility interact to affect optimal capacity decisions, and the evolution of capacity over time. Expandability and reversibility can take a variety of forms. For example, a firm might be able to expand only at particular points in time (e.g., a forest products or extractive resource firm might have to wait for the government to auction off land or resource reserves), and the ability to sell existing capital might occur unpredictably as a Poisson arrival (e.g., when there are very few potential buyers who might become interested in a specialized piece of capital). We have examined only very special forms of expandability and reversibility — namely, capital purchase and sales prices that evolve exogenously with time, or endoge-

nously with the level of installed capacity. Nonetheless, we hope that this has helped to elucidate the basic effects. Most importantly, we could see how the future rates of growth of the investment cost and the resale price of capital affected the values of the call and put options associated with expansion and disinvestment. Our numerical solutions revealed that the investment and disinvestment decisions became separated when the initial gap between the purchase and resale prices of capital, or the rate at which this gap grows, is large. This simplifies the analytical and numerical solution of these problems.

There are many ways in which this analysis can be extended or generalized. Here we list a few. [1] Various aspects of increased realism can be added to our models, but at the cost of increased complexity. For example, we took variable costs to be zero, so that we could ignore the firm's operating options. It would be messy but not too difficult to extend our model by making variable cost positive and allowing the firm to vary its capacity utilization. Similarly, other operating options can be added. [2] We considered a firm's decision problem in isolation, and therefore took the rates of variation of the purchase and resale prices, whether as functions of time or as functions of existing capacity, to be exogenous. These can be endogenized in a fuller general equilibrium analysis. [3] We confined ourselves to a setting in which the maximization problem is well-behaved. An aspect of this was our assumption in the previous section that the purchase price of capital increases with capacity. If we allowed it to decrease, the firm would enjoy increasing returns to capacity expansion and its optimal policy could consist of infrequent large jumps in its capital stock. Dixit (1995) shows how to find the optimal timing and size of such jumps, but numerical work for more specific parameterized models can provide useful further insights. [4] Physical depreciation of capital can be modeled in a more detailed way. Thus the resale price of newly installed machine would equal its purchase price, but would fall with the *age* of this machine, not with *calendar time* as in our model here. However, that would require the model to keep track of the installation dates or age profiles of the whole stock of machines. This would make the state variable infinite-dimensional, presenting daunting modeling and solution challenges.

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