

**A Markup Interpretation of Optimal Rules for
Irreversible Investment**

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**Avinash Dixit, Robert S. Pindyck and
Sigbørn Sødal**

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Avinash Dixit
Princeton University

Robert S. Pindyck
Massachusetts Institute of Technology

Sigbjørn Sødal
Agder College, Norway

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Abstract: We re-examine the basic investment problem of deciding when to incur a sunk cost to obtain a stochastically fluctuating benefit. The optimal investment rule satisfies a trade-off between a larger versus a later net benefit; we show that this trade-off is closely analogous to the standard trade-off for the pricing decision of a firm that faces a downward sloping demand curve. We reinterpret the optimal investment rule as a markup formula involving an elasticity that has exactly the same form as the formula for a firm's optimal markup of price over marginal cost. This is illustrated with several examples.

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1. Introduction.

Consider what is probably the most basic irreversible investment problem: a project can be undertaken that requires a sunk cost C and yields a benefit V . The cost is known and certain, but the benefit (measured as the present value at the time the cost is incurred) fluctuates as a stationary Markov process $\{V_t\}$ with continuous sample paths.¹ Time is continuous, and at each point the firm must decide whether to invest or to wait and reconsider later. The firm's objective is to maximize the expected present value of net benefits, with a discount rate that is constant and equal to ρ .

At time t , all of the information about the future evolution of V is summarized in the current value V_t . Therefore the optimal decision rule must be of the form: invest now if V_t is in a certain subset of possible values. otherwise wait. Also, because the process is stationary and the discount rate is constant, the optimal rule will be independent of time. So long as the process has positive persistence — i.e., a higher current value V_t shifts the distribution of the random value V_s at any future time s to the right in the sense of first-order stochastic dominance — the rule will be of the form: invest now if V_t is at or above a critical threshold V^* , otherwise wait.² The problem therefore boils down to determining the optimal choice for the threshold V^* .

As first shown by McDonald and Siegel (1986), the optimal V^* exceeds C by a “markup,” or premium, that reflects the value of waiting for new information. One can think of the firm as having an option to invest that is akin to a financial call option, and, like the call option, is optimally exercised only when “deep in the money,” i.e., when the stock price is at a premium over the exercise price. Thus one can solve the firm's investment problem (and determine the optimal markup) by finding the value of the firm's option to invest and the optimal exercise rule.³ Indeed, identifying and valuing the firm's option to invest has become the standard approach to solving irreversible investment problems.

¹ V may itself be explained in terms of other more basic economic variables like prices of output and/or inputs; we work simply with the end result.

²See Dixit and Pindyck (1996), pp. 104, 128–9.

³The option is valued assuming it is exercised optimally, so the valuation of the option yields the optimal exercise rule. See Dixit and Pindyck (1996).

However, as Baily (1995) has pointed out, an alternative way to find the optimal V^* is to examine the trade-off between larger versus later net benefits. Specifically, choosing a larger value for V^* implies that the net benefit, $V^* - C$, will be larger, but will be received at a more distant (but unknown) time in the future, and thus will be discounted more heavily. The optimal choice of V^* is that for which the additional net benefit from making V^* larger just balances the additional cost of discounting.

In this paper, we take this alternative perspective further by developing an intuitively appealing analogy with the trade-off involved in the pricing decision of a firm facing a downward-sloping demand curve — i.e., the trade-off between a higher profit margin and a lower volume of sales. We show that V can be regarded like a price, $(V - C)$ like a profit margin or markup, and the discount factor like a demand curve. The optimal V^* is then given by a markup formula involving the elasticity of the discount factor with respect to V , which has exactly the same form as the formula for a firm's optimal markup of price over marginal cost. This suggests extensions of the basic investment problem by analogy with the corresponding extensions of the monopolist's pricing problem. Here we develop one, namely the optimal choice of an ancillary expenditure in advertising or R&D which can speed up the (stochastic) passage to the threshold. The result is analogous to the formula for a monopolist's optimal advertising-to-sales ratio.

2. The Optimal Markup.

Suppose the initial level of the benefit is V_0 , and consider an arbitrary threshold $V > V_0$. Thus the firm will wait until the first time T at which the benefit V_T has reached V , and will then invest. (In technical terms, T is the first-passage time or hitting time from V_0 to V .) This time T is a random variable, and its distribution can be determined from the known probability law of the evolution of V_t . Taking expectations using this distribution, the net present value of this policy is

$$\mathcal{E}[e^{-\rho T}] (V - C).$$

Note that the expectation of the discount factor in this expression depends on both the initial value V_0 and on the threshold value V of our decision rule. We therefore denote this

discount factor as:

$$D(V_0, V) \equiv \mathcal{E}[e^{-\rho T}]. \quad (1)$$

The *optimal* threshold, V^* , is the value of V which maximizes

$$D(V_0, V) (V - C). \quad (2)$$

The first-order condition for the optimal V^* is

$$D(V_0, V^*) + D_V(V_0, V^*)V^* = D_V(V_0, V^*)C, \quad (3)$$

where D_V is the partial derivative of the discount factor D with respect to its second argument, namely the threshold value V , and we are evaluating this at $V = V^*$. This condition simply says that if the investment opportunity is to be optimally exercised, the expected marginal discounted benefit from the investment should just equal the expected marginal discounted cost.

We can rewrite eqn. (3) in the following equivalent form:

$$\frac{V^* - C}{V^*} = \left[-\frac{V^* D_V(V_0, V^*)}{D(V_0, V^*)} \right]^{-1} = 1/\epsilon_D, \quad (4)$$

where ϵ_D denotes the elasticity of the discount factor D with respect to V^* , i.e., $\epsilon_D \equiv -V^* D_V/D$. The form of this expression should be very familiar: it is just like the markup pricing rule that follows from equating marginal revenue with marginal cost:

$$\frac{p - c}{p} = 1/\epsilon_P,$$

where p is the price, c is marginal cost, and ϵ_P is the magnitude of the price elasticity of the firm's demand.

There is indeed a close connection between equation (4) for the investment markup and the markup pricing rule. To see this, compare the expression for the present value, (2), to that for the firm's profit in the usual pricing problem when marginal cost is constant, namely $(p - c)q(p)$. A higher p implies a higher profit margin $(p - c)$, but a lower volume of sales $q(p)$. The trade-off that determines the optimal price is governed by the rate at which $q(p)$ declines as p is increased, i.e., by the price elasticity of demand. In our investment problem,

a higher threshold V^* yields a higher margin ($V^* - C$) of benefits over costs, but a smaller discount factor $D(V_0, V^*)$ because the process is expected to take longer to reach the higher threshold. The investment trade-off depends on the elasticity of the discount factor with respect to the threshold.

We can put this analogy in graphical terms by considering an arbitrary threshold V , and re-writing eqn. (3) as

$$V + D(V_0, V)/D_V(V_0, V) = C. \quad (5)$$

We can think of the first term in this equation, $V = V(D, V_0)$, as the inverse of the discount factor; it is analogous to the inverse demand, or average revenue function, $p(q)$, for the price-setting firm. Likewise, the discount factor $D(V, V_0)$ is analogous to quantity for the price-setting firm, so the left-hand side of eqn. (5) — the marginal benefit from an increase in D — is analogous to the marginal revenue function.⁴ These two functions of the discount factor D are plotted in Figure 1. The optimal threshold V^* , and the corresponding optimal discount factor $D^*(V_0, V^*)$, are found at the point where the marginal benefit $D(V_0, V) + D/D_V$ is equal to the cost, C . Note that $V^* > C$; this is the markup that incorporates the option premium, or value of waiting. If the firm instead used a simple Net Present Value rule to decide when to invest, it would invest sooner, when $V = C$, so its discount factor, denoted by D^N , would be larger. (Note that in Figure 1, the current value of the benefit, V_0 , happens to be below the cost of the investment, C , so the firm would not invest immediately even if it followed a simple NPV rule, and $D^N < 1$.)

It remains to sort out one potential difficulty. It would be unfortunate if the elasticity ϵ_D depended on the initial value V_0 , as that would imply that if we reconsidered the choice after some intermediate value V_1 had been reached, we would get a different answer for the optimal V^* . To examine this, consider any three values $V_0 < V_1 < V$. Suppose that, along

⁴To see this, obtain the first-order condition for the investment problem by the discounted net payoff (2) with respect to D instead of V , recognizing that $V = V(D, V_0)$:

$$V + D \frac{dV}{dD} - C = 0.$$

This can be re-written as eqn. (5) above.

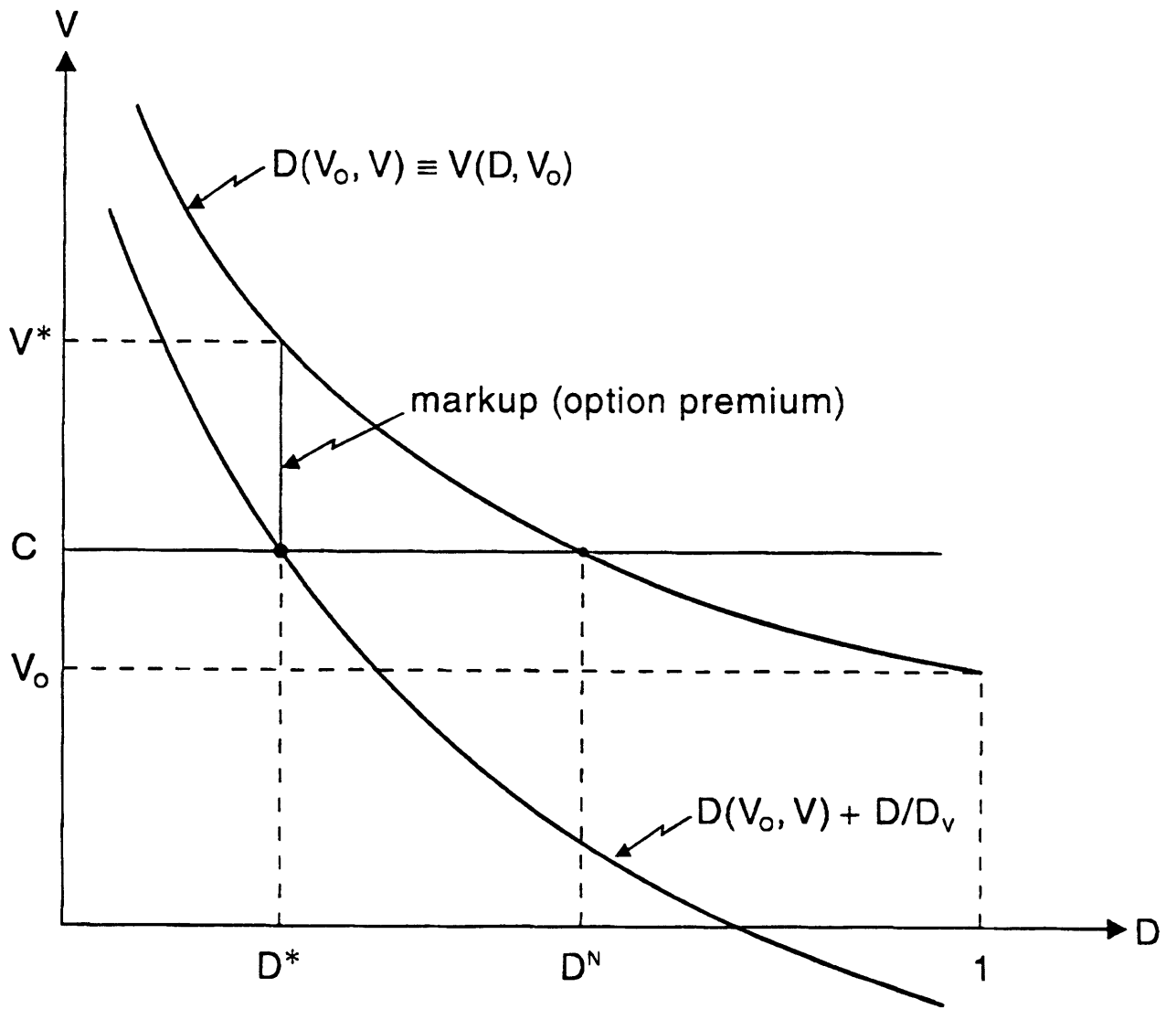


Figure 1: The Optimal Investment Markup.

any path of the process $\{V_t\}$, starting at V_0 the first time the value reaches V_1 is T_1 , and starting at V_1 the first time it reaches V^* is T_2 . Then the first time the value reaches V^* starting at V_0 is just $T = T_1 + T_2$. (In the second interval of time T_2 we have already supposed that the process does not reach V^* , and in the first interval of time T_1 the process could not have reached V^* without hitting V_1 earlier, which would contradict our definition of T_1 as the first time to V_1 .) Now

$$e^{-\rho T} = e^{-\rho T_1} e^{-\rho T_2},$$

and because of the Markov property of the process $\{V_t\}$, the random variables T_1 and T_2 are independent. Therefore we can take expectations of the two factors on the right-hand side to get

$$D(V_0, V) = D(V_0, V_1) D(V_1, V).$$

Then

$$D_V(V_0, V) = D(V_0, V_1) D_V(V_1, V),$$

and

$$\frac{V D_V(V_0, V)}{D(V_0, V)} = \frac{V D_V(V_1, V)}{D(V_1, V)}. \quad (6)$$

A similar argument can be constructed for $V_2 < V_0$, by considering paths where the process starting at V_0 first falls to V_2 before rising again and eventually reaching V^* .

This proves that the elasticity is independent of the starting value. In particular, using eqn. (6) we can write the elasticity as

$$\epsilon_D = -\frac{V^* D_V(V^*, V^*)}{D(V^*, V^*)} = -V^* D_V(V^*, V^*), \quad (7)$$

since $D(V^*, V^*) = 1$. Hence the optimal markup rule given by eqn. (4) is independent of the starting value V_0 . This can also be seen in Figure 1; although the discount factors D^* and D^N depend on V_0 , the optimal markup $V^* - C$ does not.

Finally, note that the elasticity of the discount factor, ϵ_D , can be equivalently expressed in terms of the value of the firm's option to invest. Let $F(V)$ denote the value of the firm's investment option. At the optimal exercise point, $F(V^*)$ must satisfy the value matching

condition,

$$F(V^*) = V^* - C,$$

and the smooth pasting condition,

$$F_V(V^*) = 1.$$

Combining these two conditions, we have:

$$\frac{V^*}{V^* - C} = \frac{V^* F_V(V^*)}{F(V^*)} = \epsilon_F. \quad (8)$$

The right-hand side of (8), denoted by ϵ_F , is the elasticity of the value of the investment option with respect to the value of the underlying project. Since $V^*/(V^* - C) = \epsilon_D$, at an optimum the elasticity of the discount factor coincides with the elasticity of the value of the investment option.

3. Examples.

To use this approach to finding optimal investment rules, one must find the discount factor D , given the stochastic process for V_t . This can be done as follows.

Suppose that V_t follows a general Ito process of the form

$$dV = f(V)dt + g(V)dz. \quad (9)$$

We want to find $D(V, V^*) = \mathcal{E}[e^{-\rho T}]$, where T is the hitting time to V^* , starting at $V < V^*$. Over a small time interval dt , V will change by a small random amount, dV . Hence (suppressing V^* for simplicity):

$$D(V) = e^{-\rho dt} \mathcal{E}[D(V + dV)].$$

Expanding $D(V + dV)$ using Ito's Lemma, noting that $e^{-\rho dt} = 1 - \rho dt$ for small dt , and substituting eqn. (9) for dV , we obtain the following differential equation for the discount factor:

$$\frac{1}{2} g^2(V) D_{VV} + f(V) D_V - \rho D = 0. \quad (10)$$

This equation must be solved subject to two boundary conditions: (1) $D(V^*, V^*) = 1$, and (2) $D(V, V^*) \rightarrow 0$ as $V^* - V$ becomes large.

To illustrate, we will obtain solutions using this approach for several different stochastic processes, and draw further analogies to the profit-maximizing decisions of a price-setting firm.

Geometric Brownian Motion.

First, suppose that V_t follows the geometric Brownian motion

$$dV = \alpha V dt + \sigma V dz, \tag{11}$$

with $\alpha < \rho$. Then $f(V) = \alpha V$ and $g(V) = \sigma V$, and it is easily seen that the solution to eqn. (10) is

$$D(V_0, V) = (V_0/V)^{\beta_1}, \tag{12}$$

where β_1 is the positive root (exceeding unity) of the following quadratic equation in β :

$$\frac{1}{2} \sigma^2 \beta(\beta - 1) + \alpha \beta - \rho = 0; \tag{13}$$

see Dixit and Pindyck (1996, p. 316).

In this case the elasticity of the discount factor is constant and equal to β_1 . The markup formula (4) thus implies a constant proportional mark-up,

$$(V^* - C)/V^* = 1/\beta_1,$$

or

$$V^* = \frac{\beta_1}{\beta_1 - 1} C.$$

This well-known result is analogous to the price-cost markup rule for a firm facing an isoelastic demand curve. A geometric Brownian motion for V_t implies an isoelastic discount factor because the probability distribution for percentage changes in V is independent of V , so changes in the discount factor resulting from a percentage change in V will also be independent of V .

Arithmetic Brownian Motion.

Next, suppose V follows the arithmetic Brownian motion

$$dV = \alpha dt + \sigma dz. \tag{14}$$

Then the solution to eqn. (10) is

$$D(V_0, V) = \exp[-\gamma_1 (V - V_0)],$$

where γ_1 is the positive root of the quadratic

$$\frac{1}{2} \sigma^2 \gamma^2 + \alpha \gamma - \rho = 0;$$

see Harrison (1985, p. 42). In this case, the elasticity of the discount factor is $\gamma_1 V$. Hence $(V^* - C)/V^* = 1/\gamma_1 V^*$, and we get a constant additive mark-up:

$$V^* = C + (1/\gamma_1).$$

This is analogous to the markup formula for a firm facing an exponential demand curve. (For the demand curve $q(p) = a \exp[-bp]$, the elasticity of demand is bp , and the profit-maximizing price is $p^* = c + 1/b$.)

Mean-Reverting Process.

Finally, suppose that V_t follows the mean-reverting process:

$$dV = \eta(\bar{V} - V)Vdt + \sigma Vdz. \quad (15)$$

The value, V , might follow such a process if, for example, the firm's output was a durable good so that its demand was subject to a stock adjustment process.

Then eqn. (10) for $D(V, V_1)$ becomes:

$$\frac{1}{2} \sigma^2 V^2 D_{VV}(V, V_1) + \eta(\bar{V} - V)V D_V(V, V_1) - \rho D(V, V_1) = 0.$$

This equation has the following solution (see Dixit and Pindyck, pp. 162–163):

$$F(V, V_1) = AV^\theta H\left(\frac{2\eta}{\sigma^2}V, \theta, b\right),$$

where A depends on V_1 , θ is the positive solution to the quadratic equation

$$\frac{1}{2} \sigma^2 \theta(\theta - 1) + \eta \bar{V} \theta - \rho = 0,$$

and

$$b = 2 \left(\theta + \frac{\eta \bar{V}}{\sigma^2} \right).$$

Here $H(x, \theta, b)$ is the confluent hypergeometric function, which has the following series representation:

$$H(x, \theta, b) = 1 + \frac{\theta}{b}x + \frac{\theta(\theta+1)x^2}{b(b+1)2!} + \frac{\theta(\theta+1)(\theta+2)x^3}{b(b+1)(b+2)3!} + \dots$$

The limiting behavior of the solution is used to determine A . When V approaches V_1 , the first hitting time T must approach zero, which means that $F(V, V_1) \rightarrow 1$. Thus,

$$A = \frac{1}{V_1^\theta H\left(\frac{2\eta}{\sigma^2}V_1, \theta, b\right)}.$$

Hence the discount factor becomes:

$$D(V_0, V) = \left(\frac{V_0}{V}\right)^\theta \frac{H\left(\frac{2\eta}{\sigma^2}V_0, \theta, b\right)}{H\left(\frac{2\eta}{\sigma^2}V, \theta, b\right)} \quad (16)$$

From the series representation, we obtain the following relationship between H and its derivative with respect to the first argument:

$$H_x(x, \theta, b) = \frac{\theta}{b}H(x, \theta + 1, b + 1).$$

Using this, we can determine that the elasticity of the discount factor at the optimum V^* is:

$$\epsilon_D = \theta \left[1 + \frac{2\eta V^*}{\sigma^2 b} \frac{H\left(\frac{2\eta}{\sigma^2}V^*, \theta + 1, b + 1\right)}{H\left(\frac{2\eta}{\sigma^2}V^*, \theta, b\right)} \right]. \quad (17)$$

Thus, ϵ_D is equal to a constant θ — which represents pure geometric growth — plus a term which corrects for the mean reversion effect. As the mean reversion speed η approaches zero, the second term also goes to zero, and θ approaches β_1 , as in the case of geometric Brownian motion. As η increases, mean reversion dominates.

The implications of mean reversion are easiest to see from some numerical calculations. Mean reversion implies that V is expected to stay close to \bar{V} . Hence when $V - \bar{V}$ is small, the discount factor must be larger for the mean-reverting process than for the corresponding geometric Brownian motion. Likewise if $V - \bar{V}$ is large, it can be expected to decline, so that the discount factor will be relatively small. Figure 2 illustrates this; it shows the discount

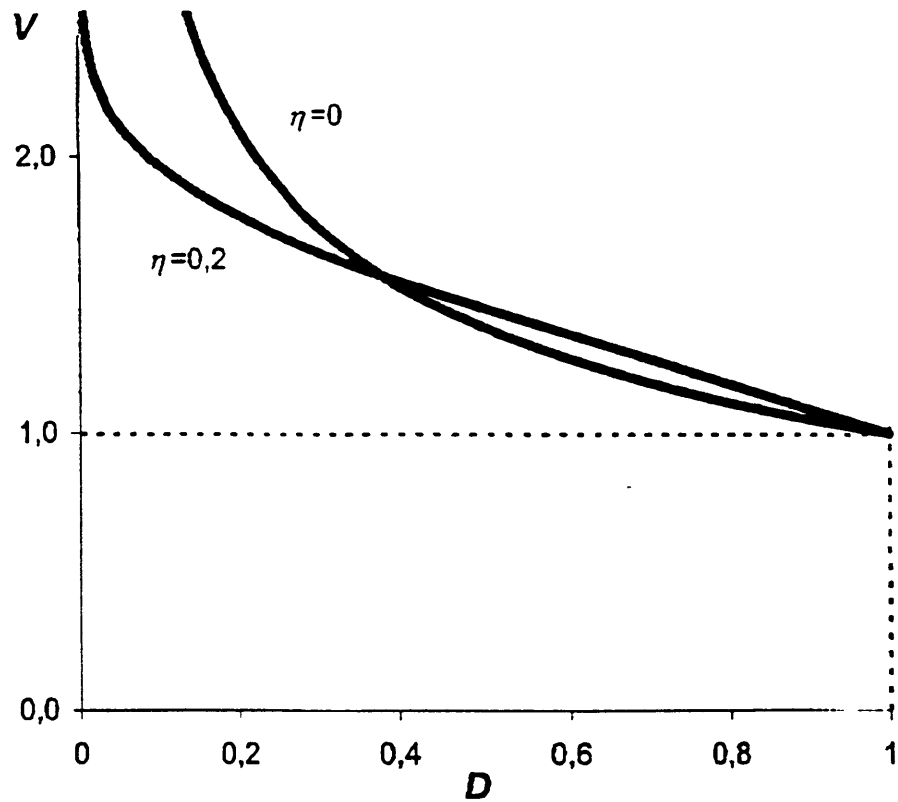


Figure 2: Discount Factor for Mean-Reverting Process and Geometric Brownian Motion ($\rho = .05, \sigma = .2, \bar{V} = 1, V_0 = 1$).

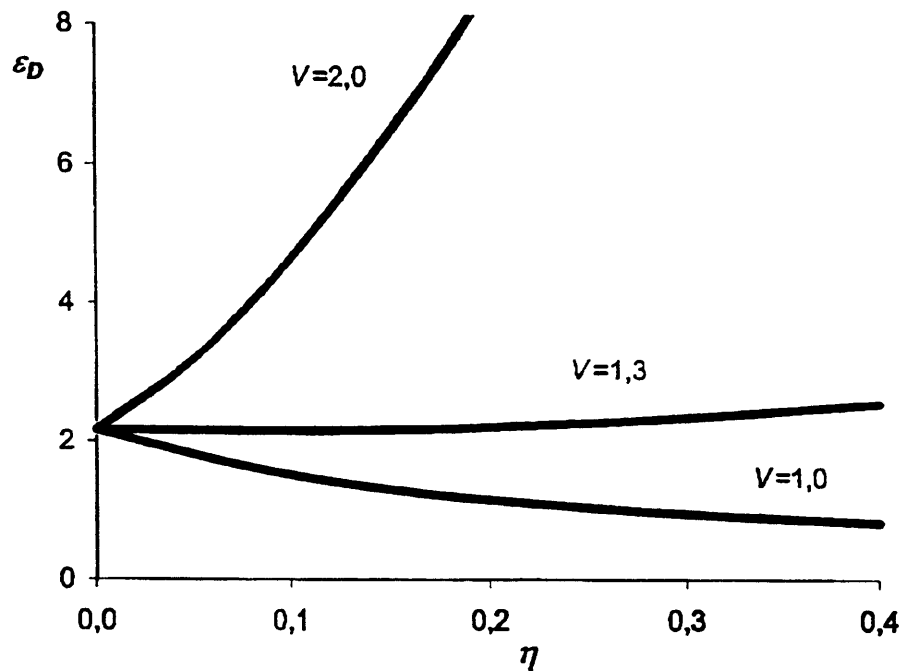


Figure 3: Elasticity of Discount Factor as a Function of the Speed of Mean Reversion.

factor as a function of V for a mean-reverting process ($\eta = 0.2$) and a geometric Brownian motion ($\eta = 0$). (In both cases, $\rho = .05$, $\sigma = 0.2$, $\bar{V} = 1$, and $V_0 = 1$.) This effect of mean reversion is also reflected in the elasticity of the discount factor, which is increasing in V . For example, $\epsilon_D(V = 1) = 1.4$ and $\epsilon_D(V = 2) = 8.54$; while the corresponding constant elasticity for the geometric Brownian motion ($\eta = 0$) is $\beta_1 = 2.16$. Figure 3 shows how the elasticity depends on the speed of mean reversion, η . When $V - \bar{V}$ is small ($V = 1.0$), ϵ_D decreases with η , but when it is large ($V = 2.0$), it increases with η .

4. Ancillary Investments in Advertising or R&D.

The close connection between investment decisions and pricing decisions has pedagogical value, but also provides insight into investment-related decisions more broadly. As an example, consider a price-setting firm that must also decide how much money, A , to spend on advertising, given its demand $q = q(p, A)$, with $\partial q/\partial A > 0$. As students are taught in intermediate microeconomics courses, the profit-maximizing advertising-to-sales ratio is given by:

$$\frac{A}{pq} = \epsilon_A/\epsilon_P, \quad (18)$$

where $\epsilon_A = (A/q)\partial q/\partial A$ is the firm's advertising elasticity of demand, and ϵ_P is the price elasticity of demand.⁵

Now let us return to our investment problem. Suppose that the firm, prior to making the sunk expenditure C in return for the benefit V , can make an ancillary investment, costing A , in advertising, marketing, or R&D activities. The exact nature of these activities is unimportant; what matters is that they lead to more rapid increases in V , and hence to an increase in the discount factor $D(V_0, V)$. We can then re-state our investment problem as:

$$\max_{V,A} [(V - C)D(V_0, V, A) - A]. \quad (19)$$

The two first-order conditions for this problem are

$$D(V_0, V, A) + (V - C)D_V(V_0, V, A) = 0, \quad (20)$$

⁵Eqn. (18) follows from maximizing profit with respect to p and A , and is sometimes referred to as the Dorfman-Steiner (1954) theorem.

and

$$(V - C)D_A(V_0, V, A) - 1 = 0. \quad (21)$$

Now define the elasticities of the discount factor with respect to V and A , respectively, as $\epsilon_V^D \equiv -VD_V/D$ and $\epsilon_A^D \equiv AD_A/D$. Then by combining the first-order conditions (20) and (21), it is easy to see that

$$\frac{A}{DV} = \epsilon_A^D / \epsilon_V^D. \quad (22)$$

Eqn. (22) is a condition for the optimal ratio of expenditures on advertising (or marketing, or R&D) to the expected discounted value of the benefit. (Remember that the *actual* discounted value of the benefit is unknown because the time until V reaches the threshold V^* is stochastic; DV is the *expected* discounted value of the threshold V^* .) It is exactly analogous to condition (18) for the advertising-to-sales ratio of a price-setting firm.

As an example, suppose a pharmaceutical firm is deciding whether to invest in a plant to produce a new drug. Suppose the benefit from this investment, V_t , follows the geometric Brownian motion of eqn. (11), and will grow over time (at expected rate α) even before the plant is built as doctors and patients learn about the drug. However, the expected growth rate α can be increased via expenditures A on advertising and marketing.⁶

To determine the optimal level of A for this example, note that the discount factor is again given by eqn. (12), with β_1 again the solution to the quadratic eqn. (13). Hence the elasticity ϵ_V^D is again equal to β_1 . But now β_1 is a function of A , since α depends on A . Differentiating the quadratic eqn. (13) with respect to A and rearranging yields the following expression for the elasticity ϵ_A^D :

$$\epsilon_A^D = -\frac{A \log D(d\alpha/dA)}{\sigma^2 \beta_1 + \alpha - \frac{1}{2}\sigma^2}. \quad (23)$$

Defining the elasticity $\epsilon_A^\alpha \equiv (A/\alpha)d\alpha/dA$, the optimal ratio of A to the discounted benefit is thus given by:

$$\frac{A}{DV} = \frac{\alpha \epsilon_A^\alpha \log(V^*/V_0)}{\sigma^2 \beta_1 + \alpha - \frac{1}{2}\sigma^2}. \quad (24)$$

⁶We treat A as a lump-sum expenditure. If the advertising and marketing expenses must be spread out over time, then A is just the present value of those expenses.

This ratio will be larger the larger is ϵ_A^α — the more productive is advertising and marketing, the more that should be done. But note that this ratio is also larger the larger is the threshold V^* . A larger V^* implies that the option to invest is more valuable (the expected net payoff $(V^* - C)$ is larger), which increases the expected return from advertising and marketing expenditures. Hence this ratio is larger if there is greater uncertainty over the evolution of V ; an increase in σ increases V^* , and (with some algebra) can be shown to reduce the denominator of (24). Finally, note that $A \rightarrow 0$ as $V^*/V_0 \rightarrow 1$; when $V^* = V_0$ there is no option premium, and thus no benefit to increasing α .

5. Conclusions.

Framing the optimal investment decision as a trade-off between larger versus later net benefits has allowed us to interpret the investment rule as a simple markup formula involving an elasticity. We have seen that the markup is exactly analogous to a firm’s optimal markup of price over marginal cost. For economists, this may be more intuitively appealing than the standard approach to irreversible investment problems in which one values the firm’s option to invest and finds the optimal exercise rule.

If the benefit, V , follows a geometric Brownian motion — as is typically assumed in applications — then the markup formula is particularly simple, since the elasticity of the discount factor is constant and equal to β_1 , the solution to the fundamental quadratic equation (13). In this case the discount factor is isoelastic with respect to V , so the investment problem is analogous to the pricing problem for a firm facing an isoelastic demand curve.

Even if V does not follow a geometric Brownian motion, this markup formulation provides a rule of thumb that can be of value to practitioners. Compared to equating marginal cost with marginal revenue, it can be easier for a manager to think about pricing in terms of a markup based on the elasticity of demand, estimates of which can be based on statistics or on judgment. Likewise, it can be easier to think about investment hurdles as a markup based on the elasticity of the discount factor, “estimates” of which can be found analytically or judgmentally.

References

- Baily, Walter Toshi, "Essays in Finance," unpublished Ph.D. dissertation, M.I.T., 1995.
- Dixit, Avinash and Robert Pindyck. 1996. *Investment Under Uncertainty*. Second printing. Princeton, NJ: Princeton University Press.
- Dorfman, Robert, and Peter O. Steiner. 1954. "Optimal Advertising and Optimal Quality." *American Economic Review*. 44: 826–836.
- Harrison, J. Michael. 1985. *Brownian Motion and Stochastic Flow Systems*. New York: John Wiley.
- McDonald, Robert, and Daniel Siegel. 1986. "The Value of Waiting to Invest." *Quarterly Journal of Economics*. 101: 707–728.