Analysis of Nonlinear Electroelastic Continua with Electric Conduction

by

John E. Harper

B.S. Aeronautical Engineering (1995)
Rensselaer Polytechnic Institute

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Author ....
Department of Aeronautics and Astronautics
May 21, 1999

Certified by ....
Nesbitt W. Hagood IV
Associate Professor
Thesis Supervisor

Accepted by ....
Jaime Peraire
Chairman, Department Graduate Committee

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Abstract

This thesis presents the nonlinear theory for large deformation electroelastic continua with electric conduction. This theory is suitable for modeling actuator and sensor devices composed of deformable, electromechanically coupled, highly insulating materials. Consistency is proven between the large deformation theory and the classical Poynting vector based piezoelectric small deformation theory, extended for electric conduction. A result is that electric body forces, realized mathematically as electric surface tractions, are retained in the small deformation approximation. A finite element formulation is presented suitable for performance analysis of deformable electromechanical actuator and sensor devices composed of highly insulating materials with nonlinear response functions, under the small deformation approximation. Results demonstrate the significant cumulative effects of a weak electric current flow for electric voltage DC offset loading of a highly electrically insulating composite device.

Thesis Supervisor: Nesbitt W. Hagood IV
Title: Associate Professor

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Chapter 1

Introduction

1.1 Motivation

Engineering analysis techniques are used in design and development of actuator and sensor devices to predict the performance of a candidate design. Accurate device analysis allows the engineer to optimize a design for a given performance objective and constraints. Alternately, inaccurate device analysis could lead to poor designs and device failure.

This thesis considers analysis of highly electrically insulating deformable bodies subject to electrical and mechanical loading. The response of such highly insulating devices can be influenced or dominated by the cumulative effects of a very weak electric current flow. Much of the engineering analysis literature is concerned with perfect electrically insulating deformable bodies. The perfect insulator approximation is only accurate provided the time scales of loading are sufficiently fast to prevent the cumulative effects of a weak electric current flow. Device designs based on perfect insulator analyses are likely to fail when subjected to sufficiently slow time scale loadings.

1.2 Objective

This thesis will report on the mathematical abstraction of deformable electromechanical actuator and sensor devices composed of highly electrically insulating materials. A first objective is to present a clear exposition with detailed proofs of the nonlinear large deformation theory of electroelastic continua with electric conduction. A second objective is to investigate the consistency between this general electroelastic continua theory and the classical small deformation piezoelectric theory based on Poynting vector interpretations, extended for electric conduction. A third objective is to develop an engineering analysis tool for deformable electromechanical actuator and sensor devices composed of highly insulating materials with nonlinear response functions (e.g., repolarizable piezoelectric ceramic material), suitable for arbitrary device geometry and loading conditions.

1.3 Background

An excellent monograph on the analysis of electroelastic perfect electrically insulating bodies subject to the assumption of small deformations is TIERSTEN's [29] *Linear Piezoelectric Plate Vibrations.* TIERSTEN presents the balance of energy equation for the classical small deformation theory based on the notion of Poynting's vector as the electric energy flux vector across a surface. The result is a local form of the energy expression containing the scalar product of electric field vector with the time derivative of electric displacement vector. The theory presented in TIERSTEN's monograph can be extended to include the effects of weak electric current density by retaining the electric current density term in Poynting's vector and appending to the system of equations the entropy inequality axiom. As noted by TIERSTEN, the effects of large deformation and electric body forces have been ignored. A natural question to ask is how accurate is this small deformation theory compared to the general large deformation theory, and what are the effects of the ignored electric body forces? This question can be answered by studying the large deformation theory of electroelastic continua and the consequences of introducing a small deformation approximation.

A large deformation theory of electroelastic continua, independent of Poynting vector interpretations, has been developed. See DIXON & ERINGEN [9], MAUGIN & ERINGEN [17] , ERINGEN & MAUGIN [11] for derivations based on a space (volume) averaging procedure, and TIERSTEN [30], TIERSTEN & TSAI [35], and DE LORENZI & TIERSTEN [8] for derivations based on a well defined model of interpenetrating continua. A necessary prerequisite would be a study of large deformation continuum mechanics theory, see ERINGEN [10], OGDEN [21], and GURTIN [15]. We remark that these two derivations result in equivalent theories, and it is worthwhile to study both approaches. An excellent monograph summarizing this large deformation electromagnetic theory is ERINGEN & MAUGIN's [11] *Electrodynamics of Continua I* which contains as a special case the electroelastic theory with electric conduction that we are interested in.

Solutions of the large deformation equations can be difficult to obtain, and therefore introduction of a small deformation approximation is frequently carried out in the literature. Examples of small deformation analyses, usually superposed on a large deformation, have been presented, for example, by TIERSTEN [31, 33, 32, 34], BAUMHAUER & TIERSTEN [3], TIERSTEN & TSAI [35], DE LORENZI & TIERSTEN [8], and also MAUGIN, ET AL [19], ERINGEN & MAUGIN [11], ERINGEN [12], MAUGIN & POUGET [18], ANI & MAUGIN [1].

In all of the above works, the starting point of the analyses are the large deformation electroelastic equations. Reduction to the small deformation equations make explicit exactly what terms are being neglected. Clearly it is desirable to begin an analysis from such a framework. A natural question to ask is, are the equations used in the classical small deformation theory, based on Poynting vector interpretations, consistent with the large deformation theory? The primary equation of concern is the balance of energy expression. Indeed, THURSTON [26] has asked this consistency question regarding the perfectly insulating electromagnetic continua theory. He introduces a total energy function as the sum of an internal energy function and a free space energy term, then transforms the energy equation in terms of the total energy function to material fields. THURSTON demonstrates that introducing a so-called thermostatic approximation, that ignores certain velocity and magnetization terms, will result in an energy expression that simplifies under the small deformation approximation to the classical energy expression derived using Poynting's vector. Use of the thermostatic approximation is not very satisfactory of a reconciliation between the two theories, and suggests the classical small deformation piezoelectric theory is not consistent with the general electroelastic theory.

THURSTON's result, however, is very interesting. Indeed, the unsatisfactory thermostatic approximation is not needed if the energy equation[1] is restricted from electromagnetic continua to electroelastic continua. Although THURSTON does not emphasize this, the transformed energy expression for electroelastic continua will be exactly consistent, under the small deformation approximation, with the classical energy expression based on Poynting's vector.

MCCARTHY & TIERSTEN [20], working with large deformation semiconducting electroelastic continua, present a transformation for the balance of energy in terms of material fields. The derived balance of energy expression[2], under the small deformation approximation and simplified to electric conduction, will be exactly consistent with the classical form of the energy expression relying on Poynting's vector when extended to include electric conduction.

An approach using THURSTON's transformed energy expression for perfectly insulating continua and MCCARTHY & TIERSTEN's transformed energy expression for semiconducting continua can be used to prove consistency between the large deformation electroelastic continua with electric conduction and the classical small deformation piezoelectric theory relying on Poynting's vector interpretation, extended for electric conduction. Such a study is certainly worthwhile, as it exposes the assumptions and approximations inherent in the classical small deformation theory, including the role of electric body forces.

TIERSTEN's monograph *Linear Piezoelectric Plate Vibrations* is concerned with the analysis of highly electrically insulating piezoelectric bodies in vibration, and therefore does not consider the effects of electric conduction. In fact, highly insulating materials are almost always assumed to be perfectly insulating. It is important to recall that all highly insulating materials, classified in the engineering literature as electrical insulators, will support non-zero electric conduction currents, usually referred to as leakage currents in elementary physics texts [39, 38, 16]. Under sufficiently fast dynamic loading of a highly insulating body, the cumulative effects of weak electric conduction currents are typically negligible, and the perfect insulator approximation may very well be an excellent one.

On the other hand, if the loading time scales are such that very weak electric currents have a cumulative effect, then an analysis based on the perfect insulator approximation could be very inaccurate. Consider the example of a highly insulating piezoelectric device under a sufficiently high frequency sinusoidal electrical loading.

[1]See eq. 13.48 on p.162 of [26]

[2]See eq. 3.16 on p.35 of [20]

In this case, the pefect insulator approximation may be an excellent one. However, consider the same device under identical high frequency electrical loading, but with an additional electric voltage DC offset. After a sufficiently long period of time the cumulative effects of the weak electric current flow will dominate the voltage offset response. This is an example of a typical loading condition on highly insulating devices when electric conduction will in general be signifcant.

1.4 Thesis Contributions

This thesis is based on the recognition that highly electrically insulating actuator and sensor devices under general electrical and mechanical loading must be analysed in the framework of electroelastic continua with electric conduction. This thesis presents a detailed account of the nonlinear theory for large deformation highly insulating electroelastic continua with electric conduction, and proves the consistency between this theory and the classical small deformation piezoelectric theory based on Poynting vector interpretations, extended for electric conduction. The essential step is proving the equivalence of the balance of energy equations in the two theories. A consequence is that electric body forces, recognized mathematically as electric surface tractions, are naturally retained in the small deformation approximation. Finally, this thesis presents a finite element formulation suitable for performance analysis of deformable electromechanical actuator and sensor devices composed of highly insulating materials with nonlinear response functions (e.g., repolarizable piezoelectric ceramic material) and arbitrary device geometries, under the small deformation approximation.

1.5 Thesis Outline

Our presentation is in the framework of continuum physics. We introduce the notion of a body as a collection of points. Deformation is a mapping of the body from some reference configuration to a new deformed configuration. The notion of change of observer and change of reference configuration is introduced. These will be needed for deducing restrictions on the constitutive functions, as required by our constitutive theory axioms. Mathematical results essential to the development are presented. Fundamental axioms of the continuum physics theory are presented in terms of spatial fields. Differential equations are derived from the integral form statements. Jump conditions are derived from integral form statements extended to include surfaces of discontinuity. Constitutive equations are derived in terms of an internal energy function and a total energy function. We introduce the notion of material fields, and systematically derive equivalent material field representations of the global and local equations. Constitutive equations are derived from the material fields, which automatically satisfy the material objectivity axiom. Jump conditions in material fields are derived from integral form statements extended to surfaces of discontinuity. These are needed to piece together solutions across material discontinuities, and specialize to boundary conditions on the bounding surface of a body. The small deformation approximation is introduced to simplify the governing equations. A weak form of the resulting small deformation equations is presented as a starting point for our finite element formulation. Solution techniques for the finite element equations are presented, with results from analysis of a piezoelectric fiber embedded in an epoxy matrix under an electric voltage DC offset loading, using a nonlinear material model for repolarization.

Chapter 2

Large Deformation Electroelastic Equations

2.1 Introduction

This chapter presents essential theorems and proofs in the nonlinear large deformation theory of electroelastic continua with electric conduction. The differential equations and jump conditions needed for device analysis are summarized in chapter 3 for convenience.

2.2 Bodies, Deformations, and Motions

[1] Bodies have the property that they occupy regions of three dimensional Euclidean point space $\mathcal{E}$. An arbitrary point x in $\mathcal{E}$ is associated with a position vector $\boldsymbol{x}$ in three dimensional Euclidean vector space E, relative to an arbitrarily choosen origin point $o \in \mathcal{E}$. For fixed o, x and $\boldsymbol{x}$ have a unique correspondence, and we can identify the point x with the vector $\boldsymbol{x}$. When o is fixed in $\mathcal{E}$, we use $\boldsymbol{x}$ to denote both a point in $\mathcal{E}$ and its corresponding position vector in E.

We define a body $\mathcal{B}$ as a regular region[1] in $\mathcal{E}$. In general, $\mathcal{B}$ will occupy different regions of $\mathcal{E}$ at different times[2] $t \in \mathbb{R}$. For convenience we choose one such region $\mathcal{B}_o$ as the reference configuration of $\mathcal{B}$. Points in the body can be identified with their positions in $\mathcal{B}_o$. We call points $\boldsymbol{X} \in \mathcal{B}_o$ material points. A deformation $\hat{\chi}$ carries the body from its reference configuration $\mathcal{B}_o$ to a deformed configuration $\mathcal{B}_d$ and carries each material point $\boldsymbol{X}$ to a point $\boldsymbol{x}$,

$$\hat{\chi} : \mathcal{B}_o \to \mathcal{E},$$
$$\hat{\chi} : \boldsymbol{X} \mapsto \boldsymbol{x},$$

[1]This section is based on OGDEN [21, pp. 77-83]

[1]A closed region is the closure of a connected, open set in $\mathcal{E}$. A regular region is a closed region with piecewise smooth boundary.

[2]The set of real numbers is denoted by $\mathbb{R}$.

where we write,

$$\mathcal{B}_d = \hat{\chi}(\mathcal{B}_o).$$

The motion χ_t of $\mathcal{B}$ is a smooth one-parameter family of deformations parameterized

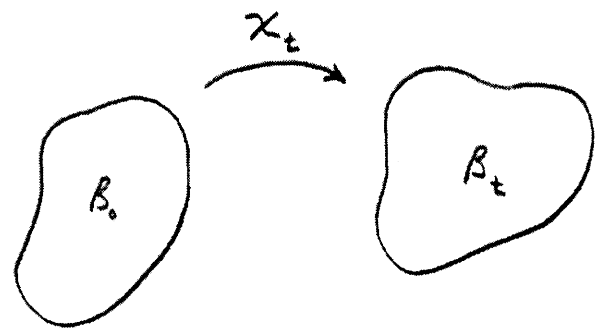

Figure 2-1: Motion χ_t from $\mathcal{B}_o$ to $\mathcal{B}_t$.

by time t. The region $\mathcal{B}_t$ is called the current configuration of $\mathcal{B}$, and the point $\boldsymbol{x}$ is called the spatial point occupied by the material point $\boldsymbol{X}$ at time t,

$$\begin{aligned} \chi_t &: \mathcal{B}_o \to \mathcal{E}, \\ \chi_t &: \boldsymbol{X} \mapsto \boldsymbol{x}. \end{aligned}$$

We write

$$\begin{aligned} \mathcal{B}_t &= \chi(\mathcal{B}_o, t), \\ \boldsymbol{x} &= \chi(\boldsymbol{X}, t). \end{aligned}$$

Axiom 2.2.1 (Axiom of Continuity) [2] *Throughout the body $\mathcal{B}$ the motion χ_t and its inverse are single-valued and as many times continuously differentiable as required.*

The inverse mapping χ_t^{-1} takes the deformed body $\mathcal{B}_t$ back to its reference configuration $\mathcal{B}_o$. We write

$$\begin{aligned} \mathcal{B}_o &= \chi^{-1}(\mathcal{B}_t, t), \\ \boldsymbol{X} &= \chi^{-1}(\boldsymbol{x}, t). \end{aligned}$$

If we choose our reference time corresponding to $\mathcal{B}_o$, at $t = 0$, then the reference configuration $\mathcal{B}_o$ necessarily satisfies

$$\begin{aligned} \mathcal{B}_o &= \chi(\mathcal{B}_o, 0), \\ \mathcal{B}_o &= \chi^{-1}(\mathcal{B}_o, 0). \end{aligned}$$

[2] TRUESDELL & TOUPIN [37, p. 243]

2.3 Observer Transformations

[3] Suppose an event in the physical world manifests itself at a point of Euclidean point space $\mathcal{E}$ and at a time in $\mathbb{R}$. This event will be recorded by an observer O as occuring at $(\boldsymbol{x}, t)$. If $\boldsymbol{x}$ and $\boldsymbol{x}_o$ are distinct points of $\mathcal{E}$ and t and t_o are distinct times in $\mathbb{R}$, then two events observed by O at $(\boldsymbol{x}_o, t_o)$ and $(\boldsymbol{x}, t)$ are seperated by a distance $\|\boldsymbol{x} - \boldsymbol{x}_o\|$ in $\mathcal{E}$ and a time interval $t - t_o$ in $\mathbb{R}$. The definition of an observer transformation is based on the notion that different observers must agree about distance and time intervals between events.

Definition 2.3.1 (Change of Observer) *An observer transformation or change of observer is defined as any transformation that takes* $(\boldsymbol{x}_o, t_o)$ *and* $(\boldsymbol{x}, t)$ *to* $(\boldsymbol{x}_o^*, t_o^*)$ *and* $(\boldsymbol{x}^*, t^*)$*, such that distances and time intervals are preserved,* [4]

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{x}_o\| &= \|\boldsymbol{x}^* - \boldsymbol{x}_o^*\| \,, \\ t - t_o &= t^* - t_o^* \,. \end{aligned}$$

The general form of such a transformation is,

$$\begin{aligned} \boldsymbol{x}^* &= \boldsymbol{Q}\,(t)\,\boldsymbol{x} + \boldsymbol{b}\,(t) \,, \qquad t^* = t - a \\ \boldsymbol{Q}'\boldsymbol{Q} &= \boldsymbol{Q}\boldsymbol{Q}' = \boldsymbol{I} \\ \det(\boldsymbol{Q}) &= \pm 1, \end{aligned}$$

where a is an arbitrary scalar, $\boldsymbol{b}(t)$ is an arbitrary vector, and $\boldsymbol{Q}(t)$ is an arbitrary orthogonal tensor. It is convenient to restrict $\boldsymbol{Q}(t)$ to arbitrary proper orthogonal tensors, such that $\det(\boldsymbol{Q}) = 1$.

Remark 2.3.2 (Observer Transformation) *For the motion* χ_t *of a body* $\mathcal{B}$*, an observer transformation or change in observer* χ_t^* *is,*

$$\begin{aligned} \chi^*(\boldsymbol{X}, t^*) &= \boldsymbol{Q}\,(t)\,\chi(\boldsymbol{X}, t) + \boldsymbol{b}\,(t) \qquad t^* = t - a. \qquad (2.1) \\ \boldsymbol{Q}'\boldsymbol{Q} &= \boldsymbol{Q}\boldsymbol{Q}' = \boldsymbol{I} \\ \det(\boldsymbol{Q}) &= 1\,. \end{aligned}$$

A transformation (2.1) that takes $(\boldsymbol{x}, t)$ to $(\boldsymbol{x}^*, t^*)$ is interpreted as a change of observer from O to O^*, such that the event recorded by O at $(\boldsymbol{x}, t)$ is the same event as that recorded by O^* at $(\boldsymbol{x}^*, t^*)$. In general, the description of a physical quantity associated with the motion χ_t of a body $\mathcal{B}$ depends on the choice of observer. Such a distinction will be important for deducing restrictions on constitutive equations for material response.

[3] This section is based on OGDEN [21, pp. 73-77] and GURTIN [15, pp. 139-145]

[4] *The 2-norm is defined by,* $\|x_i\| = \left(\sum_{i=1}^{3} x_i^2\right)^{1/2}$

2.4 Fields, Deformations, and Integral Theorems

[5] To introduce definitions of material and spatial fields, we define the reference set $\mathcal{T}_o$ and the trajectory set $\mathcal{T}$ as

$$\begin{aligned} \mathcal{T}_o &= \{(\boldsymbol{X},t) \,|\, \boldsymbol{X} \in \mathcal{B}_o,\, t \in \mathbb{R}\}\,, \\ \mathcal{T} &= \{(\boldsymbol{x},t) \,|\, \boldsymbol{x} \in \mathcal{B}_t,\, t \in \mathbb{R}\}\,. \end{aligned}$$

Definition 2.4.1 (Material and Spatial Fields) *A material field is a function with domain $\mathcal{T}_o$. A spatial field is a function with domain $\mathcal{T}$.*

Remark 2.4.2 (Fields) *All fields defined over $\mathcal{T}$ are assumed to be as many times continuously differentiable as required. Surfaces and lines of discontinuity will be addressed in section 2.14.*

Much of the theory presented involves integrals over volumes, surfaces, and lines contained in either $\mathcal{B}_o$ or $\mathcal{B}_t$. Here we introduce notation to distinguish between sets of points in the two configurations.

Definition 2.4.3 (Volumes, Surfaces, and Lines in $\mathcal{B}_o$ and $\mathcal{B}$)

- *A material volume V_o is a volume in $\mathcal{B}_o$. The material volume V is the volume in $\mathcal{B}_t$ occupied by the material points $\boldsymbol{X} \in V_o$ at time t,*

$$V = \boldsymbol{\chi}(V_o, t)\,.$$

- *A material surface S_o is a surface in $\mathcal{B}_o$. The material surface S is the surface in $\mathcal{B}_t$ occupied by the material points $\boldsymbol{X} \in S_o$ at time t,*

$$S = \boldsymbol{\chi}(S_o, t)\,.$$

- *A material line C_o is a line in $\mathcal{B}_o$. The material line C is the line in $\mathcal{B}_t$ occupied by the material points $\boldsymbol{X} \in C_o$ at time t,*

$$C = \boldsymbol{\chi}(C_o, t)\,.$$

Many of the proofs will be made more transparent by introducing a Cartesian coordinate system and manipulating vectors and tensors in their component form.

Definition 2.4.4 (Cartesian Coordinate System) *Material and spatial fields will be referred to a single Cartesian coordinate system fixed in $\mathcal{E}$. The set of basis vectors for this system will be denoted by either $\boldsymbol{i}_k$ or $\boldsymbol{i}_K$, with indices $k, K = 1, 2, 3$.*

[5] This section is based on ERINGEN [10, pp. 5-92] and GURTIN [15, pp. 41-85]

Definition 2.4.5 (Summation Convention) *Summation over once repeated indices is understood.*

For example,

$$X_M \boldsymbol{i}_M = X_1 \boldsymbol{i}_1 + X_2 \boldsymbol{i}_2 + X_3 \boldsymbol{i}_3.$$

A material point located in the reference configuration $\mathcal{B}_o$ by $\boldsymbol{P}$ and in the current configuration $\mathcal{B}_t$ by $\boldsymbol{p}$ are represented in the Cartesian coordinate system by

$$\begin{aligned} \boldsymbol{P} &= X_M \boldsymbol{i}_M, \\ \boldsymbol{p} &= x_k \boldsymbol{i}_k. \end{aligned}$$

Definition 2.4.6 (Component Notation Convention) *Components associated with the reference configuration $\mathcal{B}_o$ will consistently have capital indices. Components associated with the current configuration $\mathcal{B}_t$ will consistently have lower case indices.*

Frequently both spatial and material fields will be presented and manipulated in their component form. For example, a spatial vector field $\boldsymbol{A}$, a material tensor field $\boldsymbol{B}$, and a two-point tensor field $\boldsymbol{F}$ are referred to the Cartesian coordinate system by[1]

$$\begin{aligned} \boldsymbol{A} &= A_k \, \boldsymbol{i}_k, \\ \boldsymbol{B} &= B_{RS} \, \boldsymbol{i}_R \otimes \boldsymbol{i}_S, \\ \boldsymbol{F} &= F_{kR} \, \boldsymbol{i}_k \otimes \boldsymbol{i}_R. \end{aligned} \tag{2.2}$$

Consistent with our convention, the fields $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{F}$ can be written in component form as A_k, B_{RS}, and F_{kR} respectively. Similarly, the motion $\boldsymbol{\chi}_t$ of a body may be written in component form as

$$x_k = \chi_k \left(X_M, t \right), \tag{2.3}$$

$$X_M = \chi_M^{-1} \left(x_k, t \right). \tag{2.4}$$

We will occasionally abuse notation by not distinguishing between the function and its value in (2.3) and (2.4). For example,

$$\begin{aligned} \frac{\partial x_k}{\partial X_M} &\equiv \frac{\partial \chi_k \left(X_J, t \right)}{\partial X_M}, \\ \frac{\partial X_M}{\partial x_k} &\equiv \frac{\partial \chi_M^{-1} \left(x_j, t \right)}{\partial x_k}. \end{aligned}$$

Integral transformations will be needed to rewrite conservation laws originally defined over $\mathcal{B}_t$, in terms of fields over $\mathcal{B}_o$. For example, if boundary conditions are only known in terms of the reference configuration $\mathcal{B}_o$, then it can be useful to rewrite the governing equations in terms of fields over $\mathcal{B}_o$.

[1] The tensor product (or dyadic product) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted $\boldsymbol{a} \otimes \boldsymbol{b}$ and has Cartesian components $(\boldsymbol{a} \otimes \boldsymbol{b})_{ij} = a_i b_j$

Definition 2.4.7 (Jacobian Determinant) *The Jacobian determinant J is assumed to be strictly positive for all time t,*

$$J \;=\; \det\left(\frac{\partial x_k}{\partial X_M}\right) > 0.$$

Theorem 2.4.8 (Transformations of Arc, Area, and Volume) *A material element of arc dx_i in the current configuration $\mathcal{B}_t$ is related to its element of arc dX_J in the reference configuration $\mathcal{B}_o$ by*

$$dx_i = \frac{\partial x_i}{\partial X_J}\, dX_J. \tag{2.5}$$

A material element of area $n_i\, dS$ in $\mathcal{B}_t$ is related to its element of area $N_J\, dS_o$ in $\mathcal{B}_o$ by

$$n_i\, dS = J\frac{\partial X_J}{\partial x_i} N_J\, dS_o. \tag{2.6}$$

A material element of volume dV in $\mathcal{B}_t$ is related to its element of volume dV_o in $\mathcal{B}_o$ by

$$dV = J\, dV_o. \tag{2.7}$$

Proof. See ERINGEN [10, pp. 45-48] or OGDEN [21, pp. 83-89] for a proof. ∎

Definition 2.4.9 (Kronecker Delta) *The Kronecker delta symbol, δ_{ij}, is defined by*

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Consider a material vector field A_K and a spatial tensor field B_{ij}. It can be verified directly from definition 2.4.9 that the Kronecker delta symbol has the property of changing indices,

$$\begin{aligned} A_K\, \delta_{KM} &= A_M, \\ B_{ij}\, \delta_{jk} &= B_{ik}. \end{aligned}$$

Definition 2.4.10 (Alternating Symbol) *The alternating symbol, ε_{ijk}, is defined by*

$$\varepsilon_{ijk} = \begin{cases} 1 & \textit{if } (ijk) \textit{ is a cyclic permutation of } (123) \\ -1 & \textit{if } (ijk) \textit{ is an anti-cyclic permutation of } (123) \\ 0 & \textit{otherwise} \end{cases}$$

Consider two spatial vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$. It can be verified directly from definition 2.4.10 that the vector cross product of two vectors and the alternating symbol are related by,

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \varepsilon_{ijk} a_j b_k. \tag{2.8}$$

A direct consequence of (2.8) is an expression for the curl of a vector field in terms of the alternating symbol,

$$(\boldsymbol{\nabla} \times \boldsymbol{b})_i = \varepsilon_{ijk} b_{k,j}.$$

Useful expressions for the determinant and cofactor of a 3×3 matrix A_{ij} in terms of the alternating symbol are[6]

$$\begin{aligned} \det(A_{ij}) &= \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}A_{ip}A_{jq}A_{kr}, \\ \text{cofactor}(A_{ip}) &= \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqr}A_{jq}A_{kr}. \end{aligned} \tag{2.9}$$

Recall, the cofactor matrix and determinant satisfy[7]

$$\det(A_{ij})(A_{ij})^{-1} = \text{cofactor}(A_{ji}). \tag{2.10}$$

Consider a material tensor field T_{JK}. It can be verified directly from definition 2.4.10 that the equation

$$\varepsilon_{IJK} T_{JK} = 0$$

implies that the anti-symmetric components of T_{JK} are identically zero,

$$T_{[JK]} = 0.$$

Using (2.3) and (2.4) for fixed time t we obtain

$$dx_k = \frac{\partial x_k}{\partial X_M} dX_M, \tag{2.11}$$

$$dX_M = \frac{\partial X_M}{\partial x_k} dx_k. \tag{2.12}$$

From the chain-rule of differention, we obtain the useful relations

$$\begin{aligned} \frac{\partial x_k}{\partial X_M}\frac{\partial X_M}{\partial x_l} &= \delta_{kl}, \\ \frac{\partial X_M}{\partial x_k}\frac{\partial x_k}{\partial X_N} &= \delta_{MN}. \end{aligned}$$

[6]See SEGEL [23, pp. 14-23] for discussion on the alternating symbol and determinants
[7]See STRANG [24, pp. 211-227]

We will frequently introduce a symmetric/anti-symmetric decomposition in our proofs and statements. To simplify our presentation we introduce the following notation.

Definition 2.4.11 (S/A Decomposition) *The symmetric/anti-symmetric (S/A) decomposition of a tensor* A_{ij} *is defined as*

$$\begin{aligned} A_{(ij)} &= \frac{1}{2}(A_{ij} + A_{ji}) \\ A_{[ij]} &= \frac{1}{2}(A_{ij} - A_{ji}) \\ A_{ij} &= A_{(ij)} + A_{[ij]}. \end{aligned}$$

The deformation and strain tensors introduced next will appear when transforming our equations from spatial to material fields. They also appear when material response functions depending on the displacement gradient are required to be invariant under observer transformations.

Definition 2.4.12 (Deformation and Strain Tensors) *The deformation tensor* C_{MN} *and the strain tensor* E_{MN} *are defined as*

$$\begin{aligned} C_{MN}(X_L, t) &= \frac{\partial x_k}{\partial X_M}\frac{\partial x_k}{\partial X_N}, \\ E_{MN}(X_L, t) &= \frac{1}{2}\left(\frac{\partial x_k}{\partial X_M}\frac{\partial x_k}{\partial X_N} - \delta_{MN}\right). \end{aligned}$$

The significance of these tensors is illustrated below. Consider elements $d\boldsymbol{P}$ in $\mathcal{B}_o$ and $d\boldsymbol{p}$ in $\mathcal{B}_t$,

$$\begin{aligned} d\boldsymbol{P} &= dX_M \boldsymbol{i}_M, \\ d\boldsymbol{p} &= dx_k \boldsymbol{i}_k. \end{aligned}$$

The square of these elements are

$$\begin{aligned} dS^2 &= dX_M\, dX_M, \\ ds^2 &= dx_k\, dx_k \\ &= \frac{\partial x_k}{\partial X_M}\frac{\partial x_k}{\partial X_N}\, dX_M\, dX_N, \\ &= C_{MN}\, dX_M\, dX_N. \end{aligned}$$

The measure of change of length for the same material points in $\mathcal{B}_o$ and $\mathcal{B}_t$ is

$$\begin{aligned} ds^2 - dS^2 &= \left(\frac{\partial x_k}{\partial X_M}\frac{\partial x_k}{\partial X_N} - \delta_{MN}\right) dX_M\, dX_N, \\ &= 2E_{MN}\, dX_M\, dX_N. \end{aligned}$$

Definition 2.4.13 (Comma Notation) *A comma followed by an index denotes* *partial differentiation with respect to a coordinate. For example,*

$$\begin{aligned} x_{k,M} &= \frac{\partial x_k}{\partial X_M}, \\ X_{M,k} &= \frac{\partial X_M}{\partial x_k}. \end{aligned}$$

In continuum physics theory, the time rate of change following a material point X_k is frequently encountered. We therefore introduce the following notation.

Definition 2.4.14 (Material Time Derivative) *The material time derivative operator is defined as the time rate of change following a material particle X_M,*

$$\frac{d}{dt}(\ ) = \dot{(\ )} = \frac{\partial}{\partial t}(\ )\bigg|_{X_M}.$$

Definition 2.4.15 (Material Velocity) *The material velocity field v_k is defined as*

$$v_k = \frac{\partial \chi_k (X_J, t)}{\partial t}\bigg|_{X_M}.$$

Proposition 2.4.16 (Material Time Derivative: Spatial Fields) *The material time derivative of any spatial field $\phi(x_k, t)$ is*

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + v_k\, \phi_{,k}. \tag{2.13}$$

Proof. Introducing $x_k = \chi_k (X_K, t)$ and using the chain rule

$$\begin{aligned} \frac{d\phi}{dt} = \dot{\phi} &= \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x_k} \frac{\partial x_k}{\partial t}\bigg|_{X_M}, \\ &= \frac{\partial \phi}{\partial t} + v_k\, \phi_{,k}. \end{aligned}$$

■

In continuum mechanics it is natural to express balance laws in terms of integrals over material lines, material surfaces, and material volumes in $\mathcal{B}_t$. Below we state some results that will be useful in working with such integrals.

Lemma 2.4.17 *The material time derivative of the Jacobian determinant J is*

$$\dot{J} = J v_{k,k} \tag{2.14}$$

Proof. From (2.9) we write

$$J = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{IJK} x_{i,I} x_{j,J} x_{k,K}.$$

Differentiating J with respect to $x_{r,R}$ and using (2.9) and (2.10),

$$\begin{aligned}\frac{\partial J}{\partial x_{r,R}} &= \frac{1}{2}\varepsilon_{rjk}\varepsilon_{RJK}x_{j,J}x_{k,K}.\\ &= \text{cofactor}\,(x_{r,R})\\ &= JX_{R,r}.\end{aligned}$$

Taking the material time derivative of J

$$\dot{J} = \frac{\partial J}{\partial x_{r,R}}\frac{d}{dt}(x_{r,R}) = JX_{R,r}v_{r,R}$$

proves (2.14). ■

Lemma 2.4.18 *The material time derivative of the deformation gradient* $X_{J,k}$ *is*

$$\frac{d}{dt}(X_{J,i}) = -X_{J,k}\,v_{k,i} \tag{2.15}$$

Proof. Take the material time derivative of

$$X_{J,k}\,x_{k,K} = \delta_{JK},$$

and multiply the result by $X_{K,i}$ to obtain

$$\begin{aligned}\frac{d}{dt}(X_{J,i}) &= -X_{J,k}\,v_{k,n}\,x_{n,K}\,X_{K,i}\\ &= -X_{J,k}\,v_{k,n}\,\delta_{ni}.\end{aligned} \tag{2.16}$$

Equation (2.16) proves (2.15). ■

Below we derive three useful theorems for material time derivatives of integrals over material lines, material surfaces, and material volumes in $\mathcal{B}_t$.

Remark 2.4.19 (Integrals over Elements of $\mathcal{B}_o$) *According to definition 2.4.14, the material time derivative operates holding material points* $\boldsymbol{X} \in \mathcal{B}_o$ *constant, therefore, the material time derivative operator commutes with integrals defined over elements in* $\mathcal{B}_o$.

Theorem 2.4.20 (Material Time Derivative: Line Integral) *The material time derivative of a line integral of any spatial field* ϕ *over a material line* C *in* $\mathcal{B}_t$ *is*

$$\frac{d}{dt}\int_C \phi\,dx_i = \int_C \left(\dot{\phi}\,dx_i + \phi v_{i,j}\,dx_j\right). \tag{2.17}$$

Proof. Transform the integral over elements in $\mathcal{B}_t$ to an integral over elements in $\mathcal{B}_o$ using (2.5), commute the material time derivative operator with the integral over

C_o, and transform the integral back into an integral over elements in $\mathcal{B}_t$,

$$
\begin{aligned}
\frac{d}{dt}\int_C \phi\, dx_i &= \frac{d}{dt}\int_{C_o} \phi x_{i,J}\, dX_J \\
&= \int_{C_o} \left(\dot{\phi} x_{i,J} + \phi v_{i,J}\right) dX_J \\
&= \int_C \left(\dot{\phi} x_{i,J} + \phi v_{i,J}\right) X_{J,j}\, dx_j \\
&= \int_C \left(\dot{\phi}\, \delta_{ij} + \phi v_{i,j}\right) dx_j .
\end{aligned} \tag{2.18}
$$

Equation (2.18) proves (2.17). ∎

Theorem 2.4.21 (Material Time Derivative: Surface Integral) *The material time derivative of a surface integral of any spatial field ϕ over a material surface S in $\mathcal{B}_t$ is*

$$
\frac{d}{dt}\int_S \phi n_i\, dS = \int_S \left[\left(\dot{\phi} + \phi v_{k,k}\right) n_i - \phi v_{k,i} n_k\right] dS\,. \tag{2.19}
$$

Proof. Transform the integral over elements in $\mathcal{B}_t$ to an integral over elements in $\mathcal{B}_o$ using (2.6), commute the material time derivative operator with the integral over S_o, use the relations (2.14) and (2.15), and transform the integral back into an integral over elements in $\mathcal{B}_t$,

$$
\begin{aligned}
\frac{d}{dt}\int_S \phi n_i\, dS &= \frac{d}{dt}\int_{S_o} \phi J X_{J,i} N_J\, dS_o \\
&= \int_{S_o} \left[\dot{\phi} J X_{J,i} + \phi \dot{J} X_{J,i} + \phi J \frac{d}{dt}\left(X_{J,i}\right)\right] N_J\, dS_o \\
&= \int_{S_o} \left[\dot{\phi} J X_{J,i} + \phi J v_{k,k} X_{J,i} - \phi J X_{J,k} v_{k,i}\right] N_J\, dS_o \\
&= \int_S J\left[\left(\dot{\phi} + \phi v_{k,k}\right) X_{J,i} - \phi v_{k,i} X_{J,k}\right] J^{-1} x_{r,J}\, n_r\, dS \\
&= \int_S \left[\left(\dot{\phi} + \phi v_{k,k}\right) \delta_{ir} - \phi v_{k,i}\, \delta_{kr}\right] n_r\, dS\,.
\end{aligned} \tag{2.20}
$$

Equation (2.20) proves (2.19). ∎

Theorem 2.4.22 (Material Time Derivative: Volume Integral) *The material time derivative of a volume integral of any spatial field ϕ over a material volume V in $\mathcal{B}_t$ is*

$$
\frac{d}{dt}\int_V \phi\, dV = \int_V \left(\dot{\phi} + \phi v_{k,k}\right) dV\,. \tag{2.21}
$$

$$
\frac{d}{dt}\int_V \phi\, dV = \int_V \left[\frac{\partial \phi}{\partial t} + \left(\phi v_k\right)_{,k}\right] dV \tag{2.22}
$$

Proof. Transform the integral over elements in $\mathcal{B}_t$ to an integral over elements in $\mathcal{B}_o$ using (2.7), commute the material time derivative operator with the integral over V_o, use the relation (2.14), and transform the integral back into an integral over elements in $\mathcal{B}_t$,

$$\begin{aligned} \frac{d}{dt}\int_V \phi \, dV &= \frac{d}{dt}\int_{V_o} \phi J \, dV_o \\ &= \int_{V_o} \left(\dot{\phi} J + \phi \dot{J}\right) dV_o \\ &= \int_{V_o} \left(\dot{\phi} + \phi v_{k,k}\right) J \, dV_o \\ &= \int_V \left(\dot{\phi} + \phi v_{k,k}\right) dV \, . \end{aligned} \tag{2.23}$$

Equation (2.23) proves (2.21). Using (2.13) in (2.21) proves (2.22). ■

2.5 Fundamental Axioms of Electromagnetics and Thermomechanics

[8] This section presents the fundamental axioms of electromagnetics and thermomechanics for deformable continua, including both fluids and solids. The electromagnetic equations are presented in terms of rationalized MKS units[9]. The axioms of electromagnetism are defined over spatially fixed line, surface, and volume integrals. In our presentation below, we take these integrals to coincide at time t with the deformed body $\mathcal{B}_t$.

Definition 2.5.1 (Fields)

$$\begin{aligned} \epsilon_o &= \textit{permittivity of free space} \\ \mu_o &= \textit{permeability of free space} \\ \boldsymbol{E} &= \textit{electric field vector in } \mathcal{B}_t \\ \boldsymbol{H} &= \textit{magnetic field vector in } \mathcal{B}_t \\ \boldsymbol{P} &= \textit{polarization vector in } \mathcal{B}_t \\ \boldsymbol{D} &= \textit{electric displacement vector in } \mathcal{B}_t \\ \boldsymbol{D} &= \epsilon_o \boldsymbol{E} + \boldsymbol{P}, \\ \boldsymbol{M} &= \textit{magnetization vector in } \mathcal{B}_t \\ \boldsymbol{B} &= \textit{magnetic induction vector in } \mathcal{B}_t \\ \boldsymbol{H} &= \frac{1}{\mu_o}\boldsymbol{B} - \boldsymbol{M}, \\ q^F &= \textit{free charge per unit volume in } \mathcal{B}_t \\ \boldsymbol{J}' &= \textit{electric conduction current in } \mathcal{B}_t \textit{ (with respect to fixed frame)} \\ \boldsymbol{J} &= \textit{total electric current in } \mathcal{B}_t \textit{ (with respect to fixed frame)} \end{aligned}$$

---

[8]This section is based on ERINGEN & MAUGIN [11, pp. 72-81]

[9]See ERINGEN & MAUGIN [11, p. 406]

$$
\begin{aligned}
\boldsymbol{J} &= \boldsymbol{J}' + q^F \boldsymbol{v} \\
\rho &= \textit{mass per unit volume in } \mathcal{B}_t, \\
\rho_o &= \textit{mass per unit volume in } \mathcal{B}_o, \\
f_i^E &= \textit{electromagnetic or electric force per unit volume in } \mathcal{B}_t, \\
f_i &= \textit{non-electromagnetic force per unit mass}, \\
t_i &= \textit{force per unit area in } \mathcal{B}_t, \\
t_i &= \tau_{ji} n_j, \\
\tau_{ji} &= \textit{Cauchy stress tensor}, \\
C_k^E &= \textit{electromagnetic or electric body couple per unit volume in } \mathcal{B}_t, \\
\epsilon &= \textit{internal energy per unit mass}, \\
\Sigma &= \textit{electromagnetic or electric power per unit volume in } \mathcal{B}_t, \\
h &= \textit{heat power per unit mass}, \\
q &= \textit{heat flux per unit area in } \mathcal{B}_t, \\
\eta &= \textit{entropy per unit mass}, \\
\Theta &= \textit{absolute temperature}.
\end{aligned}
$$

Axiom 2.5.2 (Gauss' Law)

$$\oint_S \boldsymbol{D} \cdot \boldsymbol{n} \, dS = \int_V q^F \, dV . \tag{2.24}$$

Axiom 2.5.3 (Conservation of Magnetic Flux)

$$\oint_S \boldsymbol{B} \cdot \boldsymbol{n} \, dS = 0. \tag{2.25}$$

Axiom 2.5.4 (Faraday's Law)

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{x} = -\frac{\partial}{\partial t} \int_S \boldsymbol{B} \cdot \boldsymbol{n} \, dS . \tag{2.26}$$

Axiom 2.5.5 (Ampere's Law)

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{x} = \int_S \boldsymbol{J} \cdot \boldsymbol{n} \, dS + \frac{\partial}{\partial t} \int_S \boldsymbol{D} \cdot \boldsymbol{n} \, dS . \tag{2.27}$$

Axiom 2.5.6 (Conservation of Mass) *The total mass of a material body $\mathcal{B}$ is unchanged during the motion χ_t of the body.*

$$\int_V \rho \, dV = \int_{V_o} \rho_o \, dV_o \tag{2.28}$$

Axiom 2.5.7 (Balance of Momentum) *The time rate of change of momentum of the material body is equal to the resultant force acting upon the body.*

$$\frac{d}{dt} \int_V \rho v_i \, dV = \int_V \left(\rho f_i + f_i^E \right) dV + \oint_S t_i \, dS . \tag{2.29}$$

Axiom 2.5.8 (Balance of Moment of Momentum) *The time rate of moment of momentum of the material body is equal to the resultant moment of all forces and the resultant of all couples acting on the body.*

$$\frac{d}{dt}\int_V \boldsymbol{x}\times\rho\boldsymbol{v}\,dV = \int_V \left[\boldsymbol{x}\times\left(\rho\boldsymbol{f}+\boldsymbol{f}^E\right)+\boldsymbol{C}^E\right]dV + \oint_S \boldsymbol{x}\times\boldsymbol{t}\,dS\,. \tag{2.30}$$

or in component form,

$$\frac{d}{dt}\int_V \varepsilon_{knj}x_n\rho v_j\,dV = \int_V \left[\varepsilon_{knj}x_n\left(\rho f_j+f_j^E\right)+C_k^E\right]dV + \oint_S \varepsilon_{knj}x_n t_j\,dS\,. \tag{2.31}$$

Axiom 2.5.9 (Conservation of Energy) *The time rate of change of the sum of the internal and kinetic energies of a material body, considered as a closed system, is equal to the sum of the rate of work of all forces and couples and the energies that enter or leave the body per unit time.*

$$\begin{aligned}\frac{d}{dt}\int_V\left(\frac{1}{2}\rho v_i v_i+\rho\epsilon\right)dV &= \int_V\left(\Sigma+\rho h+\rho f_i v_i\right)dV\\ &+ \oint_S\left(t_i v_i-q_i n_i\right)dS\,.\end{aligned} \tag{2.32}$$

Axiom 2.5.10 (Law of Entropy) *The time rate of the total entropy is never less than the sum of the entropy supply due to body sources and the entropy influx through the surface of the body.*

$$\frac{d}{dt}\int_V \rho\eta\,dV \geq \int_V \rho\frac{h}{\Theta}\,dV - \oint_S \frac{q_i}{\Theta}n_i\,dS\,. \tag{2.33}$$

Axiom 2.5.11 (Postulate of Localization) *The axioms hold true for any volume element in V, any surface element in S, and any line element in C.*

2.6 Maxwell's Equations

This section deduces the local balance laws from the global axioms.

Theorem 2.6.1 (Maxwell's Equations) *The local equations (2.34)-(2.37) are equivalent to (2.24)-(2.27),*

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = q^F, \tag{2.34}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \tag{2.35}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \tag{2.36}$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}+\frac{\partial\boldsymbol{D}}{\partial t}. \tag{2.37}$$

Proof. Commute the partial time derivative with the spatially fixed integrals (coinciding at time t with $\mathcal{B}_t$). Applying (A.1), (A.2) to (2.24)-(2.27), and invoking the postulate of localization proves (2.34)-(2.37). ■

Proposition 2.6.2 (Conservation of Charge) *The conservation of charge equation,*

$$\nabla \cdot \boldsymbol{J} + \frac{\partial q^F}{\partial t} = 0, \tag{2.38}$$

is a consequence of Maxwell's equations (2.34) and (2.37).

Proof. Take the partial time derivative of (2.34) and the divergence of (2.37),

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{D}) = \frac{\partial q^F}{\partial t} \quad \rightarrow \quad \nabla \cdot \frac{\partial \boldsymbol{D}}{\partial t} = \frac{\partial q^F}{\partial t}, \tag{2.39}$$

$$\nabla \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right) = \nabla \cdot (\nabla \times \boldsymbol{H}) \quad \rightarrow \quad \nabla \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right) = 0. \tag{2.40}$$

Equation (2.38) follows from (2.39) and (2.40). ∎

Definition 2.6.3 (Electromagnetic Energy of Free Space) *The electromagnetic energy of free space U^F is defined as*

$$U^F = \frac{1}{2}\left(\epsilon_o \boldsymbol{E} \cdot \boldsymbol{E} + \frac{1}{\mu_o} \boldsymbol{B} \cdot \boldsymbol{B}\right). \tag{2.41}$$

Theorem 2.6.4 (Poynting's Theorem) *All fields satisfying Maxwell's equations (2.34)-(2.37) satisfy the identities*

$$\boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}), \tag{2.42}$$

$$\boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{P}}{\partial t} - \boldsymbol{M} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \frac{\partial U^F}{\partial t} = -\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}). \tag{2.43}$$

Proof. Take scalar product of (2.37) with $\boldsymbol{E}$, and using the vector identity

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}),$$

obtain

$$\boldsymbol{E} \cdot \boldsymbol{J} = \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}. \tag{2.44}$$

Using (2.36) in (2.44) proves (2.42). Using (2.41) in (2.42) proves (2.43). ∎

Remark 2.6.5 (Poynting's Vector) *The vector $(\boldsymbol{E} \times \boldsymbol{H})$ in (2.42) and (2.43) is called Poynting's vector and its surface integral is interpreted as the surface flux of electromagnetic energy. In integral form, Poynting's theorem appears mathematically as a conservation statement, with Poynting's vector as a surface flux term,*

$$\int_V \left(\boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t}\right) dV = -\oint_S (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n} \, dS.$$

Physical arguments in support of the Poynting vector interpretation can be found in STRATTON [25].

2.7 EQS Maxwell Equations

In this section the electroquasistatic (EQS) Maxwell equations are defined. These equations are Maxwell's equations (2.34)-(2.37) with the magnetic induction term assumed negligible, see HAUS & MELCHER [16], and TIERSTEN [27]. The EQS equations are a good approximation for materials with small electric current flow, where magnetic fields are presumed negligible.

Definition 2.7.1 (EQS Field) *The EQS electric displacement* $\boldsymbol{D}$ *is defined as*

$$\boldsymbol{D} = \epsilon_o \boldsymbol{E} + \boldsymbol{P}. \tag{2.45}$$

Definition 2.7.2 (Negligible Magnetic Induction) *The magnetic induction and its partial time derivative are assumed negligibly small,*

$$\frac{\partial \boldsymbol{B}}{\partial t} \approx 0, \tag{2.46}$$

$$\boldsymbol{B} \approx 0. \tag{2.47}$$

Remark 2.7.3 (Approximations) *Equation (2.46) is the usual EQS approximation, permitting a non-zero static magnetic induction field. Equation (2.47) is an additional approximation used to eliminate magnetic terms from the electromagnetic body force and body couple equations (2.73) and (2.80).*

The EQS Maxwell equations are,

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = q^F, \tag{2.48}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{2.49}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} \approx 0, \tag{2.50}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}. \tag{2.51}$$

The magnetic field term $\boldsymbol{H}$ can be eliminated by taking the divergence of (2.51) and the partial time derivative of (2.48) to form the conservation of charge statement (2.38). If $\boldsymbol{H}$ is desired it can always be determined from (2.51) once $\boldsymbol{D}$ and $\boldsymbol{E}$ are known. From this point on the EQS Maxwell equations will be defined by:

Definition 2.7.4 (EQS Maxwell Equations)

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = q^F, \tag{2.52}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0 \quad \rightarrow \quad \boldsymbol{E} = -\boldsymbol{\nabla}\phi, \tag{2.53}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial q^F}{\partial t} = 0. \tag{2.54}$$

Definition 2.7.5 (EQS Poynting Vector) *The EQS Poynting vector* $(\boldsymbol{E}\times\boldsymbol{H})^{EQS}$ *is understood as notation for a vector and is defined by*

$$(\boldsymbol{E}\times\boldsymbol{H})^{EQS} \;=\; \phi\left(\boldsymbol{J}+\frac{\partial\boldsymbol{D}}{\partial t}\right). \tag{2.55}$$

Definition 2.7.6 (Electric Energy of Free Space) *The electric energy of free space* U^F *is defined*

$$U^F \;=\; \frac{1}{2}\epsilon_o\boldsymbol{E}\cdot\boldsymbol{E}. \tag{2.56}$$

under the EQS approximation.

Theorem 2.7.7 (EQS Poynting's Theorem) *All fields satisfying the EQS Maxwell equations (2.52)-(2.54) satisfy the identities*

$$\boldsymbol{E}\cdot\boldsymbol{J}+\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}=-\boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{H})^{EQS}, \tag{2.57}$$

$$\boldsymbol{E}\cdot\boldsymbol{J}+\boldsymbol{E}\cdot\frac{\partial\boldsymbol{P}}{\partial t}+\frac{\partial U^F}{\partial t}=-\boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{H})^{EQS}. \tag{2.58}$$

Proof. Using (2.53) and the vector identity

$$\boldsymbol{\nabla}\times(\phi\boldsymbol{H})=\boldsymbol{\nabla}\phi\times\boldsymbol{H}+\phi\boldsymbol{\nabla}\times\boldsymbol{H},$$

gives

$$\begin{aligned}-\boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{H}) &= \boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}\times(\phi\boldsymbol{H})]-\boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\times\boldsymbol{H}),\\ &= -\boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\times\boldsymbol{H}).\end{aligned} \tag{2.59}$$

Using (2.37), (2.59), (2.55), and (2.46) in (2.42) proves (2.57). Using (2.45) and (2.56) in (2.57) proves (2.58). ■

We will find it convenient to work with a reduced form of the EQS Maxwell equations that do not contain free charge q^F explicitly. First we note the integral form of the EQS equations.

Proposition 2.7.8 (EQS Maxwell: Integral Form) *The integral form of the EQS Maxwell equations (2.52)-(2.54) are*

$$\oint_S D_i n_i\,dS \;=\; \int_V q^F\,dV \tag{2.60}$$

$$\oint_C E_i\,dx_i \;=\; 0 \tag{2.61}$$

$$\frac{d}{dt}\int_V q^F\,dV+\oint_S J_i' n_i\,dS \;=\; 0\,, \qquad J_i'=J_i-q^F v_i \tag{2.62}$$

Proof. Integrating (2.52) over a material volume V in spatial coordinates and using (A.1) proves (2.60). Integrating (2.53) over an open material surface S in spatial coordinates and using (A.2) proves (2.61). Integrate (2.54) over a material volume V in spatial coordinates and use (A.1) to obtain

$$\oint_S J_i n_i \, dS + \int_V \frac{\partial q^F}{\partial t} \, dV = 0. \tag{2.63}$$

Using (2.22) and (A.1) in (2.63) proves (2.62). ∎

Proposition 2.7.9 (Reduced EQS Maxwell: Integral Form) *The EQS Maxwell equations (2.60)-(2.62) are equivalent to*

$$\frac{d}{dt}\oint_S D_i n_i \, dS + \oint_S J'_i n_i \, dS = 0 \tag{2.64}$$

$$\oint_C E_i \, dx_i = 0 \tag{2.65}$$

Proof. Taking the material time derivative of (2.60) and using in (2.62) proves (2.64). ∎

It will be useful to define the following convective time derivative.

Definition 2.7.10 (Convective Time Derivative) *The convective time derivative of a spatial vector field D_i is defined as*

$$D_i^* = \dot{D}_i + D_i v_{k,k} - D_k v_{i,k} \tag{2.66}$$

This definition is motivated by (2.19) and satisfies

$$\frac{d}{dt}\oint_S D_i n_i \, dS = \oint_S D_i^* n_i \, dS\,. \tag{2.67}$$

Proposition 2.7.11 (Reduced EQS Maxwell: Local Form) *The local form of the reduced EQS Maxwell equations (2.64) and (2.65) are*

$$(D_i^* + J'_i)_{,i} = 0 \tag{2.68}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{E})_i = 0 \quad \longrightarrow \quad E_i = -\phi_{,i} \tag{2.69}$$

Proof. Equation (2.67) in (2.64) and postulate of localization proves (2.68). Equation (A.2) in (2.65) and postulate of localization proves (2.69). ∎

2.8 Conservation of Mass

This section derives the local form of conservation of mass, and also presents a useful material derivative relation.

Theorem 2.8.1 (Local Conservation of Mass) *Local conservation of mass equivalent to (2.28) is*

$$\dot{\rho} + \rho v_{k,k} = 0. \tag{2.70}$$

Proof. Taking the material time derivative of (2.28), using (2.22)

$$\int_V (\dot{\rho} + \rho v_{k,k})\, dV \;=\; 0, \tag{2.71}$$

and taking (2.71) for arbitrary volumes V (postulate of localization) proves (2.70). ■

Proposition 2.8.2 (Material Derivative Relation) *Any spatial field ϕ satisfies the following identity for integrals over a material volume V*

$$\frac{d}{dt}\int_V \rho\phi\, dV = \int_V \rho\dot{\phi}\, dV\,. \tag{2.72}$$

Proof. Consider an aribtrary function ϕ. Using (2.21) and conservation of mass (2.70) we obtain the useful relation

$$\begin{aligned}\frac{d}{dt}\int_V \rho\phi\, dV &= \int_V \frac{d}{dt}(\rho\phi) + (\rho\phi)v_{k,k}\, dV\,,\\ &= \int_V \rho\dot{\phi} + \phi\,(\dot{\rho} + \rho v_{k,k})\, dV\,,\\ &= \int_V \rho\dot{\phi}\, dV\,.\end{aligned}$$

■

2.9 Balance of Momentum

Expressions for electromagnetic body force density f_i^E acting on deformable continua have been derived in DIXON & ERINGEN [9], MAUGIN & ERINGEN [17], ERINGEN & MAUGIN [11] based on volume (space) averaging techniques, and in DE LORENZI & TIERSTEN [8] based on a well-defined interpenetrating continua model. The DE LORENZI & TIERSTEN paper is a generalization of earlier works by TIERSTEN [30] and TIERSTEN & TSAI [35].

Definition 2.9.1 (Electromagnetic Body Force) *A polarizable, magnetizable, and electrically conducting deformable media will experience an electromagnetic body force per unit volume in $\mathcal{B}_t$ (to a first approximation) of the form*[10]

$$f_i^E = q^F E_i + P_j E_{i,j} + \varepsilon_{ijk} v_j P_n B_{k,n} + \rho\varepsilon_{ijk}\dot{\Pi}_j B_k + M_n' B_{n,i} + \varepsilon_{ijk} J_j B_k. \tag{2.73}$$

$$\Pi_i = \frac{P_i}{\rho}, \qquad M_i' = M_i + (\boldsymbol{v}\times\boldsymbol{P})_i\,.$$

Introducing approximation (2.47) eliminates magnetic force density terms from (2.73) resulting in the electric body force density definition.

[10]See DE LORENZI & TIERSTEN [8, eq. 3.44, p. 944]. Equation (2.73) is equivalent to ERINGEN & MAUGIN [11, eq. 3.5.26, p. 59]

Definition 2.9.2 (Electrical Body Force) *A polarizable and electrically conducting deformable media will experience an electric body force per unit volume in $\mathcal{B}_t$ (to a first approximation) of the form*

$$f_i^E = q^F E_i + P_j E_{i,j}. \tag{2.74}$$

Proposition 2.9.3 (Electrical Stress Tensor) *The electrical body force density f_i^E defined by (2.74) can be written as the divergence of a second order tensor τ_{ji}^E*

$$\begin{aligned} f_i^E &= \tau_{ji,j}^E, & (2.75)\\ \tau_{ji}^E &= P_j E_i + \epsilon_o E_j E_i - \frac{1}{2}\epsilon_o E_k E_k \delta_{ji}, \\ &= D_j E_i - U^F \delta_{ji}. & (2.76) \end{aligned}$$

Proof. Taking divergence of (2.76) and using (2.52), (2.45), and (2.56)

$$\begin{aligned} \tau_{ji,j}^E &= (D_j E_i)_{,j} - \frac{1}{2}\epsilon_o (E_k E_k \delta_{ji})_{,j}, \\ &= D_{j,j} E_i + D_j E_{i,j} - \epsilon_o E_j E_{j,i}, \\ &= q^F E_i + P_j E_{i,j}. \end{aligned}$$

■

Theorem 2.9.4 (Local Balance of Momentum) *Local balance of momentum statements equivalent to (2.29) are*

$$\left(\tau_{ji} + \tau_{ji}^E\right)_{,j} + \rho (f_i - \dot{v}_i) = 0, \tag{2.77}$$

$$\tau_{ji,j} + f_i^E + \rho (f_i - \dot{v}_i) = 0. \tag{2.78}$$

Proof. From (2.72)

$$\frac{d}{dt}\int_V \rho v_i \, dV = \int_V \rho \dot{v}_i \, dV .$$

and noting

$$\begin{aligned} \oint_S \left[t_i(n_j) + t_i^E(n_j)\right] dS &= \oint_S \left(\tau_{ji} + \tau_{ji}^E\right) n_j \, dS, \\ &= \int_V \left(\tau_{ji} + \tau_{ji}^E\right)_{,j} dV . \end{aligned}$$

our balance of momentum statement becomes

$$\int_V \left[\left(\tau_{ji} + \tau_{ji}^E\right)_{,j} + f_i - \rho \dot{v}_i\right] dV . \tag{2.79}$$

Taking (2.79) for arbitrary volumes V (postulate of localization) proves (2.77). Using equation (2.74) in (2.77) proves (2.78). ■

2.10 Balance of Moment of Momentum

An expression for the electromagnetic body couple density C_i^E in deformable continua is derived in ERINGEN & MAUGIN [11] based on a volume (space) averaging techniques.

Definition 2.10.1 (Electromagnetic Body Couple) *A polarizable, magnetizable, and electrically conducting deformable media will experience an electrical body couple per unit volume in $\mathcal{B}_t$ (to a first approximation) of the form*[11]

$$\boldsymbol{C}^E = \boldsymbol{P} \times \boldsymbol{E} + \boldsymbol{M} \times \boldsymbol{B} + \boldsymbol{v} \times (\boldsymbol{P} \times \boldsymbol{B}) . \tag{2.80}$$

Introducing approximation (2.47) eliminates magnetic couple terms from (2.80) resulting in the electrical body couple definition:

Definition 2.10.2 (Electrical Body Couple) *A polarizable and electrically conducting deformable media will experience an electrical body couple per unit volume in $\mathcal{B}_t$ (to a first approximation) of the form*

$$\begin{aligned} C_i^E &= e_{ijk} P_j E_k , \\ \boldsymbol{C}^E &= \boldsymbol{P} \times \boldsymbol{E}. \end{aligned} \tag{2.81}$$

Theorem 2.10.3 (Local Balance of Moment of Momemtum) *Local balance of moment of momentum equivalent to (2.31) is*

$$\tau_{[nj]} = E_{[n} P_{j]} . \tag{2.82}$$

Proof. Noting

$$\begin{aligned} \oint_S \varepsilon_{knj} x_n \tau_{ij} n_i \, dS &= \int_V \left(\varepsilon_{knj} x_n \tau_{ji} \right)_{,i} dV \\ &= \int_V \varepsilon_{knj} \left(x_{n,i} \tau_{ij} + x_n \tau_{ij,i} \right) dV , \\ &= \int_V \varepsilon_{knj} \left(\delta_{ni} \tau_{ij} + x_n \tau_{ij,i} \right) dV , \\ &= \int_V \varepsilon_{knj} \left(\tau_{nj} + x_n \tau_{ij,i} \right) dV . \end{aligned} \tag{2.83}$$

Using (2.21)

$$\begin{aligned} \frac{d}{dt} \int_V \varepsilon_{knj} x_n \rho v_j \, dV &= \int_V \left[\frac{d}{dt} \left(\varepsilon_{knj} x_n \rho v_j \right) + \varepsilon_{knj} x_n \rho v_j v_{k,k} \right] dV , \\ &= \int_V \varepsilon_{knj} \left(v_n \rho v_j + x_n \dot{\rho} v_j + x_n \rho \dot{v}_j + x_n \rho v_j v_{k,k} \right) dV . \end{aligned} \tag{2.84}$$

[11]See ERINGEN & MAUGIN [11, eq. 3.5.32, p. 60]

Using (2.83) and (2.84), (2.31) becomes

$$\int_V \varepsilon_{knj} \left[x_n v_j \left(\dot{\rho} + \rho v_{k,k}\right) - x_n \left(-\rho \dot{v}_j + \rho f_j + f_j^E + \tau_{ij,i}\right) - \tau_{nj}\right] dV$$
$$= \int_V C_k^E \, dV \,. \tag{2.85}$$

Using (2.70) and (2.78), (2.85) becomes

$$\int_V \left(\varepsilon_{knj}\tau_{nj} + C_k^E\right) dV \; = 0 \quad \rightarrow \quad \varepsilon_{knj}\tau_{nj} + C_k^E = 0. \tag{2.86}$$

Recalling (2.81), (2.86) becomes

$$\begin{aligned} \varepsilon_{knj}\left(\tau_{nj} + P_n E_j\right) &= 0 \quad \rightarrow \quad \tau_{[nj]} + P_{[n}E_{j]} = 0, \\ \tau_{[nj]} &= -P_{[n}E_{j]}. \end{aligned} \tag{2.87}$$

Using the anti-symmetric relationship

$$-P_{[n}E_{j]} = E_{[n}P_{j]}.$$

in (2.87) proves (2.82). ■

Definition 2.10.4 (Partial and Total Stress Tensors) *The partial and total stress tensors τ_{ji}^P and τ_{ji}^T are defined as*

$$\tau_{ji}^P = \tau_{ji} + P_j E_i, \tag{2.88}$$
$$\tau_{ji}^T = \tau_{ji} + \tau_{ji}^E, \tag{2.89}$$
$$= \tau_{ji} + D_j E_i - U^F \delta_{ij}. \tag{2.90}$$

The stress tensors τ_{ji}^P and τ_{ji}^T defined above appear in our presentation of the constitutive equations.

Proposition 2.10.5 (Symmetry of Partial and Total Tensors) *Tensors τ_{ji}^P and τ_{ji}^T are symmetric*

$$\tau_{[ji]}^P = 0 \tag{2.91}$$
$$\tau_{[ji]}^T = 0 \tag{2.92}$$

Proof. Recalling the fact

$$E_{[j}P_{i]} = -P_{[j}E_{i]}$$

and using (2.82) we obtain the anti-symmetric part of τ_{ji} as

$$\tau_{[ji]} = -P_{[j}E_{i]}. \tag{2.93}$$

Introducing the S/A decomposition of τ^P_{ji} and using (2.93)

$$\begin{aligned}
\tau^P_{ji} &= \tau^P_{(ji)} + \tau^P_{[ji]} \\
&= \tau_{(ji)} + \tau_{[ji]} + P_{(j}E_{i)} + P_{[j}E_{i]} \\
&= \tau_{(ji)} + P_{(j}E_{i)} \qquad (2.94)
\end{aligned}$$

Equation (2.94) proves (2.91). Similarly decomposing τ^T_{ji} and using (2.93)

$$\begin{aligned}
\tau^T_{ji} &= \tau^T_{(ji)} + \tau^T_{[ji]} \\
&= \tau_{(ji)} + \tau_{[ji]} + P_{(j}E_{i)} + P_{[j}E_{i]} + \epsilon_o E_j E_i - U^F \delta_{ij} \\
&= \tau_{(ji)} + P_{(j}E_{i)} + \epsilon_o E_j E_i - U^F \delta_{ij} \qquad (2.95)
\end{aligned}$$

Equation (2.95) proves (2.92). ■

2.11 Electromagnetic Power

[12] Expressions for the electromagnetic power density Σ for deformable continua have been derived in DIXON & ERINGEN [9], MAUGIN & ERINGEN [17], ERINGEN & MAUGIN [11] based on volume (space) averaging technqiues, and in DE LORENZI & TIERSTEN [8] based on a well-defined interpenetrating continua model. The DE LORENZI & TIERSTEN paper is a generalization of earlier works by TIERSTEN [30] and TIERSTEN & TSAI [35].

Definition 2.11.1 (Electromagnetic Power Density) *A polarizable, magnetizable, and electrically conducting deformable media has electromagnetic power per unit volume in $\mathcal{B}_t$ (to a first approximation) of the form*[13].

$$\begin{aligned}
\Sigma &= \left(P_i E_{j,i} + q^F E_j\right) v_j + E_i \rho \dot{\Pi}_i + J'_i E_i - M'_i \frac{\partial B_i}{\partial t}, \qquad (2.96) \\
\Pi_i &= \frac{P_i}{\rho}, \\
J'_i &= J_i - q^F v_i, \qquad (2.97) \\
M'_i &= M_i + (\boldsymbol{v} \times \boldsymbol{P})_i .
\end{aligned}$$

Proposition 2.11.2 (Equivalent Electromagnetic Power) *Electromagnetic power density Σ is equivalent to*

$$\Sigma = E_i \frac{\partial P_i}{\partial t} - M_i \frac{\partial B_i}{\partial t} + J_i E_i + (E_i P_i v_j)_{,j}. \qquad (2.98)$$

[12] This section is based on THURSTON [26, pp. 157-163]

[13] See DE LORENZI & TIERSTEN [8, eq. 4.4, p.944]. Equation 2.96 is equivalent to ERINGEN & MAUGIN [11, eq. 3.5.41, p. 61]

Proof.

$$\dot{\Pi}_i = \frac{d}{dt}\left(\frac{P_i}{\rho}\right) = \rho^{-1}\dot{P}_i - \rho^{-2}P_i\dot{\rho}.$$

Using (2.70)

$$\begin{aligned}
\dot{\Pi}_i &= \rho^{-1}\dot{P}_i + \rho^{-2}P_i\rho v_{j,j}, \\
\rho\dot{\Pi}_i &= \dot{P}_i + P_i v_{k,k}, && (2.99) \\
&= \frac{\partial P_i}{\partial t} + v_k P_{i,k} + P_i v_{k,k}. && (2.100)
\end{aligned}$$

Recalling the Maxwell equation (2.36),

$$-M_i'\frac{\partial B_i}{\partial t} = -M_i\frac{\partial B_i}{\partial t} + (\boldsymbol{v}\times\boldsymbol{P})_i(\boldsymbol{\nabla}\times\boldsymbol{E})_i.$$

Noting the vector identity,

$$(\boldsymbol{v}\times\boldsymbol{P})_i(\boldsymbol{\nabla}\times\boldsymbol{E})_i = v_k E_{i,k}P_i - P_k E_{i,k}v_i,$$

we obtain

$$\begin{aligned}
-M_i'\frac{\partial B_i}{\partial t} &= -M_i\frac{\partial B_i}{\partial t} + v_k E_{i,k}P_i - P_k E_{i,k}v_i, \\
\Sigma &= E_i\frac{\partial P_i}{\partial t} + E_i P_i v_{j,j} + J_i E_i - M_i\frac{\partial B_i}{\partial t} \\
&+ v_j E_{i,j}P_i + E_i v_j P_{i,j}. && (2.101)
\end{aligned}$$

Grouping terms to obtain $(E_iP_iv_j)_{,j}$ in (2.101) proves (2.98). ∎

2.12 Conservation of Energy: Electroelastic Continua

In this section we establish a series of equivalent statements for conservation of energy.

Theorem 2.12.1 (Conservation of Energy I) *Conservation of energy for electroelastic continua is*

$$\begin{aligned}
\frac{d}{dt}\int_V\left(\frac{1}{2}\rho v_i v_i + \rho\epsilon\right)dV &= \int_V(\Sigma + \rho h + \rho f_i v_i)\,dV \\
&+ \oint_S (t_j v_j - q_i n_i)\,dS\,, && (2.102) \\
\Sigma &= \rho E_i\dot{\Pi}_i + \left(P_iE_{j,i} + q^F E_j\right)v_j + J_i'E_i, && (2.103) \\
\Pi_i &= \frac{P_i}{\rho}, && (2.104) \\
J_i' &= J_i - q^F v_i, && (2.105)
\end{aligned}$$

$$t_j = \tau_{ij} n_i. \tag{2.106}$$

Proof. Theorem 2.12.1 follows immediately from axiom 2.5.9 and definition 2.96 by introducing the EQS approximation (2.46). ■

Proposition 2.12.2 (Equivalent Electric Power Densities) *Electric power density Σ is equivalent to*

$$\Sigma = \rho E_i \dot{\Pi}_i + P_i E_{j,i} v_j + J_i E_i, \tag{2.107}$$

$$\Sigma = E_i \dot{P}_i + E_i P_i v_{j,j} + P_i E_{j,i} v_j + J_i E_i, \tag{2.108}$$

$$\Sigma = E_i \frac{\partial P_i}{\partial t} + (P_i E_i v_j)_{,j} + J_i E_i, \tag{2.109}$$

$$\Sigma = -\frac{\partial U^F}{\partial t} - (\boldsymbol{E} \times \boldsymbol{H})^{EQS}_{i,i} + (P_i E_i v_j)_{,j}. \tag{2.110}$$

Proof. Equation (2.107) follows immediately from (2.103) by using (2.105) and canceling $q^F v_i$ terms. From (2.99)

$$\rho E_i \dot{\Pi}_i = E_i \dot{P}_i + E_i P_i v_{j,j}.$$

Using this in (2.107) proves (2.108). Using (2.46) in (2.98) proves (2.109). ■

Remark 2.12.3 *We can verify that (2.109) is equivalent to (2.108) in the electroelastic approximation by noting $E_i = -\phi_{,i}$ implies*

$$E_{i,j} = -\phi_{,ij} = -\phi_{,ji} = E_{j,i}.$$

Proposition 2.12.4 (Conservation of Energy II) *Conservation of energy statement equivalent to (2.102) is*

$$\begin{aligned} &\frac{d}{dt}\int_V \left(\frac{1}{2}\rho v_i v_i + \rho\epsilon + U^F\right) dV = \int_V \rho\,(h + f_i v_i)\, dV \\ &+ \oint_S \left[U^F v_i - (\boldsymbol{E} \times \boldsymbol{H})^{EQS}_i + P_j E_j v_i\right] n_i \, dS \\ &+ \oint_S (t_j v_j - q_i n_i)\, dS. \end{aligned} \tag{2.111}$$

$$t_j = \tau_{ij} n_i. \tag{2.112}$$

Proof. Adding the term

$$\frac{d}{dt}\int_V U^F \, dV$$

to both sides of (2.102) with (2.110), use (2.19) to obtain

$$\frac{d}{dt}\int_V U^F \, dV = \int_V \left[\frac{\partial U^F}{\partial t} + (U^F v_i)_{,i}\right] dV.$$

Canceling like terms and using the divergence theorem (A.1) proves (2.111). ■

Definition 2.12.5 (Total Energy) *The total energy density U is defined*

$$U = \epsilon + \rho^{-1}U^F \quad \rightarrow \quad \rho U = \rho\epsilon + U^F. \tag{2.113}$$

Theorem 2.12.6 (Conservation of Energy III) *Conservation of energy statement equivalent to (2.102) is* [14]

$$\begin{aligned} \frac{d}{dt}\int_V \left(\frac{1}{2}\rho v_i v_i + \rho U\right) dV \;&=\; \int_V \left[E_i\dot{D}_i + \left(U^F + E_i P_i\right) v_{j,j} + P_i E_{j,i} v_j\right] dV \\ &+\; \int_V \left(J_i E_i + \rho h + \rho f_i v_i\right) dV \\ &+\; \oint_S \left(t_i v_i - q_i n_i\right) dS\,. \end{aligned} \tag{2.114}$$

Proof. Using (2.17) and (2.102) with (2.108) we can write

$$\begin{aligned} \frac{d}{dt}\int_V \left(\frac{1}{2}\rho v_i v_i + \rho U\right) dV &= \int_V \left(\dot{U^F} + U^F v_{i,i}\right) dV + \frac{d}{dt}\int_V \left(\frac{1}{2}\rho v_i v_i + \rho\epsilon\right) dV\,, \\ &= \int_V \left(\dot{U^F} + U^F v_{i,i} + E_i\dot{P}_i + E_i P_i v_{j,j} + P_i E_{j,i} v_j\right) dV \\ &+ \int_V \left(J_i E_i + \rho h + \rho f_i v_i\right) dV + \oint_S \left(t_i v_i - q_i n_i\right) dS\,. \end{aligned} \tag{2.115}$$

Using (2.45) and (2.56) in (2.115) proves (2.114). ∎

Next we are interested in simplifying these integral expression and arriving at local forms of energy balance.

Theorem 2.12.7 (Local Conservation of Energy Statements) *Local conservation of energy statements equivalent to (2.102) with (2.108) and to (2.114) are*

$$\rho\dot{\epsilon} \;=\; E_i\dot{P}_i + \left[\tau_{ji} + E_k P_k \delta_{ij}\right] v_{i,j} + J_i' E_i + \rho h - q_{i,i}, \tag{2.116}$$

$$\rho\dot{U} \;=\; E_i\dot{D}_i + \left[\tau_{ji} + (U^F + E_k P_k)\delta_{ij}\right] v_{i,j} + J_i' E_i + \rho h - q_{i,i}. \tag{2.117}$$

Proof. Using equation (2.72)

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\rho v_i v_i + \rho U\right) dV = \int_V \left(\rho v_i \dot{v}_i + \rho\dot{U}\right) dV\,. \tag{2.118}$$

From divergence theorem

$$\begin{aligned} \oint_S \left(t_i v_i - q_i n_i\right) dS \;&=\; \oint_S \left(\tau_{ji} n_j v_i - q_i n_i\right) dS\,, \\ &=\; \int_V \left[(\tau_{ji} v_i)_{,j} - q_{i,i}\right] dV\,. \end{aligned} \tag{2.119}$$

Using (2.118), (2.119), and balance of momentum

$$\tau_{ji,j} + q^F E_i + P_j E_{i,j} + \rho\left(f_i - \dot{v}_i\right) = 0.$$

[14] This form of the energy expression is motivated by THURSTON [26, eq. 13.47, p. 162]

in (2.102) with (2.108) and (2.114) we obtain

$$\int_V \rho\dot{\epsilon}\, dV = \int_V \left[E_i\dot{P}_i + E_kP_kv_{i,i} + \tau_{ji}v_{i,j} + J_i'E_i + \rho h - q_{i,i}\right] dV\,, \quad (2.120)$$

$$\int_V \rho\dot{U}\, dV = \int_V \left[E_i\dot{D}_i + (U^F + E_iP_i)v_{j,j} + \tau_{ji}v_{i,j} + J_i'E_i + \rho h - q_{i,i}\right] dV\,. \quad (2.121)$$

respectively. Requiring (2.120) and (2.121) to hold for arbitrary volumes V (postulate of localization) proves (2.116) and (2.117). ■

The next integral form of the energy statement will be particularly useful when deriving jump conditions of the energy across a moving surface of discontinuity.

Theorem 2.12.8 (Conservation of Energy IV: Global Form) *Conservation of energy statement equivalent to (2.102) is* [15]

$$\begin{aligned}\frac{d}{dt}\int_V \left(\frac{1}{2}\rho v_iv_i + \rho\epsilon + U^F\right) dV &= \int_V \rho\,(h + f_iv_i)\, dV \\ &+ \oint_S \left[\left(\tau_{ij} + \tau_{ij}^E\right)v_j - \phi\,(J_i' + D_i^*) - q_i\right] dS \end{aligned} \quad (2.122)$$

where the convective derivative D^* is defined in (2.66).

Proof. From (2.111), (2.56), and (2.55),

$$\begin{aligned}\frac{d}{dt}\int_V \left(\frac{1}{2}\rho v_iv_i + \rho\epsilon + U^F\right) dV &= \int_V \rho\,(h + f_iv_i)\, dV \\ &+ \oint_S \left[\frac{1}{2}\epsilon_oE_kE_kv_i - \phi\left(J_i + \frac{\partial D_i}{\partial t}\right) + P_jE_jv_i\right] n_i\, dS \\ &+ \oint_S (t_{ij}v_j - q_i)\, n_i\, dS\,. \end{aligned} \quad (2.123)$$

Noting $\tau_{ij}^E = D_iE_j - \left(\frac{1}{2}\right)\epsilon_oE_kE_k\delta_{ij}$,

$$\begin{aligned}\frac{1}{2}\epsilon_oE_kE_kv_i + P_kE_kv_i &= \left(\frac{1}{2}\epsilon_oE_k + P_k + \frac{1}{2}\epsilon_oE_k\right)E_kv_i - \frac{1}{2}\epsilon_oE_kE_kv_i \\ &= D_kE_kv_i - \frac{1}{2}\epsilon_oE_kE_kv_i \\ &= \tau_{ij}^Ev_j + D_kE_kv_i - D_iE_jv_j. \end{aligned} \quad (2.124)$$

Next consider the term we temporarily define as t,

$$\begin{aligned} t &= \oint_S (D_kE_kv_i - D_iE_jv_j)\, n_i\, dS \\ &= \int_V (D_kE_kv_i - D_iE_jv_j)_{,i}\, dV\,. \end{aligned}$$

[15] This conservation of energy equation was derived by McCarthy & Tiersten [20, eq. 3.13, p. 35], in the context of semiconducting continua.

Introducing $E_k = -\phi_{,k}$,

$$E_{k,j} = -\phi_{,jk} = -\phi_{,kj} = E_{j,k}.$$

then,

$$\begin{aligned} t &= \int_V (D_k v_i)_{,i} E_k + D_i v_j E_{i,j} - E_j (D_i v_j)_{,i} - D_i v_j E_{j,i} \, dV \\ &= \int_V (D_k v_i)_{,i} E_k - E_k (D_i v_k)_{,i} \, dV \\ &= \int_V \phi_{,k} (D_i v_k - D_k v_i)_{,i} \, dV \\ &= \int_V \left[\phi (D_i v_k - D_k v_i)_{,i}\right]_{,k} - \phi (D_i v_k - D_k v_i)_{,ik} \, dV \end{aligned}$$

Noting that $(D_i v_k - D_k v_i)_{,ik} = 0$ and using (2.52),

$$\begin{aligned} t &= \int_V \left[\phi (D_i v_k - D_k v_i)_{,i}\right]_{,k} dV \\ &= \oint_S \phi (D_i v_k - D_k v_i)_{,i} n_k \, dS \\ &= \oint_S \phi \left(q^F v_k + D_i v_{k,i} - D_{k,i} v_i - D_k v_{i,i}\right) n_k \, dS \,. \end{aligned} \tag{2.125}$$

Using (2.124) and (2.125) in (2.123) gives

$$\begin{aligned} \frac{d}{dt}\int_V \left(\frac{1}{2}\rho v_i v_i + \rho\epsilon + U^F\right) dV &= \int_V \rho (h + f_i v_i) \, dV + \oint_S \left[\left(\tau_{ij} + \tau_{ij}^E\right) v_j\right] n_i \, dS \\ + \oint_S \left[\phi \left(q^F v_i - J_i - \frac{\partial D_i}{\partial t} - D_{i,k} v_k + D_k v_{i,k} - D_i v_{k,k}\right) - q_i\right] n_i \, dS \,. \end{aligned} \tag{2.126}$$

Using $J_i' = J_i - q^F v_i$ and (2.13) in (2.126) gives,

$$\begin{aligned} \frac{d}{dt}\int_V \left(\frac{1}{2}\rho v_i v_i + \rho\epsilon + U^F\right) dV &= \int_V \rho (h + f_i v_i) \, dV \\ + \oint_S \left[\left(\tau_{ij} + \tau_{ij}^E\right) v_j + \phi \left(-J_i' - \dot{D}_i + D_k v_{i,k} - D_i v_{k,k}\right) - q_i\right] n_i \, dS \,. \end{aligned} \tag{2.127}$$

Finally using (2.66) in (2.127) proves (2.122). ■

Theorem 2.12.9 (Conservation of Energy IV: Local Form) *The local form of conservation of energy statement for (2.122), in terms of the function $U = \epsilon + \rho^{-1} U^F$, is*

$$\rho \dot{U} = \rho h + \left(\tau_{ij} + \tau_{ij}^E\right) v_{j,i} + E_i \left(J_i' + D_i^*\right) - q_{i,i} \tag{2.128}$$

Proof. Using (A.1), (2.72) and the postulate of localization with (2.122) gives,

$$\rho \dot{v}_i v_i + \rho \dot{U} = \rho h + \rho f_i v_i + \left(\tau_{ij} + \tau_{ij}^E\right)_{,i} v_j + \left(\tau_{ij} + \tau_{ij}^E\right) v_{j,i}$$

$$- \quad \phi_{,i}(J_i' + D_i^*) - \phi\,(J_i' + D_i^*)_{,i} - q_{i,i}. \tag{2.129}$$

Using (2.77) and (2.64) in (2.129) proves (2.128). ∎

2.13 Entropy Inequality

Theorem 2.13.1 (Local Entropy Inequality) *Local entropy inequality statement equivalent to (2.33) is*

$$\rho\dot{\eta} + \left(\frac{q_i}{\Theta}\right)_{,i} - \rho\frac{h}{\Theta} \geq 0, \tag{2.130}$$

$$\rho\Theta\dot{\eta} + q_{i,i} - q_i\Theta^{-1}\Theta_{,i} - \rho h \geq 0. \tag{2.131}$$

Proof. Using (2.72)

$$\frac{d}{dt}\int_V \rho\eta\, dV = \int_V \rho\dot{\eta}\, dV\,.$$

and noting

$$\oint_S \frac{q_i}{\Theta} n_i\, dS = \int_V \left(\frac{q_i}{\Theta}\right)_{,i} dV\,.$$

we obtain

$$\int_V \left[\rho\dot{\eta} + \left(\frac{q_i}{\Theta}\right)_{,i} - \rho\frac{h}{\Theta}\right] dV \geq 0. \tag{2.132}$$

Taking (2.132) for arbitrary volumes V (postulate of localization) proves (2.130). Expanding divergence term in (2.130) and multiplying by Θ proves (2.131). ∎

2.14 Surfaces of Discontinuity

[16] In this section we generalize our global balance laws to include moving surfaces of discontinuity.

Definition 2.14.1 (Discontinuity Surfaces and Lines) *A surface (line) on which a material or spatial field is not continuous, is called a surface (line) of discontinuity.*

Discontinuity surfaces such as shock waves in nonlinear wave propagation problems frequently arise in applications. In this section we derive generalized expressions for Gauss, Stokes, and material time derivative theorems. These will allow us to extend our global balance laws to include surfaces and lines of discontinuity. Application of the postulate of localization will result in jump conditions that fields must satisfy across such surfaces and lines.

[16] This section is based on Eringen [10, pp. 427-429]

Recall the Gauss theorem for a spatial vector field A_k,

$$\int_V A_{k,k}\, dV = \oint_S A_k n_k\, dS$$

where S is the material surface enclosing the material volume V in spatial coordinates. We extend this theorem to regions containing a discontinuity surface with the following theorem.

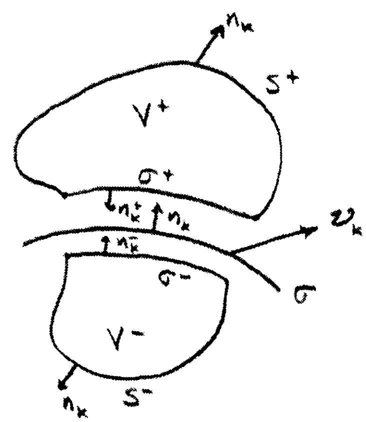

Figure 2-2: Surface of discontinuity for generalized Gauss' theorem.

Theorem 2.14.2 (Generalized Gauss) *A spatial field A_k defined over a material volume V in $\mathcal{B}_t$, bounded by surface S, and containing a surface of discontinuity σ satisfies the integral statement,*

$$\begin{aligned} \oint_{S-\sigma} A_k n_k\, dS &= \int_{V-\sigma} A_{k,k}\, dV + \int_\sigma [\![A_k]\!]\, n_k\, dS \\ [\![A_k]\!] &= A_k^+ - A_k^- \\ V - \sigma &= V^+ + V^- \\ S - \sigma &= S^+ + S^- \end{aligned} \tag{2.133}$$

where A_k^+ and A_k^- are the limiting values of A_k as the discontinuity surface σ is approached from the positive n_k side and negative n_k side, respectively.

Proof. From the Gauss theorem,

$$\begin{aligned} \int_{V^+} A_{k,k}\, dV &= \int_{S^+} A_k n_k\, dS + \int_{\sigma^+} A_k n_k^+\, dS \\ \int_{V^-} A_{k,k}\, dV &= \int_{S^-} A_k n_k\, dS + \int_{\sigma^-} A_k n_k^-\, dS \end{aligned}$$

In the limit as $\sigma^+ \to \sigma$, $n_k^+ \to -n_k$. In the limit as $\sigma^- \to \sigma$, $n_k^- \to n_k$. Then taking the limit as σ^+ and σ^- approach the discontinuity surface σ,

$$\int_{V^+ + V^-} A_{k,k}\, dV = \int_{S^+ + S^-} A_k n_k\, dS - \int_\sigma [\![A_k]\!]\, n_k\, dS\,. \tag{2.134}$$

Equation (2.134) proves (2.133). ∎

We use this generalized Gauss theorem to obtain the following useful generalized integral theorem.

Theorem 2.14.3 (Generalized Material Derivative Statement) *A spatial field ϕ defined over a material volume V in $\mathcal{B}_t$ and containing a surface of discontinuity σ moving with absolute velocity ν_k satisfies the integral statement,*

$$\begin{aligned}
\frac{d}{dt}\int_{V-\sigma}\phi\,dV &= \int_{V-\sigma}\dot{\phi}+\phi v_{k,k}\,dV+\int_{\sigma}[\![\,\phi\,(v_k-\nu_k)\,]\!]\,n_k\,dS \qquad (2.135)\\
[\![\,A_k\,]\!] &= A_k^+ - A_k^-\\
V-\sigma &= V^+ + V^-\\
S-\sigma &= S^+ + S^-.
\end{aligned}$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\int_{V^+}\phi\,dV &= \int_{V^+}\frac{\partial\phi}{\partial t}\,dV+\int_{S^+}\phi v_k n_k\,dS+\int_{\sigma^+}\phi\nu_k n_k^+\,dS\\
\frac{d}{dt}\int_{V^-}\phi\,dV &= \int_{V^-}\frac{\partial\phi}{\partial t}\,dV+\int_{S^-}\phi v_k n_k\,dS+\int_{\sigma^-}\phi\nu_k n_k^-\,dS\,.
\end{aligned}$$

In the limit as $\sigma^+\to\sigma$, $n_k^+\to -n_k$. In the limit as $\sigma^-\to\sigma$, $n_k^-\to n_k$. Then taking the limit as σ^+ and σ^- approach the discontinuity surface σ,

$$\begin{aligned}
\frac{d}{dt}\int_{V^++V^-}\phi\,dV &= \int_{V^++V^-}\frac{\partial\phi}{\partial t}\,dV+\int_{S^++S^-}\phi v_k n_k\,dS\\
&- \int_{\sigma}[\![\,\phi\,]\!]\,\nu_k n_k\,dS\,.
\end{aligned}$$

From the generalized Gauss theorem,

$$\int_{S-\sigma}\phi v_k n_k\,dS=\int_{V-\sigma}(\phi v_k)_{,k}\,dV+\int_{\sigma}[\![\,\phi v_k\,]\!]\,n_k\,dS\,.$$

Then noting that ν_k commutes with $[\![\,\cdot\,]\!]$ operator,

$$\frac{d}{dt}\int_{V-\sigma}\phi\,dV=\int_{V-\sigma}\left(\frac{\partial\phi}{\partial t}+\phi_{,k}v_k+\phi v_{k,k}\right)dV+\int_{\sigma}[\![\,\phi\,(v_k-\nu_k)\,]\!]\,n_k\,dS\,. \qquad (2.136)$$

Using (2.13) in (2.136) proves (2.135). ∎

We will require a generalized version of Stokes's theorem to account for a line of discontinuity in a surface S.

Theorem 2.14.4 (Generalized Stokes') *A spatial field A_k defined over an open material surface S in $\mathcal{B}_t$, bounded by the line C, and containing a line of discontinuity γ satisfies the integral statement,*

$$\oint_{C-\gamma}A_i\,dx_i = \int_{S-\gamma}(\boldsymbol{\nabla}\times\boldsymbol{A})_i\,n_i\,dS+\int_{\gamma}[\![\,A_i\,]\!]\,dx_i \qquad (2.137)$$

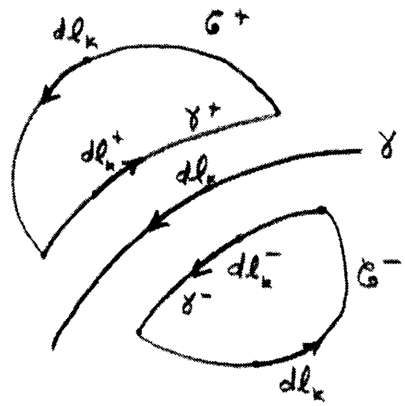

Figure 2-3: Line of discontinuity for generalized Stokes' theorem.

$$\begin{aligned} [\![A_i]\!] &= A_i^+ - A_i^- \\ S - \gamma &= S^+ + S^- \\ C - \gamma &= C^+ + C^-. \end{aligned}$$

Proof. From Stokes' theorem,

$$\begin{aligned} \int_{S+} (\boldsymbol{\nabla} \times \boldsymbol{A})_i \, n_i \, dS &= \int_{C+} A_i \, dx_i + \int_{\gamma^+} A_i \, dx_i^+ \\ \int_{S-} (\boldsymbol{\nabla} \times \boldsymbol{A})_i \, n_i \, dS &= \int_{C^-} A_i \, dx_i + \int_{\gamma^-} A_i \, dx_i^- \end{aligned}$$

In the limit as $\gamma^+ \to \gamma$, $dx_k^+ \to -dx_k$. In the limit as $\gamma^- \to \gamma$, $dx_k^- \to dx_k$. Then taking the limit as γ^+ and γ^- approach the discontinuity line γ,

$$\int_{S^+ + S^-} (\boldsymbol{\nabla} \times \boldsymbol{A})_i \, n_i \, dS = \int_{C^+ + C^-} A_i \, dx_i - \int_\gamma [\![A_i]\!] \, dx_i$$

proves (2.137). ∎

2.15 Jump Conditions

In this section we use the generalized theorems derived earlier to obtain jump conditions across surfaces of discontinuity. First we summarize in convenient integral form the electroelastic EQS equations.

Proposition 2.15.1 (Integral EQS Electroelastic Equations: Spatial Form)

$$\begin{aligned} \oint_{S-\sigma} (D_i^* + J_i') \, n_i \, dS &= 0 \\ \oint_{C-\gamma} E_i \, dx_i &= 0 \end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\int_{V-\sigma} \rho \, dV &= 0 \\
\frac{d}{dt}\int_{V-\sigma} \rho v_i \, dV &= \oint_{S-\sigma} \left(\tau_{ji} + \tau_{ji}^E\right) dS + \int_{V-\sigma} \rho f_i \, dV \\
\frac{d}{dt}\int_{V-\sigma} \left(\frac{1}{2}\rho v_i v_i + \rho\epsilon + U^F\right) dV &= \int_{V-\sigma} \rho \left(h + f_i v_i\right) dV \\
&\quad + \oint_{S-\sigma} \left[\left(\tau_{ij} + \tau_{ij}^E\right) v_j - \phi \left(J_i' + D_i^*\right) - q_i\right] dS \\
\frac{d}{dt}\int_{V-\sigma} \rho\eta \, dV &\geq \int_{V-\sigma} \rho \frac{h}{\Theta} dV - \oint_{S-\sigma} \frac{q_i}{\Theta} n_i \, dS
\end{aligned}
\tag{2.138}
$$

Jump conditions are obtained by applying the three generalized integral theorems derived earlier and valid for volumes containing a surface discontinuity moving at velocity ν_k. Applying the postulate of localization then results in differential equations derived earlier and new jump conditions across the discontinuity surface and discontinuity line.

The above procedure is straight forward and will be illustrated for the balance of momentum equation.

Example 2.15.2 (Jump Condition Calculations) Applying (2.133) and (2.135) gives

$$
\begin{aligned}
\frac{d}{dt}\int_{V-\sigma} \left(\rho v_i\right) dV &= \int_{V-\sigma} \left[\frac{d}{dt}\left(\rho v_i\right) + \left(\rho v_i\right) v_{k,k}\right] dV \\
&\quad + \int_{\sigma} [\![\rho v_i \left(v_k - \nu_k\right)]\!] n_k \, dS
\end{aligned}
\tag{2.139}
$$

$$
\oint_{S-\sigma} \left(\tau_{ji} + \tau_{ji}^E\right) n_j \, dS = \int_{V-\sigma} \left(\tau_{ji} + \tau_{ji}^E\right) dV + \int_{\sigma} [\![\tau_{ji} + \tau_{ji}^E]\!] n_j \, dS \tag{2.140}
$$

Using (2.139) and (2.140) in (2.138), the global momentum balance laws for a volume containing a discontinuity surface moving with absolute velocity ν_k, becomes

$$
\begin{aligned}
\int_{V-\sigma} \left[\frac{d}{dt}\left(\rho v_i\right) + \left(\rho v_i\right) v_{k,k}\right] dV &+ \int_{\sigma} [\![\rho v_i \left(v_j - \nu_j\right) - \left(\tau_{ji} + \tau_{ji}^E\right)]\!] n_j \, dS \\
&= \int_{V-\sigma} \left(\tau_{ji} + \tau_{ji}^E\right)_{,j} + \rho f_i \, dV
\end{aligned}
$$

From the postulate of localization we obtain

$$
\begin{aligned}
\frac{d}{dt}\left(\rho v_i\right) + \left(\rho v_i\right) v_{k,k} &= \left(\tau_{ji} + \tau_{ji}^E\right)_{,j} + \rho f_i && \text{in } V - \sigma \\
[\![\rho v_i \left(v_j - \nu_j\right)]\!] n_j &= [\![\tau_{ji} + \tau_{ji}^E]\!] n_j && \text{across } \sigma.
\end{aligned}
$$

The differential equation is equivalent to local form for balance of momentum we derived earlier. The jump condition across the moving discontinuity surface σ is new.

Here we summarize the corresponding jump conditions for the EQS electroelastic equations in spatial fields.

Proposition 2.15.3 (Jump Conditions: Moving Surface of Discontinuity)

$$\begin{aligned}
[\![D_i^* + J_i']\!]\, n_i &= 0 && \textit{across } \sigma \\
[\![\phi]\!] &= 0 && \textit{across } \gamma \\
[\![x_k]\!] &= 0 && \textit{across } \sigma \\
[\![\rho\,(v_k - \nu_k)]\!]\, n_k &= 0 && \textit{across } \sigma \\
[\![\rho v_i\,(v_k - \nu_k)]\!]\, n_j &= [\![\tau_{ji} + \tau_{ji}^E]\!]\, n_j && \textit{across } \sigma \\
[\![\left(\frac{1}{2}\rho v_j v_j + \rho\epsilon + U^F\right)(v_i - \nu_i)]\!]\, n_i &= \\
[\![\left(\tau_{ij} + \tau_{ij}^E\right) v_j - \phi\,(J_i' + D_i^*) - q_i]\!]\, n_i & && \textit{across } \sigma \\
[\![\rho\eta\,(v_j - \nu_j)]\!]\, n_j &\geq -[\![\frac{q_j}{\Theta}]\!]\, n_j && \textit{across } \sigma
\end{aligned}$$

Proposition 2.15.4 (Jump Conditions: Material Surface of Discontinuity)

$$\begin{aligned}
[\![D_i^* + J_i']\!]\, n_i &= 0 && \textit{across } \sigma \\
[\![\phi]\!] &= 0 && \textit{across } \gamma \\
[\![x_k]\!] &= 0 && \textit{across } \sigma \\
[\![\tau_{ji} + \tau_{ji}^E]\!]\, n_j &= 0 && \textit{across } \sigma \\
[\![\left(\tau_{ij} + \tau_{ij}^E\right) v_j - \phi\,(J_i' + D_i^*) - q_i]\!]\, n_i &= 0 && \textit{across } \sigma \\
[\![\frac{q_j}{\Theta}]\!]\, n_j &\geq 0 && \textit{across } \sigma
\end{aligned}$$

2.16 Objective Fields and Reference Configurations

[17] In this section we define the notion of an objective field. Objective fields are quantities that are independent of observer or form invariant under observer transformations. In the following section we introduce three axioms of constitutive theory. One of these requires that material response functions be independent of observer, that is, objective. This notion is then used to deduce restrictions on the allowable class of material response functions. For example, material objectivity restricts a scalar function depending on the deformation gradient $x_{i,K}$, which is not an objective function, to a scalar function depending on the strain tensor E_{JK}, which is an objective function. In this manner, the strain tensor arises naturally in the mathematical description of material response. Next we introduce change of reference configuration. Material symmetries can be characterized by a required invariance under a class of reference configuration transformations.

Prior to stating our definition of objective fields, we define arbitrary line elements $d\boldsymbol{x}_1, d\boldsymbol{x}_2$ in $\mathcal{B}_t$ and line elements $d\boldsymbol{X}_1, d\boldsymbol{X}_2$ in $\mathcal{B}_o$. Under an observer transformation

[17] This section is based on OGDEN [21, pp. 133-137] and GURTIN [15, pp. 165-175]

(2.1) the material line elements remain unchanged and spatial line elements transform as,

$$d\boldsymbol{x}_k^* = \boldsymbol{Q}(t)\, d\boldsymbol{x}_k \quad \text{for} \quad k = 1, 2.$$

Definition 2.16.1 (Objective Fields) *Objective scalar, vector, tensor, and two-point tensor fields with domains both in $\mathcal{T}$ and $\mathcal{T}_o$ are defined according to transformations under a change in observer. Consider the following fields,*

- *A spatial scalar field ϕ*
- *A material scalar field Φ*
- *A spatial vector field $\boldsymbol{A} = A_j \boldsymbol{i}_j$*
- *A material vector field $\boldsymbol{H} = H_K \boldsymbol{i}_K$*
- *A spatial tensor field $\boldsymbol{B} = B_{ij} \boldsymbol{i}_i \otimes \boldsymbol{i}_j$*
- *A material tensor field $\boldsymbol{M} = M_{RS} \boldsymbol{i}_R \otimes \boldsymbol{i}_S$*
- *A spatial two-point tensor field $\boldsymbol{G} = G_{Ki} \boldsymbol{i}_K \otimes \boldsymbol{i}_i$*
- *A material two-point tensor field $\boldsymbol{F} = F_{iK} \boldsymbol{i}_i \otimes \boldsymbol{i}_K$*

These fields are said to be objective if,

$$\begin{aligned}
\phi^*(\boldsymbol{x}^*, t^*) &= \phi(\boldsymbol{x}, t), \\
\Phi^*(\boldsymbol{X}, t^*) &= \Phi(\boldsymbol{X}, t) \\
d\boldsymbol{x}_1^{*\prime} \boldsymbol{A}^*(\boldsymbol{x}^*, t^*) &= d\boldsymbol{x}_1{}' \boldsymbol{A}(\boldsymbol{x}, t), \\
d\boldsymbol{X}_1{}' \boldsymbol{H}^*(\boldsymbol{X}, t^*) &= d\boldsymbol{X}_1{}' \boldsymbol{H}(\boldsymbol{X}, t), \\
d\boldsymbol{x}_1^{*\prime} \boldsymbol{B}^*(\boldsymbol{x}^*, t^*)\, d\boldsymbol{x}_2^* &= d\boldsymbol{x}_1{}' \boldsymbol{B}(\boldsymbol{x}, t)\, d\boldsymbol{x}_2, \\
d\boldsymbol{X}_1{}' \boldsymbol{M}^*(\boldsymbol{X}^*, t^*)\, d\boldsymbol{X}_2 &= d\boldsymbol{X}_1{}' \boldsymbol{M}(\boldsymbol{X}, t)\, d\boldsymbol{X}_2, \\
d\boldsymbol{X}_1{}' \boldsymbol{G}^*(\boldsymbol{x}^*, t^*)\, d\boldsymbol{x}_1^* &= d\boldsymbol{X}_1{}' \boldsymbol{G}(\boldsymbol{x}, t)\, d\boldsymbol{x}_1, \\
d\boldsymbol{x}_1^{*\prime} \boldsymbol{F}^*(\boldsymbol{X}, t^*)\, d\boldsymbol{X}_1 &= d\boldsymbol{x}_1{}' \boldsymbol{F}(\boldsymbol{X}, t)\, d\boldsymbol{X}_1.
\end{aligned}$$

Recall, for example, that $\phi^*(\boldsymbol{x}^*, t^*)$ and $\phi(\boldsymbol{x}, t)$ are quantities associated with the same event as recorded by two different observers, O^* and O respectively.

Remark 2.16.2 (Objective Fields) *Definition 2.16.1 implies the following transformation requirements for objective fields under a change in observer (2.1):*

$$\phi^*(\boldsymbol{x}^*, t^*) = \phi(\boldsymbol{x}, t). \tag{2.141}$$

$$\Phi^*(\boldsymbol{X}, t^*) = \Phi(\boldsymbol{X}, t). \tag{2.142}$$

$$\boldsymbol{A}^*(\boldsymbol{x}^*, t^*) = \boldsymbol{Q}(t)\boldsymbol{A}(\boldsymbol{x}, t). \tag{2.143}$$

$$\boldsymbol{H}^*(\boldsymbol{X}, t^*) = \boldsymbol{H}(\boldsymbol{X}, t). \tag{2.144}$$

$$\boldsymbol{B}^*(\boldsymbol{x}^*, t^*) = \boldsymbol{Q}(t)\boldsymbol{B}(\boldsymbol{x}, t)\, \boldsymbol{Q}(t)'. \tag{2.145}$$

$$M^*(X^*,t^*) = M(X,t). \tag{2.146}$$
$$G^*(x^*,t^*) = G(x,t)\,Q(t)'. \tag{2.147}$$
$$F^*(X,t^*) = Q(t)F(X,t). \tag{2.148}$$

Definition 2.16.3 (Change of Reference Configuration) *A change of reference configuration is a deformation κ_o that takes $\mathcal{B}_o$ to a new reference configuration $\overline{\mathcal{B}}_o$ and takes material points X to new material points $\overline{X}$,*

$$\kappa_o : \mathcal{B}_o \to \mathcal{E},$$
$$\kappa_o : X \mapsto \overline{X}.$$

We write

$$\overline{\mathcal{B}}_o = \kappa_o(\mathcal{B}_o)$$
$$\overline{X} = \kappa_o(X)$$

2.17 Axioms of Constitutive Theory

[18] We introduce three axioms of constitutive theory that must be satisfied by any material response or constitutive equations. These axioms will be used to restrict the class of equations that may describe material response.

Axiom 2.17.1 (Admissibility) *The constitutive equations must be consistent with the fundamental axioms of electromagnetics and thermomechanics.*

Axiom 2.17.2 (Material Objectivity) *The constitutive response functions must be independent of observer.*

Axiom 2.17.3 (Material Symmetry) *The constitutive response functions must be scalar invariant with respect to a group of transformations of the reference configuration representing the material symmetry conditions.*

Recall, an observer transformation (2.1) that takes (x,t) to (x^*,t^*) is interpreted as a change in observer from O to O^*, such that the event recorded by O at (x,t) is the same event as that recorded by O^* at (x^*,t^*). Consider a scalar ϵ, vector f, and tensor A defined over $\mathcal{B}_t$ corresponding to a particular event at (x,t) as recorded by O. If O^* observes this same event, measured at (x^*,t^*) according to O^*, then O^* must necessarily measure the scalar as $\epsilon^* = \epsilon$, the vector as $f^* = Q(t)f$, and the tensor as $A^* = Q(t)AQ(t)'$. In other words, the observers O and O^* measure the same event. An event is necessarily independent of observer, and therefore the scalar, vector, and tensor quantities corresponding to the event are objective fields, and transform according to remark 2.16.2 under an observer transformation. Consider an event, with corresponding fields $\epsilon, \Theta, E, D, J', \tau^T$ measured by O at (x,t). These

[18] The axioms are based on Eringen & Maugin [11, pp. 133-144]

fields are necessarily objective fields, and transform according to remark 2.16.2 under a change of observer.

The axiom of material objectivity requires that material response *functions* are independent of observer. Here it is important to distinguish between the material response function, and its value. Material objectivity is a powerful axiom that imposes restrictions on the set of admissible material response functions. The next section clarifies this discussion with some examples.

2.18 Constitutive Function Restrictions

[19] Suppose we deduce or assume the following constitutive equations relative to the reference configuration $\mathcal{B}_o$,

$$\begin{aligned} \epsilon(\boldsymbol{x},t) &= \hat{\epsilon}(\chi(\boldsymbol{X},t)), \\ \boldsymbol{E}(\boldsymbol{x},t) &= \hat{\boldsymbol{E}}(\chi(\boldsymbol{X},t)), \\ \boldsymbol{\tau}(\boldsymbol{x},t) &= \hat{\boldsymbol{\tau}}(\chi(\boldsymbol{X},t)). \end{aligned}$$

As remarked in the previous section, the fields ϵ, $\boldsymbol{E}$, and $\boldsymbol{\tau}$ transform objectively under a change in observer,

$$\begin{aligned} \epsilon^*(\boldsymbol{x}^*,t^*) &= \epsilon(\boldsymbol{x},t), \\ \boldsymbol{E}^*(\boldsymbol{x}^*,t^*) &= \boldsymbol{Q}(t)\boldsymbol{E}(\boldsymbol{x},t), \\ \boldsymbol{\tau}^*(\boldsymbol{x}^*,t^*) &= \boldsymbol{Q}(t)\boldsymbol{\tau}(\boldsymbol{x},t)\boldsymbol{Q}'(t). \end{aligned}$$

Using the constitutive equations we obtain,

$$\begin{aligned} \epsilon^*(\boldsymbol{x}^*,t^*) &= \hat{\epsilon}(\chi(\boldsymbol{X},t)), \\ \boldsymbol{E}^*(\boldsymbol{x}^*,t^*) &= \boldsymbol{Q}(t)\hat{\boldsymbol{E}}(\chi(\boldsymbol{X},t)), \\ \boldsymbol{\tau}^*(\boldsymbol{x}^*,t^*) &= \boldsymbol{Q}(t)\hat{\boldsymbol{\tau}}(\chi(\boldsymbol{X},t))\boldsymbol{Q}'(t). \end{aligned}$$

The axiom of material objectivity requires the material response functions $\hat{\epsilon}$, $\hat{\boldsymbol{E}}$, and $\hat{\boldsymbol{\tau}}$ to be the independent of observer. This implies

$$\begin{aligned} \epsilon^*(\boldsymbol{x}^*,t^*) &= \hat{\epsilon}(\chi^*(\boldsymbol{X},t^*)), \\ \boldsymbol{E}^*(\boldsymbol{x}^*,t^*) &= \hat{\boldsymbol{E}}(\chi^*(\boldsymbol{X},t^*)), \\ \boldsymbol{\tau}^*(\boldsymbol{x}^*,t^*) &= \hat{\boldsymbol{\tau}}(\chi^*(\boldsymbol{X},t^*)). \end{aligned}$$

and therefore,

$$\begin{aligned} \hat{\epsilon}(\chi(\boldsymbol{X},t)) &= \hat{\epsilon}(\chi^*(\boldsymbol{X},t^*)), \\ \boldsymbol{Q}(t)\hat{\boldsymbol{E}}(\chi(\boldsymbol{X},t)) &= \hat{\boldsymbol{E}}(\chi^*(\boldsymbol{X},t^*)), \end{aligned}$$

[19] See COLEMAN & NOLL [7, pp. 170-173], TRUESDELL & NOLL [36, pp. 41-47], and GURTIN [15, pp. 143-145] for a discussion of constitutive function restrictions and the principle of objectivity

$$\boldsymbol{Q}(t)\hat{\boldsymbol{\tau}}\left(\boldsymbol{\chi}\left(\boldsymbol{X},t\right)\right)\boldsymbol{Q}'(t) = \hat{\boldsymbol{\tau}}\left(\boldsymbol{\chi}^*\left(\boldsymbol{X},t^*\right)\right).$$

These equations are the mathematical statement of material objectivity, and must be satisfied by $\hat{\epsilon}$, $\hat{\boldsymbol{E}}$, and $\hat{\boldsymbol{\tau}}$ for all possible proper orthogonal $\boldsymbol{Q(t)}$.

Next, we write the deformation gradient in direct notation as

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}\left(\boldsymbol{X},t\right)}{\partial \boldsymbol{X}}.$$

To make the restrictions more explicit, suppose we deduce or assume the constitutive equations are

$$\begin{aligned} \epsilon\left(\boldsymbol{x},t\right) &= \hat{\epsilon}\left(\boldsymbol{F}\right), \\ \boldsymbol{E}\left(\boldsymbol{x},t\right) &= \hat{\boldsymbol{E}}\left(\boldsymbol{F}\right), \\ \boldsymbol{\tau}\left(\boldsymbol{x},t\right) &= \hat{\boldsymbol{\tau}}\left(\boldsymbol{F}\right). \end{aligned}$$

Under a change of observer, $\boldsymbol{F}$ transforms as

$$\begin{aligned} \boldsymbol{F}^* &= \frac{\partial \boldsymbol{\chi}^*\left(\boldsymbol{X},t\right)}{\partial \boldsymbol{X}} \\ &= \boldsymbol{Q}(t)\boldsymbol{F}. \end{aligned}$$

From our previous result, the axiom of objectivity imposes the following restrictions on the material response functions,

$$\begin{aligned} \hat{\epsilon}\left(\boldsymbol{F}\right) &= \hat{\epsilon}\left(\boldsymbol{Q}(t)\boldsymbol{F}\right), \\ \boldsymbol{Q}(t)\hat{\boldsymbol{E}}\left(\boldsymbol{F}\right) &= \hat{\boldsymbol{E}}\left(\boldsymbol{Q}(t)\boldsymbol{F}\right), \\ \boldsymbol{Q}(t)\hat{\boldsymbol{\tau}}\left(\boldsymbol{F}\right)\boldsymbol{Q}'(t) &= \hat{\boldsymbol{\tau}}\left(\boldsymbol{Q}(t)\boldsymbol{F}\right). \end{aligned}$$

Next we consider the restrictions imposed by the axiom of material symmetry on the constitutive functions. Consider the change in reference configuration from $\mathcal{B}_o$ and $\overline{\mathcal{B}}_o$,

$$\begin{aligned} \boldsymbol{\kappa}_o\left(\boldsymbol{X}\right) &= \boldsymbol{S}'\boldsymbol{X} \\ \boldsymbol{S}'\boldsymbol{S} &= \boldsymbol{S}\boldsymbol{S}' = \boldsymbol{I} \\ \det\left(\boldsymbol{S}\right) &= 1. \end{aligned}$$

The motion with respect to the new reference configuration is

$$\begin{aligned} \overline{\boldsymbol{\chi}}_t : \overline{\mathcal{B}}_o &\rightarrow \mathcal{B}_t \\ \overline{\boldsymbol{\chi}}_t : \overline{\boldsymbol{X}} &\mapsto \boldsymbol{x} \end{aligned}$$

We write

$$\mathcal{B}_t = \overline{\boldsymbol{\chi}}\left(\overline{\mathcal{B}}_o, t\right)$$

$$x = \overline{\chi}\left(\overline{X}, t\right)$$

The deformation gradient with respect to the new reference configuration is

$$\begin{aligned} \overline{F} &= \frac{\partial \overline{\chi}\left(\overline{X}, t\right)}{\partial \overline{X}} \\ &= FS \end{aligned}$$

With respect to the two reference configurations $\mathcal{B}_o$ and $\overline{\mathcal{B}}_o$, the constitutive equations are

$$\begin{aligned} \epsilon(x, t) &= \hat{\epsilon}(F), \\ E(x, t) &= \hat{E}(F), \\ \tau(x, t) &= \hat{\tau}(F) \end{aligned}$$

$$\begin{aligned} \epsilon(x, t) &= \overline{\epsilon}\left(\overline{F}\right), \\ E(x, t) &= \overline{E}\left(\overline{F}\right), \\ \tau(x, t) &= \overline{\tau}\left(\overline{F}\right). \end{aligned}$$

Suppose that the response of the material relative to $\mathcal{B}_o$ is always indistinguishable from that relative to $\overline{\mathcal{B}}_o$ for all proper orthogonal tensors S in the set of transformations $\mathcal{S}$, such that

$$\begin{aligned} \hat{\epsilon} \equiv \overline{\epsilon} &\rightarrow \hat{\epsilon}(F) = \hat{\epsilon}(FS), \\ \hat{E} \equiv \overline{E} &\rightarrow \hat{E}(F) = \hat{E}(FS), \\ \hat{\tau} \equiv \overline{\tau} &\rightarrow \hat{\tau}(F) = \hat{\tau}(FS). \end{aligned}$$

The set $\mathcal{S}$ is said to characterize the symmetry of the material relative to the reference configuration $\mathcal{B}_o$. If we replace F with FS' we obtain

$$\begin{aligned} \hat{\epsilon}(FS') &= \hat{\epsilon}(F), \\ \hat{E}(FS') &= \hat{E}(F), \\ \hat{\tau}(FS') &= \hat{\tau}(F). \end{aligned}$$

This shows that if $S \in \mathcal{S}$ then $S' \in \mathcal{S}$. Combining the imposed restriction from material symmetry with the imposed restriction from material objectivity and choosing $Q = S$, implies the restriction that the response functions are invariant under $\mathcal{S}$,

$$\begin{aligned} \hat{\epsilon}(F) &= \hat{\epsilon}(SFS'), \\ S\hat{E}(F) &= \hat{E}(SFS'), \\ S\hat{\tau}S'(F) &= \hat{\tau}(SFS'). \end{aligned}$$

2.19 Constitutive Equations: Spatial Fields

This section uses the conservation of energy and entropy inequality equations, under the axiom of admissibility, to derive restrictions on material response functions. The result is a set of relations between material response functions and partial derivatives of a scalar energy response function.

The axiom of admissibility requires material response functions to be consistent with the fundamental axioms of electromagnetics and thermomechanics, in particular the conservation of energy and entropy inequality equations. It is useful to form the so-called Clausius-Duhem (C-D) inequality by eliminating the heat flux and heat source terms from the entropy inequality (2.131) and conservation of energy equation, either (2.116) or (2.117). The resulting inequality is required to be satisfied for all independent processes. The resulting necessary and sufficient conditions yield general constitutive equations that govern material response. These constitutive equations are in terms of first derivatives of energy functions, $\hat{\epsilon}$ or $\hat{U}$. The axioms of material objectivity and material symmetry can then be applied to deduce further restrictions on these equations.

Proposition 2.19.1 (Local C-D Inequality Statements: Spatial Fields) *The local form of the C-D inequality statements are*

$$\rho\Theta\dot{\eta} - \rho\dot{\epsilon} + E_i\dot{P}_i + (\tau_{ji} + E_kP_k\delta_{ij})\,v_{i,j} + J_i'E_i - q_i\Theta^{-1}\Theta_{,i} \geq 0 \quad (2.149)$$

$$\rho\Theta\dot{\eta} - \rho\dot{U} + E_i\dot{D}_i + \left[\tau_{ji} + (U^F + E_kP_k)\delta_{ij}\right]v_{i,j} + J_i'E_i - q_i\Theta^{-1}\Theta_{,i} \geq 0 \quad (2.150)$$

Proof. The C-D inequalities are obtained by eliminating $(q_{i,i} - \rho h)$ from the entropy inequality (2.131) using conservation of energy statements. Reodering (2.116) and (2.117)

$$\begin{aligned} q_{i,i} - \rho h &= -\rho\dot{\epsilon} + E_i\dot{P}_i + (\tau_{ji} + E_kP_k\delta_{ij})\,v_{i,j} + J_i'E_i \\ q_{i,i} - \rho h &= -\rho\dot{U} + E_i\dot{D}_i + \left[\tau_{ji} + (U^F + E_kP_k)\delta_{ij}\right]v_{i,j} + J_i'E_i \end{aligned}$$

and using in (2.131) proves (2.149) and (2.150). ∎

Consider the following,

$$v_{i,j} = X_{K,j}v_{i,K} = X_{K,j}\frac{d}{dt}(x_{i,K})\,. \quad (2.151)$$

Then equations (2.149) and (2.150) together with (2.151) motivate the assumption that

$$\epsilon = \hat{\epsilon}(\eta, P_i, x_{i,K}) \quad (2.152)$$

$$U = \hat{U}(\eta, D_i, x_{i,K}) \quad (2.153)$$

Taking the material time derivatives of (2.152) and (2.153),

$$\dot{\epsilon} = \frac{\partial \hat{\epsilon}}{\partial \eta}\dot{\eta} + \frac{\partial \hat{\epsilon}}{\partial P_i}\dot{P}_i + \frac{\partial \hat{\epsilon}}{\partial (x_{i,K})}\frac{d}{dt}(x_{i,K}) \tag{2.154}$$

$$\dot{U} = \frac{\partial \hat{U}}{\partial \eta}\dot{\eta} + \frac{\partial \hat{U}}{\partial D_i}\dot{D}_i + \frac{\partial \hat{U}}{\partial (x_{i,K})}\frac{d}{dt}(x_{i,K}). \tag{2.155}$$

Using (2.154) and (2.155) in equations (2.149) and (2.150) respectively we obtain

$$\begin{aligned} &\rho\left(\Theta - \frac{\partial \hat{\epsilon}}{\partial \eta}\right)\dot{\eta} + \left(E_i - \rho\frac{\partial \hat{\epsilon}}{\partial P_i}\right)\dot{P}_i + J_i' E_i - q_i \Theta^{-1}\Theta_{,i} \\ &+ \left[(\tau_{ji} + E_n P_n \delta_{ij}) X_{K,j} - \rho\frac{\partial \hat{\epsilon}}{\partial (x_{i,K})}\right]\frac{d}{dt}(x_{i,K}) \geq 0 \end{aligned} \tag{2.156}$$

$$\begin{aligned} &\rho\left(\Theta - \frac{\partial \hat{U}}{\partial \eta}\right)\dot{\eta} + \left(E_i - \rho\frac{\partial \hat{U}}{\partial D_i}\right)\dot{D}_i + J_i' E_i - q_i \Theta^{-1}\Theta_{,i} \\ &+ \left(\left[\tau_{ji} + \left(U^F + E_n P_n\right)\delta_{ij}\right] X_{K,j} - \rho\frac{\partial \hat{U}}{\partial (x_{i,K})}\right)\frac{d}{dt}(x_{i,K}) \geq 0. \end{aligned} \tag{2.157}$$

Inequalities (2.156) and (2.157) are linear in $\dot{\eta}, \dot{P}_i, \dot{\overline{x_{i,J}}}$, and $\dot{\eta}, \dot{D}_i, \dot{\overline{x_{i,J}}}$ respectively, and must be satisfied for all independent variations of these quantities. Necessary and sufficient conditions are

$$J_i' E_i - q_i \Theta^{-1}\Theta_{,i} \geq 0$$

$$\Theta = \frac{\partial \hat{\epsilon}}{\partial \eta} \tag{2.158}$$

$$E_i = \rho\frac{\partial \hat{\epsilon}}{\partial P_i} \tag{2.159}$$

$$(\tau_{ji} + E_n P_n \delta_{ij}) X_{K,j} = \rho\frac{\partial \hat{\epsilon}}{\partial (x_{i,K})} \tag{2.160}$$

and

$$J_i' E_i - q_i \Theta^{-1}\Theta_{,i} \geq 0$$

$$\Theta = \frac{\partial \hat{U}}{\partial \eta} \tag{2.161}$$

$$E_i = \rho\frac{\partial \hat{U}}{\partial D_i} \tag{2.162}$$

$$\left[\tau_{ji} + \left(U^F + E_n P_n\right)\delta_{ij}\right] X_{K,j} = \rho\frac{\partial \hat{U}}{\partial (x_{i,K})} \tag{2.163}$$

Equations (2.158) - (2.160) and (2.161) - (2.163) are restrictions imposed by combined statements of balance of energy and entropy inequality, as required by the axiom of admissibility. The energy functions $\hat{\epsilon}$ and $\hat{U}$ can not be arbitrary functions of their arguments, they must satisfy the axiom of material objectivity and material symmetry. Instead of deriving these conditions here, it turns out the material objectivity will be satisfied if the constitutive equations are derived from the C-D inequality written in terms of material fields, as introduced in the next section. See TIERSTEN [28, pp. 1309-1310] for an example of material objectivity calculations.

Next we derive the heat conduction equation by simplifying the conservation of energy equation with the constitutive equations. Substitute (2.158)-(2.160), (2.154) and (2.161)-(2.163), (2.155) into the energy equations (2.156) and (2.157), respectively. Both result in a heat conduction equation,

$$\rho\Theta\dot{\eta} + q_{i,i} = \rho h + J_i' E_i.$$

2.20 Material Fields

In this section we introduce material fields. In typical continuum mechanics problems, the material points x_k in the deformed body $\mathcal{B}_t$ are part of the solution, and therefore unknown a priori. Loading and boundary conditions are usually known at material points X_M on the undeformed body $\mathcal{B}_o$. The fundamental axioms and resulting differential equations and jump conditions are stated in terms of spatial fields defined over the deformed and a priori unknown body B_t. It can be convenient, in particular for approximate theories, to rewrite the equations in terms of fields defined over the known reference configuration $\mathcal{B}_o$. Transformations of arc, area, and volume (2.5)-(2.7) relating elements in $\mathcal{B}_t$ to elements in $\mathcal{B}_o$ can be used to introduce relevant fields defined over $\mathcal{B}_o$. These fields are called material fields and are introduced below.

Consider a spatial scalar field q^F. A corresponding material field $\mathcal{Q}^F$ can be defined such that the volume integral of $\mathcal{Q}^F$ over a material volume V_o in $\mathcal{B}_o$ is equal to the volume integral of q^F over the corresponding material volume V in $\mathcal{B}_t$,

$$\int_{V_o} \mathcal{Q}^F \, dV_o = \int_V q^F \, dV \,. \tag{2.164}$$

The required relationship between $\mathcal{Q}^F$ and q^F can be derived using (2.7),

$$\begin{aligned} \int_{V_o} \mathcal{Q}^F \, dV_o &= \int_V q^F \, dV \\ &= \int_{V_o} q^F J \, dV_o \,. \end{aligned}$$

The material field $\mathcal{Q}^F$ required to satisfy (2.164) is then,

$$\mathcal{Q}^F = Jq^F.$$

Consider a spatial vector field D_i. A corresponding material field $\mathcal{D}_J$ can be defined such that the surface integral of $\mathcal{D}_J$ over a material surface S_o in $\mathcal{B}_o$ is equal to the surface integral of D_i over the corresponding material surface S in $\mathcal{B}_t$,

$$\int_{S_o} \mathcal{D}_J N_J \, dS_o = \int_S D_i n_i \, dS. \tag{2.165}$$

The required relationship between $\mathcal{D}_J$ and D_i can be derived using (2.6),

$$\begin{aligned}\int_{S_o} \mathcal{D}_J N_J \, dS_o &= \int_S D_i n_i \, dS \\ &= \int_{S_o} D_i J X_{J,i} N_J \, dS_o.\end{aligned}$$

The material field $\mathcal{D}_J$ required to satisfy (2.165) is then,

$$\mathcal{D}_J = J X_{J,i} D_i.$$

Consider another spatial vector field E_i. A corresponding material field $\mathcal{E}_J$ can be defined such that the line integral of $\mathcal{E}_J$ over a material line C_o in $\mathcal{B}_o$ is equal to the line integral of E_i over the corresponding material line C in $\mathcal{B}_t$,

$$\int_{C_o} \mathcal{E}_J \, dX_J = \int_C E_i \, dx_i. \tag{2.166}$$

The required relationship between $\mathcal{E}_J$ and E_i can be derived using (2.5),

$$\begin{aligned}\int_{C_o} \mathcal{E}_J \, dX_J &= \int_C E_i \, dx_i \\ &= \int_{C_o} E_i x_{i,J} \, dX_J.\end{aligned}$$

The material field $\mathcal{E}_J$ required to satisfy (2.166) is then,

$$\mathcal{E}_J = x_{i,J} E_i.$$

We note, the same defintion for $\mathcal{E}_J$ would have been obtained if it were defined in terms of $\mathcal{D}_K$, E_i, and D_j by,

$$\int_{V_o} \mathcal{E}_J \mathcal{D}_J \, dV_o = \int_V E_i D_i \, dV,$$

Definition 2.20.1 (Material Fields) *For convenience we define all material fields below. Their definitions will be motivated in the proofs that follow.*

$$\mathcal{Q}^F = J q^F \quad \rightarrow \quad q^F = J^{-1} \mathcal{Q}^F \tag{2.167}$$

$$\mathcal{P}_J = J X_{J,i} P_i \quad \rightarrow \quad P_i = J^{-1} x_{i,J} \mathcal{P}_J \tag{2.168}$$

$$\mathcal{D}_J = J X_{J,i} D_i \quad \rightarrow \quad D_i = J^{-1} x_{i,J} \mathcal{D}_J \tag{2.169}$$

$$\mathcal{E}_J = x_{i,J} E_i \quad \rightarrow \quad E_i = X_{J,i} \mathcal{E}_J \tag{2.170}$$

$$\mathcal{J}'_J = J X_{J,i} J'_i \quad \rightarrow \quad J'_i = J^{-1} x_{i,J} \mathcal{J}'_J \tag{2.171}$$

$$
\begin{aligned}
\mathcal{Q}_J = JX_{J,i}q_i &\quad\rightarrow\quad q_i = J^{-1}x_{i,J}\mathcal{Q}_J && (2.172)\\
T^P_{Si} = JX_{S,j}\tau^P_{ji} &\quad\rightarrow\quad \tau^P_{ji} = J^{-1}T^P_{Si}x_{j,S} && (2.173)\\
T^E_{Si} = JX_{S,j}\tau^E_{ji} &\quad\rightarrow\quad \tau^E_{ji} = J^{-1}x_{j,S}T^E_{Si} && (2.174)\\
\mathcal{T}^P_{SR} = JX_{S,j}X_{R,i}\tau^P_{ji} &\quad\rightarrow\quad \tau^P_{ji} = J^{-1}x_{j,S}x_{i,R}\mathcal{T}^P_{SR} && (2.175)\\
\mathcal{T}^T_{SR} = JX_{S,j}X_{R,i}\tau^T_{ji} &\quad\rightarrow\quad \tau^T_{ji} = J^{-1}x_{j,S}x_{i,R}\mathcal{T}^T_{SR} && (2.176)
\end{aligned}
$$

2.21 Equations in Material Fields

[20] The material form of the thermomechanical and EQS Maxwell equations will be systematically derived by integrating the local equations over a material surface or volume in $\mathcal{B}_t$ and introducing the appropriate transformations of elements of arc, area, or volume in $\mathcal{B}_o$. The resulting expressions are the global material forms of the original local spatial equations.

Theorem 2.21.1 (Global EQS Maxwell Equations: Material Fields) *The global form of the EQS Maxwell equations is,*

$$
\begin{aligned}
\oint_{S_o} \mathcal{D}_J N_J \, dS_o &= \int_{V_o} \mathcal{Q}^F \, dV_o , && (2.177)\\
\int_{C_o} \mathcal{E}_J \, dX_J &= 0, && (2.178)\\
\frac{d}{dt}\int_{V_o} \mathcal{Q}^F \, dV_o + \oint_{S_o} \mathcal{J}'_J N_J \, dS_o &= 0. && (2.179)
\end{aligned}
$$

Proof. Integrate (2.52) and (2.54) over a material volume V in spatial coordinates and integrate (2.53) over a material surface S in spatial coordinates, and using (2.22), (A.1), and (A.2),

$$
\begin{aligned}
\oint_S D_i n_i \, dS &= \int_V q^F \, dV , && (2.180)\\
\int_C E_i \, dx_i &= 0, && (2.181)\\
\oint_S J_i n_i \, dS + \frac{d}{dt}\int_V q^F \, dV - \oint_S q^F v_i n_i \, dS &= 0. && (2.182)
\end{aligned}
$$

Using (2.97) and introducing transformations (2.5)-(2.7) in (2.180)-(2.182) gives

$$
\begin{aligned}
\oint_{S_o} D_i J X_{J,i} N_J \, dS_o &= \int_{V_o} q^F J \, dV_o , && (2.183)\\
\int_{C_o} E_i \frac{\partial x_i}{\partial X_J} \, dX_J &= 0, && (2.184)\\
\frac{d}{dt}\int_{V_o} q^F J \, dV_o + \oint_{S_o} J'_i X_{J,i} N_J \, dS_o &= 0. && (2.185)
\end{aligned}
$$

[20] Conservation of energy equations in material fields are based on THURSTON [26, pp. 157-165]

Using (2.167), (2.169), (2.170), and (2.171) in (2.183) - (2.185) proves (2.177) - (2.179). ■

Theorem 2.21.2 (Local EQS Maxwell Equations: Material Fields)

$$\mathcal{D}_{J,J} = \mathcal{Q}^F, \tag{2.186}$$

$$\epsilon_{IJK}\mathcal{E}_{K,J} = 0 \quad \longrightarrow \quad \mathcal{E}_J = -\phi,_J, \tag{2.187}$$

$$\dot{\mathcal{Q}}^F + \mathcal{J}'_{J,J} = 0. \tag{2.188}$$

Proof. Noting the material time derivative commutes with the volume integral over a material volume and using (A.1) and (A.2) in (2.177)-(2.179) and requiring the integrals hold for arbitrary volumes V_o and surfaces S_o in the appropriate relations (postulate of localization) proves (2.186)-(2.188). ■

It will be convenient to work with a reduced form of the EQS equations.

Theorem 2.21.3 (Reduced EQS Maxwell: Integral Form and Material Fields)

$$\oint_{S_o} \left(\dot{\mathcal{D}}_J + \mathcal{J}'_J\right) N_J \, dS_o = 0$$

$$\int_{C_o} \mathcal{E}_J \, dX_J = 0$$

Proof. Immediate consequence of (2.180)-(2.182). ■

Theorem 2.21.4 (Reduced EQS Maxwell: Local Form and Material Fields)

$$\left(\dot{\mathcal{D}}_J + \mathcal{J}'_J\right)_{,J} = 0$$

$$\epsilon_{IJK}\mathcal{E}_{K,J} = 0 \quad \longrightarrow \quad \mathcal{E}_J = -\phi,_J.$$

Theorem 2.21.5 (Conservation of Mass: Material Fields) *Global and local conservation of mass in material form is equivalent to (2.28) is*

$$\int_V \rho \, dV = \int_{V_o} \rho_o \, dV_o, \tag{2.189}$$

$$J\rho = \rho_o. \tag{2.190}$$

Proof. Equation (2.189) is a restatement of axiom 2.5.6. Using (2.7), (2.189) becomes

$$\int_{V_o} (J\rho - \rho_o) \, dV_o = 0.$$

Using postulate of localization proves (2.190). ■

Theorem 2.21.6 (Balance of Momentum: Material Fields) *Global and local balance of momentum in material form is*

$$\oint_{S_o} \left(T_{Ji} + T^E_{Ji}\right) N_J \, dS_o + \int_{V_o} \rho_o \left(f_i - \dot{v}_i\right) dV_o \;=\; 0, \tag{2.191}$$

$$\left(T_{Ji} + T^E_{Ji}\right)_{,J} + \rho_o \left(f_i - \dot{v}_i\right) \;=\; 0. \tag{2.192}$$

Proof. Integrate (2.77) over a material volume V in spatial (deformed) coordinates and use the divergence theorem (A.1),

$$\oint_S \left(\tau_{ji} + \tau^E_{ji}\right) n_i \, dS + \int_V \rho \left(f_i - \dot{v}_i\right) dV = 0. \tag{2.193}$$

Using the transformations (2.6) and (2.7) in (2.193),

$$\oint_{S_o} \left(\tau_{ji} + \tau^E_{ji}\right) J X_{J,i} N_J \, dS_o + \int_{V_o} \rho \left(f_i - \dot{v}_i\right) J \, dV_o = 0. \tag{2.194}$$

Using (2.173) and (2.174) in (2.194) proves (2.191). Using divergence theorem (A.1) in (2.191) and requiring to hold for arbitrary volumes V_o (postulate of localization) proves (2.192). ■

Theorem 2.21.7 (Symmetry of Partial and Total Tensors: Material Fields) *The material stress tensors $\mathcal{T}^T_{SR}$ and $\mathcal{T}^P_{SR}$ are symmetric,*

$$\mathcal{T}^T_{[SR]} \;=\; 0, \tag{2.195}$$

$$\mathcal{T}^P_{[SR]} \;=\; 0 \tag{2.196}$$

Proof. From (2.91),

$$\tau^T_{ji} = \tau^T_{ij}. \tag{2.197}$$

Introducing (2.176) into (2.197),

$$\mathcal{T}^T_{SR} x_{j,S} x_{i,R} J^{-1} \;=\; \mathcal{T}^T_{MN} x_{i,M} x_{j,N} J^{-1} \tag{2.198}$$

Multiplying (2.198) by $J X_{S,j} X_{R,i}$ gives,

$$\begin{aligned} \mathcal{T}^T_{SR} &= X_{S,j} X_{R,i} x_{i,M} x_{j,N} \mathcal{T}^T_{MN}, \\ &= \delta_{SN} \delta_{RM} \mathcal{T}^T_{MN}, \\ &= \mathcal{T}^T_{RS}. \end{aligned} \tag{2.199}$$

Equation (2.199) proves (2.195). Repeating the calculations using (2.91) and (2.175) proves (2.196). ■

Prior to establishing the material forms of local conservation of energy expressions, we must derive total derivative expressions.

Proposition 2.21.8 (Material Derivative: Spatial and Material Field) *Defining $\mathcal{A}_J = JX_{J,i}A_i$, the material derivative of A_i is*

$$\dot{A}_i = \dot{\mathcal{A}}_J x_{i,J} J^{-1} + A_k v_{i,k} - A_i v_{k,k}. \qquad (2.200)$$

Proof. Taking the material derivative of A_i

$$\dot{A}_i = \dot{\mathcal{A}}_J x_{i,J} J^{-1} + \mathcal{A}_J \frac{d}{dt}(x_{i,J}) J^{-1} - \mathcal{A}_J x_{i,J} J^{-2} \dot{J}$$

Using (2.14) and (2.201),

$$\frac{d}{dt}(x_{i,J}) = v_{i,J} = v_{i,k} x_{k,J}, \qquad (2.201)$$

proves (2.200). ■

Lemma 2.21.9 (Local Conservation of Energy) *Local conservation of energy equations equivalent to (2.116) and (2.117) are*

$$\rho\dot{\epsilon} = \mathcal{E}_J \dot{\mathcal{P}}_J J^{-1} + \tau^P_{ji} v_{i,j} + J'_i E_i + \rho h - q_{i,i} \qquad (2.202)$$
$$\rho\dot{U} = \mathcal{E}_J \dot{\mathcal{D}}_J J^{-1} + \tau^T_{ji} v_{i,j} + J'_i E_i + \rho h - q_{i,i} \qquad (2.203)$$

Proof. Using theorem 2.21.8, $\dot{P}_i$ and $\dot{D}_i$ are,

$$\dot{P}_i = \dot{\mathcal{P}}_J x_{i,J} J^{-1} + P_k v_{i,k} - P_i v_{k,k}, \qquad (2.204)$$
$$\dot{D}_i = \dot{\mathcal{D}}_J x_{i,J} J^{-1} + D_k v_{i,k} - D_i v_{k,k}. \qquad (2.205)$$

Using (2.170),(2.190),(2.204), and (2.205) in (2.116) and (2.117) obtain

$$\rho\dot{\epsilon} = \mathcal{E}_J \dot{\mathcal{P}}_J J^{-1} + [\tau_{ji} + P_j E_i] v_{i,j} + J'_i E_i + \rho h - q_{i,i} \qquad (2.206)$$
$$\rho\dot{U} = \mathcal{E}_J \dot{\mathcal{D}}_J J^{-1} + \left[\tau_{ji} + D_j E_i - U^F \delta_{ij}\right] v_{i,j} + J'_i E_i + \rho h - q_{i,i} \qquad (2.207)$$

Rewriting (2.206) and (2.207) using (2.88) and (2.90) proves (2.206) and (2.207) ■

Definition 2.21.10 (Strain Rate Decomposition) *The S/A decomposition of $v_{i,j}$ is defined as*

$$d_{ij} = v_{(i,j)}, \qquad w_{ij} = v_{[i,j]}$$
$$v_{i,j} = d_{ij} + w_{ij} \qquad (2.208)$$

Proposition 2.21.11 (Partial and Total Stress Power Densities) *Stress power densities $\tau^P_{ji} v_{i,j}$ and $\tau^T_{ji} v_{i,j}$ simplify to*

$$\tau^P_{ji} v_{i,j} = \tau^P_{ji} d_{ij} \qquad (2.209)$$
$$\tau^T_{ji} v_{i,j} = \tau^T_{ji} d_{ij} \qquad (2.210)$$

Proof. Using (2.91) in the form $\tau^P_{ji} = \tau^P_{(ji)}$

$$\begin{aligned}\tau^P_{ji} v_{i,j} &= \tau^P_{(ji)} v_{i,j} \\ &= \tau^P_{(ij)} \left(d_{ij} + w_{ij}\right) \\ &= \tau^P_{ji} d_{ij}\end{aligned}$$

Equation (2.210) is similarly proved using (2.92) in the form $\tau^T_{ji} = \tau^T_{(ji)}$. ■

Proposition 2.21.12 (Symmetric Strain Rate: Material Fields)

$$d_{pq} = X_{R,p} X_{S,q} \dot{E}_{RS}$$

Proof.

$$\begin{aligned}C_{RS} &= x_{k,R} x_{k,S} = \delta_{RS} + 2E_{RS} \\ \dot{C}_{RS} &= v_{k,R} x_{k,S} + x_{k,R} v_{k,S} = 2\dot{E}_{RS}\end{aligned}$$

Multiplying through by $X_{R,p} X_{S,q}$

$$X_{R,p} X_{S,q} \dot{C}_{RS} = v_{k,p} \delta_{kq} + \delta_{kp} v_{k,q} = 2X_{R,p} X_{S,q} \dot{E}_{RS}$$

or

$$v_{q,p} + v_{p,q} = X_{R,p} X_{S,q} \dot{C}_{RS} = 2X_{R,p} X_{S,q} \dot{E}_{RS}$$

From (2.208)

$$\begin{aligned}d_{qp} &= \frac{1}{2}\left(v_{q,p} + v_{p,q}\right) \\ &= \frac{1}{2} X_{R,p} X_{S,q} \dot{C}_{RS} && (2.211) \\ &= X_{R,p} X_{S,q} \dot{E}_{RS} && (2.212)\end{aligned}$$

■

Proposition 2.21.13 (Partial and Total Stress Power Densities: Material Fields) *Stress power densities $\tau^P_{ji} v_{i,j}$ and $\tau^T_{ji} v_{i,j}$ are identically*

$$\begin{aligned}\tau^P_{ji} v_{i,j} &= \tau^P_{ji} X_{R,i} X_{S,j} \dot{E}_{RS} && (2.213) \\ \tau^T_{ji} v_{i,j} &= \tau^T_{ji} X_{R,i} X_{S,j} \dot{E}_{RS} && (2.214)\end{aligned}$$

Proof. Equation (2.209) with (2.212) proves (2.213). Similarly, (2.210) with (2.212) proves (2.214). ■

Theorem 2.21.14 (Local Conservation of Energy: Material Fields) *Global and local onservation of energy statements equivalent to (2.116) and (2.117) in material* *fields are*

$$\rho_o\dot{\epsilon} = \mathcal{E}_J\dot{\mathcal{P}}_J + \mathcal{T}^P_{SR}\dot{E}_{RS} + \mathcal{J}'_J\mathcal{E}_J + \rho_o h - \mathcal{Q}_{J,J} \quad (2.215)$$

$$\rho_o\dot{U} = \mathcal{E}_J\dot{\mathcal{D}}_J + \mathcal{T}^T_{SR}\dot{E}_{RS} + \mathcal{J}'_J\mathcal{E}_J + \rho_o h - \mathcal{Q}_{J,J} \quad (2.216)$$

Proof. Integrate (2.202) and (2.203) over a material domain V in spatial coordinates and use the divergence theorem (A.1),

$$\int_V \rho\dot{\epsilon}\,dV = \int_V \left(\mathcal{E}_J\dot{\mathcal{P}}_J J^{-1} + \tau^P_{ji}v_{i,j} + J'_iE_i + \rho h\right)dV - \oint_S q_i n_i\,dS\,, \quad (2.217)$$

$$\int_V \rho\dot{U}\,dV = \int_V \left(\mathcal{E}_J\dot{\mathcal{D}}_J J^{-1} + \tau^T_{ji}v_{i,j} + J'_iE_i + \rho h\right)dV - \oint_S q_i n_i\,dS\,. \quad (2.218)$$

Introducing transformations (2.6) and (2.7) gives,

$$\int_{V_o} \rho\dot{\epsilon}J\,dV_o = \int_{V_o} \left(\mathcal{E}_J\dot{\mathcal{P}}_J J^{-1} + \tau^P_{ji}v_{i,j} + J'_iE_i + \rho h\right)J\,dV_o - \oint_{S_o} q_iJX_{J,i}N_J\,dS_o \quad (2.219)$$

$$\int_{V_o} \rho\dot{U}J\,dV_o = \int_{V_o} \left(\mathcal{E}_J\dot{\mathcal{D}}_J J^{-1} + \tau^T_{ji}v_{i,j} + J'_iE_i + \rho h\right)J\,dV_o - \oint_{S_o} q_iJX_{J,i}N_J\,dS_o\,. \quad (2.220)$$

Using (2.190), (2.170), (2.171), (2.172), (2.175), (2.176), (2.213), and (2.214) in (2.219) and (2.220) gives,

$$\int_{V_o} \rho_o\dot{\epsilon}\,dV_o = \int_{V_o} \left(\mathcal{E}_J\dot{\mathcal{P}}_J + \mathcal{T}^P_{SR}\dot{E}_{RS} + \mathcal{J}'_J\mathcal{E}_J + \rho_o h\right)dV_o - \oint_{S_o} \mathcal{Q}_JN_J\,dS_o \quad (2.221)$$

$$\int_{V_o} \rho_o\dot{U}\,dV_o = \int_{V_o} \left(\mathcal{E}_J\dot{\mathcal{D}}_J + \mathcal{T}^T_{SR}\dot{E}_{RS} + \mathcal{J}'_J\mathcal{E}_J + \rho_o h\right)dV_o - \oint_{S_o} \mathcal{Q}_JN_J\,dS_o\,. \quad (2.222)$$

Using the divergence theorem (A.1) in (2.221) and (2.222) and requiring the result to hold for arbitrary V_o proves (2.215) and (2.216). ∎

Theorem 2.21.15 (Global Conservation of Energy: Material Fields)

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\rho_o v_iv_i + \rho_o U\right)dV = \int_V \rho_o\left(h + f_iv_i\right)dV + \oint_{S_o}\left[\left(T_{Jj} + T^E_{Jj}\right)v_j - \phi\left(\mathcal{J}'_J + \dot{\mathcal{D}}_J\right) - \mathcal{Q}_J\right]N_J\,dS_o \quad (2.223)$$

Proof. Using (2.200) and (2.66),

$$D^*_i = \dot{D}_i + D_iv_{k,k} - D_kv_{i,k}$$

$$\dot{\mathcal{D}}_J = D^*_iJX_{J,i}. \quad (2.224)$$

Using (2.224), and appropriate entries from definition 2.20.1 in (2.122) proves (2.223). ■

Theorem 2.21.16 (Entropy Inequality: Material Fields) *Global and local entropy inequality statements equivalent to (2.130) in material fields are*

$$\int_{V_o} \rho_o \dot{\eta}\, dV_o + \oint_{S_o} \frac{Q_J}{\Theta} N_J\, dS_o - \int_{V_o} \rho_o \frac{h}{\Theta}\, dV_o \;\geq\; 0, \tag{2.225}$$

$$\rho_o \Theta \dot{\eta} + Q_{J,J} - Q_J \Theta^{-1} \Theta_{,J} - \rho_o h \;\geq\; 0. \tag{2.226}$$

Proof. Integrate the product of Θ with (2.131) over a material volume V in spatial coordinates and using the divergence theorem (A.1) obtain

$$\int_V \rho \dot{\eta}\, dV + \oint_S \frac{q_i}{\Theta} n_i\, dS - \int_V \rho \frac{h}{\Theta}\, dV \;\geq\; 0. \tag{2.227}$$

Using transformations (2.6) and (2.7) in (2.227),

$$\int_{V_o} \rho \dot{\eta} J\, dV_o + \oint_{S_o} \frac{q_i}{\Theta} J X_{J,i} N_J\, dS_o - \int_{V_o} \rho \frac{h}{\Theta} J\, dV_o \;\geq\; 0. \tag{2.228}$$

Using (2.190) and (2.172) in (2.228) proves (2.225). Using the divergence theorem (A.1) in (2.225) and requiring the statement to hold for arbitrary volumes V_o, and multiplying the result by Θ proves (2.226). ■

2.22 Surfaces of Discontinuity: Material Fields

In this section we generalize our global balance statements in material fields to include surfaces of discontinuity moving with absolute velocity ν_k. We note that Gauss and Stokes' theorems remain unchanged for material coordinates. Therefore the derivations for generalized integral theorems are valid.

Theorem 2.22.1 (Generalized Gauss: Material Fields) *A material field A_M defined over a material volume V_o in $\mathcal{B}_o$, bounded by surface S_o, and containing a surface of discontinuity σ_o satisfies the integral statement,*

$$\oint_{S_o - \sigma_o} A_M N_M\, dS_o = \int_{V_o - \sigma_o} A_{M,M}\, dV_o + \int_{\sigma_o} [\![A_M]\!] N_M\, dS_o \tag{2.229}$$

$$[\![A_K]\!] = A_K^+ + A_K^-$$

$$V_o - \sigma_o = V_o^+ + V_o^-$$

$$S_o - \sigma_o = S_o^+ + S_o^-$$

Proof. Equation (2.229) is a restatement in material coordinates of (2.133). ■

Theorem 2.22.2 (Generalized Stokes: Material Fields) *A material field A_M defined over an open material surface S_o in $\mathcal{B}_o$, bounded by the line C_o, and con-*

taining a line of discontinuity γ_o satisfies the integral statement,

$$\begin{aligned}
\oint_{C_o-\gamma_o} A_K\, dX_K &= \int_{S_o-\gamma_o} \epsilon_{IJK} A_{K,J} N_I\, dS_o + \int_{\gamma_o} [\![\, A_K \,]\!]\, dX_K \qquad (2.230)\\
[\![\, A_K \,]\!] &= A_K^+ + A_K^- \\
S_o - \gamma_o &= S_o^+ + S_o^- \\
C_o - \gamma_o &= C_o^+ + C_o^-
\end{aligned}$$

Proof. Equation (2.230) is a restatement in material coordinates of (2.137). ■

Theorem 2.22.3 (Generalized Total Derivative: Material Fields) *A material field ϕJ defined over a material volume V_o in $\mathcal{B}_o$ and containing a surface of discontinuity σ_o moving with absolute velocity ν_k satisfies the integral statement,*

$$\begin{aligned}
\frac{d}{dt}\int_{V_o-\sigma_o} \phi J\, dV_o &= \int_{V_o-\sigma_o} \left(\dot{\phi} + \phi v_{k,k}\right) J\, dV_o \\
&+ \int_{\sigma_o} [\![\, \phi\left(v_k - \nu_k\right) J X_{J,k} \,]\!]\, N_J\, dS_o \qquad (2.231)\\
[\![\, A_K \,]\!] &= A_K^+ + A_K^- \\
V_o - \sigma_o &= V_o^+ + V_o^- \\
S_o - \sigma_o &= S_o^+ + S_o^-
\end{aligned}$$

Proof. Using theorem 2.4.8 in (2.135) and noting that normals n_k and N_J commute with the jump operator $[\![\,\cdot\,]\!]$ proves (2.231). ■

2.23 Jump Conditions: Material Fields

In this section we generalize global balance statements to include surfaces of discontinuity moving with velocity ν_k.

Proposition 2.23.1 (Integral EQS Electroelastic Equations: Material Form)

$$\begin{aligned}
\oint_{S_o-\sigma_o} \left(\dot{\mathcal{D}}_J + \mathcal{J}_J'\right) N_J\, dS_o &= 0 \\
\int_{C_o-\gamma_o} \mathcal{E}_J\, dX_J &= 0 \\
\frac{d}{dt}\int_{V_o-\sigma_o} \rho J\, dV_o &= 0 \\
\frac{d}{dt}\int_{V_o-\sigma_o} \rho v_i J\, dV_o = \int_{V_o-\sigma_o} \rho_o f_i\, dV_o &+ \oint_{S_o-\sigma_o} \left(T_{Ji} + T_{Ji}^E\right) N_J\, dS_o \\
\frac{d}{dt}\int_{V_o-\sigma_o} \left(\frac{1}{2}\rho v_i v_i + \rho U\right) J\, dV_o &= \int_{V_o-\sigma_o} \rho_o\left(h + f_i v_i\right) dV_o \\
+ \oint_{S_o-\sigma_o} \left[\left(T_{Jj} + T_{Jj}^E\right) v_j - \phi\left(\mathcal{J}_J' + \dot{\mathcal{D}}_J\right) - \mathcal{Q}_J\right] N_J\, dS_o \\
\frac{d}{dt}\int_{V_o-\sigma_o} \rho\eta J\, dV_o \geq \int_{V_o-\sigma_o} \rho_o h\, dV_o &- \oint_{S_o-\sigma_o} \frac{\mathcal{Q}_J}{\Theta} N_J\, dS_o .
\end{aligned}$$

We obtain local differential equations and jump conditions by applying the generalized integral theorems followed by the postulate of localization to the global balance laws. The resulting local equations will be identical to those derived earlier, however the jump conditions across the moving surface of discontinuity will be new. We present an example considering the balance of momentum equation.

Example 2.23.2 (Jump Condition Calculation: Material Fields) Consider the integral balance of momentum equations and apply the general integral theorems (2.231) and (2.229),

$$\begin{aligned}
\frac{d}{dt}\int_{V_o-\sigma_o}(\rho v_i)\,J\,dV_o &= \int_{V_o-\sigma_o}\left[\frac{d}{dt}(\rho v_i)+(\rho v_i)\,v_{k,k}\right]J\,dV_o \\
&+ \int_{\sigma_o}[\![\,\rho v_i\,(v_k-\nu_k)\,JX_{J,k}\,]\!]\,N_J\,dS_o \\
\oint_{S_o-\sigma_o}\left(T_{Ji}+T^E_{Ji}\right)N_J\,dS_o &= \int_{V_o-\sigma_o}\left(T_{Ji}+T^E_{Ji}\right)_{,J}dV_o \\
&+ \int_{\sigma_o}[\![\,T_{Ji}+T^E_{Ji}\,]\!]\,N_J\,dS_o\,.
\end{aligned}$$

Substituting these in integral balance law we obtain,

$$\begin{aligned}
&\int_{V_o-\sigma_o}\left[\frac{d}{dt}(\rho v_i)+(\rho v_i)\,v_{k,k}\right]J\,dV_o+\int_{\sigma_o}[\![\,\rho_o v_i\,(v_k-\nu_k)\,X_{J,k}\,]\!]\,N_J\,dS_o \\
&\quad=\int_{V_o-\sigma_o}\left[\rho_o h+\left(T_{Ji}+T^E_{Ji}\right)_{,J}\right]dV_o+\int_{\sigma_o}[\![\,T_{Ji}+T^E_{Ji}\,]\!]\,N_J\,dS_o\,.
\end{aligned}$$

applying the postulate of localization we obtain

$$\begin{aligned}
J\frac{d}{dt}(\rho v_i)+J\,(\rho v_i)\,v_{k,k}=\rho_o h+\left(T_{Ji}+T^E_{Ji}\right)_{,J} &\qquad \text{in } V_o-\sigma_o \\
[\![\,\rho_o v_i\,(v_k-\nu_k)\,X_{J,k}\,]\!]\,N_J=[\![\,T_{Ji}+T^E_{Ji}\,]\!]\,N_J &\qquad \text{across } \sigma_o
\end{aligned}$$

The local equations are equivalent to those derived before, however the jump conditions that must be satisfied across a moving surface of discontinuity are new.

Proposition 2.23.3 (Jump Conditions: Moving Surface of Discontinuity)

$$\begin{aligned}
[\![\,\dot{\mathcal{D}}_J+\mathcal{J}'_J\,]\!]\,N_J=0 &\qquad \textit{across } \sigma_o \\
[\![\,\phi\,]\!]=0 &\qquad \textit{across } \gamma_o \\
[\![\,x_k\,]\!]=0 &\qquad \textit{across } \sigma_o \\
[\![\,\rho_o\,(v_k-\nu_k)\,X_{J,k}\,]\!]\,N_J=0 &\qquad \textit{across } \sigma_o \\
[\![\,\rho_o v_i\,(v_k-\nu_k)\,X_{J,k}\,]\!]\,N_J=[\![\,T_{Ji}+T^E_{Ji}\,]\!]\,N_J &\qquad \textit{across } \sigma_o \\
[\![\,\left(\frac{1}{2}\rho_o v_i v_i+\rho_o U\right)(v_k-\nu_k)\,X_{J,k}\,]\!]\,N_J= & \\
[\![\,\left(T_{Jj}+T^E_{Jj}\right)v_j-\phi\left(\mathcal{J}'_J+\dot{\mathcal{D}}_J\right)-\mathcal{Q}_J\,]\!]\,N_J &\qquad \textit{across } \sigma_o
\end{aligned}$$

$$[\![\rho_o \eta (v_k - \nu_k) X_{J,k}]\!] N_J \geq - [\![\frac{\mathcal{Q}_J}{\Theta}]\!] N_J \qquad \textit{across } \sigma_o$$

Proposition 2.23.4 (Jump Conditions: Material Surface of Discontinuity)

$$\begin{aligned}
[\![\dot{\mathcal{D}}_J + \mathcal{J}'_J]\!] N_J &= 0 && \textit{across } \sigma_o \\
[\![\phi]\!] &= 0 && \textit{across } \gamma_o \\
[\![x_k]\!] &= 0 && \textit{across } \sigma_o \\
[\![T_{Ji} + T^E_{Ji}]\!] N_J &= 0 && \textit{across } \sigma_o \\
[\![\left(T_{Jj} + T^E_{Jj}\right) v_j - \phi \left(\mathcal{J}'_J + \dot{\mathcal{D}}_J\right) - \mathcal{Q}_J]\!] N_J &= 0 && \textit{across } \sigma_o \\
[\![\frac{\mathcal{Q}_J}{\Theta}]\!] N_J &\geq 0 && \textit{across } \sigma_o
\end{aligned}$$

2.24 Constitutive Equations in Material Fields

In this section we derive restrictions on material response functions imposed by the conservation of energy and entropy inequality equations, in terms of material fields. The resulting material response functions satisfy the axiom of material objectivity without further restriction. Additional restrictions will be imposed by the axiom of material symmetry.

Theorem 2.24.1 (Local C-D Inequality Statements in Material Fields) *The Clausius-Duhem inequalities in material fields are*

$$\rho_o \Theta \dot{\eta} - \rho_o \dot{\epsilon} + \mathcal{E}_J \dot{\mathcal{P}}_J + \mathcal{T}^P_{SR} \dot{E}_{RS} + \mathcal{J}'_J \mathcal{E}_J - \mathcal{Q}_J \Theta^{-1} \Theta_{,J} \geq 0 \tag{2.232}$$
$$\rho_o \Theta \dot{\eta} - \rho_o \dot{U} + \mathcal{E}_J \dot{\mathcal{D}}_J + \mathcal{T}^T_{SR} \dot{E}_{RS} + \mathcal{J}'_J \mathcal{E}_J - \mathcal{Q}_J \Theta^{-1} \Theta_{,J} \geq 0 \tag{2.233}$$

Proof. Using (2.215) and (2.216) in (2.226) proves (2.232) and (2.233). ■
Equations (2.232) and (2.233) motivate the assumed functional forms

$$\epsilon = \hat{\epsilon} (\eta, \mathcal{P}_J, E_{RS}) \tag{2.234}$$
$$U = \hat{U} (\eta, \mathcal{D}_J, E_{RS}) \tag{2.235}$$

for ϵ and U. We note that $\hat{\epsilon}$ and $\hat{U}$ in this assumed form satisfy the axiom of material objectivity, they are objective functions. Procedures identical to section 2.19 can be used to obtain constitutive equations in terms of derivatives on the potential functions. The result is a material characterization based on either $\hat{\epsilon}$ or $\hat{U}$, described as functions of $(\eta, \mathcal{P}_J, E_{RS})$ or $(\eta, \mathcal{D}_J, E_{RS})$. Characterizing a material in terms of these fields may be inconvenient. Legendre transformations can be used to change independent variables in the material response functions.

We introduce the following Legendre transformation to switch independent variables from η to the absolute temperature Θ. Inspection of (2.232) and (2.233) motivates

$$\epsilon = \Psi_1 + \Theta \eta \quad \rightarrow \quad \dot{\epsilon} = \dot{\Psi}_1 + \dot{\Theta} \eta + \Theta \dot{\eta} \tag{2.236}$$

$$U = \Psi_2 + \Theta\eta \quad \rightarrow \quad \dot{U} = \dot{\Psi}_2 + \dot{\Theta}\eta + \Theta\dot{\eta}. \tag{2.237}$$

Based on this inspection we define the following Legendre transformations.

Definition 2.24.2 (Legendre Transforms I) *Legendre transformations Ψ_1 and Ψ_2 are defined as*

$$\begin{aligned} \Psi_1 &= \epsilon - \Theta\eta \\ \Psi_2 &= U - \Theta\eta \end{aligned}$$

Theorem 2.24.3 (Legendre Transformed C-D Inequality Statements) *Local Clausius-Duhem inequalities equivalent to (2.232) and (2.233) are*

$$-\rho_o\left(\dot{\Psi}_1 + \dot{\Theta}\eta\right) + \mathcal{E}_J\dot{\mathcal{P}}_J + \mathcal{T}^P_{SR}\dot{E}_{RS} + \mathcal{J}'_J\mathcal{E}_J - \mathcal{Q}_J\Theta^{-1}\Theta_{,J} \geq 0 \tag{2.238}$$
$$-\rho_o\left(\dot{\Psi}_2 + \dot{\Theta}\eta\right) + \mathcal{E}_J\dot{\mathcal{D}}_J + \mathcal{T}^T_{SR}\dot{E}_{RS} + \mathcal{J}'_J\mathcal{E}_J - \mathcal{Q}_J\Theta^{-1}\Theta_{,J} \geq 0 \tag{2.239}$$

Proof. Using (2.236) and (2.237) in (2.232) and (2.233) proves (2.238) and (2.239). ∎

Equations (2.238) and (2.239) motivate the objective functional forms

$$\begin{aligned} \rho_o\Psi_1 &= \overline{\Psi}_1\left(\Theta, \mathcal{P}_J, E_{RS}\right) \\ \rho_o\Psi_2 &= \overline{\Psi}_2\left(\Theta, \mathcal{D}_J, E_{RS}\right) \end{aligned}$$

Assuming these true

$$\rho_o\dot{\Psi}_1 = \frac{\partial\overline{\Psi}_1}{\partial\Theta}\dot{\Theta} + \frac{\partial\overline{\Psi}_1}{\partial\mathcal{P}_J}\dot{\mathcal{P}}_J + \frac{\partial\overline{\Psi}_1}{\partial E_{RS}}\dot{E}_{RS} \tag{2.240}$$
$$\rho_o\dot{\Psi}_2 = \frac{\partial\overline{\Psi}_2}{\partial\Theta}\dot{\Theta} + \frac{\partial\overline{\Psi}_2}{\partial\mathcal{D}_J}\dot{\mathcal{D}}_J + \frac{\partial\overline{\Psi}_2}{\partial E_{RS}}\dot{E}_{RS} \tag{2.241}$$

Using (2.240) and (2.241) in (2.238) and (2.239)

$$\begin{aligned} -\left(\rho_o\eta + \frac{\partial\overline{\Psi}_1}{\partial\Theta}\right)\dot{\Theta} + \left(\mathcal{E}_J - \frac{\partial\overline{\Psi}_1}{\partial\mathcal{P}_J}\right)\dot{\mathcal{P}}_J &+ \left(\mathcal{T}^P_{SR} - \frac{\partial\overline{\Psi}_1}{\partial E_{RS}}\right)\dot{E}_{RS} \\ &+ \mathcal{J}'_J\mathcal{E}_J - \mathcal{Q}_J\Theta^{-1}\Theta_{,J} \geq 0 \end{aligned} \tag{2.242}$$

$$\begin{aligned} -\left(\rho_o\eta + \frac{\partial\overline{\Psi}_2}{\partial\Theta}\right)\dot{\Theta} + \left(\mathcal{E}_J - \frac{\partial\overline{\Psi}_2}{\partial\mathcal{D}_J}\right)\dot{\mathcal{D}}_J &+ \left(\mathcal{T}^T_{SR} - \frac{\partial\overline{\Psi}_2}{\partial E_{RS}}\right)\dot{E}_{RS} \\ &+ \mathcal{J}'_J\mathcal{E}_J - \mathcal{Q}_J\Theta^{-1}\Theta_{,J} \geq 0 \end{aligned} \tag{2.243}$$

Necessary and sufficient conditions are

$$\rho_o\eta = -\frac{\partial\overline{\Psi}_1}{\partial\Theta}, \qquad \mathcal{E}_J = \frac{\partial\overline{\Psi}_1}{\partial\mathcal{P}_J}, \qquad \mathcal{T}^P_{SR} = \frac{\partial\overline{\Psi}_1}{\partial E_{RS}} \tag{2.244}$$
$$\mathcal{J}'_J\mathcal{E}_J - \mathcal{Q}_J\Theta^{-1}\Theta_{,J} \geq 0$$

and

$$
\begin{aligned}
\rho_o \eta &= -\frac{\partial \overline{\Psi}_2}{\partial \Theta}, \quad \mathcal{E}_J = \frac{\partial \overline{\Psi}_2}{\partial \mathcal{D}_J}, \quad \mathcal{T}^T_{SR} = \frac{\partial \overline{\Psi}_2}{\partial E_{RS}} \qquad (2.245) \\
\mathcal{J}'_J \mathcal{E}_J &- \mathcal{Q}_J \Theta^{-1} \Theta_{,J} \geq 0
\end{aligned}
$$

Theorem 2.24.4 (Legendre Transformed Conservation of Energy Statements) *Local conservation of energy statements equivalent to (2.215) and (2.216) are*

$$
\rho_o \left(\dot{\Psi}_1 + \dot{\Theta}\eta + \Theta\dot{\eta} \right) = \mathcal{E}_J \dot{\mathcal{P}}_J + \mathcal{T}^P_{SR} \dot{E}_{RS} + \mathcal{J}'_J \mathcal{E}_J + \rho_o h - \mathcal{Q}_{J,J} \qquad (2.246)
$$

$$
\rho_o \left(\dot{\Psi}_2 + \dot{\Theta}\eta + \Theta\dot{\eta} \right) = \mathcal{E}_J \dot{\mathcal{D}}_J + \mathcal{T}^T_{SR} \dot{E}_{RS} + \mathcal{J}'_J \mathcal{E}_J + \rho_o h - \mathcal{Q}_{J,J} \qquad (2.247)
$$

Proof. Using (2.236) and (2.237) in (2.215) and (2.216) proves (2.246) and (2.247). ■

We obtain our equations of heat conduction by simplifying the conservation of energy equations (2.246) and (2.247).

Theorem 2.24.5 (Local Heat Conduction in Material Fields) *The local heat conduction equation in material fields is*

$$
\rho_o \Theta \dot{\eta} = \mathcal{J}'_J \mathcal{E}_J + \rho_o h - \mathcal{Q}_{J,J} \qquad (2.248)
$$

Proof. Using (2.244) and (2.245) in (2.246) and (2.247) proves (2.248). ■

Another useful Legendre transformation changes the independent variables from $(\eta, \mathcal{P}_J, E_{RS})$ or $(\eta, \mathcal{D}_J, E_{RS})$ to $(\Theta, \mathcal{E}_J, E_{RS})$. Inspection of (2.232) and (2.233) motivates

$$
\epsilon = \Psi_3 + \Theta\eta + \mathcal{E}_J \mathcal{P}_J \rho_o^{-1} \qquad (2.249)
$$

$$
U = \Psi_4 + \Theta\eta + \mathcal{E}_J \mathcal{D}_J \rho_o^{-1} \qquad (2.250)
$$

which gives

$$
\begin{aligned}
\dot{\epsilon} &= \dot{\Psi}_3 + \dot{\Theta}\eta + \Theta\dot{\eta} + \dot{\mathcal{E}}_J \mathcal{P}_J \rho_o^{-1} + \mathcal{E}_J \dot{\mathcal{P}}_J \rho_o^{-1} \\
\dot{U} &= \dot{\Psi}_4 + \dot{\Theta}\eta + \Theta\dot{\eta} + \dot{\mathcal{E}}_J \mathcal{D}_J \rho_o^{-1} + \mathcal{E}_J \dot{\mathcal{D}}_J \rho_o^{-1}.
\end{aligned}
$$

Based on this inspection we define the Legendre transformations

Definition 2.24.6 (Legendre Transforms II) *Legendre transformations* Ψ_1 *and* Ψ_2 *are defined as*

$$
\begin{aligned}
\Psi_3 &= \epsilon - \Theta\eta - \mathcal{E}_J \mathcal{P}_J \rho_o^{-1} \\
\Psi_4 &= U - \Theta\eta - \mathcal{E}_J \mathcal{D}_J \rho_o^{-1}
\end{aligned}
$$

Theorem 2.24.7 (Legendre Transformed Local C-D Inequality Statements II) *Local Clausius-Duhem inequalities equivalent to (2.232) and (2.233) are*

$$
-\rho_o \left(\dot{\Psi}_3 + \dot{\Theta}\eta \right) - \mathcal{P}_J \dot{\mathcal{E}}_J + \mathcal{T}^P_{SR} \dot{E}_{RS} + \mathcal{J}'_J \mathcal{E}_J - \mathcal{Q}_J \Theta^{-1} \Theta_{,J} \geq 0 \qquad (2.251)
$$

$$-\rho_o\left(\dot{\Psi}_4+\dot{\Theta}\eta\right)-\mathcal{D}_J\dot{\mathcal{E}}_J+\mathcal{T}^T_{SR}\dot{E}_{RS}+\mathcal{J}'_J\mathcal{E}_J-\mathcal{Q}_J\Theta^{-1}\Theta_{,J} \geq 0 \tag{2.252}$$

Proof. Using (2.249) and (2.250) in (2.232) and (2.233) proves (2.251) and (2.252). ■

Equations (2.251) and (2.252) motivate the objective functional forms

$$\begin{aligned}\rho_o\Psi_3 &= \overline{\Psi}_3\left(\Theta,\mathcal{E}_J,E_{RS}\right)\\ \rho_o\Psi_4 &= \overline{\Psi}_4\left(\Theta,\mathcal{E}_J,E_{RS}\right)\end{aligned}$$

Assuming these true

$$\rho_o\dot{\Psi}_3 = \frac{\partial\overline{\Psi}_3}{\partial\Theta}\dot{\Theta}+\frac{\partial\overline{\Psi}_3}{\partial\mathcal{E}_J}\dot{\mathcal{E}}_J+\frac{\partial\overline{\Psi}_3}{\partial E_{RS}}\dot{E}_{RS} \tag{2.253}$$

$$\rho_o\dot{\Psi}_4 = \frac{\partial\overline{\Psi}_4}{\partial\Theta}\dot{\Theta}+\frac{\partial\overline{\Psi}_4}{\partial\mathcal{E}_J}\dot{\mathcal{E}}_J+\frac{\partial\overline{\Psi}_4}{\partial E_{RS}}\dot{E}_{RS} \tag{2.254}$$

Using (2.253) and (2.254) in (2.251) and (2.252)

$$\begin{aligned}-\left(\rho_o\eta+\frac{\partial\overline{\Psi}_3}{\partial\Theta}\right)\dot{\Theta}+\left(\mathcal{P}_J+\frac{\partial\overline{\Psi}_3}{\partial\mathcal{E}_J}\right)\dot{\mathcal{E}}_J &+ \left(\mathcal{T}^P_{SR}-\frac{\partial\overline{\Psi}_3}{\partial E_{RS}}\right)\dot{E}_{RS}\\ &+ \mathcal{J}'_J\mathcal{E}_J-\mathcal{Q}_J\Theta^{-1}\Theta_{,J}\geq 0\end{aligned} \tag{2.255}$$

$$\begin{aligned}-\left(\rho_o\eta+\frac{\partial\overline{\Psi}_4}{\partial\Theta}\right)\dot{\Theta}+\left(\mathcal{D}_J+\frac{\partial\overline{\Psi}_4}{\partial\mathcal{E}_J}\right)\dot{\mathcal{E}}_J &+ \left(\mathcal{T}^T_{SR}-\frac{\partial\overline{\Psi}_4}{\partial E_{RS}}\right)\dot{E}_{RS}\\ &+ \mathcal{J}'_J\mathcal{E}_J-\mathcal{Q}_J\Theta^{-1}\Theta_{,J}\geq 0\end{aligned} \tag{2.256}$$

Necessary and sufficient conditions are

$$\begin{aligned}\rho_o\eta &= -\frac{\partial\overline{\Psi}_3}{\partial\Theta},\qquad \mathcal{P}_J=-\frac{\partial\overline{\Psi}_3}{\partial\mathcal{E}_J},\qquad \mathcal{T}^P_{SR}=\frac{\partial\overline{\Psi}_3}{\partial E_{RS}}\\ \mathcal{J}'_J\mathcal{E}_J &- \mathcal{Q}_J\Theta^{-1}\Theta_{,J}\geq 0\end{aligned} \tag{2.257}$$

and

$$\begin{aligned}\rho_o\eta &= -\frac{\partial\overline{\Psi}_4}{\partial\Theta},\qquad \mathcal{D}_J=-\frac{\partial\overline{\Psi}_4}{\partial\mathcal{E}_J},\qquad \mathcal{T}^T_{SR}=\frac{\partial\overline{\Psi}_4}{\partial E_{RS}}\\ \mathcal{J}'_J\mathcal{E}_J &- \mathcal{Q}_J\Theta^{-1}\Theta_{,J}\geq 0\end{aligned} \tag{2.258}$$

Theorem 2.24.8 (Legendre Transformed Local Conservation of Energy II) *Local conservation of energy statements equivalent to (2.215) and (2.216) are*

$$\rho_o\left(\dot{\Psi}_1+\dot{\Theta}\eta+\Theta\dot{\eta}\right) = -\mathcal{P}_J\dot{\mathcal{E}}_J+\mathcal{T}^P_{SR}\dot{E}_{RS}+\mathcal{J}'_J\mathcal{E}_J+\rho_o h-\mathcal{Q}_{J,J} \tag{2.259}$$

$$\rho_o\left(\dot{\Psi}_2+\dot{\Theta}\eta+\Theta\dot{\eta}\right) = -\mathcal{D}_J\dot{\mathcal{E}}_J+\mathcal{T}^T_{SR}\dot{E}_{RS}+\mathcal{J}'_J\mathcal{E}_J+\rho_o h-\mathcal{Q}_{J,J} \tag{2.260}$$

Proof. Using (2.249) and (2.250) in (2.215) and (2.216) proves (2.259) and (2.260). ■

We obtain our equations of heat conduction by simplifying the conservation of energy equations (2.259) and (2.260).

Theorem 2.24.9 (Local Heat Conduction in Material Fields) *The local heat conduction equation in material fields is*

$$\rho_o \Theta \dot{\eta} = \mathcal{J}_J' \mathcal{E}_J + \rho_o h - \mathcal{Q}_{J,J} \tag{2.261}$$

Proof. Using (2.257) and (2.258) in (2.259) and (2.260) proves (2.261). ■

2.25 Equation Summary: Spatial Fields

Here we summarize the local equations and jump conditions for deformable electroelastic continua with electric conduction, in terms of spatial fields [21].

$$\begin{aligned}
(D_i^* + J_i')_{,i} &= 0 \\
E_i + \phi_{,i} &= 0 \\
\dot{\rho} + \rho v_{k,k} &= 0 \\
\left(\tau_{ji} + \tau_{ji}^E\right)_{,j} + \rho\left(f_i - \dot{v}_i\right) &= 0 \\
\tau_{[ji]}^T = 0 \qquad \text{where } \tau_{ji}^T &= \tau_{ji} + \tau_{ji}^E \\
\rho \dot{U} = \left(\tau_{ij} + \tau_{ij}^E\right) v_{j,i} + E_i\left(J_i' + D_i^*\right) &+ \rho h - q_{i,i} \\
\rho \dot{\eta} \geq \rho \frac{h}{\Theta} &- \left(\frac{q_i}{\Theta}\right)_{,i} \\
\rho \Theta \dot{\eta} = E_i J_i' &+ \rho h - q_{i,i}
\end{aligned}$$

Jump conditions across a material surface of discontinuity, such as the bounding surface of a material body are,

$$\begin{aligned}
[\![D_i^* + J_i']\!] n_i &= 0 && \text{across } \sigma \\
[\![\phi]\!] &= 0 && \text{across } \gamma \\
[\![x_k]\!] &= 0 && \text{across } \sigma \\
[\![\tau_{ji} + \tau_{ji}^E]\!] n_j &= 0 && \text{across } \sigma \\
[\![\left(\tau_{ij} + \tau_{ij}^E\right) v_j - \phi\left(J_i' + D_i^*\right) - q_i]\!] n_i &= 0 && \text{across } \sigma \\
[\![\frac{q_j}{\Theta}]\!] n_j &\geq 0 && \text{across } \sigma
\end{aligned}$$

[21]The convective time derivative D_i^* is defined in (2.66)

The system of equations is not closed without constitutive equations describing the material response.

$$\begin{aligned}\Psi_2 &= U - \Theta\eta \\ \rho_o\Psi_2 &= \overline{\Psi}_2(\Theta, \mathcal{D}_J, E_{RS})\end{aligned}$$

Then from our results obtained earlier,

$$\eta = \frac{1}{\rho_o}\frac{\partial\overline{\Psi}_2}{\partial\Theta}, \qquad E_i = X_{J,i}\frac{\partial\overline{\Psi}_2}{\partial\mathcal{D}_J}, \qquad \tau^T_{ji} = J^{-1}x_{j,S}x_{i,R}\frac{\partial\overline{\Psi}_2}{\partial E_{RS}}$$

Additionally we have,

$$\begin{aligned}J'_i &= J^{-1}x_{i,J}\overline{\mathcal{J}'_J}(\mathcal{D}_J, E_{RS}, \Theta, \Theta_{,K}) \\ q_i &= J^{-1}x_{i,J}\overline{\mathcal{Q}_J}(\mathcal{D}_J, E_{RS}, \Theta, \Theta_{,K})\end{aligned}$$

subject to the restriction

$$J'_iE_i - q_i\Theta^{-1}\Theta_{,i} \geq 0.$$

Remark 2.25.1 (Perfect Electrically Insulating Bodies) *The above system of equations specializes to perfectly insulating bodies by constraining conduction current density $J'_i = 0$ inside the body. The bounding surface of the body may have a non-zero prescribed current density. We can append this modified system of equations with the original EQS Maxwell equations if we introduce a surface charge density w^F on the discontinuity surface σ,*

$$\begin{aligned}\oint_S D_in_i\,dS &= \int_V q^F\,dV \\ D_{i,i} &= q^F \\ [\![D_i]\!]\, n_i &= w^F \qquad \textit{across } \sigma.\end{aligned}$$

2.26 Equation Summary: Material Fields

Here we summarize the local equations and jump conditions for deformable electroelastic continua with electric conduction, in terms of material fields.

$$\begin{aligned}\left(\dot{\mathcal{D}}_J + \mathcal{J}'_J\right)_{,J} &= 0 \\ \mathcal{E}_J + \phi_{,J} &= 0 \\ J\rho &= \rho_o \\ \left(\mathcal{T}_{JK}x_{i,K} + \mathcal{T}^E_{JK}x_{i,K}\right)_{,J} + \rho_o(f_i - \dot{v}_i) &= 0 \\ \mathcal{T}^T_{[JK]} = 0\,, \qquad \mathcal{T}^T_{JK} &= \mathcal{T}_{JK} + \mathcal{T}^E_{JK} \\ \rho_o\dot{U} = \left(\mathcal{T}_{SR} + \mathcal{T}^E_{SR}\right)\dot{E}_{RS} + \mathcal{E}_J\left(\mathcal{J}'_J + \dot{\mathcal{D}}_J\right) &+ \rho_o h - \mathcal{Q}_{J,J}\end{aligned}$$

$$\rho\dot{\eta} \geq \rho_o \frac{h}{\Theta} - \left(\frac{\mathcal{Q}_J}{\Theta}\right)_{,J}$$
$$\rho\Theta\dot{\eta} = \mathcal{E}_J \mathcal{J}'_J + \rho_o h - \mathcal{Q}_{J,J}$$

Jump conditions across a material surface of discontinuity, such as the bounding surface of a material body are,

$$\begin{aligned}
[\![\, \dot{\mathcal{D}}_J + \mathcal{J}'_J \,]\!]\, N_J &= 0 && \text{across } \sigma_o \\
[\![\, \phi \,]\!] &= 0 && \text{across } \gamma_o \\
[\![\, x_k \,]\!] &= 0 && \text{across } \sigma_o \\
[\![\, \mathcal{T}_{JK} x_{i,K} + \mathcal{T}^E_{JK} x_{i,K} \,]\!]\, N_J &= 0 && \text{across } \sigma_o \\
[\![\, \left(\mathcal{T}_{JK} x_{j,K} + \mathcal{T}^E_{JK} x_{j,K}\right) v_j - \phi \left(\mathcal{J}'_J + \dot{\mathcal{D}}_J\right) - \mathcal{Q}_J \,]\!]\, N_J &= 0 && \text{across } \sigma_o \\
[\![\, \frac{\mathcal{Q}_J}{\Theta} \,]\!]\, N_J &\geq 0 && \text{across } \sigma_o
\end{aligned}$$

The system of equations is not closed without constitutive equations describing the material response.

$$\Psi_2 = U - \Theta\eta \,, \qquad \rho_o \Psi_2 = \overline{\Psi}_2 \left(\Theta, \mathcal{D}_J, E_{RS}\right)$$
$$\eta = \frac{1}{\rho_o}\frac{\partial \overline{\Psi}_2}{\partial \Theta} \,, \qquad \mathcal{E}_J = \frac{\partial \overline{\Psi}_2}{\partial \mathcal{D}_J} \,, \qquad \mathcal{T}^T_{SR} = \frac{\partial \overline{\Psi}_2}{\partial E_{RS}}.$$

Additionally, subject to the inequality constraint,

$$\begin{aligned}
\mathcal{J}'_J &= \overline{\mathcal{J}'_J} \left(\mathcal{D}_J, E_{RS}, \Theta, \Theta_{,K}\right) \\
\mathcal{Q}_J &= \overline{\mathcal{Q}_J} \left(\mathcal{D}_J, E_{RS}, \Theta, \Theta_{,K}\right) \\
\mathcal{J}'_J \mathcal{E}_J &- \mathcal{Q}_J \Theta^{-1} \Theta_{,J} \geq 0.
\end{aligned}$$

Remark 2.26.1 (Perfect Electrically Insulating Bodies) *The general system of equations specializes to perfectly insulating bodies by constraining conduction current density $\mathcal{J}'_J = 0$ inside the body. The bounding surface of the body may have a non-zero prescribed current density. We can append this modified system of equations with the original EQS Maxwell equations if we introduce a surface charge density $\mathcal{W}^F$ on the discontinuity surface σ_o,*

$$\begin{aligned}
\oint_{S_o} \mathcal{D}_J N_J \, dS_o &= \int_{V_o} \mathcal{Q}^F \, dV_o \\
\mathcal{D}_{J,J} &= \mathcal{Q}^F \\
[\![\, \mathcal{D}_J \,]\!]\, N_J &= \mathcal{W}^F \qquad \textit{across } \sigma_o.
\end{aligned}$$

Remark 2.26.2 (Rigid and Static Bodies) *The general system of equations specializes to rigid and static bodies by constraining the strain tensor $E_{RS} = 0$ and the velocity vector $v_i = 0$. We can append this modified system of equations with the original EQS Maxwell equations if we introduce a surface charge density $\mathcal{W}^F$ on the* *discontinuity surface* σ_o.

$$\oint_{S_o} \mathcal{D}_J N_J \, dS_o = \int_{V_o} \mathcal{Q}^F \, dV_o$$

$$\oint_{S_o} \mathcal{J}'_J N_J \, dS_o = -\frac{d}{dt} \int_{V_o} \mathcal{Q}^F \, dV_o$$

$$\mathcal{D}_{J,J} = \mathcal{Q}^F$$

$$\mathcal{J}'_{J,J} = -\dot{\mathcal{Q}}^F$$

$$[\![\mathcal{D}_J]\!] N_J = \mathcal{W}^F \qquad \textit{across } \sigma_o.$$

$$[\![\mathcal{J}'_J]\!] N_J = -\dot{\mathcal{W}}^F \qquad \textit{across } \sigma_o.$$

Chapter 3

Small Deformation Approximations

3.1 Introduction

The purpose of this chapter is to introduce the small deformation approximation (SDA) into the general EQS electroelastic equations summarized below. The resulting SDA equations are greatly simplified and specialize to the classical linear piezoelectric equations, extended to include electrical conduction. A result is that electric body forces, realized mathematically as electric surface tractions, are retained in the small deformation approximation.

3.2 Large Deformation Equations: Material Fields

Below we present a summary of the general EQS electroelastic equations with electric conduction in material fields.

$$
\begin{aligned}
\left(\dot{\mathcal{D}}_J + \mathcal{J}'_J\right)_{,J} &= 0 \\
\mathcal{E}_J + \phi_{,J} &= 0 \\
J\rho &= \rho_o \\
\left(\mathcal{T}_{JK} x_{i,K} + \mathcal{T}^E_{JK} x_{i,K}\right)_{,J} + \rho_o\left(f_i - \dot{v}_i\right) &= 0 \\
\mathcal{T}^T_{[JK]} = 0 \ , \qquad \mathcal{T}^T_{JK} &= \mathcal{T}_{JK} + \mathcal{T}^E_{JK} \\
\rho_o \dot{U} = \left(\mathcal{T}_{SR} + \mathcal{T}^E_{SR}\right)\dot{E}_{RS} + \mathcal{E}_J\left(\mathcal{J}'_J + \dot{\mathcal{D}}_J\right) &+ \rho_o h - \mathcal{Q}_{J,J} \\
\rho_o \dot{\eta} \geq \rho_o \frac{h}{\Theta} &- \left(\frac{\mathcal{Q}_J}{\Theta}\right)_{,J} \\
\rho_o \Theta \dot{\eta} = \mathcal{E}_J \mathcal{J}'_J &+ \rho_o h - \mathcal{Q}_{J,J}
\end{aligned}
$$

Jump conditions across material surfaces of discontinuity, such as the bounding surface of a material body are,

$$
\begin{aligned}
[\![\dot{\mathcal{D}}_J + \mathcal{J}'_J]\!] N_J &= 0 \qquad \text{across } \sigma_o \\
[\![\phi]\!] &= 0 \qquad \text{across } \gamma_o
\end{aligned}
$$

$$[\![\mathcal{T}_{JK} x_{i,K} + \mathcal{T}^E_{JK} x_{i,K}]\!] N_J = 0 \qquad \text{across } \sigma_o$$
$$[\![\left(\mathcal{T}_{JK} x_{j,K} + \mathcal{T}^E_{JK} x_{j,K}\right) v_j - \phi \left(\mathcal{J}'_J + \dot{\mathcal{D}}_J\right) - \mathcal{Q}_J]\!] N_J = 0 \qquad \text{across } \sigma_o$$
$$[\![\frac{\mathcal{Q}_J}{\Theta}]\!] N_J \geq 0 \qquad \text{across } \sigma_o$$

The system of equations is not closed without constitutive equations describing the material response.

$$\Psi_2 = U - \Theta\eta \,, \qquad \rho_o \Psi_2 = \overline{\Psi}_2 \left(\Theta, \mathcal{D}_J, E_{RS}\right)$$
$$\eta = \frac{1}{\rho_o} \frac{\partial \overline{\Psi}_2}{\partial \Theta} \,, \qquad \mathcal{E}_J = \frac{\partial \overline{\Psi}_2}{\partial \mathcal{D}_J} \,, \qquad \mathcal{T}^T_{SR} = \frac{\partial \overline{\Psi}_2}{\partial E_{RS}}.$$

Additionally, subject to the inequality constraint,

$$\mathcal{J}'_J = \overline{\mathcal{J}'_J} \left(\mathcal{D}_J, E_{RS}, \Theta, \Theta_{,K}\right)$$
$$\mathcal{Q}_J = \overline{\mathcal{Q}_J} \left(\mathcal{D}_J, E_{RS}, \Theta, \Theta_{,K}\right)$$
$$\mathcal{J}'_J \mathcal{E}_J - \mathcal{Q}_J \Theta^{-1} \Theta_{,J} \geq 0.$$

Remark 3.2.1 (Material Time Derivative: Material Fields) *Material fields such as $\mathcal{D}_J$ are defined over material points X_j in $\mathcal{B}_o$. The material time derivative of a material field is simply a partial derivative with respect to time,*[1]

$$\dot{\mathcal{D}}_J = \frac{\partial \mathcal{D}_J \left(X_J, t\right)}{\partial t}$$
$$v_i = \frac{\partial \chi_i \left(X_J, t\right)}{\partial t}.$$

3.3 Small Deformation Equations: Material Fields

The material form of the large deformation equations summarized above are particularly useful for deriving approximate theories. The difficulty with the above expression is that $x_{j,K}$ is part of the solution, and unknown a priori. We can greatly simplify the above equations by introducing the small deformation approximation. First we define the mechanical displacement vector.

Definition 3.3.1 (Mechanical Displacement) *A mechanical displacement vector u_M is defined as*

$$x_k = \left(X_M + u_M\right) \delta_{Mk}, \tag{3.1}$$

Introducing u_M into $x_{j,K}$, E_{MN}, and v_k gives

$$x_{j,K} = \delta_{jK} + \frac{\partial u_M}{\partial X_K} \delta_{Mj}$$

[1] *Compare to the spatial field description where a nonlinear convective term that arises*

$$
\begin{aligned}
E_{MN} &= \frac{1}{2}\left(\frac{\partial u_M}{\partial X_N} + \frac{\partial u_N}{\partial X_M} + \frac{\partial u_K}{\partial X_M}\frac{\partial u_K}{\partial X_N}\right) \\
v_k &= \dot{u}_M \delta_{Mk}.
\end{aligned}
$$

Definition 3.3.2 (Small Deformation Approximation (SDA)) *The displacement gradient and its material derivative are assumed small,*

$$
\left|\frac{\partial u_K}{\partial X_J}\right| \ll 1 \quad \textit{for each } K, J = 1..3, \tag{3.2}
$$

$$
\left|\frac{\partial \dot{u}_K}{\partial X_J}\right| \ll 1 \quad \textit{for each } K, J = 1..3. \tag{3.3}
$$

Remark 3.3.3 (Simplifications Under SDA) *The SDA implies the following approximations,*

$$
\begin{aligned}
x_{j,K} &= \delta_{jK} + \frac{\partial u_M}{\partial X_K}\delta_{Mj} \approx \delta_{jK} \\
J &= \det\left(\frac{\partial x_i}{\partial X_J}\right) \approx 1. \\
E_{MN} &= \frac{1}{2}\left(\frac{\partial u_M}{\partial X_N} + \frac{\partial u_N}{\partial X_M} + \frac{\partial u_K}{\partial X_M}\frac{\partial u_K}{\partial X_N}\right) \\
&\approx \frac{1}{2}\left(\frac{\partial u_M}{\partial X_N} + \frac{\partial u_N}{\partial X_M}\right)
\end{aligned} \tag{3.4}
$$

It is interesting to note that $x_{i,K} \approx \delta_{iK}$ and $J \approx 1$ imply that all spatial and material fields in definition 2.20.1 are indistinguishable in the SDA approximation. This suggests the extreme nature of the simplification.

Theorem 3.3.4 (EQS Electroelastic SDA Equations) *The EQS electroelastic equations under the SDA approximation simplify to the following:*

$$
\begin{aligned}
\left(\dot{\mathcal{D}}_J + \mathcal{J}'_J\right)_{,J} &= 0 \\
\mathcal{E}_J + \phi_{,J} &= 0 \\
\left(\mathcal{T}_{JK}\delta_{iK} + \mathcal{T}^E_{JK}\delta_{iK}\right)_{,J} + \rho_o\left(f_i - \dot{v}_i\right) &= 0 \\
\mathcal{T}^T_{[JK]} = 0\ , \qquad \mathcal{T}^T_{JK} &= \mathcal{T}_{JK} + \mathcal{T}^E_{JK} \\
\rho_o\dot{U} = \left(\mathcal{T}_{SR} + \mathcal{T}^E_{SR}\right)\dot{\overline{E}}_{RS} + \mathcal{E}_J\left(\mathcal{J}'_J + \dot{\mathcal{D}}_J\right) &+ \rho_o h - \mathcal{Q}_{J,J} \\
\rho_o\dot{\eta} &\geq \rho_o\frac{h}{\Theta} - \left(\frac{\mathcal{Q}_J}{\Theta}\right)_{,J} \\
\rho_o\Theta\dot{\eta} &= \mathcal{E}_J\mathcal{J}'_J + \rho_o h - \mathcal{Q}_{J,J} \\
\overline{E}_{MN} &= \frac{1}{2}\left(\frac{\partial u_M}{\partial X_N} + \frac{\partial u_N}{\partial X_M}\right)
\end{aligned}
$$

The corresponding jump conditions across material surfaces of discontinuity, such as the bounding surface of a material body, are

$$
\begin{aligned}
[\![\dot{\mathcal{D}}_J + \mathcal{J}'_J]\!] N_J &= 0 && \textit{across } \sigma_o \\
[\![\phi]\!] &= 0 && \textit{across } \gamma_o \\
[\![u_M]\!] &= 0 && \textit{across } \sigma_o \\
[\![\mathcal{T}_{JK}\delta_{iK} + \mathcal{T}^E_{JK}\delta_{iK}]\!] N_J &= 0 && \textit{across } \sigma_o \\
[\![\left(\mathcal{T}_{JK}\delta_{jK} + \mathcal{T}^E_{JK}\delta_{jK}\right) v_j - \phi\left(\mathcal{J}'_J + \dot{\mathcal{D}}_J\right) - \mathcal{Q}_J]\!] N_J &= 0 && \textit{across } \sigma_o \\
[\![\frac{\mathcal{Q}_J}{\Theta}]\!] N_J &\geq 0 && \textit{across } \sigma_o
\end{aligned}
$$

The system of equations is not closed without constitutive equations describing the material response.

$$
\Psi_2 = U - \Theta\eta \ , \qquad \rho_o\Psi_2 = \overline{\Psi}_2\left(\Theta, \mathcal{D}_J, \overline{E}_{RS}\right)
$$

$$
\eta = \frac{1}{\rho_o}\frac{\partial\overline{\Psi}_2}{\partial\Theta} \ , \qquad \mathcal{E}_J = \frac{\partial\overline{\Psi}_2}{\partial\mathcal{D}_J} \ , \qquad \mathcal{T}^T_{SR} = \frac{\partial\overline{\Psi}_2}{\partial\overline{E}_{RS}}.
$$

Additionally, subject to the inequality constraint,

$$
\begin{aligned}
\mathcal{J}'_J &= \overline{\mathcal{J}'_J}\left(\mathcal{D}_J, \overline{E}_{RS}, \Theta, \Theta_{,K}\right) \\
\mathcal{Q}_J &= \overline{\mathcal{Q}_J}\left(\mathcal{D}_J, \overline{E}_{RS}, \Theta, \Theta_{,K}\right) \\
&\mathcal{J}'_J\mathcal{E}_J - \mathcal{Q}_J\Theta^{-1}\Theta_{,J} \geq 0.
\end{aligned}
$$

Remark 3.3.5 (Perfect Electrically Insulating Bodies) *The general system of equations specializes to perfectly insulating bodies by constraining conduction current density $\mathcal{J}'_J = 0$ inside the body. The bounding surface of the body may have a non-zero prescribed current density. We can append this modified system of equations with the original EQS Maxwell equations if we introduce a surface charge density $\mathcal{W}^F$ on the discontinuity surface σ_o,*

$$
\begin{aligned}
\oint_{S_o} \mathcal{D}_J N_J \, dS_o &= \int_{V_o} \mathcal{Q}^F \, dV_o \\
\mathcal{D}_{J,J} &= \mathcal{Q}^F \\
[\![\mathcal{D}_J]\!] N_J &= \mathcal{W}^F \qquad \textit{across } \sigma_o.
\end{aligned}
\tag{3.5}
$$

Remark 3.3.6 (Rigid and Static Bodies) *The general system of equations specializes to rigid and static bodies by constraining the strain tensor $E_{RS} = 0$ and the velocity vector $v_i = 0$. We can append this modified system of equations with the original EQS Maxwell equations if we introduce a surface charge density $\mathcal{W}^F$ on the discontinuity surface σ_o.*

$$
\oint_{S_o} \mathcal{D}_J N_J \, dS_o = \int_{V_o} \mathcal{Q}^F \, dV_o
$$

$$\oint_{S_o} \mathcal{J}'_J N_J \, dS_o = -\frac{d}{dt} \int_{V_o} \mathcal{Q}^F \, dV_o$$

$$\mathcal{D}_{J,J} = \mathcal{Q}^F$$

$$\mathcal{J}'_{J,J} = -\dot{\mathcal{Q}}^F$$

$$[\![\mathcal{D}_J]\!] N_J = \mathcal{W}^F \qquad \textit{across } \sigma_o.$$

$$[\![\mathcal{J}'_J]\!] N_J = -\dot{\mathcal{W}}^F \qquad \textit{across } \sigma_o.$$

Chapter 4

Finite Element Formulations

4.1 Introduction

The current and following chapters will consider solution of the EQS electroelastic SDA equations presented in theorem 3.3.4. Restricting ourself to the simplified equations, it is convenient to introduce new notation.

4.2 Electroelastic SDA Equations

Definition 4.2.1 (Notation) *The notation in theorem 3.3.4 is changed as follows:*

$$\begin{aligned} u_M &\to u_i & \mathcal{D}_J &\to D_i \\ \mathcal{Q}^F &\to q^F & \mathcal{W}^F &\to w^F \\ \mathcal{J}'_J &\to J_i & \mathcal{E}_J &\to E_i \\ \phi &\to \phi & \mathcal{T}_{JK} &\to \tau_{ij} \\ \mathcal{T}^E_{JK} &\to \tau^E_{ij} & \rho_o f_i &\to f_i \\ v_i &\to v_i & \mathcal{T}^T_{JK} &\to \tau^T_{ij} \\ \rho_o U &\to U & \overline{E}_{RS} &\to \epsilon_{ij} \\ h &\to h & \mathcal{Q}_J &\to q_i \\ \eta &\to \eta & \Theta &\to \Theta \\ N_J &\to n_j & V_o &\to V \\ S_o &\to S & C_o &\to C \end{aligned}$$

Next we present the SDA equations in the new notation.

Theorem 4.2.2 (EQS Electroelastic SDA Equations)

$$\left(\dot{D}_i + J_i\right)_{,i} = 0$$

$$E_i + \phi_{,i} = 0$$

$$\left(\tau_{jk} + \tau^E_{jk}\right)_{,j} + f_i - \rho_o \dot{v}_i = 0$$

$$
\begin{aligned}
\tau^T_{[jk]} = 0 \ , \qquad \tau^T_{jk} &= \tau_{jk} + \tau^E_{jk} \\
\dot{U} = \left(\tau_{ji} + \tau^E_{ji}\right)\dot{\epsilon}_{ij} + E_k\left(J_k + \dot{D}_k\right) &+ \rho_o h - q_{k,k} \\
\rho_o \dot{\eta} \geq \rho_o \frac{h}{\Theta} &- \left(\frac{q_j}{\Theta}\right)_{,j} \\
\rho_o \Theta \dot{\eta} = E_k J_k &+ \rho_o h - q_{k,k} \\
\epsilon_{ij} = \frac{1}{2}&\left(u_{i,j} + u_{j,i}\right)
\end{aligned}
$$

Jump conditions across a material surface of discontinuity, such as the bounding surface of a material body are,

$$
\begin{aligned}
[\![\dot{D}_i + J_i]\!] n_i &= 0 && \textit{across } \sigma_o \\
[\![\phi]\!] &= 0 && \textit{across } \gamma_o \\
[\![u_i]\!] &= 0 && \textit{across } \sigma_o \\
[\![\tau_{jk} + \tau^E_{jk}]\!] n_j &= 0 && \textit{across } \sigma_o \\
[\![\left(\tau_{jk} + \tau^E_{jk}\right) v_k - \phi\left(J_j + \dot{D}_j\right) - q_j]\!] n_j &= 0 && \textit{across } \sigma_o \\
[\![\frac{q_j}{\Theta}]\!] n_j &\geq 0 && \textit{across } \sigma_o
\end{aligned}
$$

The system of equations is not closed without the constitutive equations describing the material response.

$$
\begin{aligned}
&\Psi_2 = U - \Theta\eta \ , \qquad \rho_o \Psi_2 = \overline{\Psi}_2\left(\Theta, D_j, \epsilon_{ij}\right) \\
&\eta = \frac{1}{\rho_o}\frac{\partial \overline{\Psi}_2}{\partial \Theta} \ , \qquad E_j = \frac{\partial \overline{\Psi}_2}{\partial D_j} \ , \qquad \tau^T_{ji} = \frac{\partial \overline{\Psi}_2}{\partial \epsilon_{ij}}.
\end{aligned}
$$

Additionally, subject to the inequality constraint,

$$
\begin{aligned}
J_j &= \overline{J_j}\left(D_j, \epsilon_{ij}, \Theta, \Theta_{,k}\right) \\
q_j &= \overline{q_j}\left(D_j, \epsilon_{ij}, \Theta, \Theta_{,k}\right) \\
J_j E_j &- q_j \Theta^{-1} \Theta_{,j} \geq 0.
\end{aligned}
$$

Remark 4.2.3 (Perfect Electrically Insulating Bodies) *The general system of equations specializes to perfectly insulating bodies by constraining conduction current density $J_i = 0$ inside the body. The bounding surface of the body may have a non-zero prescribed current density. We can append this modified system of equations with the original EQS Maxwell equations if we introduce a surface charge density w^F on the discontinuity surface σ_o,*

$$
\begin{aligned}
\oint_S D_i n_i \, dS &= \int_V q^F \, dV \\
D_{i,i} &= q^F \\
[\![D_i]\!] n_i &= w^F \qquad \textit{across } \sigma_o.
\end{aligned}
\tag{4.1}
$$

Remark 4.2.4 (Rigid and Static Bodies) *The general system of equations spe-*

cializes to rigid and static bodies by constraining the strain tensor $\epsilon_{ij} = 0$ *and the velocity vector* $v_i = 0$. *We can append this modified system of equations with the original EQS Maxwell equations if we introduce a surface charge density* w^F *on the discontinuity surface* σ_o.

$$\begin{aligned}
\oint_S D_i n_i \, dS &= \int_V q^F \, dV \\
\oint_S J_i n_i \, dS &= -\frac{\partial}{\partial t} \int_V q^F \, dV \\
D_{i,i} &= q^F \\
J_{i,i} &= -\dot{q}^F \\
[\![D_i]\!] n_i &= w^F \qquad \text{across } \sigma_o. \\
[\![J_i]\!] n_i &= -\dot{w}^F \qquad \text{across } \sigma_o.
\end{aligned}$$

4.3 Weak Forms of Equations

In this section we obtain the weak form of the balance of momentum and Maxwell EQS equations suitable for a finite element analysis.

Theorem 4.3.1 (Balance of Momentum: Weak Form) *A material volume* V *bounded by surface* $S = S_f + S_u$ *is subject to mechanical surface tractions* $f_i^{S_f}$ *on* S_f, *mechanical displacement constraints* $u_i^{S_u}$ *on* S_u, *and body force density* f_i^B *in* V. *The electric stress tractions* t_i^E *outside the material volume* V *are assumed negligible, consistent with an assumption of zero electric fields* E_i *outside* V. *The balance of momentum equations and corresponding jump conditions across* S *are*

$$\tau^T_{ji,j} + f_i^B = \rho_o \frac{\partial^2 u_i}{\partial t^2} \qquad \tau^T_{[ji]} = 0$$

$$\tau^T_{ji} n_j = f_i^{S_f} \text{ on } S_f \qquad u_i = u_i^{S_u} \text{ on } S_u.$$

Consider a weighting function $\overline{u}_i$ *and tensor* $\overline{\epsilon}_{ij}$, *defined over the material volume* V, *such that*

$$\overline{u}_i = 0 \text{ on } S_u \qquad \overline{\epsilon}_{ij} = \frac{1}{2}\left(\overline{u}_{i,j} + \overline{u}_{j,i}\right).$$

Then the balance of momentum equations and jump conditions have the equivalent weak form,

$$\begin{aligned}
\int_V \tau^T_{ji} \overline{\epsilon}_{ij} \, dV + \int_V \rho_o \frac{\partial^2 u_i}{\partial t^2} \overline{u}_i \, dV &= \int_V f_i^B \overline{u}_i \, dV + \int_{S_f} f_i^{S_f} \overline{u}_i \, dS_f, \qquad (4.2) \\
u_i &= u_i^{S_u} \text{ on } S_u
\end{aligned}$$

Proof. Multiply the balance of momentum equation through by $\overline{u}_i$ and integrate over V,

$$\int_V \left(\tau^T_{ji,j} + f^B_i\right) \overline{u}_i \, dV \; = \int_V \rho_o \frac{\partial^2 u_i}{\partial t^2} \overline{u}_i \, dV \, .$$

Use the chain rule to eliminate derivatives from τ^T_{ji},

$$(\tau^T_{ji}\overline{u}_i)_{,j} = \tau^T_{ji,j}\overline{u}_i + \tau^T_{ji}\overline{u}_{i,j}.$$

Use the identity

$$\int_V (\tau^T_{ji}\overline{u}_i)_{,j} \, dV \; = \oint_S \tau^T_{ji}\overline{u}_i n_j \, dS \, ,$$

and $\tau^T_{[ji]} = 0 \to \tau^T_{ji} = \tau^T_{ij}$, such that

$$\tau^T_{ji}\overline{u}_{i,j} = \tau^T_{ji}\frac{1}{2}(\overline{u}_{i,j} + \overline{u}_{j,i}) = \tau^T_{ji}\overline{\epsilon}_{ij},$$

to obtain

$$\oint_S \tau^T_{ji}\overline{u}_i n_j \, dS \; - \int_V \tau^T_{ji}\overline{\epsilon}_{ij} \, dV \; + \int_V \left(f^B_i - \rho_o \frac{\partial^2 u_i}{\partial t^2}\right) \overline{u}_i \, dV \; = 0.$$

Use the jump condition and constraint

$$\tau^T_{ji} n_j = f^{S_f}_i \text{ on } S_f \qquad \overline{u}_i = 0 \text{ on } S_u$$

to obtain a weak form of the linear momentum balance

$$\int_V \tau^T_{ji}\overline{\epsilon}_{ij} \, dV \; + \int_V \rho_o \frac{\partial^2 u_i}{\partial t^2} \overline{u}_i \, dV \; = \int_V f^B_i \overline{u}_i \, dV \; + \int_S f^{S_f}_i \overline{u}_i \, dS \, .$$

■

Theorem 4.3.2 (EQS Maxwell: Weak Form) *A material volume V bounded by surface $S = S_q + S_\phi$ is subject to electric current density $\boldsymbol{J}^{\text{ext}} \cdot \boldsymbol{n} = J^{S_q}$ on S_q and electric voltage constraints ϕ^{S_ϕ} on S_ϕ. The electric displacement fields $\boldsymbol{D}$ and their time derivatives are assumed negligible outside the material volume V. The conservation of charge equation and corresponding jump conditions across S are*

$$\boldsymbol{\nabla} \cdot \left(\dot{\boldsymbol{D}} + \boldsymbol{J}\right) = 0 \qquad \left(\dot{\boldsymbol{D}} + \boldsymbol{J}\right) \cdot \boldsymbol{n} = J^{S_q} \text{ on } S_q \qquad \phi = \phi^{S_\phi} \text{ on } S_\phi$$

Consider a weighting function $\overline{\phi}$ and vector $\overline{\boldsymbol{E}}$, defined over the material volume V, such that

$$\overline{\phi} = 0 \text{ on } S_\phi \qquad \overline{\boldsymbol{E}} = -\boldsymbol{\nabla}\phi$$

Then the conservation of charge equation and jump conditions have the equivalent

equivalent weak form,

$$\begin{aligned}\int_V \left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\cdot\overline{\boldsymbol{E}}\,dV &= -\int_{S_q} J^{S_q}\overline{\phi}\,dS_q\\ \phi &= \phi^{S_\phi} \text{ on } S_\phi\end{aligned}$$

Proof. Multiply the conservation of charge equation through by $\overline{\phi}$ and integrate over V,

$$\int_V \boldsymbol{\nabla}\cdot\left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\overline{\phi}\,dV = 0$$

Use the chain rule to eliminate derivatives from $\left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)$,

$$\boldsymbol{\nabla}\cdot\left[\left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\overline{\phi}\right] = \boldsymbol{\nabla}\cdot\left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\overline{\phi} + \left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\cdot\boldsymbol{\nabla}\overline{\phi}.$$

Use the identity

$$\int_V \boldsymbol{\nabla}\cdot\left[\left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\overline{\phi}\right]\,dV = \oint_S \left[\left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\overline{\phi}\right]\cdot\boldsymbol{n}\,dS\,,$$

to obtain

$$\oint_S \left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\cdot\boldsymbol{n}\overline{\phi}\,dS + \int_V \left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\cdot\overline{\boldsymbol{E}}\,dV = 0.$$

Use the jump condition and constraint,

$$\left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\cdot\boldsymbol{n} = J^{S_q} \text{ on } S_q \qquad \overline{\phi} = 0 \text{ on } S_\phi$$

to obtain

$$\int_V \left(\dot{\boldsymbol{D}}+\boldsymbol{J}\right)\cdot\overline{\boldsymbol{E}}\,dV = -\oint_{S_q} J^{S_q}\overline{\phi}\,dS_q\,.$$

■

4.4 Solution Technique

Standard techniques have been developed for the finite element solution of a system of differential equations. These take as a starting point the weak form of the governing equations, see BATHE [2] for finite element procedures in the context of continuum mechanics of solids. Finite element procedures will be used to spatially discretize the weak form equations, and finite difference techniques will be used to discretize the equations over time. The result is a finite degree of freedom system of algebraic equations. In this thesis we are interested in presenting a finite element formulation suitable for nonlinear material response functions. In this case, the resulting system of algebraic equations will also be nonlinear. Below we present our notation for solving the nonlinear system of algebraic equations using Newton's method. We present this to motivate the form of our finite element formulation in the following sections, which anticipate a Newton method solution.

Our finite element formulations[1] will result in the following system of nonlinear ordinary differential equations

$$Res\left(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}, \ddot{\boldsymbol{\xi}}\right) = 0,$$

where $\boldsymbol{\xi}$ is our vector of nodal unknowns. Anticipating a finite difference discretization in time, we write the equations at some specified time $t + \Delta t$,

$$\boldsymbol{Res}(\boldsymbol{\xi}_{t+\Delta t}, \dot{\boldsymbol{\xi}}_{t+\Delta t}, \ddot{\boldsymbol{\xi}}_{t+\Delta t}) = 0.$$

This system will be discretized using an implicit finite difference technique,

$$\begin{aligned} \dot{\boldsymbol{\xi}}_{t+\Delta t} &= \dot{\boldsymbol{\xi}}_{t+\Delta t}(\boldsymbol{\xi}_{t+\Delta t}, \boldsymbol{\xi}_t, \dot{\boldsymbol{\xi}}_t, \ddot{\boldsymbol{\xi}}_t), \\ \ddot{\boldsymbol{\xi}}_{t+\Delta t} &= \ddot{\boldsymbol{\xi}}_{t+\Delta t}(\boldsymbol{\xi}_{t+\Delta t}, \boldsymbol{\xi}_t, \dot{\boldsymbol{\xi}}_t, \ddot{\boldsymbol{\xi}}_t). \end{aligned}$$

Dropping the explicit dependence on variables at time t, because they are known at time $t + \Delta t$, we obtain

$$\boldsymbol{Res}(\boldsymbol{\xi}_{t+\Delta t}, \dot{\boldsymbol{\xi}}_{t+\Delta t}(\boldsymbol{\xi}_{t+\Delta t}), \ddot{\boldsymbol{\xi}}_{t+\Delta t}(\boldsymbol{\xi}_{t+\Delta t})) = 0.$$

Redefining such that

$$\boldsymbol{\xi}_{t+\Delta t} \longrightarrow \boldsymbol{\xi}$$

we introduce

$$\begin{aligned} \delta\boldsymbol{\xi} &\stackrel{\triangle}{=} \boldsymbol{\xi}^{\nu+1} - \boldsymbol{\xi}^{\nu}, \\ \boldsymbol{Res}^{\nu} &\stackrel{\triangle}{=} \boldsymbol{Res}(\boldsymbol{\xi}^{\nu}) \end{aligned}$$

where ν is the iteration level for a Newton method solution technique. Consider our equations at iteration level $\nu + 1$ and introduce a first order Taylor series expansion,

$$\begin{aligned} \boldsymbol{Res}^{\nu+1} &\simeq \boldsymbol{Res}^{\nu} + \left.\frac{\partial \boldsymbol{Res}}{\partial \boldsymbol{\xi}}\right|^{\nu} \delta\boldsymbol{\xi} + \left.\frac{\partial \boldsymbol{Res}}{\partial \dot{\boldsymbol{\xi}}}\right|^{\nu} \left.\frac{\partial \dot{\boldsymbol{\xi}}}{\partial \boldsymbol{\xi}}\right|^{\nu} \delta\boldsymbol{\xi} + \left.\frac{\partial \boldsymbol{Res}}{\partial \ddot{\boldsymbol{\xi}}}\right|^{\nu} \left.\frac{\partial \ddot{\boldsymbol{\xi}}}{\partial \boldsymbol{\xi}}\right|^{\nu} \delta\boldsymbol{\xi} + \cdots \\ &= \boldsymbol{Res}^{\nu} + \left[\frac{\partial \boldsymbol{Res}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{Res}}{\partial \dot{\boldsymbol{\xi}}}\frac{\partial \dot{\boldsymbol{\xi}}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{Res}}{\partial \ddot{\boldsymbol{\xi}}}\frac{\partial \ddot{\boldsymbol{\xi}}}{\partial \boldsymbol{\xi}}\right]^{\nu} \delta\boldsymbol{\xi} + \cdots \end{aligned}$$

or

$$\boldsymbol{Res}^{\nu+1} \simeq \boldsymbol{Res}^{\nu} + J^{\nu}\delta\boldsymbol{\xi}$$

where we have implicitly defined the Jacobian matrix J^{ν}. Newton's method is entirely

[1]See BATHE [2] for details of standard finite element procedures

based on the following two statements

$$\begin{aligned} \boldsymbol{Res}^{\nu+1} &\simeq \boldsymbol{Res}^{\nu} + J^{\nu}\delta\boldsymbol{\xi} \\ \boldsymbol{Res}^{\nu+1} &= 0 \end{aligned}$$

resulting in Newton's method equations that are solved iteratively

$$J^{\nu}\delta\boldsymbol{\xi} = -\boldsymbol{Res}^{\nu}.$$

until components of the residual vector $\boldsymbol{Res}^{\nu}$ are sufficiently small.

4.4.1 Choice of Independent Variables

Boundary conditions and the functional dependence of available constitutive relations are what determines the choice of independent variables in our problem formulation. For convenience we introduce Voight notation to replace the symmetric tensors τ_{ji}^{T} and ϵ_{ij} with $\boldsymbol{T}$ and $\boldsymbol{S}$,

$$\begin{aligned} \boldsymbol{T} &= \left[\begin{array}{cccccc} \tau_{11}^{T} & \tau_{22}^{T} & \tau_{33}^{T} & \tau_{32}^{T} & \tau_{13}^{T} & \tau_{12}^{T} \end{array}\right]' \\ \boldsymbol{S} &= \left[\begin{array}{cccccc} \epsilon_{11} & \epsilon_{22} & \epsilon_{33} & 2\epsilon_{32} & 2\epsilon_{13} & 2\epsilon_{12} \end{array}\right]' \end{aligned}$$

The jump conditions involve mechanical displacement $\boldsymbol{u}$ and voltage ϕ specification on the corresponding domain boundaries. Therefore, our finite element formulation must have $\boldsymbol{u}$ and ϕ for independent variables. In the conventional formulation we would have available,

$$\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{S}, \boldsymbol{E}) \qquad \boldsymbol{S} = \boldsymbol{S}(\boldsymbol{S}, \boldsymbol{E}).$$

From our definitions of strain and electric field,

$$\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{u}) \qquad \boldsymbol{E} = \boldsymbol{E}(\phi).$$

Therefore our constitutive relations and jump conditions are known in terms of the same independent variables. In this case the weak form of the equations presented earlier are suitable.

It turns out however that constitutive relations may not be avaialable in the above form for a given material. Electroelastic materials with hysteresis may have constitutive relations of the form

$$\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{S}, \boldsymbol{D}) \qquad \boldsymbol{E} = \boldsymbol{E}(\boldsymbol{S}, \boldsymbol{D}),$$

where these nonlinear relations may not be easily inverted. In this case, our boundary conditions require $\boldsymbol{u}, \phi$ as independent variables and our constitutive equations are in terms of $\boldsymbol{S}(\boldsymbol{u}), \boldsymbol{D}$.

Consider the weak formulations presented earlier. Prior to introducing our constitutive relations, the weak forms have $\boldsymbol{T}, \boldsymbol{u}$, and $\boldsymbol{D}$ as independent variables. For closure of the equations we require constitutive relations, without them our system of equations is underdetermined. We can introduce $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{S}, \boldsymbol{D})$, but not $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{S}, \boldsymbol{D})$. To accomodate this second constitutive relation we introduce the electric potential definition

$$\boldsymbol{E} + \boldsymbol{\nabla}\phi = \boldsymbol{0}.$$

This equation can be made suitable for finite element procedures by puting it in a weak form. This is accomplished by multiplying the electric potential equation by $\overline{\boldsymbol{D}}$ and integrating over the volume V[2],

$$\int_V (\boldsymbol{E} + \boldsymbol{\nabla}\phi) \cdot \overline{\boldsymbol{D}}\, dV = 0.$$

4.5 Finite Element Formulation

This section starts from the mixed weak form of the small deformation electroelastic equations with electric conduction. A finite element formulation is presented that anticipates a Newton method solution of the resulting nonlinear discretized equations. The formulation can be implemented using four node, three dimensional, isoparametric finite elements presented in BATHE [2, pp. 375, 979-987] and a backward Euler finite difference (implicit) time integration[3]. Note that similar finite element formulations for perfect electrically insulating materials are presented in appendix B.

4.5.1 Mixed Weak Form

$$\int_V \boldsymbol{T} \cdot \overline{\boldsymbol{S}}\, dV + \int_V \rho_o \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \overline{\boldsymbol{u}}\, dV - \int_V \boldsymbol{f}^B \cdot \overline{\boldsymbol{u}}\, dV - \int_{S_f} \boldsymbol{f}^{S_f} \cdot \overline{\boldsymbol{u}}^{S_f}\, dS_f = 0$$

$$\int_V (\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}) \cdot \overline{\boldsymbol{E}}\, dV + \oint_S J^{S_q} \overline{\phi}^{S_q}\, dS = 0$$

$$\int_V (\boldsymbol{E} + \boldsymbol{\nabla}\phi) \cdot \overline{\boldsymbol{D}}\, dV = 0$$

4.5.2 Mixed Weak Form Rewritten

$$\int_V \overline{\boldsymbol{S}}' \boldsymbol{T}\, dV + \int_V \overline{\boldsymbol{u}}' \rho_o \frac{\partial^2 \boldsymbol{u}}{\partial t^2}\, dV - \int_V \overline{\boldsymbol{u}}' \boldsymbol{f}^B\, dV - \int_{S_f} \overline{\boldsymbol{u}}^{Sf\,\prime} \boldsymbol{f}^{S_f}\, dS_f = (0)$$

$$\int_V \overline{\boldsymbol{E}}' (\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J})\, dV + \int_{S_q} \overline{\phi}^{S_q\,\prime} J^{S_q}\, dS_q$$

$$\int_V \overline{\boldsymbol{D}}' (\boldsymbol{E} + \boldsymbol{\nabla}\phi)\, dV = 0$$

[2]This procedure is equivalent to a mixed variational equation derived by GHANDI [13]

[3]See BATHE [2, pp. 830-835]

4.5.3 Test Functions Defined

$$
\begin{aligned}
\overline{\boldsymbol{u}}(x_i,t) &= H_u(x_i)\hat{\overline{\boldsymbol{u}}}(t) \\
\overline{\boldsymbol{u}}^{S_f}(x_i,t) &= H_u^{S_f}(x_i)\hat{\overline{\boldsymbol{u}}}(t) \\
\overline{\boldsymbol{S}}(\overline{\boldsymbol{u}}(x_i,t)) &= B_u(x_i)\hat{\overline{\boldsymbol{u}}}(t) \\
\overline{\phi}(x_i,t) &= H_\phi(x_i)\hat{\overline{\boldsymbol{\phi}}}(t) \\
\overline{\phi}^{S_q}(x_i,t) &= H_\phi^{S_q}(x_i)\hat{\overline{\boldsymbol{\phi}}}(t) \\
\overline{\boldsymbol{E}}(\overline{\phi}(x_i,t)) &= -B_\phi(x_i)\hat{\overline{\boldsymbol{\phi}}}(t) \\
\overline{\boldsymbol{D}}(x_i,t) &= H_D(x_i)\hat{\overline{\boldsymbol{D}}}(t)
\end{aligned}
$$

4.5.4 Introducing Test Functions

$$
\hat{\overline{\boldsymbol{u}}}'\left\{\int_V B_u{}'\boldsymbol{T}\,dV + \int_V H_u{}'\rho_o\frac{\partial^2\boldsymbol{u}}{\partial t^2}\,dV - \int_V H_u{}'\boldsymbol{f}^B\,dV - \int_{S_f} H_u^{S_f}{}'\boldsymbol{f}^{S_f}\,dS_f\right\} = 0
$$

$$
\hat{\overline{\boldsymbol{\phi}}}'\left\{\int_V -B_\phi{}'(\frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{J})\,dV + \int_{S_q} H_\phi^{S_q}{}'J^{S_q}\,dS_q\right\} = 0
$$

$$
\hat{\overline{\boldsymbol{D}}}'\left\{\int_V H_D{}'(\boldsymbol{E} + \boldsymbol{\nabla}\phi)\,dV\right\} = 0
$$

We require the weak form to hold for all $\hat{\overline{\boldsymbol{u}}}(t), \hat{\overline{\boldsymbol{\phi}}}(t), \hat{\overline{\boldsymbol{D}}}(t)$. A necessary condition is that $\{\,\} = 0$,

$$
\begin{bmatrix} \int_V \rho_o H_u{}'\boldsymbol{u}_{,tt}\,dV \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \int_V B_\phi{}'\boldsymbol{D}_{,t}\,dV \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \int_V B_u{}'\boldsymbol{T}\,dV \\ \int_V B_\phi{}'\boldsymbol{J}\,dV \\ \int_V H_D{}'(\boldsymbol{E} + \boldsymbol{\nabla}\phi)\,dV \end{bmatrix}
$$

$$
- \begin{bmatrix} \int_V H_u{}'\boldsymbol{f}^B\,dV \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \int_{S_f} H_u^{S_f}{}'\boldsymbol{f}^{S_f}\,dS_f \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{0} \\ \int_{S_q} H_\phi^{S_q}{}'J^{S_q}\,dS_q \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{0}
$$

or

$$
\boldsymbol{FI}^m + \boldsymbol{FI}^c + \boldsymbol{FI}^k - \boldsymbol{FE}^b - \boldsymbol{FE}^{S_f} - \boldsymbol{FE}^{S_q} = \boldsymbol{0}
$$

4.5.5 Shape Functions Defined

$$
\begin{aligned}
\boldsymbol{u}(x_i,t) &= H_u(x_i)\hat{\boldsymbol{u}}(t) \\
\boldsymbol{S}(\boldsymbol{u}(x_i,t)) &= B_u(x_i)\hat{\boldsymbol{u}}(t) \\
\phi(x_i,t) &= H_\phi(x_i)\hat{\boldsymbol{\phi}}(t) \\
\boldsymbol{\nabla}\phi(\phi(x_i,t)) &= B_\phi(x_i)\hat{\boldsymbol{\phi}}(t) \\
\boldsymbol{D}(x_i,t) &= H_D(x_i)\hat{\boldsymbol{D}}(t)
\end{aligned}
$$

The shape functions have been chosen identical to the corresponding test functions (Galerkin's method).

4.5.6 Introducing Constitutive and Shape Functions

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{T}(\boldsymbol{S}(\boldsymbol{\xi}), \boldsymbol{D}(\boldsymbol{\xi})) \\ \boldsymbol{E} &= \boldsymbol{E}(\boldsymbol{S}(\boldsymbol{\xi}), \boldsymbol{D}(\boldsymbol{\xi})) \\ \boldsymbol{J} &= \boldsymbol{J}(\boldsymbol{E}) \end{aligned}$$

and using the chain rule

$$\begin{aligned} \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{\xi}} &= \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{S}}\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{D}}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\xi}} \\ &= \begin{bmatrix} C_{uu} \end{bmatrix}\begin{bmatrix} B_u & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_{ud} \end{bmatrix}\begin{bmatrix} 0 & 0 & H_d \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \frac{\partial \boldsymbol{J}}{\partial \boldsymbol{\xi}} &= \frac{\partial \boldsymbol{J}}{\partial \boldsymbol{\nabla}\phi}\frac{\partial \boldsymbol{\nabla}\phi}{\partial \boldsymbol{\xi}} \\ &= \begin{bmatrix} C_{q\phi} \end{bmatrix}\begin{bmatrix} 0 & B_\phi & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \frac{\partial(\boldsymbol{E} + \boldsymbol{\nabla}\phi)}{\partial \boldsymbol{\xi}} &= \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{S}}\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{D}}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{\nabla}\phi}{\partial \boldsymbol{\xi}} \\ &= \begin{bmatrix} C_{du} \end{bmatrix}\begin{bmatrix} B_u & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_{dd} \end{bmatrix}\begin{bmatrix} 0 & 0 & H_d \end{bmatrix} + \begin{bmatrix} 0 & B_\phi & 0 \end{bmatrix} \end{aligned}$$

or

$$\begin{aligned} \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{\xi}} &= \begin{bmatrix} C_{uu}B_u & 0 & C_{ud}H_d \end{bmatrix} \\ \frac{\partial \boldsymbol{J}}{\partial \boldsymbol{\xi}} &= \begin{bmatrix} 0 & C_{q\phi}B_\phi & 0 \end{bmatrix} \\ \frac{\partial(\boldsymbol{E} + \boldsymbol{\nabla}\phi)}{\partial \boldsymbol{\xi}} &= \begin{bmatrix} C_{du}B_u & B_\phi & C_{dd}H_d \end{bmatrix} \end{aligned}$$

4.5.7 Jacobian Matrices

$$\begin{aligned} \frac{\partial \boldsymbol{FI}^m}{\partial \ddot{\boldsymbol{\xi}}} &= \int_V \begin{bmatrix} \rho_o {H_u}' H_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} dV \\ &= \int_V \rho_o {H_m}' H_m \, dV = J^m \end{aligned}$$

$$\frac{\partial \boldsymbol{FI}^c}{\partial \dot{\boldsymbol{\xi}}} = \int_V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & {B_\phi}' H_d \\ 0 & 0 & 0 \end{bmatrix} dV$$

$$
\begin{aligned}
&= \int_V H_{c1}{}' H_{c2} \, dV = J^c \\
\frac{\partial \boldsymbol{FI}^k}{\partial \boldsymbol{\xi}} &= \int_V \begin{bmatrix} B_u{}' C_{uu} B_u & 0 & B_u{}' C_{ud} H_d \\ 0 & B_\phi{}' C_{q\phi} B_\phi & 0 \\ H_d{}' C_{du} B_u & H_d{}' B_\phi & H_d{}' C_{dd} H_d \end{bmatrix} dV \\
&= \int_V \begin{bmatrix} B_u & 0 & 0 \\ 0 & B_\phi & 0 \\ 0 & 0 & H_d \end{bmatrix}' \begin{bmatrix} C_{uu} & 0 & C_{ud} \\ 0 & C_{q\phi} & 0 \\ C_{du} & I & C_{dd} \end{bmatrix} \begin{bmatrix} B_u & 0 & 0 \\ 0 & B_\phi & 0 \\ 0 & 0 & H_d \end{bmatrix} dV \\
&= \int_V B' C B \, dV \\
&= J^k
\end{aligned}
$$

4.5.8 Define Loading Interpolation Functions

$$
\begin{aligned}
\boldsymbol{f}^B(x_i, t) &= H_{bf}(x_i) \boldsymbol{B_f}(t) \\
\boldsymbol{f}^{S_f}(x_i, t) &= H_{sf}(x_i) \boldsymbol{S_f}(t) \\
J^{S_q}(x_i, t) &= H_{sq}(x_i) \boldsymbol{S_q}(t)
\end{aligned}
$$

4.5.9 Residual Vectors

$$
\begin{aligned}
\boldsymbol{FI}^m &= \int_V \rho_o H_m{}' H_m \, dV \, \ddot{\boldsymbol{\xi}} = J^m \ddot{\boldsymbol{\xi}} \\
\boldsymbol{FI}^c &= \int_V H_{c1}{}' H_{c2} \, dV \, \dot{\boldsymbol{\xi}} = J^c \dot{\boldsymbol{\xi}} \\
\boldsymbol{FI}^k &= \int_V B' \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{J} \\ (\boldsymbol{E} + \boldsymbol{\nabla} \phi) \end{bmatrix} dV \\
\boldsymbol{FE}^b &= \int_V \begin{bmatrix} H_u{}' \\ 0 \\ 0 \end{bmatrix} \boldsymbol{f}^B \, dV = \int_V H' \boldsymbol{f}^B \, dV \\
&= \int_V H' H_{bf} \, dV \, \boldsymbol{B_f} = FE_{bf} \boldsymbol{B_f} \\
\boldsymbol{FE}^{S_f} &= \int_{S_f} \begin{bmatrix} H_u^{S_f}{}' \\ 0 \\ 0 \end{bmatrix} \boldsymbol{f}^{S_f} \, dS_f = \int_{S_f} H_{su}{}' \boldsymbol{f}^{S_f} \, dS_f \\
&= \int_{S_f} H_{su}{}' H_{sf} \, dS_f \, \boldsymbol{S_f} = FE_{sf} \boldsymbol{S_f}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{FE}^{S_q} &= \int_{S_q} \begin{bmatrix} 0 \\ H_{\phi}^{S_q}{}' \\ 0 \end{bmatrix} J^{S_q}\, dS_q = \int_{S_q} H_{s\phi}{}' J^{S_q}\, dS_q \\
&= \int_{S_q} H_{s\phi}{}' H_{sq}\, dS_q\, \boldsymbol{S_q} = FE_{sq} \boldsymbol{S_q}
\end{aligned}
$$

Chapter 5

Results

5.1 Introduction

In this chapter we present some classical examples of electric conduction in rigid and static bodies to establish significance of the charge relaxation time constant for linear materials. Analysis results are then presented under a typical loading condition where electric conduction in a highly insulating electromechanical device will eventually dominate the device response. Specifically, we present results for electric voltage DC offset loading of an active fiber composite (AFC) device using a nonlinear material model for repolarizable piezoelectric ceramic. DC offset loading is common in applications to maximize the effective linear range of device operation.

5.2 Charge Relaxation and Time Scales

Example 5.2.1 (Rigid and Static Body) We consider a rigid $\epsilon_{ij} = 0$ and static $v_i = 0$ body. Then the governing equation is

$$\left(\dot{D}_i + J_i\right)_{,i} = 0. \tag{5.1}$$

Assuming a steady state response exists, then at steady state,

$$\frac{d}{dt} = 0 \rightarrow J_{i,i} = 0 \tag{5.2}$$

the response of the body is completely dominated by electrical conduction.

Example 5.2.2 (Rigid and Static Body, Uniform and Linear Material) We consider a rigid $\epsilon_{ij} = 0$ and static $v_i = 0$ body. Then the governing differential equations are

$$D_{i,i} = q^F \qquad J_{i,i} + \frac{\partial q^F}{\partial t} = 0. \tag{5.3}$$

Additionally, consider the body composed of a homogeneous and linear material,

$$\begin{aligned} J_i(x_i,t) &= \sigma E_i(x_i,t) && \sigma \in \mathbb{R}_{>0} \\ D_i(x_i,t) &= \epsilon E_i(x_i,t) && \epsilon \in \mathbb{R}_{>0}. \end{aligned} \tag{5.4}$$

Differentiating these constitutive relations,

$$J_{i,i} = \sigma E_{i,i} \qquad E_{i,i} = \frac{1}{\epsilon} D_{i,i} \qquad \rightarrow J_{i,i} = \frac{\sigma}{\epsilon} D_{i,i} \tag{5.5}$$

Using (5.5) and (5.3) gives a first order differential equation for free charge density $q^F(x_i,t)$

$$\frac{\partial q^F}{\partial t} + \left(\frac{\sigma}{\epsilon}\right) q^F = 0 \qquad q^F(x_i,0) = q_o^F(x_i)$$

This differential equation has solution,

$$q^F(x_i,t) = q_o^F(x_i)\, e^{-t/\tau} \qquad \tau = \frac{\epsilon}{\sigma} \tag{5.6}$$

Therefore charge relaxes exponentially with time constant $\tau = \epsilon/\sigma$.

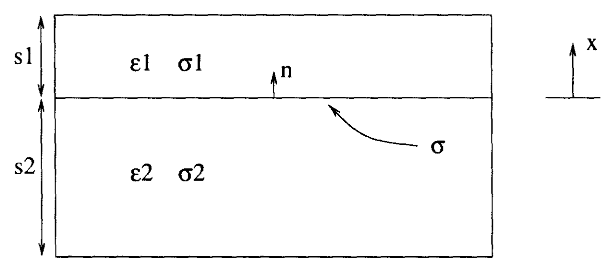

Figure 5-1: Two layer body geometry: Example 5.2.3.

Example 5.2.3 (Rigid and Static Body, Piecewise Uniform and Linear Material) We consider a rigid $\epsilon_{ij} = 0$ and static $v_i = 0$ body. Then the governing differential equations and jump conditions are

$$\begin{aligned} & & D_{i,i} = q^F \qquad & E_i + \phi_{,i} = 0 \\ [\![D_i]\!] n_i = w^F \qquad & & [\![\dot{D}_i + J_i]\!] n_i = 0 \qquad & [\![\phi]\!] = 0. \end{aligned} \tag{5.7}$$

Consider the electric voltage boundary conditions,

$$\phi^1(s_1,t) = 0 \qquad \phi^2(-s_2,t) = V(t). \tag{5.8}$$

The body geometry and loading conditions are consistent with a one dimensional spatial fields approximation,

$$D_i(x_i,t) \rightarrow D(x,t)$$

$$\begin{aligned} E_i(x_i, t) &\to E(x, t) \\ \phi(x_i, t) &\to \phi(x, t) \end{aligned} \tag{5.9}$$

The body is composed of two layers of uniform and linear material, distinguished by superscripts,

$$\begin{aligned} D^k(x, t) &= \epsilon^k E^k(x, t) \qquad && \epsilon^k \in \mathbb{R}_{>0} \\ J^k(x, t) &= \sigma^k E^k(x, t) \qquad && \sigma^k \in \mathbb{R}_{>0} \end{aligned} \tag{5.10}$$

Consider the initial condition on free charge density $q^F(x, t) = 0$. Then, from example 5.2.2 we see that $q^F(x, t) = 0$ for all time $t \in \mathbb{R}_{>0}$. Then in each uniform region $k \in \{1, 2\}$ we have

$$\frac{\partial D^k(x, t)}{\partial x} = 0. \tag{5.11}$$

Therefore the fields D^k and E^k are spatially uniform

$$D^k(x, t) \to D^k(t) \qquad E^k(x, t) \to E^k(t) \qquad J^k(x, t) \to J^k(t)$$

Imposing the boundary conditions

$$\phi^1(s_1, t) = 0 \qquad \phi^2(-s_2, t) = V(t) \tag{5.12}$$

and integrating the electric field/potential equation through the thickness

$$\begin{aligned} \int_{-s_2}^{0} E^2(t)\, dx + \int_0^{s_1} E^1(t)\, dx &= -\int_{-s_2}^{0} \frac{\partial \phi^2(x, t)}{\partial x}\, dx - \int_0^{s_1} \frac{\partial \phi^1(x, t)}{\partial x}\, dx \\ &= -\phi^1(s_1, t) + \phi^2(-s_2, t) = V(t) \end{aligned} \tag{5.13}$$

gives

$$E^1(t)^1 + E^2(t) s_2 = V(t). \tag{5.14}$$

Using the jump condition across σ_o,

$$\dot{D}^1 + J^1 = \dot{D}^2 + J^2 \to \epsilon^1 \dot{E}^1 + \sigma^1 E^1 = \epsilon^2 \dot{E}^2 + \sigma^2 E^2 \tag{5.15}$$

Using (5.14) in (5.15) yields the following differential equations for E^1 and E^2,

$$\begin{aligned} \left(\epsilon^2 s_1 + \epsilon^1 s_2\right) \dot{E}^1(t) + \left(\sigma^2 s_1 + \sigma^1 s_2\right) E^1(t) &= \frac{\epsilon^2}{s_2} \dot{V}(t) + \frac{\sigma^2}{s_2} V(t) \\ \left(\epsilon^2 s_1 + \epsilon^1 s_2\right) \dot{E}^2(t) + \left(\sigma^2 s_1 + \sigma^1 s_2\right) E^2(t) &= \frac{\epsilon^1}{s_1} \dot{V}(t) + \frac{\sigma^1}{s_1} V(t) \end{aligned}$$

Defining the following constants,

$$\tau = \frac{\epsilon^2 s_1 + \epsilon^1 s_2}{\sigma^2 s_1 + \sigma^1 s_2} \qquad \begin{aligned} \alpha^1 &= \frac{\epsilon^2}{s_2\left(\epsilon^2 s_1 + \epsilon^1 s_2\right)} \\ \alpha^2 &= \frac{\epsilon^1}{s_1\left(\epsilon^2 s_1 + \epsilon^1 s_2\right)} \end{aligned} \qquad \begin{aligned} \beta^1 &= \frac{\sigma^2}{s_2\left(\epsilon^2 s_1 + \epsilon^1 s_2\right)} \\ \beta^2 &= \frac{\sigma^1}{s_1\left(\epsilon^2 s_1 + \epsilon^1 s_2\right)} \end{aligned}$$

we obtain

$$\begin{aligned} \dot{E}^1(t) + \frac{1}{\tau} E^1(t) &= \alpha^1 \dot{V}(t) + \beta^1 V(t) \\ \dot{E}^2(t) + \frac{1}{\tau} E^2(t) &= \alpha^2 \dot{V}(t) + \beta^2 V(t). \end{aligned}$$

Consider response to a step input $V(t) = \hat{V}$. Then at time $t = 0^+$ we have $w^F = 0$. Using the jump condition across σ_o

$$[\![\, D_i \,]\!]\, n_i = 0 \rightarrow \epsilon^1 E^1(0^+) = \epsilon^2 E^2(0^+) \tag{5.16}$$

and (5.14) gives initial conditions at $t = 0^+$,

$$E^1(0^+) = \frac{\epsilon^2 \hat{V}}{\epsilon^2 s_1 + \epsilon^1 s_2} \qquad E^2(0^+) = \frac{\epsilon^1 \hat{V}}{\epsilon^2 s_1 + \epsilon^1 s_2} \tag{5.17}$$

The step response of the two layer body is

$$\begin{aligned} E^1(t) &= \frac{\sigma^2 \hat{V}}{\sigma^2 s_1 + \sigma^1 s_2}\left(1 - e^{-t/\tau}\right) + \frac{\epsilon^2 \hat{V}}{\epsilon^2 s_1 + \epsilon^1 s_2} e^{-t/\tau} \\ E^2(t) &= \frac{\sigma^1 \hat{V}}{\sigma^2 s_1 + \sigma^1 s_2}\left(1 - e^{-t/\tau}\right) + \frac{\epsilon^1 \hat{V}}{\epsilon^2 s_1 + \epsilon^1 s_2} e^{-t/\tau} \end{aligned} \tag{5.18}$$

For t/τ small the electrical response is that of a perfectly insulating material. For t/τ large the electrical response is dominated by electrical conduction. Most importantly, the electrical response of the body is not determined by the electrical conductivity alone, but rather the geometry weighted ratio of electrical conductivities to permittivities. It can be shown that the electrical time constant for the system is bounded by the time constants for the individual materials, $\tau^1 \leq \tau \leq \tau^2$ [1].

[1] Form the ratios τ/τ^1 and τ/τ^2

Remark 5.2.4 (Time Constants of Engineering Materials) *The effect of electric conduction on the electrical response for linear materials is determined not by electrical conductivity alone, but the ratio of permittivity to conductivity. It is useful to look at order of magnitude values for some common engineering materials* [2].

Table 5.1: Charge relaxation time constants.

Material	σ *[ohm-m]*$^{-1}$	ϵ/ϵ_o	τ *[sec]*
Copper	*5.7* 10^{7}	*1.0*	*1.6* 10^{-19}
Seawater	*4.0* 10^{0}	*80*	*1.8* 10^{-10}
Water	*4.0* 10^{-6}	*80*	*1.8* 10^{-4}
Epoxy, $23°C$	*2.6* 10^{-13}	*4.4*	*150*
Epoxy, $100°C$	*2.8* 10^{-12}	*4.4*	*14*
PZT-5H, $23°C$	*3.4* 10^{-12}	*3400*	*8850*
PZT-5H, $100°C$	*8.2* 10^{-12}	*3400*	*3670*
Corn oil	*4.0* 10^{-11}	*2.7*	*.60*
Glass	*1.0* 10^{-12}	*8.9*	*79*
Teflon	*1.0* 10^{-16}	*5.0*	*4.4* 10^{5}
Quartz	*1.0* 10^{-17}	*4.0*	*3.5* 10^{6}

5.3 Example: Active Fiber Composite

This section presents an analysis of electric voltage DC offset loading of a highly electrically insulating actuator/sensor device. The device is composed of a piezoelectric ceramic fiber embedded in an epoxy matrix, an active fiber composite or AFC device[3]. The analysis considers a nonlinear material model for repolarizable piezoelectric ceramic due to GHANDI [13] and described in the next section. Results demonstrate significant cumulative effects of weak electric conduction currents.

5.3.1 Material Model: Polarizable Piezoelectric

The section describes a nonlinear material model for polarizable piezoelectric ceramics developed by GHANDI [13]. Our first step is to derive constitutive equations with appropriate independent variables from the energy balance and entropy equations. This is accomplished using the balance of energy and entropy inequality equations to form the Clausius-Duhem (C-D) inequality.

$$\dot{U} = \tau_{ji}^{T}\dot{\epsilon}_{ij} + E_k\dot{D}_k + E_kJ_k + \rho_o h - q_{k,k}$$

[2] The material parameters are from HAUS & MELCHER [16, p. 222,251]. The values for epoxy and PZT-5H were measured using a bridge circuit, see appendix C

[3] See BENT [4, 5], RODGERS [22], and BENT, RODGERS, & HAGOOD [6]

$$\Theta\rho_o\dot{\eta} + q_{j,j} - \rho_o h - q_j\Theta^{-1}\Theta_{,j} \geq 0. \tag{5.19}$$

We form the C-D inequality by eliminating $q_{j,j} - \rho_o h$,

$$\Theta\rho_o\dot{\eta} - \dot{U} + \tau^T_{ji}\dot{\epsilon}_{ij} + E_k\dot{D}_k + E_kJ_k - q_j\Theta^{-1}\Theta_{,j} \geq 0. \tag{5.20}$$

Introduce the Legendre transform $G = U - \rho_o\Theta\eta - \tau^T_{ji}\epsilon_{ij}$, then

$$-\dot{U} = -\dot{G} - \rho_o\dot{\Theta}\eta - \rho_o\Theta\dot{\eta} - \dot{\tau}^T_{ji}\epsilon_{ij} - \tau^T_{ji}\dot{\epsilon}_{ij}. \tag{5.21}$$

The C-D inequality becomes

$$-\dot{G} - \rho_o\dot{\Theta}\eta - \dot{\tau}^T_{ji}\epsilon_{ij} + E_k\dot{D}_k + E_kJ_k - q_j\Theta^{-1}\Theta_{,j} \geq 0. \tag{5.22}$$

Introducing the isothermal approximation $\Theta(x_j, t) = \text{constant}$,

$$-\dot{G} - \dot{\tau}^T_{ji}\epsilon_{ij} + E_k\dot{D}_k + E_kJ_k \geq 0. \tag{5.23}$$

Next we define G as

$$\begin{aligned} G &= \hat{G}\left(\tau^T_{ji}, D_k\right) \\ -\dot{G} &= -\frac{\partial \hat{G}}{\partial D_k}\dot{D}_k - \frac{\partial \hat{G}}{\partial \tau^T_{ji}}\dot{\tau}^T_{ji}. \end{aligned} \tag{5.24}$$

Introducing into the C-D inequality,

$$\left(E_k - \frac{\partial \hat{G}}{\partial D_k}\right)\dot{D}_k - \left(\frac{\partial \hat{G}}{\partial \tau^T_{ji}}\right)\dot{\tau}^T_{ji} + E_kJ_k \geq 0. \tag{5.25}$$

This expression will hold for all independent processes $D_k(x_j, t)$ and $\tau^T_{ji}(x_k, t)$ if and only if,

$$E_k = \frac{\partial \hat{G}}{\partial D_k} \qquad \epsilon_{ij} = -\frac{\partial \hat{G}}{\partial \tau^T_{ji}} \qquad E_kJ_k \geq 0. \tag{5.26}$$

We write,

$$E_k = \hat{E}_k\left(\tau^T_{ji}, D_k\right) \qquad \epsilon_{ij} = \hat{\epsilon}_{ij}\left(\tau^T_{ji}, D_k\right) \qquad J_k = \sigma E_k \qquad \sigma \in \mathbb{R}_{\geq 0}, \tag{5.27}$$

where we have assumed for simplicity a linear and isotropic electrical conduction law that clearly satisfies the required inequality. The material response functions are then completely determined by the scalar function $G = \hat{G}\left(D_k, \tau^T_{ji}\right)$.

GHANDI introduces a memory vector[4] $D^*_i(D_k)$ that by definition does not evolve thermodynamically, such that $\dot{D}^*_i = 0$. This vector is defined to evolve according to

[4]The vector D^*_i is not to be confused, in this small deformation context, with the convective time derivative

an evolution rule, but thermodynamically $D^*(D_k)$ is a constant. Then G is defined by,

$$G = \hat{G}\left(\tau^T_{ji}, D_k, D^*_i(D_k)\right)\Big|_{D^*_i} \tag{5.28}$$

where it is indicated that thermodynamically, D^*_i is held constant. Then we have,

$$\begin{aligned} E_k &= \hat{E}_k\left(\tau^T_{ji}, D_k, D^*_k(D_j)\right) = \frac{\partial}{\partial D_k}\left(\hat{G}\Big|_{D^*_i}\right) \\ \epsilon_{ij} &= \hat{\epsilon}_{ij}\left(\tau^T_{ji}, D_k, D^*_k(D_j)\right) = -\frac{\partial}{\partial \tau^T_{ji}}\left(\hat{G}\Big|_{D^*_i}\right) \\ J_k &= \sigma E_k \qquad \sigma \in \mathbb{R}_{\geq 0}, \end{aligned} \tag{5.29}$$

Symmetry of ϵ_{ij} and τ^T_{ji} permit introduction of Voight notation,

$$\begin{array}{ll} (11) \to 1 & (32) = (23) \to 4 \\ (22) \to 2 & (31) = (13) \to 5 \\ (33) \to 3 & (12) = (21) \to 6 \\ \epsilon_{ij} \to S_k & \tau^T_{ji} \to T_k \end{array}$$

where $k \in \{1, \cdots, 6\}$. Redefining $\hat{E}_k, \hat{S}_k$ consistent with $\hat{E}_k, \hat{\epsilon}_{ij}$ we finally obtain

$$\begin{aligned} E_k &= \hat{E}_k\left(T_k, D_k, D^*_k(D_j)\right) \\ S_k &= \hat{S}_k\left(T_k, D_k, D^*_k(D_j)\right). \end{aligned} \tag{5.30}$$

At this point we will drop the component notation and adopt vector notation. Then,

$$\begin{aligned} \boldsymbol{E} &= \hat{\boldsymbol{E}}\left(\boldsymbol{T}, \boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right) \\ \boldsymbol{S} &= \hat{\boldsymbol{S}}\left(\boldsymbol{T}, \boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right) \\ &= \boldsymbol{S}^\alpha\left(\boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right) + \boldsymbol{S}^\beta\left(\boldsymbol{T}, \boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right) \\ &= \boldsymbol{S}^\alpha\left(\boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right) + \frac{\partial \boldsymbol{S}^\beta}{\partial \boldsymbol{T}}\left(\boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right)\boldsymbol{T} \\ \boldsymbol{J} &= \sigma \boldsymbol{E} \qquad \sigma \in \mathbb{R}_{\geq 0} \end{aligned} \tag{5.31}$$

emphasizing the linear dependence of $\boldsymbol{S}^\beta$ on $\boldsymbol{T}$. Solving for $\boldsymbol{T}$,

$$\boldsymbol{T} = \hat{\boldsymbol{T}}\left(\boldsymbol{S}, \boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right) = \left[\frac{\partial \boldsymbol{S}^\beta}{\partial \boldsymbol{T}}\left(\boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right)\right]^{-1}\left(\boldsymbol{S} - \boldsymbol{S}^\alpha\left(\boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right)\right) \tag{5.32}$$

The following matrices will be required in the finite element implementation,

$$C_{uu} = \left.\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{S}}\right|_{\boldsymbol{D}} \qquad C_{ud} = \left.\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{S}} \qquad C_{q\phi} = \frac{\partial \boldsymbol{J}}{\partial \boldsymbol{\nabla}\phi} = -\frac{\partial \boldsymbol{J}}{\partial \boldsymbol{E}}$$

$$C_{du} = \left.\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{S}}\right|_{\boldsymbol{D}} \qquad C_{dd} = \left.\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{S}} \tag{5.33}$$

Remark 5.3.1 (Material Model: Jacobian Matrices)

$$\begin{aligned}
C_{uu} &= \left[\frac{\partial \boldsymbol{S}^{\beta}}{\partial \boldsymbol{T}}\right]^{-1} \\
C_{ud} &= -C_{uu} \left.\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T}} \\
C_{du} &= \left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{T}}\right|_{\boldsymbol{D}} C_{uu} \\
C_{dd} &= \left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{T}}\right|_{\boldsymbol{D}} C_{ud} + \left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T}} \\
C_{q\phi} &= -\sigma \boldsymbol{I}
\end{aligned} \tag{5.34}$$

where,

$$\begin{aligned}
\left.\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T}} &= \left.\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T},\boldsymbol{D}^*} + \left.\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{D}^*}\right|_{\boldsymbol{T},\boldsymbol{D}} \frac{\partial \boldsymbol{D}^*}{\partial \boldsymbol{D}} \\
\left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T}} &= \left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T},\boldsymbol{D}^*} + \left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{D}^*}\right|_{\boldsymbol{T},\boldsymbol{D}} \frac{\partial \boldsymbol{D}^*}{\partial \boldsymbol{D}}
\end{aligned} \tag{5.35}$$

Proof. Calculation of C_{uu}, C_{du}, and $C_{q\phi}$ are immediate. Consider C_{ud},

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{D}}\left[\left.\left(\boldsymbol{S} = \boldsymbol{S}^{\alpha}(\boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})) + \frac{\partial \boldsymbol{S}^{\beta}}{\partial \boldsymbol{T}}(\boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D}))\,\boldsymbol{T}\right)\right|_{\boldsymbol{S}}\right] \\
\rightarrow \quad & 0 = \frac{\partial \boldsymbol{S}^{\alpha}}{\partial \boldsymbol{D}}(\boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})) + \frac{\partial}{\partial \boldsymbol{D}}\left(\frac{\partial \boldsymbol{S}^{\beta}}{\partial \boldsymbol{T}}(\boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D}))\right)\boldsymbol{T} \\
\rightarrow \quad & 0 = \left.\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T}} + \frac{\partial \boldsymbol{S}^{\beta}}{\partial \boldsymbol{T}} \left.\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{S}} \\
\rightarrow \quad & \left.\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{S}} = -C_{uu} \left.\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T}},
\end{aligned} \tag{5.36}$$

proves the C_{ud} relation. Next consider C_{dd},

$$\begin{aligned}
\left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{S}} &= \left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{T}}\right|_{\boldsymbol{D}} \left.\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{S}} + \left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T}} \\
&= \left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{T}}\right|_{\boldsymbol{D}} C_{ud} + \left.\frac{\partial \hat{\boldsymbol{E}}}{\partial \boldsymbol{D}}\right|_{\boldsymbol{T}},
\end{aligned} \tag{5.37}$$

proves the C_{dd} relation. ■

Definition 5.3.2 (Material Model: Memory Variable Evolution Rule)

$$
\begin{aligned}
\overline{\boldsymbol{\Delta}} &= {}^{t+\Delta t}\boldsymbol{D} - {}^{t}\boldsymbol{D}^* \\
\left\|\overline{\boldsymbol{\Delta}}\right\| &= \left(\sum_{k=1}^{3} \overline{\Delta}_k^2\right)^{\frac{1}{2}}
\end{aligned}
$$

if $\left\|\overline{\boldsymbol{\Delta}}\right\| < \Delta_{cr}$ *then*

$$
\begin{aligned}
{}^{t+\Delta t}\boldsymbol{D}^* &= {}^{t}\boldsymbol{D}^* \\
\frac{\partial({}^{t+\Delta t}D_i^*)}{\partial D_j} &= 0
\end{aligned}
$$

if $\left\|\overline{\boldsymbol{\Delta}}\right\| \geq \Delta_{cr}$ *then*

$$
\begin{aligned}
{}^{t+\Delta t}\boldsymbol{D}^* &= {}^{t}\boldsymbol{D}^* + \overline{\boldsymbol{\Delta}}\left(1 - \Delta_{cr}\left\|\overline{\boldsymbol{\Delta}}\right\|^{-1}\right) \\
\frac{\partial({}^{t+\Delta t}D_i^*)}{\partial D_j} &= \delta_{ij}\left(1 - \Delta_{cr}\left\|\overline{\boldsymbol{\Delta}}\right\|^{-1}\right) + \Delta_{cr}\overline{\Delta}_i\overline{\Delta}_j\left\|\overline{\boldsymbol{\Delta}}\right\|^{-3}
\end{aligned}
$$

Definition 5.3.3 (Material Model: Response Function)

$$
\begin{aligned}
G &= \hat{G}\left(\tau_{ji}^T, D_k, D_i^*(D_k)\right)\Big|_{D_i^*} \\
\hat{G} &= \sum_{k=1}^{27} \alpha_k G_k\left(\boldsymbol{T}, \boldsymbol{D}, \boldsymbol{D}^*(\boldsymbol{D})\right) \\
E_k &= \hat{E}_k\left(\tau_{ji}^T, D_k, D_k^*(D_j)\right) = \frac{\partial}{\partial D_k}\left(\hat{G}\Big|_{D_i^*}\right) \\
\epsilon_{ij} &= \hat{\epsilon}_{ij}\left(\tau_{ji}^T, D_k, D_k^*(D_j)\right) = -\frac{\partial}{\partial \tau_{ji}^T}\left(\hat{G}\Big|_{D_i^*}\right)
\end{aligned} \tag{5.38}
$$

The α_k are real constants and the G_k are polynomial functions of the following set of tensor invariants of the set $\{\tau_{ji}^T, D_i, D_i^*\}$ and for convenience we write $a_i = D_i^*$.

$$
\begin{aligned}
I_1 &= a_i a_i \\
I_2 &= a_i D_i \\
I_3 &= D_i D_i \\
J_0 &= \tau_{ii}^T \\
J_1 &= \tau_{ij}^T a_i a_j \\
J_2 &= \tau_{ij}^T (a_i D_j + a_j D_i) \\
J_3 &= \tau_{ij}^T D_i D_j \\
K_0 &= \tau_{ij}^T \tau_{ji}^T \\
K_1 &= \tau_{ik}^T \tau_{kj}^T a_i a_j \\
K_2 &= \tau_{ik}^T \tau_{kj}^T (a_i D_j + a_j D_i) \\
K_3 &= \tau_{ik}^T \tau_{kj}^T D_i D_j
\end{aligned}
$$

The following coefficients produce stresses τ_{ji}^{T} in $[Pa]*10^6$ and electric displacements D_k, D_k^* in $[C/m^2]$.

$$
\begin{aligned}
\overline{\Delta} &= 0.02 & & \\
G_1 &= I_2 & \alpha_1 &= -2.781*10^7 \\
G_2 &= I_3 & \alpha_2 &= 1.410*10^7 \\
G_3 &= I_3I_3 & \alpha_3 &= 4.918*10^6 \\
G_4 &= J_0I_1 & \alpha_4 &= 1.540*10^5 \\
G_5 &= J_0I_2 & \alpha_5 &= -3.220*10^5 \\
G_6 &= J_0I_3 & \alpha_6 &= 1.824*10^5 \\
G_7 &= J_0I_3I_3 & \alpha_7 &= -3.299*10^3 \\
G_8 &= J_1 & \alpha_8 &= -3.048*10^5 \\
G_9 &= J_1I_2 & \alpha_9 &= 7.927*10^5 \\
G_{10} &= J_2 & \alpha_{10} &= 3.286*10^5 \\
G_{11} &= J_3 & \alpha_{11} &= -3.990*10^5 \\
G_{12} &= J_3I_2 & \alpha_{12} &= -2.221*10^6 \\
G_{13} &= J_3I_3 & \alpha_{13} &= 1.477*10^6 \\
G_{14} &= J_0J_0 & \alpha_{14} &= 9.135*10^0 \\
G_{15} &= J_0J_0I_1 & \alpha_{15} &= 1.691*10^3 \\
G_{16} &= J_0J_0I_2 & \alpha_{16} &= -3.088*10^3 \\
G_{17} &= J_0J_0I_3 & \alpha_{17} &= 1.349*10^3 \\
G_{18} &= J_0J_1 & \alpha_{18} &= -5.336*10^3 \\
G_{19} &= J_0J_2 & \alpha_{19} &= 4.965*10^3 \\
G_{20} &= J_0J_3 & \alpha_{20} &= -4.676*10^3 \\
G_{21} &= K_0 & \alpha_{21} &= -3.157*10^1 \\
G_{22} &= K_0I_1 & \alpha_{22} &= 1.235*10^3 \\
G_{23} &= K_0I_2 & \alpha_{23} &= -3.316*10^3 \\
G_{24} &= K_0I_3 & \alpha_{24} &= 2.270*10^3 \\
G_{25} &= K_1 & \alpha_{25} &= -5.070*10^2 \\
G_{26} &= K_2 & \alpha_{26} &= 1.490*10^3 \\
G_{27} &= K_3 & \alpha_{27} &= -2.510*10^3
\end{aligned}
\tag{5.39}
$$

5.3.2 Analysis Results

In this section we present analysis results demonstrating the transient response of a highly electrically insulating active fiber composite (AFC) device under an electric voltage DC offset loading. Figures 5-4 - 5-10 demonstrate electrical charging response

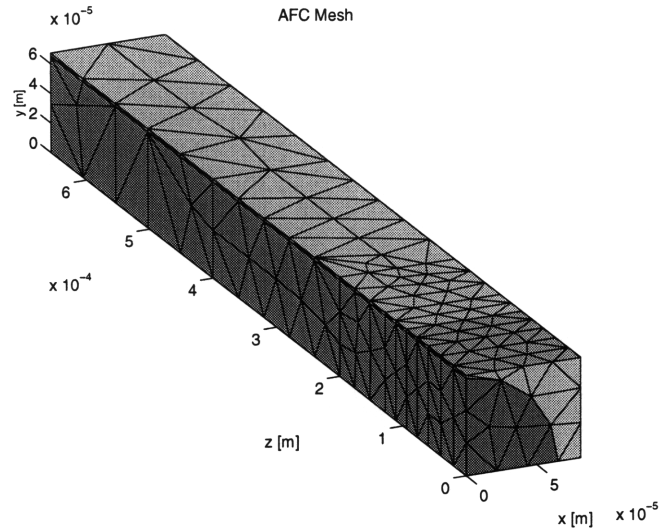

Figure 5-2: Active fiber composite: finite element mesh.

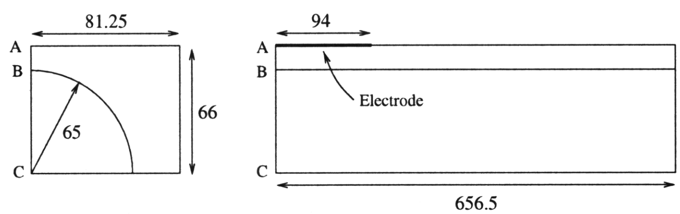

Figure 5-3: Active fiber composite dimensions, in $[m] * 10^{-6}$.

of the AFC device under an electric voltage DC offset loading. This loading condition is used frequently in applications to maximize the linear range of operation. Consider the following loading and boundary conditions.

Definition 5.3.4 (Loading and Boundary Conditions) *The components u_1, u_2, u_3 correspond to mechanical displacement u_i along the coordinate axes x, y, z, respectively. Dimensions for x, y, z are specified in $[m] * 10^{-6}$. Boundary conditions imposed are[5],*

$$
\begin{aligned}
x = 0 &\ :\ u_1 = 0 \\
y = 0 &\ :\ u_2 = 0 \\
z = 0 &\ :\ u_3 = 0 \\
z = 656.5 &\ :\ \phi = 0 \\
y = 66,\ 0 \leq z \leq 94 &\ :\ \phi = \phi^+(t) = \begin{cases} 1050\ [V] & 0^+ \leq t \leq 1200\ [s] \\ 0\ [V] & 1200 < t \leq 1500\ [s] \end{cases}
\end{aligned}
$$

All surfaces not specified have $\left(t_j + t_j^E\right)^+ = 0$ and $\left(\dot{D}_i + J_i\right)^+ = 0$. No mechanical

[5] *These boundary conditions are not symmetry conditions for an AFC device. Symmetry conditions were not enforced due to numerical difficulties.*

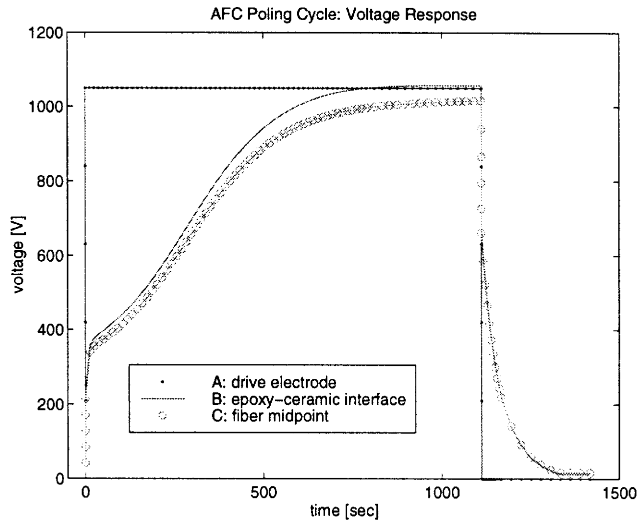

Figure 5-4: Voltage response through thickness.

body forces or inertial terms are included in the analysis. Isothermal conditions are assumed. Electrical conduction values for epoxy and PZT-5H at 100°*C from table 5.1 are used.*

These results demonstrate that electric and stress fields vary considerably over time as a result of weak electric current flow within the material. The results demonstrate a transition between capacitance dominated response during the initial seconds of voltage loading, to an electric conduction dominated response. Over the order of minutes the electric field levels change by a factor of four due to the cumulative effect of weak electric current flow.

The figures demonstrate three interesting points. Weak electric current flow in the highly insulating materials is not negligible, as made clear by the transient response of the device[6]. After the electric field is unloaded to zero, non-zero mechanical displacements, strains, and stresses remain. This is a result of the nonlinear polarization model for the piezoelectric ceramic. Another important feature is the difference in time scales between the initial loading and unloading of electric voltage. Inspection of the figures indicate a charging transient of 1000 seconds for the initial voltage loading. However, the charging transient for voltage unloading is 400 seconds. During the initial voltage loading, the piezoelectric ceramic is repoling, and the effective permittivity is much large than during the voltage unloading. The result of a larger permittivity is a longer effective (linear response) time constant for charge relaxation. This example has demonstrated that a highly electrically insulating body can be dominated by electric conduction under slow time electrical loadings.

[6] A perfectly insulating device would maintain the response at time 0^+ under a constant electric voltage DC offset (not including mechanical inertia effects)

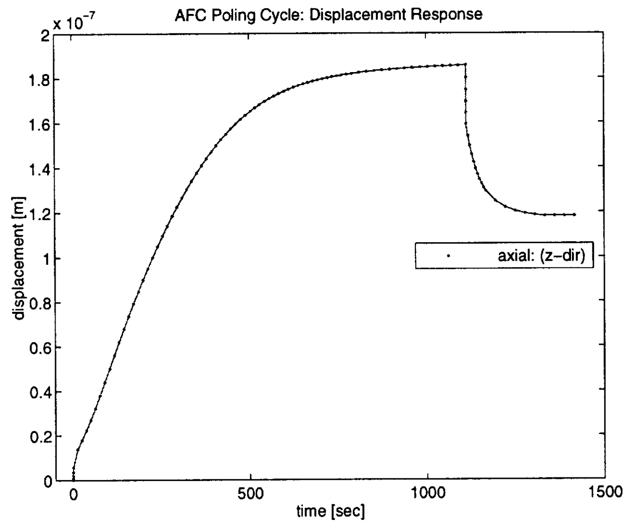

Figure 5-5: Axial end face displacement.

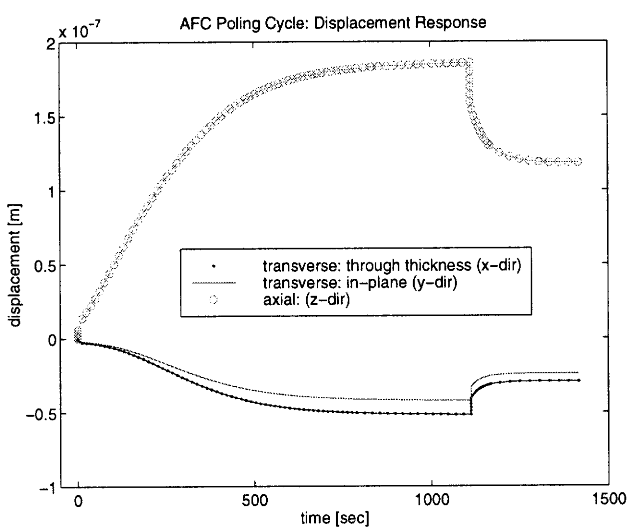

Figure 5-6: Transverse centerline displacements.

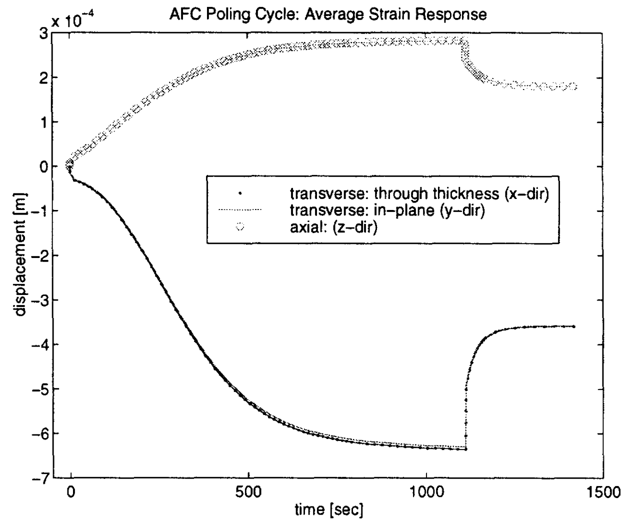

Figure 5-7: Average axial and transverse strains.

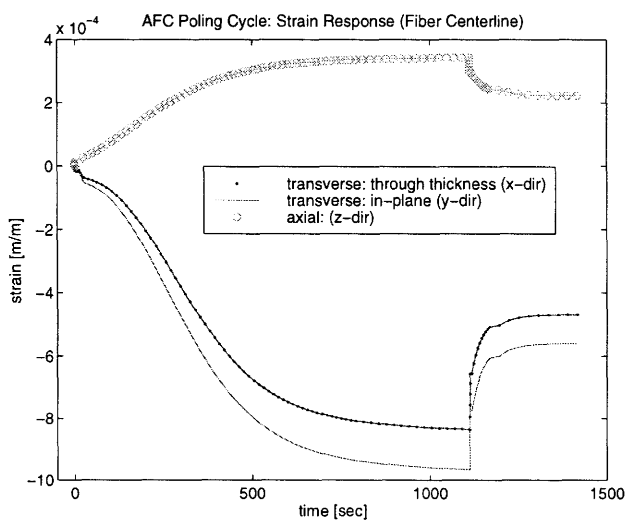

Figure 5-8: Fiber centerline strains.

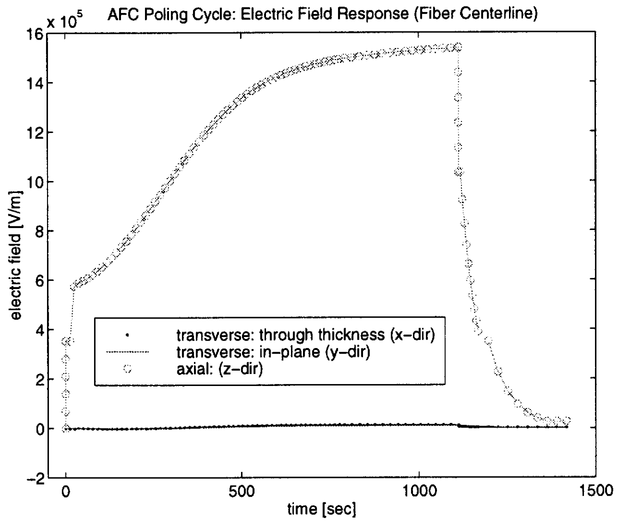

Figure 5-9: Fiber centerline electric field response.

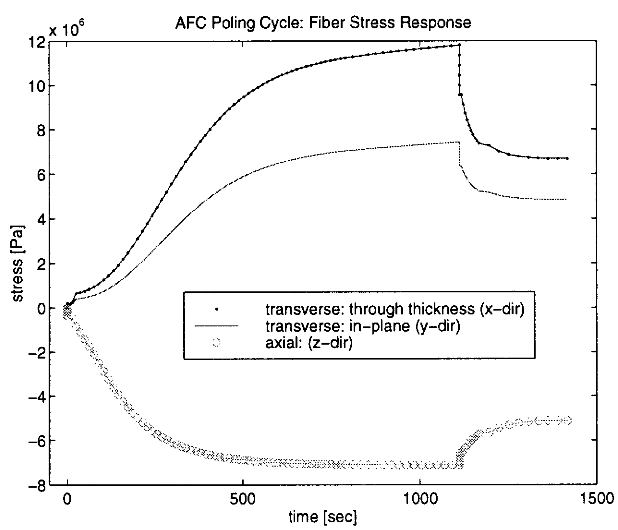

Figure 5-10: Fiber centerline stress response.

Chapter 6

Conclusions

This thesis has reported on the mathematical abstraction of deformable electromechanical actuator and sensor devices composed of highly electrically insulating materials. The presentation included detailed proofs of the nonlinear large deformation theory of electroelastic continua with electric conduction.

Consistency was proven between the large deformation theory and the classical Poynting vector based piezoelectric small deformation theory, extended for electric conduction. A result was that electric body forces, realized mathematically as electric surface tractions, are retained in the small deformation approximation.

A finite element formulation, suitable as an engineering analysis tool, was developed for deformable electromechanical actuator and sensor devices composed of highly insulating materials with nonlinear response functions (e.g., repolarizable piezoelectric ceramic material). The finite element formulation was demonstrated by analyzing the loading response of a highly electrically insulating active fiber composite device. Results demonstrated significant cumulative effects of a weak electric current flow under electric voltage DC offset loading.

Appendix A

Integral Theorems

Theorem A.0.5 (Green-Gauss) *A field* $\boldsymbol{A}$ *defined over a material volume* V *in* $\mathcal{B}$ *and bounded by surface* S*, satisfies the integral statement,*

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{A}\, dV = \oint_S \boldsymbol{A} \cdot \boldsymbol{n}\, dS \tag{A.1}$$

See ERINGEN [10, p. 427].

Theorem A.0.6 (Stokes) *A field* $\boldsymbol{A}$ *defined over an open material surface* S *in* $\mathcal{B}$ *and bounded by the line* C*, satisfies the integral statement,*

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \boldsymbol{n}\, dS = \oint_l \boldsymbol{A} \cdot d\boldsymbol{l} \tag{A.2}$$

See ERINGEN [10, p. 427].

Appendix B

Finite Element Formulations

B.1 Weak Forms: No Electric Conduction

The equations and finite element formulations presented in this appendix are consistent with notation presented in chapter 4. The equations presented are for small deformation electroelastic continua with no electric conduction.

Theorem B.1.1 (EQS Maxwell w/o Electric Conduction: Weak Form I) *A material volume V bounded by surface $S = S_q + S_\phi$ is subject to surface electric charge density loading q^{S_q} on S_q and electric voltage constraints ϕ^{S_ϕ} on S_ϕ. The electric displacement fields $\boldsymbol{D}$ are assumed negligible outside the material volume V. The EQS Maxwell equation and corresponding jump conditions across S are*

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = q^F \qquad \boldsymbol{D} \cdot \boldsymbol{n} = -q^{S_q} \text{ on } S_q \qquad \phi = \phi^{S_\phi} \text{ on } S_\phi$$

Consider a weighting function $\overline{\phi}$ and vector $\overline{\boldsymbol{E}}$, defined over the material volume V, such that

$$\overline{\phi} = 0 \text{ on } S_\phi \qquad \overline{\boldsymbol{E}} = -\boldsymbol{\nabla}\phi$$

Then the EQS Maxwell equation and jump conditions have the equivalent weak form,

$$\int_V \boldsymbol{D} \cdot \overline{\boldsymbol{E}}\, dV = \int_V q^F \overline{\phi}\, dV + \int_{S_q} q^{S_q} \overline{\phi}\, dS_q$$

Proof. Multiply the EQS Maxwell equation through by $\overline{\phi}$ and integrate over V,

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{D} \overline{\phi}\, dV = \int_V q^F \overline{\phi}\, dV\,.$$

Use the chain rule to eliminate derivatives from $\boldsymbol{\nabla} \cdot \boldsymbol{D}$,

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{D}\overline{\phi}\right) = \boldsymbol{\nabla} \cdot \boldsymbol{D}\overline{\phi} + \boldsymbol{D} \cdot \boldsymbol{\nabla}\overline{\phi}$$

Use the identity

$$\int_V \boldsymbol{\nabla} \cdot \left(\boldsymbol{D}\overline{\phi}\right) dV = \oint_S \left(\boldsymbol{D}\overline{\phi}\right) \cdot \boldsymbol{n}\, dS$$

to obtain

$$\oint_S \boldsymbol{D}\overline{\phi} \cdot \boldsymbol{n}\, dS + \int_V \boldsymbol{D} \cdot \overline{\boldsymbol{E}}\, dV = \int_V q^F \overline{\phi}\, dV$$

Use the jump condition and constraint,

$$\boldsymbol{D} \cdot \boldsymbol{n} = -q^{S_q} \text{ on } S_q \qquad \overline{\phi} = 0 \text{ on } S_\phi$$

to obtain

$$\int_V \boldsymbol{D} \cdot \overline{\boldsymbol{E}}\, dV = \int_V q^F \overline{\phi}\, dV + \int_{S_q} q^{S_q} \overline{\phi}\, dS_q$$

■

For convenience we repeat the weak form theorem for balance of momentum from chapter 4.

Theorem B.1.2 (Balance of Momentum: Weak Form) *A material volume V bounded by surface $S = S_f + S_u$ is subject to mechanical surface tractions $f_i^{S_f}$ on S_f, mechanical displacement constraints $u_i^{S_u}$ on S_u, and body force density f_i^B in V. The electric stress tractions t_i^E outside the material volume V are assumed negligible, consistent with an assumption of zero electric fields E_i outside V. The balance of momentum equations and corresponding jump conditions across S are*

$$\tau^T_{ji,j} + f_i^B = \rho_o \frac{\partial^2 u_i}{\partial t^2} \qquad \tau^T_{[ji]} = 0$$
$$\tau^T_{ji} n_j = f_i^{S_f} \textit{ on } S_f \qquad u_i = u_i^{S_u} \textit{ on } S_u.$$

Consider a weighting function $\overline{u}_i$ and tensor $\overline{\epsilon}_{ij}$, defined over the material volume V, such that

$$\overline{u}_i = 0 \textit{ on } S_u \qquad \overline{\epsilon}_{ij} = \frac{1}{2}\left(\overline{u}_{i,j} + \overline{u}_{j,i}\right).$$

Then the balance of momentum equations and jump conditions have the weak form,

$$\begin{aligned} \int_V \tau^T_{ji}\overline{\epsilon}_{ij}\, dV + \int_V \rho_o \frac{\partial^2 u_i}{\partial t^2}\overline{u}_i\, dV &= \int_V f_i^B \overline{u}_i\, dV + \int_{S_f} f_i^{S_f}\overline{u}_i\, dS_f\,, \\ u_i &= u_i^{S_u} \textit{ on } S_u \end{aligned}$$

B.2 Finite Element Formulation I

This section presents a finite element formulation of perfectly electrically insulating electroelastic materials under the small deformation approximation. The formulation is suitable for nonlinear material response functions and anticipates a Newton method solution technique. The formulation is suitable for constitutive equations with strain $\boldsymbol{S}$ and electric field $\boldsymbol{E}$ as independent variables.

B.2.1 Weak Form

$$\int_V \tau^T_{ji}\overline{\epsilon}_{ij}\, dV + \int_V \rho_o \frac{\partial^2 u_i}{\partial t^2}\overline{u}_i\, dV = \int_V f_i^B \overline{u}_i\, dV + \oint_{S_f} f_i^{S_f}\overline{u}_i^{S_f}\, dS_f$$

$$\int_V D_i \overline{E}_i\, dV = \int_V q^B \overline{\phi}\, dV + \int_{S_q} q^{S_q}\overline{\phi}\, dS_q$$

Noting the symmetric of τ^T_{ji} and ϵ_{ij}, we introduce Voight notation

$$\boldsymbol{T} = \begin{bmatrix} \tau^T_{11} & \tau^T_{22} & \tau^T_{33} & \tau^T_{32} & \tau^T_{13} & \tau^T_{12} \end{bmatrix}'$$

$$\boldsymbol{S} = \begin{bmatrix} \epsilon_{11} & \epsilon_{22} & \epsilon_{33} & 2\epsilon_{32} & 2\epsilon_{13} & 2\epsilon_{12} \end{bmatrix}'$$

we obtain

$$\int_V \boldsymbol{T}\cdot\overline{\boldsymbol{S}}\, dV + \int_V \rho_o \frac{\partial^2 \boldsymbol{u}}{\partial t^2}\cdot\overline{\boldsymbol{u}}\, dV = \int_V \boldsymbol{f}^B\cdot\overline{\boldsymbol{u}}\, dV + \int_{S_f} \boldsymbol{f}^{S_f}\cdot\overline{\boldsymbol{u}}^{Sf}\, dS_f$$

$$\int_V \boldsymbol{D}\cdot\overline{\boldsymbol{E}}\, dV = \int_V q^B\overline{\phi}\, dV + \int_{S_q} q^{S_q}\overline{\phi}\, dS_q$$

B.2.2 Weak Form Rewritten

$$\int_V \overline{\boldsymbol{S}}'\boldsymbol{T}\, dV + \int_V \overline{\boldsymbol{u}}'\rho_o \frac{\partial^2 \boldsymbol{u}}{\partial t^2}\, dV = \int_V \overline{\boldsymbol{u}}'\boldsymbol{f}^B\, dV + \int_{S_f} \overline{\boldsymbol{u}}^{Sf\,\prime}\boldsymbol{f}^{S_f}\, dS_f$$

$$\int_V \overline{\boldsymbol{E}}'\boldsymbol{D}\, dV = \int_V \overline{\phi}'q^B\, dV + \int_{S_q} \overline{\phi}'q^{S_q}\, dS_q$$

B.2.3 Test Functions Defined

$$\begin{aligned}
\overline{\boldsymbol{u}}(x_i,t) &= H_u(x_i)\hat{\overline{\boldsymbol{u}}}(t)\\
\overline{\boldsymbol{u}}^{S_f}(x_i,t) &= H_u^{S_f}(x_i)\hat{\overline{\boldsymbol{u}}}(t)\\
\overline{\boldsymbol{S}}(\overline{\boldsymbol{u}}(x_i,t)) &= B_u(x_i)\hat{\overline{\boldsymbol{u}}}(t)\\
\overline{\phi}(x_i,t) &= H_\phi(x_i)\hat{\overline{\boldsymbol{\phi}}}(t)\\
\overline{\phi}^{S_q}(x_i,t) &= H_\phi^{S_q}(x_i)\hat{\overline{\boldsymbol{\phi}}}(t)\\
\overline{\boldsymbol{E}}(\overline{\phi}(x_i,t)) &= B_\phi(x_i)\hat{\overline{\boldsymbol{\phi}}}(t)
\end{aligned}$$

B.2.4 Shape Functions Defined

$$\begin{aligned}
\boldsymbol{u}(x_i,t) &= H_u(x_i)\hat{\boldsymbol{u}}(t) \\
\boldsymbol{u}^{S_f}(x_i,t) &= H_u^{S_f}(x_i)\hat{\boldsymbol{u}}(t) \\
\boldsymbol{S}(\overline{\boldsymbol{u}}(x_i,t)) &= B_u(x_i)\hat{\boldsymbol{u}}(t) \\
\phi(x_i,t) &= H_\phi(x_i)\hat{\boldsymbol{\phi}}(t) \\
\phi^{S_q}(x_i,t) &= H_\phi^{S_q}(x_i)\hat{\boldsymbol{\phi}}(t) \\
\boldsymbol{E}(\overline{\phi}(x_i,t)) &= B_\phi(x_i)\hat{\boldsymbol{\phi}}(t)
\end{aligned}$$

where we have chosen the shape functions identical to corresponding test functions (Galerkin's method).

B.2.5 Introducing Test Functions

$$\hat{\overline{\boldsymbol{u}}}'\left\{\int_V B_u{}'\boldsymbol{T}\,dV + \int_V H_u{}'\rho_o\frac{\partial^2\boldsymbol{u}}{\partial t^2}\,dV - \int_V H_u{}'\boldsymbol{f}^B\,dV - \int_{S_f} H_u^{S_f}{}'\boldsymbol{f}^{S_f}\,dS_f\right\} = 0$$

$$\hat{\overline{\boldsymbol{\phi}}}'\left\{\int_V B_\phi{}'\boldsymbol{D}\,dV - \int_V H_\phi{}'q^B\,dV - \int_{S_q} H_\phi^{S_q}{}'q^{S_q}\,dS_q\right\} = 0$$

We require the weak form to hold for all $\hat{\boldsymbol{u}}(t), \hat{\boldsymbol{\phi}}(t)$. A necessary condition is $\{\ \} = 0$. We obtain

$$\begin{bmatrix} \int_V \rho_o H_u{}'\frac{\partial^2\boldsymbol{u}}{\partial t^2}\,dV \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \int_V B_u{}'\boldsymbol{T}\,dV \\ \int_V B_\phi{}'\boldsymbol{D}\,dV \end{bmatrix} - \begin{bmatrix} \int_V H_u{}'\boldsymbol{f}^B\,dV \\ \int_V H_\phi{}'q^B\,dV \end{bmatrix} +$$

$$- \begin{bmatrix} \int_{S_f} H_u^{S_f}{}'\boldsymbol{f}^{S_f}\,dS_f \\ \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{0} \\ \int_{S_q} H_\phi^{S_q}{}'q^{S_q}\,dS_q \end{bmatrix} = \boldsymbol{0}$$

or

$$\boldsymbol{FI}^m + \boldsymbol{FI}^k - \boldsymbol{FE}^b - \boldsymbol{FE}^{S_f} - \boldsymbol{FE}^{S_q} = \boldsymbol{0}$$

B.2.6 Newton's Method at Time t

$$\boldsymbol{\xi} \triangleq \begin{bmatrix} \hat{\boldsymbol{u}} \\ -\hat{\boldsymbol{\phi}} \end{bmatrix}$$

$$\delta\boldsymbol{\xi} \triangleq \boldsymbol{\xi}^{\nu+1} - \boldsymbol{\xi}^{\nu}$$

$$\boldsymbol{Res}^{\nu} \triangleq \boldsymbol{Res}(\boldsymbol{\xi}^{\nu})$$

For simplicity we assume $\boldsymbol{f}^B, \boldsymbol{f}^{S_f}, q^B, q^{S_f}$ are independent of $\hat{\boldsymbol{u}}(t), \hat{\boldsymbol{\phi}}(t)$.

$$\boldsymbol{Res}(\boldsymbol{\xi},\ddot{\boldsymbol{\xi}}) \triangleq \boldsymbol{FI}^m + \boldsymbol{FI}^k - \boldsymbol{FE}^b - \boldsymbol{FE}^{S_f} - \boldsymbol{FE}^{S_q} \tag{B.1}$$

$$\begin{aligned}
\boldsymbol{Res}^{\nu+1} &\simeq \boldsymbol{Res}^{\nu} + \left.\frac{\partial \boldsymbol{Res}}{\partial \boldsymbol{\xi}}\right|_{\nu} \delta\boldsymbol{\xi} + \left.\frac{\partial \boldsymbol{Res}}{\partial \ddot{\boldsymbol{\xi}}}\right|_{\nu} \delta\ddot{\boldsymbol{\xi}} + \cdots \\
\boldsymbol{Res}^{\nu+1} &= \mathbf{0}
\end{aligned}$$

We obtain the equations for Newton's method

$$\left.\frac{\partial \boldsymbol{Res}}{\partial \boldsymbol{\xi}}\right|_{\nu} \delta\boldsymbol{\xi} + \left.\frac{\partial \boldsymbol{Res}}{\partial \ddot{\boldsymbol{\xi}}}\right|_{\nu} \delta\ddot{\boldsymbol{\xi}} = -\boldsymbol{Res}^{\nu} \tag{B.2}$$

where we will introduce a temporal approximation such that

$$\delta\ddot{\boldsymbol{\xi}} = \boldsymbol{f}(\boldsymbol{\xi}^{\nu}) + F(\boldsymbol{\xi}^{\nu})\delta\boldsymbol{\xi}$$

This linear system must be solved iteratively, starting from an initial guess, until some norm measure of error is achieved.

B.2.7 Introducing Constitutive and Shape Function

$$\begin{aligned}
\boldsymbol{T} &= \boldsymbol{T}(\boldsymbol{S}(\boldsymbol{\xi}), \boldsymbol{E}(\boldsymbol{\xi})) \\
\boldsymbol{D} &= \boldsymbol{D}(\boldsymbol{S}(\boldsymbol{\xi}), \boldsymbol{E}(\boldsymbol{\xi}))
\end{aligned}$$

In anticipation of calculating Jacobian matrices for Newton's method we use the chain rule to define the following matrices.

$$\begin{aligned}
\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{\xi}} &= \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{S}}\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{E}}\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\xi}} \\
&= \left[\begin{array}{c} C_{uu} \end{array}\right]\left[\begin{array}{cc} B_u & 0 \end{array}\right] + \left[\begin{array}{c} C_{u\phi} \end{array}\right]\left[\begin{array}{cc} 0 & -B_\phi \end{array}\right]
\end{aligned}$$

similarly

$$\begin{aligned}
\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\xi}} &= \frac{\partial \boldsymbol{D}}{\partial \boldsymbol{S}}\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{D}}{\partial \boldsymbol{E}}\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\xi}} \\
&= \left[\begin{array}{c} C_{\phi u} \end{array}\right]\left[\begin{array}{cc} B_u & 0 \end{array}\right] + \left[\begin{array}{c} C_{\phi\phi} \end{array}\right]\left[\begin{array}{cc} 0 & -B_\phi \end{array}\right]
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{\xi}} &= \left[\begin{array}{cc} C_{uu}B_u & -C_{u\phi}B_\phi \end{array}\right] \\
\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\xi}} &= \left[\begin{array}{cc} C_{\phi u}B_u & -C_{\phi\phi}B_\phi \end{array}\right]
\end{aligned}$$

B.2.8 Jacobian Matrices

$$\frac{\partial \boldsymbol{FI}^m}{\partial \ddot{\boldsymbol{\xi}}} = \left[\begin{array}{cc} \int_V \rho_o {H_u}' H_u \, dV & 0 \\ 0 & 0 \end{array}\right]$$

$$
\begin{aligned}
&= \int_V \rho_o H_m{}' H_m \, dV \\
&= J^m
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{FI}^k}{\partial \boldsymbol{\xi}} &= \begin{bmatrix} \int_V B_u{}' C_{uu} B_u \, dV & -\int_V B_u{}' C_{u\phi} B_\phi \, dV \\ \int_V B_\phi{}' C_{\phi u} B_u \, dV & -\int_V B_\phi{}' C_{\phi\phi} B_\phi \, dV \end{bmatrix} \\
&= \int_V \begin{bmatrix} B_u & 0 \\ 0 & B_\phi \end{bmatrix}' \begin{bmatrix} C_{uu} & -C_{u\phi} \\ C_{\phi u} & -C_{\phi\phi} \end{bmatrix} \begin{bmatrix} B_u & 0 \\ 0 & B_\phi \end{bmatrix} dV \\
&= \int_V B' C B \, dV \\
&= J^k
\end{aligned}
$$

B.2.9 Define Loading Interpolation Functions

$$
\begin{aligned}
\begin{bmatrix} \boldsymbol{f}^B(x_i, t) \\ q^B(x_i, t) \end{bmatrix} &= H_{bfq}(x_i) \boldsymbol{B_{fq}}(t) \\
\boldsymbol{f}^{S_f}(x_i, t) &= H_{sf}(x_i) \boldsymbol{S_f}(t) \\
q^{S_q} &= H_{sq}(x_i) \boldsymbol{S_q}(t)
\end{aligned}
$$

B.2.10 Residual Vectors

$$
\begin{aligned}
\boldsymbol{FI}^m &= \int_V \rho_o H_m{}' H_m \, dV \, \ddot{\boldsymbol{\xi}} \\
&= J^m \ddot{\boldsymbol{\xi}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{FI}^k &= \int_V B' \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{D} \end{bmatrix} dV \\
\boldsymbol{FE}^b &= \int_V \begin{bmatrix} H_u{}' & 0 \\ 0 & H_\phi{}' \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^B \\ q^B \end{bmatrix} dV \\
&= \int_V H' \begin{bmatrix} \boldsymbol{f}^B \\ q^B \end{bmatrix} dV \\
&= \int_V H' H_{bfq} \, dV \, \boldsymbol{B_{fq}} \\
&= FE_{bfq} \boldsymbol{B_{fq}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{FE}^{S_f} &= \int_{S_f} \begin{bmatrix} H_u^{S_f}{}' \\ 0 \end{bmatrix} \boldsymbol{f}^{S_f} \, dS_f \\
&= \int_{S_f} H_{su}{}' \boldsymbol{f}^{S_f} \, dS_f \\
&= \int_{S_f} H_{su}{}' H_{sf} \, dS_f \, \boldsymbol{S_f}
\end{aligned}
$$

$$= FE_{sf}\boldsymbol{S_f}$$

$$\begin{aligned}
\boldsymbol{FE}^{S_q} &= \int_{S_q} \begin{bmatrix} 0 \\ H_\phi^{S_q}{}' \end{bmatrix} q^{S_q}\, dS_q \\
&= \int_{S_q} H_{s\phi}{}' q^{S_q}\, dS_q \\
&= \int_{S_q} H_{s\phi}{}' H_{sq}\, dS_q\, \boldsymbol{S_q} \\
&= FE_{sq}\boldsymbol{S_q}
\end{aligned}$$

B.3 Finite Element Formulation II

This section presents a finite element formulation of perfectly electrically insulating electroelastic materials under the small deformation approximation. The formulation is suitable for nonlinear material response functions and anticipates a Newton method solution technique. The formulation is suitable for constitutive equations with strain $\boldsymbol{S}$ and electric displacement $\boldsymbol{D}$ as independent variables.

B.3.1 Mixed Weak Form

$$\begin{aligned}
\int_V \boldsymbol{T}\cdot\overline{\boldsymbol{S}}\, dV + \int_V \rho_o \frac{\partial^2 \boldsymbol{u}}{\partial t^2}\cdot\overline{\boldsymbol{u}}\, dV - \int_V \boldsymbol{f}^B\overline{\boldsymbol{u}}\, dV - \int_{S_f} \boldsymbol{f}^{S_f}\overline{\boldsymbol{u}}^{S_f}\, dS_f &= 0 \\
\int_V \boldsymbol{D}\cdot\overline{\boldsymbol{E}}\, dV - \int_V q^B\overline{\phi}\, dV - \int_{S_q} q^{S_q}\overline{\phi}\, dS_q &= 0 \\
\int_V (\boldsymbol{E} + \boldsymbol{\nabla}\phi)\cdot\overline{\boldsymbol{D}}\, dV &= 0
\end{aligned}$$

B.3.2 Mixed Weak Form Rewritten

$$\begin{aligned}
\int_V \overline{\boldsymbol{S}}'\boldsymbol{T}\, dV + \int_V \overline{\boldsymbol{u}}'\rho_o \frac{\partial^2 \boldsymbol{u}}{\partial t^2}\, dV - \int_V \overline{\boldsymbol{u}}'\boldsymbol{f}^B\, dV - \int_{S_f} \overline{\boldsymbol{u}}^{Sf}{}'\boldsymbol{f}^{S_f}\, dS_f &= (0) \\
\int_V \overline{\boldsymbol{E}}'\boldsymbol{D}\, dV - \int_V \overline{\phi}' q^B\, dV - \int_{S_q} \overline{\phi}' q^{S_q}\, dS_q &= \boldsymbol{0} \\
\int_V \overline{\boldsymbol{D}}'(\boldsymbol{E} + \boldsymbol{\nabla}\phi)\, dV &= 0
\end{aligned}$$

B.3.3 Test Functions Defined

$$\begin{aligned}
\overline{\boldsymbol{u}}(x_i,t) &= H_u(x_i)\hat{\overline{\boldsymbol{u}}}(t) \\
\overline{\boldsymbol{u}}^{S_f}(x_i,t) &= H_u^{S_f}(x_i)\hat{\overline{\boldsymbol{u}}}(t) \\
\overline{\boldsymbol{S}}(\overline{\boldsymbol{u}}(x_i,t)) &= B_u(x_i)\hat{\overline{\boldsymbol{u}}}(t) \\
\overline{\phi}(x_i,t) &= H_\phi(x_i)\hat{\overline{\boldsymbol{\phi}}}(t)
\end{aligned}$$

$$
\begin{aligned}
\overline{\phi}^{S_q}(x_i,t) &= H_\phi^{S_q}(x_i)\hat{\overline{\boldsymbol{\phi}}}(t) \\
\overline{\boldsymbol{E}}(\overline{\phi}(x_i,t)) &= -B_\phi(x_i)\hat{\overline{\boldsymbol{\phi}}}(t) \\
\overline{\boldsymbol{D}}(x_i,t) &= H_D(x_i)\hat{\overline{\boldsymbol{D}}}(t)
\end{aligned}
$$

B.3.4 Shape Functions Defined

$$
\begin{aligned}
\boldsymbol{u}(x_i,t) &= H_u(x_i)\hat{\boldsymbol{u}}(t) \\
\boldsymbol{S}(\boldsymbol{u}(x_i,t)) &= B_u(x_i)\hat{\boldsymbol{u}}(t) \\
\phi(x_i,t) &= H_\phi(x_i)\hat{\boldsymbol{\phi}}(t) \\
\boldsymbol{\nabla}\phi(\phi(x_i,t)) &= B_\phi(x_i)\hat{\boldsymbol{\phi}}(t) \\
\boldsymbol{D}(x_i,t) &= H_D(x_i)\hat{\boldsymbol{D}}(t)
\end{aligned}
$$

where we have chosen the shape functions identical to corresponding test functions (Galerkin's method).

B.3.5 Introducing Test Functions

$$
\hat{\overline{\boldsymbol{u}}}'\left\{\int_V B_u{}'\boldsymbol{T}\,dV + \int_V H_u{}'\rho_o\frac{\partial^2\boldsymbol{u}}{\partial t^2}\,dV - \int_V H_u{}'\boldsymbol{f}^B\,dV - \int_{S_f} H_u^{S_f}{}'\boldsymbol{f}^{S_f}\,dS_f\right\} = 0
$$

$$
\hat{\overline{\boldsymbol{\phi}}}'\left\{\int_V B_\phi{}'\boldsymbol{D}\,dV - \int_V H_\phi{}'q^B\,dV - \int_{S_q} H_\phi^{S_q}{}'q^{S_q}\,dS_q\right\} = 0
$$

$$
\hat{\overline{\boldsymbol{D}}}'\left\{\int_V H_D{}'(\boldsymbol{E}+\boldsymbol{\nabla}\phi)\,dV\right\} = 0
$$

We require the weak form to hold for all $\hat{\overline{\boldsymbol{u}}}(t), \hat{\overline{\boldsymbol{\phi}}}(t), \hat{\overline{\boldsymbol{D}}}(t)$. A necessary condition $\longrightarrow \{\} = 0$

$$
\begin{bmatrix} \int_V \rho_o H_u{}'\frac{\partial^2\boldsymbol{u}}{\partial t^2}\,dV \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \int_V B_u{}'\boldsymbol{T}\,dV \\ \int_V B_\phi{}'\boldsymbol{D}\,dV \\ \int_V H_D{}'(\boldsymbol{E}+\boldsymbol{\nabla}\phi)\,dV \end{bmatrix} - \begin{bmatrix} \int_V H_u{}'\boldsymbol{f}^B\,dV \\ \int_V -H_\phi{}'q^B\,dV \\ \boldsymbol{0} \end{bmatrix} +
$$

$$
- \begin{bmatrix} \int_{S_f} H_u^{S_f}{}'\boldsymbol{f}^{S_f}\,dS_f \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{0} \\ \int_{S_q} -H_\phi^{S_q}{}'q^{S_q}\,dS_q \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{0}
$$

or

$$
\boldsymbol{FI}^m + \boldsymbol{FI}^k - \boldsymbol{FE}^b - \boldsymbol{FE}^{S_f} - \boldsymbol{FE}^{S_q} = \boldsymbol{0}
$$

B.3.6 Introducing Constitutive and Shape Functions

$$
\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{S}(\boldsymbol{\xi}), \boldsymbol{D}(\boldsymbol{\xi}))
$$

$$\boldsymbol{E} = \boldsymbol{D}(\boldsymbol{S}(\boldsymbol{\xi}), \boldsymbol{D}(\boldsymbol{\xi}))$$

and using the chain rule

$$\begin{aligned}
\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{\xi}} &= \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{S}}\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{D}}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\xi}} \\
&= \begin{bmatrix} C_{uu} \end{bmatrix}\begin{bmatrix} B_u & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_{ud} \end{bmatrix}\begin{bmatrix} 0 & 0 & H_d \end{bmatrix} \\
\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\xi}} &= \begin{bmatrix} 0 & 0 & H_d \end{bmatrix} \\
\frac{\partial(\boldsymbol{E} + \boldsymbol{\nabla}\phi)}{\partial \boldsymbol{\xi}} &= \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{S}}\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{D}}\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\xi}} + \frac{\partial \boldsymbol{\nabla}\phi}{\partial \boldsymbol{\xi}} \\
&= \begin{bmatrix} C_{du} \end{bmatrix}\begin{bmatrix} B_u & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_{dd} \end{bmatrix}\begin{bmatrix} 0 & 0 & H_d \end{bmatrix} + \begin{bmatrix} 0 & B_\phi & 0 \end{bmatrix}
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{\xi}} &= \begin{bmatrix} C_{uu}B_u & 0 & C_{ud}H_d \end{bmatrix} \\
\frac{\partial \boldsymbol{D}}{\partial \boldsymbol{\xi}} &= \begin{bmatrix} 0 & 0 & H_d \end{bmatrix} \\
\frac{\partial(\boldsymbol{E} + \boldsymbol{\nabla}\phi)}{\partial \boldsymbol{\xi}} &= \begin{bmatrix} C_{du}B_u & B_\phi & C_{dd}H_d \end{bmatrix}
\end{aligned}$$

B.3.7 Jacobian Matrices

$$\begin{aligned}
\frac{\partial \boldsymbol{FI}^m}{\partial \ddot{\boldsymbol{\xi}}} &= \begin{bmatrix} \int_V \rho_o H_u{}' H_u \, dV & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \int_V \rho_o H_m{}' H_m \, dV = J^m \\
\frac{\partial \boldsymbol{FI}^k}{\partial \boldsymbol{\xi}} &= \begin{bmatrix} \int_V B_u{}' C_{uu} B_u \, dV & 0 & \int_V B_u{}' C_{ud} H_d \, dV \\ 0 & 0 & \int_V B_\phi{}' H_d \, dV \\ \int_V H_d{}' C_{du} B_u \, dV & \int_V H_d{}' B_\phi \, dV & \int_V H_d{}' C_{dd} H_d \, dV \end{bmatrix} \\
&= \int_V \begin{bmatrix} B_u & 0 & 0 \\ 0 & B_\phi & 0 \\ 0 & 0 & H_d \end{bmatrix}' \begin{bmatrix} C_{uu} & 0 & C_{ud} \\ 0 & 0 & I \\ C_{du} & I & C_{dd} \end{bmatrix} \begin{bmatrix} B_u & 0 & 0 \\ 0 & B_\phi & 0 \\ 0 & 0 & H_d \end{bmatrix} dV \\
&= \int_V B' C B \, dV \\
&= J^k
\end{aligned}$$

B.3.8 Define Loading Interpolation Functions

$$\begin{bmatrix} f_i^B(x_i,t) \\ q^B \end{bmatrix} = H_{bfq}(x_i)\boldsymbol{B_{fq}}(t)$$

$$\boldsymbol{f}^{S_f}(x_i,t) = H_{sq}(x_i)\boldsymbol{S_f}(t)$$

$$q^{S_q}(x_i,t) = H_{sq}(x_i)\boldsymbol{S_q}(t)$$

B.3.9 Residual Vectors

$$\boldsymbol{FI}^m = \int_V \rho_o H_m{}' H_m \, dV \, \ddot{\boldsymbol{\xi}} = J^m \ddot{\boldsymbol{\xi}}$$

$$\boldsymbol{FI}^k = \int_V B' \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{D} \\ (\boldsymbol{E} + \boldsymbol{\nabla}\phi) \end{bmatrix} dV$$

$$\begin{aligned} \boldsymbol{FE}^b &= \int_V \begin{bmatrix} H_u{}' & 0 \\ 0 & -H_\phi{}' \\ 0 & 0 \end{bmatrix} dV = \int_V H' \begin{bmatrix} \boldsymbol{f}^B \\ q^b \end{bmatrix} dV \\ &= \int_V H' H_{bfq} \, dV \, \boldsymbol{B_{fq}} = FE_{bfq}\boldsymbol{B_{fq}} \end{aligned}$$

$$\begin{aligned} \boldsymbol{FE}^{S_f} &= \int_{S_f} \begin{bmatrix} H_u^{S_f}{}' \\ 0 \\ 0 \end{bmatrix} \boldsymbol{f}^{S_f} \, dS_f = \int_{S_f} H_{su}{}' \boldsymbol{f}^{S_f} \, dS_f \\ &= \int_{S_f} H_{su}{}' H_{sf} \, dS_f \, \boldsymbol{S_f} = FE_{sf}\boldsymbol{S_f} \end{aligned}$$

$$\begin{aligned} \boldsymbol{FE}^{S_q} &= \int_{S_q} \begin{bmatrix} 0 \\ -H_\phi^{S_q}{}' \\ 0 \end{bmatrix} q^{S_q} \, dS_q = \int_{S_q} H_{s\phi}{}' q^{S_q} \, dS_q \\ &= \int_{S_q} H_{s\phi}{}' H_{sq} \, dS_q \, \boldsymbol{S_q} = FE_{sq}\boldsymbol{S_q} \end{aligned}$$

Appendix C

Electric Conduction Measurements

Electric conduction values were measured by tuning a potentiometer to zero the voltage V. The measurement is a zero load measurement.

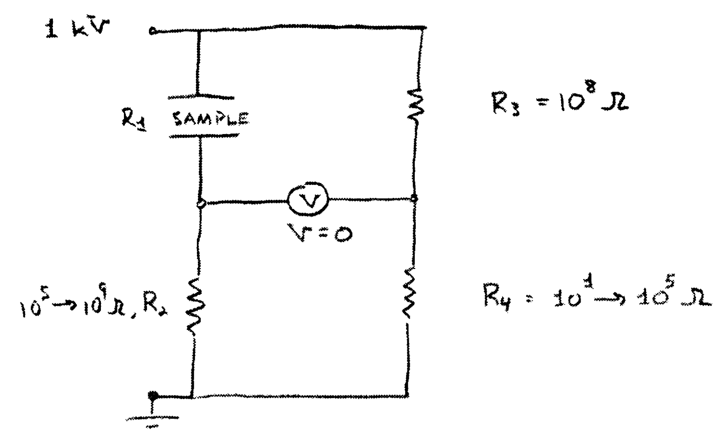

Figure C-1: Electric conduction measurement circuit.

Appendix D

Classical Small Deformation Derivation

D.1 Introduction

This appendix presents the classical small deformation piezoelectric equations, extended to include electric conduction and electric body forces. They are based on postulating a conservation of energy statement using Poynting's vector as presented in TIERSTEN [29, pp. 25-39]. We modify the classical equations by including electrical body forces in their surface traction form. This formulation starts from notions of small deformation 'built in'. The main assumption is that Poynting's vector represents the electric energy flux through a surface.

D.1.1 Conservation of energy

After postulating the conservation of energy statement we enforce invariance requirements after GREEN & RIVLIN [14] to obtain linear and angular momentum equations. We could have used momentum and angular momentum equations following TIERSTEN [29] instead of the invariance arguments. The balance of energy equation is postulated in the form[1],

$$\begin{aligned}\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\rho v_i v_i + U\right) dV \quad &= \quad \int_V f_i v_i \, dV + \oint_S t_i v_i \, dS + \int_V \rho h \, dV - \oint_S q_i n_i \, dS + \\ &\oint_S t_i^E v_i \, dS - \oint_S (\boldsymbol{E}\times\boldsymbol{H})_i \, n_i \, dS \,.\end{aligned}$$

Noting that surface tractions survive a limit process as volume approaches zero and surface remains finite, where a body force like gravity does not, we have exploited this property of electric 'body forces' and written them as electric surface tractions. Note the traction vectors t_i, t_i^E are related to their respective stress tensors by

$$t_i = \tau_{ji} n_j$$

[1]See definition 2.5.1 for field definitions

$$t_i^E = \tau_{ji}^E n_j$$

Also, we utilize the EQS approximation to the Poynting vector

$$(\boldsymbol{E} \times \boldsymbol{H})_i = \phi(J_i + \dot{D}_i)$$

Noting the following

$$\begin{aligned}
\oint_S \tau_{ji} v_i n_j \, dS &= \int_V (\tau_{ji} v_i)_{,j} \, dV \\
\oint_S \tau_{ji}^E v_i n_j \, dS &= \int_V \left(\tau_{ji}^E v_i\right)_{,j} \, dV \\
\oint_S q_i n_i \, dS &= \int_V q_{i,i} \, dV \\
\oint_S \phi(J_i + \dot{D}_i) n_i \, dS &= \int_V \left[\phi(J_i + \dot{D}_i)\right]_{,i} \, dV
\end{aligned}$$

we obtain

$$\begin{aligned}
\int_V \left(\rho v_i \dot{v}_i + \dot{U}\right) \, dV = \int_V f_i v_i \, dV &+ \int_V (\tau_{ji} v_i)_{,j} \, dV + \int_V \rho h \, dV - \int_V q_{i,i} \, dV \\
&+ \int_V \left(\tau_{ji}^E v_i\right)_{,j} \, dV - \int_V \left[\phi(J_i + \dot{D}_i)\right]_{,i} \, dV
\end{aligned}$$

Expanding the divergence terms and noting from EQS Maxwell

$$\begin{aligned}
(J_i + \dot{D}_i)_{,i} &= 0 \\
E_i &= -\phi_{,i}
\end{aligned}$$

and by noting this holds for arbitrary volume V we obtain the local for of our energy statement

$$\begin{aligned}
\dot{U} = \left(\tau_{ji,j} + \tau_{ji,j}^E + f_i - \rho \dot{v}_i\right) v_i + \left(\tau_{ji} + \tau_{ji}^E\right) v_{i,j} + \rho h - q_{i,i} \\
+ E_i (J_i + \dot{D}_i)
\end{aligned}$$

We obtain the governing equations of motion by enforcing invariance of our energy expression w.r.t. rigid motion of the body. We first consider a rigid dispacement where b_i is an arbitrary vector

$$v_i \to v_i + b_i$$

We assume during the rigid dispacement that all terms $U, \rho, f_i, \tau_{ji}, \tau_{ji}^E, \rho h, q_i, \phi, J_i, D_i$ remain constant

$$\begin{aligned}
\dot{U} = \left(\tau_{ji,j} + \tau_{ji,j}^E + f_i - \rho \dot{v}_i\right)(v_i + b_i) + \left(\tau_{ji} + \tau_{ji}^E\right) v_{i,j} + \rho h - q_{i,i} \\
+ E_i (J_i + \dot{D}_i)
\end{aligned}$$

Using our energy statement we obtain

$$\left(\tau_{ji,j} + \tau^E_{ji,j} + f_i - \rho\dot{v}_i\right) b_i = 0$$

Requiring this to hold for arbitrary b_i, we obtain our equations of linear momentum

$$(\tau_{ji} + \tau^E_{ji})_{,j} + f_i - \rho\dot{v}_i = 0$$

Using our linear momentum equations in our energy statement it simplifies considerably to

$$\dot{U} = \left(\tau_{ji} + \tau^E_{ji}\right) v_{i,j} + \rho h - q_{i,i} + E_i(J_i + \dot{D}_i)$$

Next we superpose an arbitrary rigid body rotation where Ω_k is arbitrary vector

$$v_i \to v_i + \varepsilon_{ijk} e_j \Omega_k v_{i,j} \to v_{i,j} + \varepsilon_{ijk}\Omega_k$$

we obtain

$$\dot{U} = \left(\tau_{ji} + \tau^E_{ji}\right) (v_{i,j} + \varepsilon_{ijk}\Omega_k)\rho h - q_{i,i} + E_i(J_i + \dot{D}_i)$$

Simplifying using our energy expression

$$(\tau_{ji} + \tau^E_{ji})\varepsilon_{ijk}\Omega_k = 0$$

we require this to hold for arbitrary Ω_k. This requires the anti-symmetric part of $(\tau_{ji} + \tau^E_{ji}) = 0$,or introducing the symmetric and anti-symmetic operators

$$\begin{aligned} A_{(i,j)} &= \frac{1}{2}(A_{ij} + A_{ij}) \\ A_{[i,j]} &= \frac{1}{2}(A_{ij} - A_{ji}), \end{aligned}$$

we rewrite this condition as

$$(\tau + \tau^E)_{[j,i]} = 0$$

Rewriting $v_{i,j}$ as the sum of its symmetric and anti-symmetric parts

$$\begin{aligned} v_{i,j} &= \dot{u}_{(i,j)} + \dot{u}_{[i,j]} \\ &= \dot{\epsilon}_{ij} + \dot{\omega}_{ij} \end{aligned}$$

and noting that contraction of a symmetric and anti-symmetric tensor is zero we obtain

$$\begin{aligned} (\tau_{ji} + \tau^E_{ji})v_{i,j} &= (\tau_{ji} + \tau^E_{ji})(\dot{\epsilon}_{ij} + \dot{\omega}_{ij}) \\ &= (\tau_{ji} + \tau^E_{ji})\dot{\epsilon}_{ij} \end{aligned}$$

Finally our energy expression simplifies to

$$\dot{U} = (\tau_{ji} + \tau_{ji}^{E})\dot{\epsilon}_{ij} + \rho h - q_{i,i} + E_i(J_i + \dot{D}_i)$$

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