Methods for a systematic analysis of power converters

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Abstract
This contribution aims at presenting fundamental principles and theoretical tools for a comprehensive study and systematic analysis of generic power conversion circuits. The contents are divided into three main parts. Part I presents a new graphical technique for the state analysis of the so-called 'fundamental LCEI type circuit', which is found in sequences of many practical power converter topologies. In Part II, this graphical technique is used within a straightforward sequence-by-sequence algorithm to study the intrinsic functionality of power converters and to describe the evolution in time of their main quantities (state variables) and the conditions governing switch commutations and state transitions. Part III is devoted to power converter modelling techniques, a major advance for stable control loop design.

1 Introduction

In many practical situations the power electronics engineer is confronted with the hard task of understanding how a given new electronic circuit works. This is for instance the case in some 'reverse engineering' studies or when someone not familiar with new power conversion topologies wishes to acquire a deep understanding thereof. A typical problem can be stated as follows: how to define the status evolution of the switches over time and to draw the waveforms of the voltages and currents across the main elements in a power electronic circuit, consisting of DC sources (voltage and/or current) reactive passive components (inductances/capacitors) and electronic semiconductor switches (thyristors, diodes, IGBTs). Answering this kind of question is of course mandatory not only to understand the circuit functionality but also for dimensioning purposes. During the 1970s the LEEI laboratory in Toulouse, France, developed a technique to systematically analyse any power conversion circuit, based on a sequence-by-sequence approach. The study starts with a known or 'guessed' sequence, corresponding to the state of a particular set of switches. For that sequence, different steps (up to six as shown later on) will be followed in the specific order given by a flowchart. This procedure includes, for instance, in one step a test of compatibility to find out if a given sequence really exists, i.e., if all the voltages and currents on the switches are compatible with the initial 'guess' and no contradictions occur (such as voltage and current sources found to be short circuited and open circuited, respectively; diodes found in open state with a positive voltage or thyristors with negative currents). In another step, the evolution of the so-called 'state variables' of the circuit is obtained, leading to the step in which the events that may result in a switch to the next sequence will be considered. All the steps are based on simple logical procedures and require little mathematical calculations except for the one corresponding to the computation of the evolution of the state variables with time. Fortunately, for the majority of the practical cases, each sequence corresponds either to a first- or second-order circuit which can be modelled by a differential equation of the first or second order, taking into account the right initial conditions. In order to further simplify this task, LEEI has developed a graphical method called 'phase plane representation'. Any second-order circuit can therefore be represented graphically by using this method. An elementary second-order circuit, called LCEI circuit, formed by one DC voltage source, one DC current source, one inductor and one capacitor, can be found in many practical sequences and its thorough study is essential for understanding the basics of any phase plane representation. Part I of this paper deals with the theoretical background of the phase plane representation of the elementary LCEI circuit. Several practical examples of direct use of this technique will also be given. In Part II, after a brief classification of the different methods of study of power converters, the sequential analytical method will be presented including the flowchart of the systematic analysis. A case study, based on a thyristor chopper circuit, will be developed as an example of the proposed methodology.

Another important issue in power converter studies is the techniques for modelling them. Most of the power converters need feedback-based control loops. In order to analyse the stability of these loops and to tune the controllers' parameters to obtain the best possible performance, a mathematical model of the power converter circuit has to be developed. The particularity of power converter modelling is that the system mixes time-discrete and continuous signals. For instance, the output of the plant may be the voltage across a capacitor (continuous variable), whereas the input of the plant may be the control signal of one transistor (digital signal). In part III, a generic modelling technique (called 'state space modelling') will be explained in detail using state space theory. An alternative, more direct and practical method, entitled 'equivalent average circuit modelling' will also be presented.

Part I – The phase plane representation

2 Forced state and free state in first- and second-order circuits—recall

2.1 State variables

Any linear system behaviour can be described by a set of main variables, whose values directly define its state. Any other quantities of the system can be expressed as a function of these state variables and system inputs.

An important property of state variables is that they cannot change value instantaneously. They have to be continuous functions in time. In any electrical circuit, the state variables are

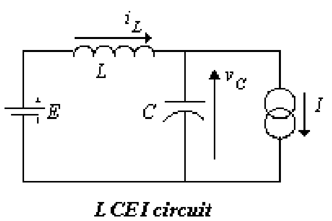
– the currents flowing through each independent inductor,

– the voltages across each independent capacitor.

Of course, if two inductors are connected in series in one branch of the circuit, this leads to only one state variable because the current flowing through the first inductor is the same as the current flowing through the second one. Likewise, the direct connection of two capacitors in parallel results in a single state variable as the voltage across both units has to be the same.

2.2 Response of a linear system

Any linear system (electrical circuit) can be described mathematically by a set of differential equations. For instance, the elementary LCEI circuit shown in Fig. 1 can be described by the differential equation system shown on the right.



$$\begin{cases} E = L\dfrac{di_L}{dt} + v_C \\ i_L = C\dfrac{dv_C}{dt} + I \end{cases} \Rightarrow E = LC\frac{d^2v_C}{dt^2} + v_C$$

Fig. 1: LCEI type circuit with the corresponding differential equation system

The full response of any linear system in the time domain ***R(t)*** is equal to the sum of the free-state response $\boldsymbol{R_{\Phi}(t)}$ and of the forced state response $\boldsymbol{R_{F}(t)}$ assuming the forced-state response is finite:

$$R(t) = R_{\Phi}(t) + R_{\mathrm{F}}(t)\,. \tag{1}$$

In the case of an electrical circuit:

- The free-state response $R_{\Phi}(t)$ is the response of the circuit without excitation sources, i.e., voltage sources short-circuited and current sources in open circuit.
- The forced-state response $R_F(t)$ is the response of the circuit in steady state. If all the excitation sources are DC, all capacitors in the circuit behave like open circuits and all inductors like short circuits.

In the case of the LCEI circuit of Fig. 1, the free-state and the forced-state solutions of the differential equation with regard to the variable $v_c(t)$ are, respectively:

$$v_{C\Phi}(t) = A\cos(\omega t)\,, \quad v_{CF}(t) = Const = E\,. \tag{2}$$

The full response is therefore:

$$v_{C}(t) = v_{C\Phi}(t) + v_{CF}(t) = A\cos(\omega t) + E\,. \tag{3}$$

The constant A in Eq. (3) can be computed from the initial conditions statement:

$$v_{C}(0) = v_{C\Phi}(0) + v_{CF}(0) = A + E \quad \Rightarrow \quad A = v_{C}(0) - E \tag{4}$$

where $v_c(0)$ is the initial charge of the capacitor.

In any linear circuit, the order of the system (circuit) is equal to the total count of state variables, which is the sum of independent capacitors and of independent inductors.

In Fig. 2 several examples of linear circuit responses with the equivalent circuits giving either the free state or the forced state responses are shown.

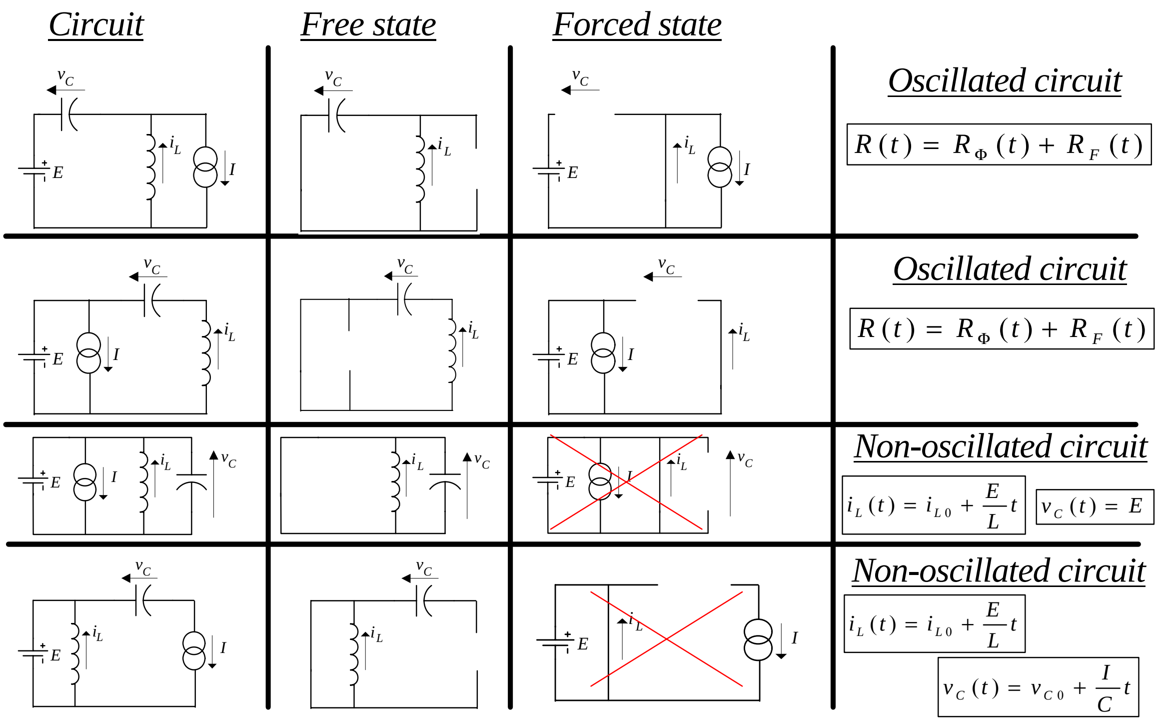


Fig. 2: Examples of free-state and forced-state circuit responses

In the first case, the free state corresponds to a series LC circuit, yielding a second-order circuit with oscillating response. In the second case, the configuration of the free-state circuit is the same, therefore giving also a second-order circuit with oscillating response.

However, in the third case, the free-state circuit leads to both the capacitor and the inductor being in short circuit. Thus it is a non-oscillating circuit, and the response in time is quite easy to obtain by direct inspection: the current in the inductor increases linearly with time and the voltage across the capacitor is constant and equal to *E*.

The fourth case is also a non-oscillating circuit because the free-state circuit has an inductor in short circuit and a capacitor in open circuit. It is also obvious to determine the time domain response.

3 Response of a LCEI type circuit to voltage and current steps: the phase plane method

The objective is now to study the response of the elementary LCEI circuit with two DC excitation sources (voltage, *E*, and current, *I*) and given initial conditions for the inductor current, i_{L0}, and capacitor voltage, v_{c0}. At first all power losses in the circuit will be neglected. In practice, there are losses especially in the internal resistance of the inductor. However, the time constant of the inductor is often so much larger than the switching period that the damping effect can be neglected at the switching period scale. Damping effect due to losses will be discussed later in Section 3.3.

3.1 Theoretical analysis: state equations

The free-state and steady-state circuits of the LCEI circuit are shown in Fig. 3:

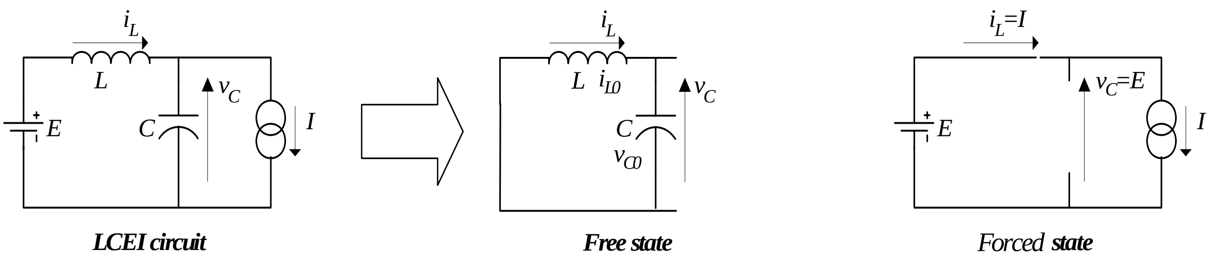


Fig. 3: Free-state and forced-state circuits of the LCEI circuit

The forced-state response, $[i_{LF}(t), v_{CF}(t)]$, can be directly expressed by:

$$\begin{cases} i_{LF}(t) = I \\ v_{CF}(t) = E \end{cases} . \tag{5}$$

The free-state circuit can be modelled by the following differential equations:

$$\begin{cases} i_{L\varphi} = C\dfrac{dv_{C\varphi}}{dt} \\ v_{C\varphi} = -L\dfrac{di_{L\varphi}}{dt} \end{cases} \tag{6}$$

which can be merged into the following one:

$$i_{L\varphi} + LC\frac{d^2 i_{L\varphi}}{d^2 t} = 0 . \tag{7}$$

The solution to this equation system, giving the free-state response of the circuit, $[i_{L\Phi}(t),\ v_{C\Phi}(t)]$, is:

$$\begin{cases} i_{L\varphi}(t) = A\cos(\omega t) + B\sin(\omega t) \\ v_{C\varphi}(t) = -BL\omega\cos(\omega t) + AL\omega\sin(\omega t) \end{cases} \tag{8}$$

with ω being the resonance frequency of the circuit, $\omega = 1/\sqrt{LC}$.

The full response $[i_L(t),\ v_C(t)]$ is therefore:

$$\begin{cases} i_L(t) = I + A\cos(\omega t) + B\sin(\omega t) \\ v_C(t) = E - B\omega L\cos(\omega t) + A\omega L\sin(\omega t) \end{cases} . \tag{9}$$

Constants A and B depend on the initial conditions of the circuit, and can be obtained by evaluating Eq. (9) for $t = 0$:

$$\begin{cases} i_L(0) = i_{L0} = I + A \\ v_C(0) = v_{C0} = E - B\omega L \end{cases} ; \qquad \begin{cases} A = i_{L0} - I \\ B = \dfrac{E - v_{C0}}{\omega L} = \sqrt{\dfrac{C}{L}}(E - v_{C0}) \end{cases} . \tag{10}$$

By replacing A and B from Eq. (10) in Eq. (9), the full response equation is obtained:

$$\begin{cases} i_L(t) = I + (i_{L0} - I)\cos(\omega t) - \sqrt{\dfrac{C}{L}}(v_{C0} - E)\sin(\omega t) \\ v_C(t) = E + (v_{C0} - E)\cos(\omega t) + \sqrt{\dfrac{L}{C}}(i_{L0} - I)\sin(\omega t) \end{cases} . \tag{11}$$

It is shown below that, with the appropriate scale, Eq. (11) gives, in polar coordinates, the equation of a circle. In order to demonstrate this statement, the following expression is derived from Eq. (11):

$$\begin{aligned} &\frac{L}{C}\left(i_L(t) - I\right)^2 + \left(v_C(t) - E\right)^2 = \\ &= \frac{L}{C}(i_{L0} - I)^2\cos^2(\omega t) - \cancel{2\sqrt{\frac{L}{C}}(i_{L0} - I)(v_{C0} - E)\cos(\omega t)\sin(\omega t)} + (v_{C0} - E)^2\sin^2(\omega t) + \\ &+ (v_{C0} - E)^2\cos^2(\omega t) + \cancel{2\sqrt{\frac{L}{C}}(v_{C0} - E)(i_{L0} - I)\cos(\omega t)\sin(\omega t)} + \frac{L}{C}(i_{L0} - I)^2\sin^2(\omega t) = \\ &= \frac{L}{C}(i_{L0} - I)^2 + (v_{C0} - E)^2 . \end{aligned} \tag{12}$$

Or, which is equivalent:

$$\left(\sqrt{\frac{L}{C}}i_L(t) - \sqrt{\frac{L}{C}}I\right)^2 + \left(v_C(t) - E\right)^2 = \frac{L}{C}(i_{L0} - I)^2 + (v_{C0} - E)^2 . \tag{13}$$

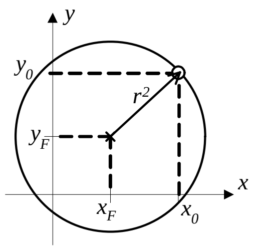
Considering the following relationships:

$$y = \sqrt{\frac{L}{C}} i_{\mathrm{L}}(t) \; ; \quad x = v_{\mathrm{C}}(t)$$

$$y_{\mathrm{F}} = \sqrt{\frac{L}{C}} I \; ; \quad x_{\mathrm{F}} = E$$

$$y_0 = \sqrt{\frac{L}{C}} i_{\mathrm{L0}} \; ; \quad x_0 = v_{\mathrm{C0}}$$

Eq. (13) can be re-written according to (14):



$$\left(y - y_{\mathrm{F}}\right)^2 + \left(x - x_{\mathrm{F}}\right)^2 = (y_0 - y_{\mathrm{F}})^2 + (x_0 - x_{\mathrm{F}})^2 \; . \tag{14}$$

This corresponds to the equation of a circle in Cartesian coordinates. The circle is centred in the point given by the forced state response (x_{F}, y_{F}) and it passes through the point corresponding to the initial conditions (x_0, y_0). Note that in order to obtain a circle, the ordinate (inductor current) has to be multiplied by the scale factor $\sqrt{L/C}$.

3.2 Graphical representation

In conclusion, the LCEI type circuit generates an oscillating response which can be represented graphically in the phase plane ($i_{\mathrm{L}} \sqrt{L/C}$ versus v_{C}) as follows (see Fig. 4):

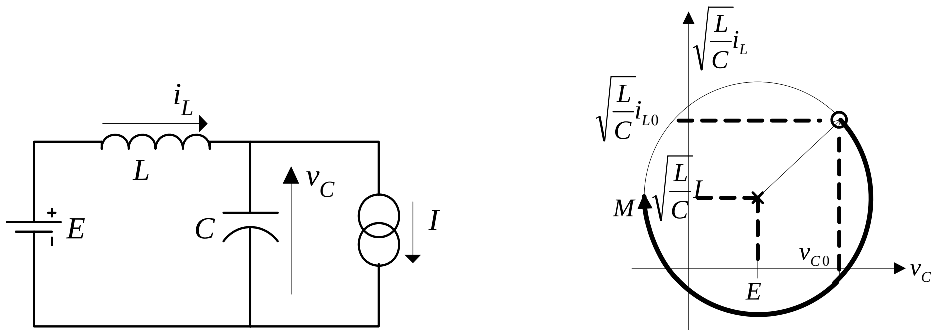


Fig. 4: Phase plane representation of the LCEI elementary circuit

– First, the point corresponding to the forced state is plotted in the graph (symbol *x*). In this case the point is *E* for the abscissas and $\sqrt{L/C}I$ for the ordinates;

– Second, the point corresponding to the initial conditions (symbol 0). is plotted: v_{C0} for the abscissas and $\sqrt{L/C}i_{\mathrm{L0}}$ for the ordinates;

– Third, a circle is centred on the forced state point and drawn through the initial conditions point. The rotation is clockwise. Note that for this sense of rotation to be correct, the capacitor voltage has to be in the same sense as the DC voltage source *E*, and the inductor current in the same sense as the DC current source *I* (current and voltage arrows oriented tip-to-tip, see Fig. 4).

The waveforms for the inductor current and the capacitor voltage in the time domain can be obtained directly from the phase plane representation (see Fig. 5).

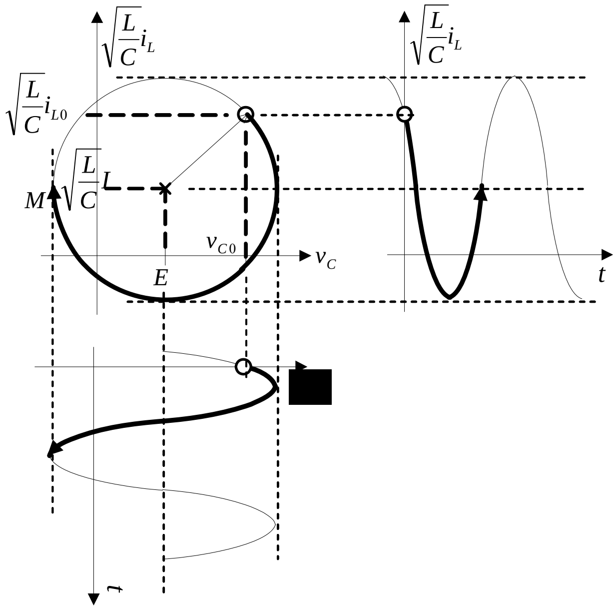


Fig. 5: Obtaining the waveforms in the time domain

As the point M moves clockwise around the circle with constant speed ω,

- the inductor current corresponds to the points in the ordinate;
- the voltage across the capacitor to the points in the abscissa.

The resulting waveforms are sinusoids with average values equal to the steady state current and voltages, and with a frequency equal to the oscillation frequency of the circuit, $\omega = 1/\sqrt{LC}$.

Some waveform quantities, such as the maximum and minimum values of inductor current and capacitor voltage can also be directly computed from the phase plane:

$$\text{oscillating magnitude (radius), } r = \sqrt{(v_{C0} - E)^2 + \frac{L}{C}(i_{L0} - I)^2} \tag{15}$$

$$\text{maximum inductor current, } i_{L_{max}} = I + \sqrt{\frac{C}{L}}r = I + \sqrt{\frac{C}{L}(v_{C0} - E)^2 + (i_{L0} - I)^2} \tag{16}$$

$$\text{minimum inductor current, } i_{L_{min}} = I - \sqrt{\frac{C}{L}}r = I - \sqrt{\frac{C}{L}(v_{C0} - E)^2 + (i_{L0} - I)^2} \tag{17}$$

$$\text{maximum capacitor voltage, } v_{C_{max}} = E + r = E + \sqrt{(v_{C0} - E)^2 + \frac{L}{C}(i_{L0} - I)^2} \tag{18}$$

$$\text{minimum capacitor voltage, } v_{C_{min}} = E - r = E - \sqrt{(v_{C0} - E)^2 + \frac{L}{C}(i_{L0} - I)^2} \ . \tag{19}$$

3.3 Response of a LCEI circuit with damping effect

If the damping effect has to be considered, owing for instance to the internal resistance in series of the inductance, the differential equation given by (7) now becomes:

$$i_{L\varphi} + RC\frac{\mathrm{d}i_{L\varphi}}{\mathrm{d}t} + LC\frac{\mathrm{d}^2 i_{L\varphi}}{\mathrm{d}^2 t} = 0\,. \tag{20}$$

The corresponding full response, in the case of a light damping effect ($R/L << 1/\sqrt{LC}$), is given by:

$$\begin{cases} i_L(t) = I + e^{-\alpha t}\left[C_1\cos(\omega t) + C_2\sin(\omega t)\right] \\ v_C(t) = E + e^{-\alpha t}\left[C_3\cos(\omega t) + C_4\sin(\omega t)\right] \end{cases} \tag{21}$$

where C_1, C_2, C_3, and C_4 are constants which depend on the initial conditions.

The phase plane representation of Eq. (21) gives a spiral instead of a circle (see Fig. 6).

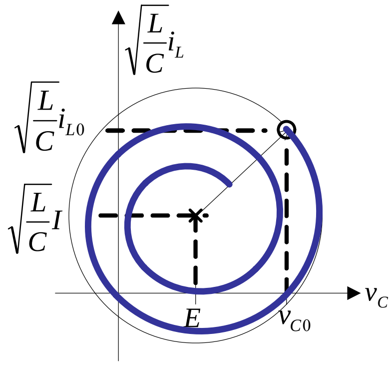


Fig. 6: Phase plane representation of the elementary LCEI circuit in case of 'light' damping

The spiral starts at the initial conditions point, runs clockwise, initially along the undamped circle, and converges towards the steady state operating point.

3.4 Practical examples

In this subsection, some simple examples of phase plane representations of power electronic circuits, including power switches, are given.

3.4.1 Charging a capacitor from a DC voltage source

The following circuit can be used to charge a capacitor with a limited current peak, from a constant DC voltage source (see Fig. 7). The charging process can be launched by firing the thyristor.

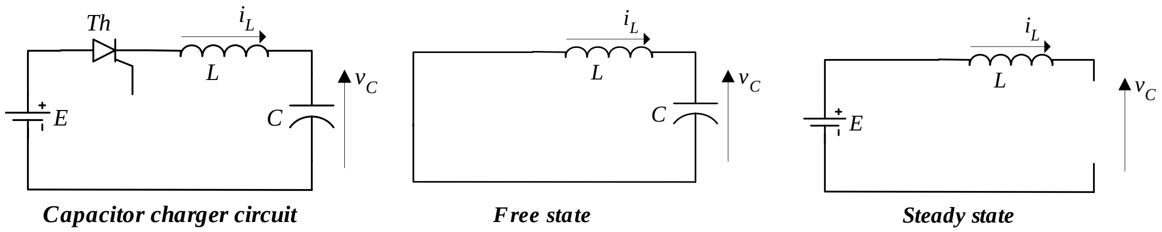


Fig. 7: Capacitor charger circuit with associated free-state and steady-state circuits, when the thyristor is 'on'

The phase plane corresponding to the charging process is depicted in Fig. 8, together with the plots of the voltage and current waveforms in the time domain.

The forced state point coordinates, corresponding to the steady-state circuit of Fig. 7, are (E, 0) because the DC current source is non-existent. Assuming the capacitor initially fully discharged (thyristor ‘off’), the initial conditions point is (0, 0). These points are marked in the phase plane of Fig. **8** by x and o, respectively.

Once the thyristor is fired, the operating point moves clockwise in a half circle centred at point (E, 0) and starting at point (0, 0). The capacitor voltage increases up to twice the value of the DC voltage source E. The inductor current starts by increasing, reaches the maximum value once the capacitor voltage is equal to E, and decreases back to zero at the end of the charging period.

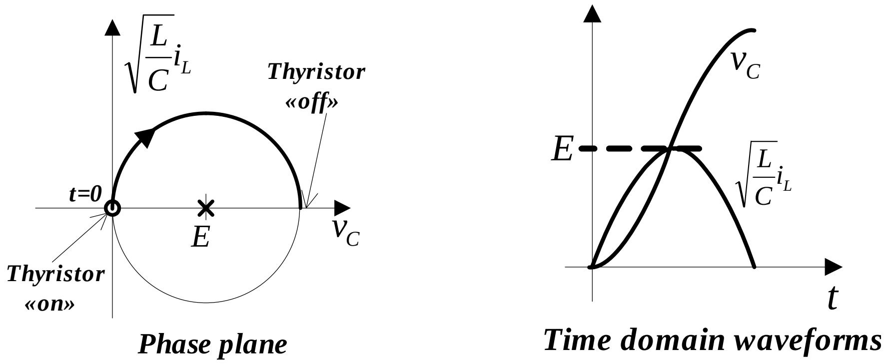


Fig. 8: Phase plane representation and time domain waveforms during the charge period, assuming zero initial conditions (thyristor off, capacitor fully discharged)

3.4.2 Thyristor rectifier with free-wheel thyristor CROWBAR

This example corresponds to a practical case study at CERN, the SPS main power converters. In many bipolar power converters, the three-phase network is rectified by a thyristor full bridge. The voltage ripple resulting from this rectification process is smoothed by a passive LC filter. Because the supply is supposed to be bipolar, free-wheeling diodes can not be used to discharge the magnet current in case of any converter’s fault. Instead, a thyristor is typically used, mounted according to a free-wheel thyristor (FWT) CROWBAR configuration. For the sake of simplicity and reliability, this thyristor is often triggered by a BOD (break-over diode) circuit connected between the anode and the gate. The idea is that when the voltage across the FWT reaches the break over voltage the BOD collapses, injecting a current pulse in the gate which fires the thyristor. This thyristor switches ‘off’ naturally, once its current reaches zero.

Most of the time, the CROWBAR system is placed as shown in Fig. 9, case 1. However, in this case a high current spike is generated in the capacitor when the FWT switches on. In order to bound this spike, the FWT can be located as shown in case 2, thus the capacitor current due to the discharge is limited by the filtering inductor at no extra cost.

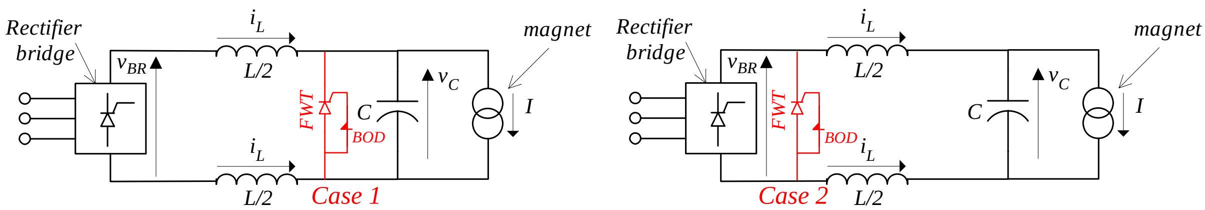


Fig. 9: Free-Wheel Thyristor (FWT) CROWBAR system to discharge the magnet in case of power converter failure: two different possible locations

It was observed that when the main network shuts down, the FWT occasionally switches ‘on’ and ‘off’ periodically (BANG BANG mode), instead of switching ‘on’ and remaining so until the magnet is discharged.

What is the explanation for this phenomenon and when does it occur? This question can be easily answered by using the phase plane representation.

In case 2, after the mains network shuts down, the last two conducting thyristors of the rectifier bridge remain 'on' for a short time. At this point, the FWT is still 'off'. The equivalent circuit with its free state and forced (or steady) state sub circuits is shown in Fig. 10.

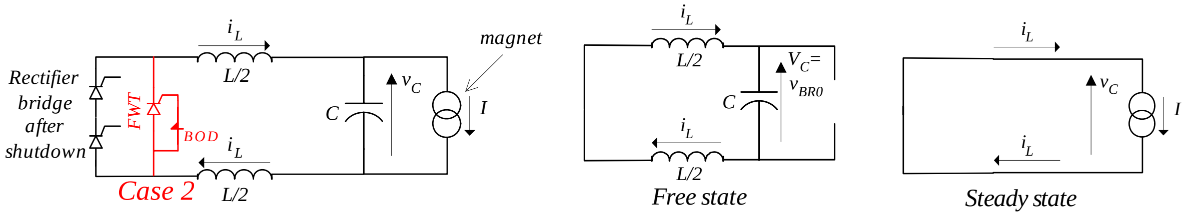


Fig. 10: Equivalent circuit after mains network shutdown; free-state and forced- (or steady-) state subcircuits

The forced state point is (0, $\sqrt{L/C}I$) and, assuming the initial voltage on the capacitor is equal to v_{BR0}, the initial conditions point is therefore (v_{BR0}, $\sqrt{L/C}I$). The circuit response is oscillating, therefore corresponding to part of a circle in the phase plane.

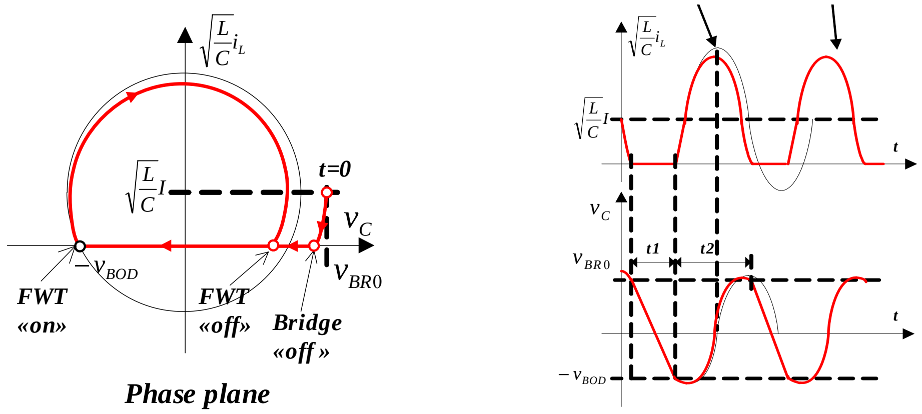


Fig. 11: Phase plane and waveforms in the time domain. Deviations from circle represent a light damping effect.

As shown in Fig. 11, once the inductor current reaches zero, both the rectifier bridge thyristors switch 'off' (gating signals are inhibited after mains network shutdown detection), and because the capacitor voltage remains positive, the FWT remains 'off'. The magnet current, which is considered to be constant at this time scale, will then flow through the capacitor. The inductor current is then zero and the capacitor voltage decreases linearly in time and becomes negative (straight line in the phase plane). Once the voltage reaches the break-over voltage of the BOD in the negative sense ($-v_{BOD}$), the FWT will be triggered 'on'. The free-state and forced-state circuits are the same as the former ones, as the FWT is in parallel with the rectifier bridge. The operating point follows another part of a new circle, cantered in (0, $\sqrt{L/C}I$) and starting in the new initial conditions point ($-v_{BOD}$, 0).

Once the inductor current reaches zero again, the FWT switches 'off', and the process is repeated several times until the magnet is discharged.

The voltage and currents in the time domain can be directly obtained from the phase plane, by employing the process depicted in Fig. 5.

As can be seen from the phase plane, the above-mentioned process takes place only if the magnet current is low enough for the rectifier bridge thyristors to switch off once the inductor current reaches zero (condition A in Fig. 12). If the magnet current is high enough, the inductor current remains always positive and therefore the two last conducting rectifier bridge thyristors will assure the full discharge of the magnet. This case (identified in Fig. 12 by condition B) may be destructive for those thyristors which are not dimensioned to resist the full magnet discharge, while starting with the high junction temperature, resulting from their previous normal operation.

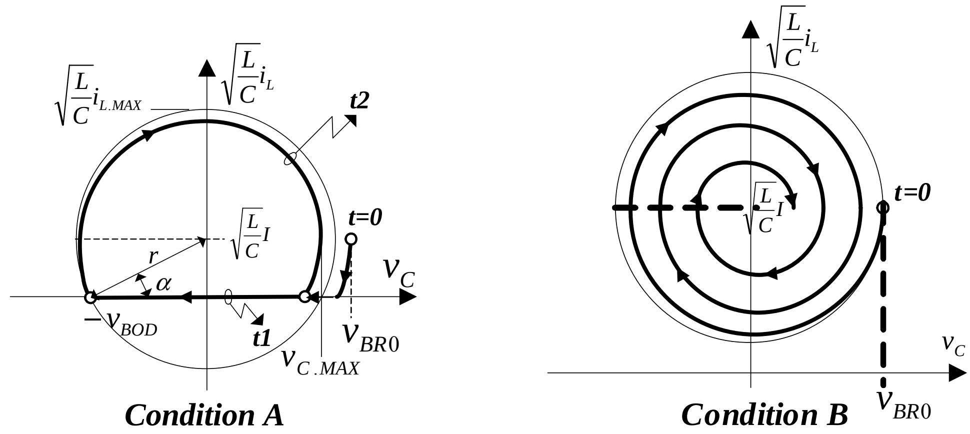


Fig. 12: Two possible conditions for magnet discharging

In summary:

<u>Condition A (as explained above)</u>, $v_{\mathrm{BR0}} > \sqrt{L/C}\,I$:

The rectifier bridge thyristors switch ‘off’. The FWT assures the magnet discharge in BANG-BANG mode. The maximum inductor current, maximum capacitor voltage and oscillating frequency can be expressed, in this order and neglecting damping effect, by the following equations:

$$i_{L_{\max}} \approx I + \sqrt{I^2 + \frac{C}{L} v_{\mathrm{BOD}}^2} \tag{22}$$

$$v_{\mathrm{C}_{\max}} \approx \sqrt{\frac{L}{C} I^2 + v_{\mathrm{BOD}}^2} \tag{23}$$

$$t1 \approx \frac{2Cv_{\mathrm{BOD}}}{I} \quad ; \qquad t2 \approx \left[\pi + 2a\tan\left(\frac{I\sqrt{L/C}}{v_{\mathrm{BOD}}} \right) \right] \sqrt{LC} \quad ; \qquad f_{\mathrm{OSC}} = \frac{1}{t1 + t2} \quad . \tag{24}$$

<u>Condition B</u>, $v_{\mathrm{BR0}} < \sqrt{L/C}\,I$

The rectifier bridge thyristors remain switched ‘on’ and assure full magnet discharge. The FWT never switches ‘on’ because it remains short circuited by the rectifier bridge thyristors.

Part II – Methods of study of power converters

4 Classification of the methods of study

Different methodologies apply to the study, analysis, simulation, or functional understanding of power converter circuits. Some of them are well suited to approximate quick engineering understanding and dimensioning, others are more oriented toward detailed simulation purposes. One possible classification is briefly presented below.

4.1 Analytical methods

These methods are based on the mathematical modelling of the circuit by a set of differential equations. Once the differential equations system is defined for a certain state of the power switches (sequence), the response of the circuit can be obtained by solving the differential equations system, taking into account the correct initial conditions of the state variables (inductor currents, capacitor voltages), which correspond to the values obtained at the end of the former sequence.

4.2 Graphical representations

This method consists in direct utilization of pre-computed look-up-tables, abacus, characteristics plots in p.u. units, etc.

4.3 Graphical/analytical methods: the phase plane

This is a mixture of the above two methods. It corresponds to the application of mathematical differential equations, supported by a graphical approach (like, for instance, time-domain waveform drawing) or tool (like the phase plane representation).

4.4 Simulation methods

These are based on specific computer software and CAD tools for complex analog and digital circuitry simulation. Some examples are: PSPICE, SABER, MATHLAB/SIMULINK, SIMPLORER, and PSIM.

This category of methods can be divided into two sub-groups.

4.4.1 Functional based methods

The power converter is considered as a 'black-box': only the input/output relationship or function is taken into account as a block diagram. No internal behaviour is analysed. This method presupposes that the output depends on the input variables only. It is not valid, for instance, in discontinuous mode, because in this case the output also depends on the internal state variables.

4.4.2 Sequential analytical methods

The converter operation is broken down into different sequences. In each sequence, all the power switches remain in the same state so that an equivalent linear circuit can be redrawn by replacing closed and opened switches by short circuits and open circuits, respectively. The method consists therefore in studying, in parallel, each individual sequence and the transition conditions between sequences. This sub-group can again be broken down into two classes: the methods without *a priori* knowledge where all possible sequences of the power converter are extensively analysed, and the methods with *a priori* knowledge in which additional information, often acquired by experience, is taken into account to discard some sequences that are *a priori* known to be unfeasible. This last method, which is one of the most efficient for numerous practical cases, will be detailed below.

5 Sequential analytical methods

5.1 The principle

The principle is based on the description of the power converter in a sequence-by-sequence approach. In order to accomplish this procedure, two major tasks have to be accomplished in parallel for each sequence:

a) analysing the conditions for a sequence transition;

b) computing the state evolutions during each sequence.

A sequence transition (task a) is accomplished whenever one or several switches change state.

Examples of conditions for switching 'on' events are: diodes: $V_T > 0$; thyristors: $V_T > 0$ and gate signal 'on'; transistors: gate signal 'on'.

Examples of conditions for switching 'off' events are: diodes and thyristors: $I_T = 0$; transistors: gate signal 'off'.

In order to accomplish task b, the voltages across each independent capacitor and the current in each independent inductor have to be derived by any analytical/graphical method. The initial conditions are the same as the final state values of the former sequence.

Knowing the expressions for these state variables, the expressions for the currents on each closed switch and for the voltage across each open switch can be derived.

At this stage, the analysis flow leads back to task a where the conditions for transition to the next sequence are evaluated.

5.2 Flowchart for a systematic analysis — remarks

The full procedure can be systematized by the following flowchart where each operation is done in the order indicated.

For a given sequence

1. Search for the order of the system

As explained in Section 2.2, the order of the system is equal to the number of state variables, which is the sum of independent capacitors and independent inductors.

2. Compute the expressions for:

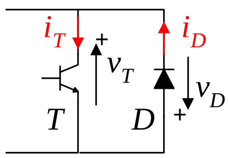
voltage across open switches and current flowing through closed switches, as a function of sources values and state variables (even though the expressions of the latter are not known at this stage).

Remarks:

1. The current flowing through each switch is taken as positive when it flows in the sense of the semiconductor conduction.

2. The voltage across each switch is taken in the opposite sense of the current flow (reception convention).

3. Each switch at closed state is considered as a voltage source, $V_D = \Delta$ (pn junction directly polarized).



3. Tests of compatibility: check for the existence of the given sequence

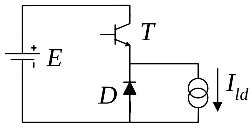
For all switches **except the last one changing state**, check for instance that the voltage across each open diode is negative, the current flowing through each closed switch is positive.

Remarks:

1. A state change of a switch may induce an instantaneous change on another switch, leading to several switching events at the same instant.

2. This new switching event produces another sequence.

*3. A new test of compatibility has to be performed in order to find the **stable sequence.***



T on => D off ; instantaneously
T off => D on ; instantaneously
T on and D on: inexistent sequence

4. Compute the expressions for the state variables

Find the solution of the equation system in forced and free states, taking as initial conditions the final conditions of the former sequence. In the case of LCEI type circuits, the phase plane representation (as will be seen in the next example) is an extremely powerful tool. It avoids finding the analytical equations at each point in time, by instead just drawing the curves in the phase plane.

5. Check for the events that may generate a switching

Types of events leading to a possible sequence transition are

Switches with natural switching:

natural turning off: current on the switch at closed state = 0;

natural turning on: voltage across the switch at open state ≥ 0.

Switches with forced switching:

forced turning off: current on the switch at closed state > 0 & gate signal 'off';

forced turning on: voltage across the switch at open state > 0 & gate signal 'on'.

6. Selection of the event leading to a sequence transition

Find the first event occurring among those listed in point 5.

This task often demands deep thought and knowledge of the system operating conditions.

5.3 Choice of the first sequence

In any problem of circuit analysis, the first question often asked is "Where to start?" In some cases, a poor choice of first sequence may lead to a 'divergent' result in the circuit analysis (turn-around). This is the case of circuits where initial conditions for a steady state cycle are not obvious to determine: they depend on the way transient operation behaves, pre-charging procedures, etc.

However, when the process starts with a 'good sequence', then the former algorithm leads, step-by-step, directly to the end of the study, without any possibility for errors like missing sequence or deadlocks.

This is the only part of the algorithm which is not fully automatic and where some prior knowledge or feeling about the circuit can be tremendously helpful.

It is not possible to systematically define rules for this starting point. However, there are some practical rules for choosing the first sequence, which result in consistency of the analysis and convergence:

1. choose a sequence where the load is connected to the source or (active phase);
2. choose a sequence corresponding to a free-wheeling state;
3. choose a sequence corresponding to a discontinuous conduction state.

In some cases, one of these rules is initially applied and if any mismatch occurs later, others are tried.

5.4 Example: study of a thyristor chopper

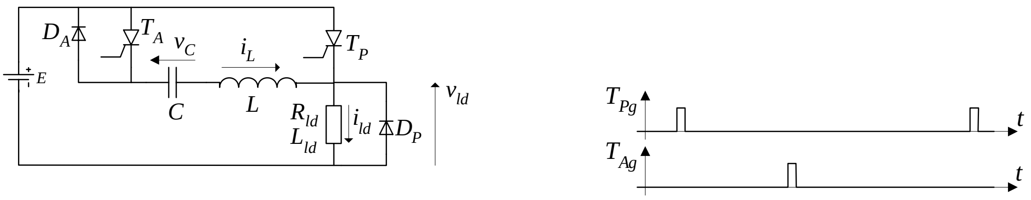
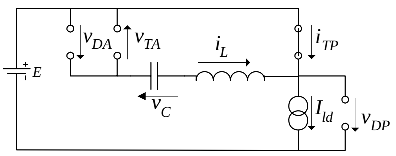


Fig. 13: Thyristor chopper circuit and firing signals

The example of the power electronics circuit shown in Fig. 13 illustrates the principle of the method. It is called a thyristor chopper because it generates a chopped voltage from a constant DC source E with a variable average value. The thyristor T_P is fired 'on' when the gate signal T_{Pg} is applied and it is switched 'off' once the auxiliary thyristor T_A is fired 'on' by the signal T_{Ag}. The analysis procedure given by the flowchart of Section 5.2 is described sequence-by-sequence below. The time domain waveforms of the main variables are drawn further on Fig. 14.

Sequence 1

For the initial sequence, an active phase is considered (T_P 'on' and D_P 'off') and the voltage across the capacitor v_C is assumed to be negative.



1) Search for the order of the system
LC circuit is open $\Rightarrow$ No evolution of the state variables

2) Voltages & currents on semiconductors
$v_{TA} = -v_C$; $v_{DA} = +v_C$; $v_{DP} = -E$; $i_{TP} = I_{ld}$

3) Test of compatibility
$v_{DA} < 0$; $v_{DP} < 0$; $i_{TP} > 0$; $v_{TA} > 0$ (*no gate signal*)
Seq. OK

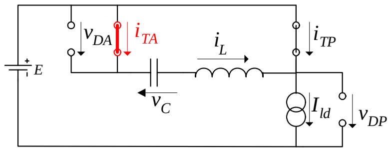
4) Evolution of state variables
No evolution

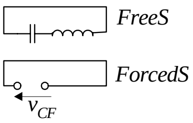
5) Events that may generate a switching
$v_C < 0 \Rightarrow v_{TA} > 0 \Rightarrow$ ***T*** $\boldsymbol{T_A}$ **switches 'on'** if gate signal applied

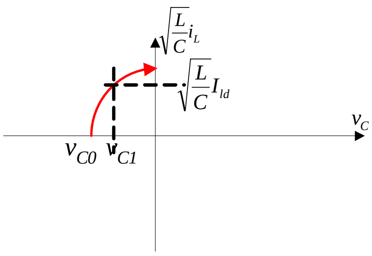
6) The event leading to a sequence transition
$\boldsymbol{T_A}$ **switches 'on'** when a gate signal is applied
Final condition: $v_C = v_{C0}$; $i_L = 0$

Note:
In (3) the voltage across D_P is negative. Because $v_C < 0$, the voltage across D_A is also negative. The current flowing through the closed thyristor T_P is positive. The voltage across the open thyristor T_A is positive, but no gate signal is applied yet. The sequence exists.

Sequence 2 (T_A has switched 'on')







1) Search for the order of the system
Free-state circuit of 2[nd] order ⇒ Circle in the phase plane

2) Voltages & currents on semiconductors
$i_{TA} = i_L$; $v_{DA} = -\Delta$; $v_{DP} = -E$; $i_{TP} = I_{ld} - i_L$

3) Test of compatibility
$v_{DA} < 0$; $v_{DP} < 0$; $i_{TP} > 0$; **Seq. OK**

4) Evolution of state variables
Circle: Centre @ (v_{CF}, i_{LF}) = (0, 0);
Starting point @ (v_{C0}; i_{L0}) = (v_{C0}, 0)

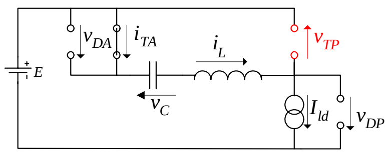
5) Events that may generate a switching
T_A may switch off if i_{TA} = 0 => i_L = 0; D_A and D_P keep off;
T_P may switch off if i_{TP} = 0 => i_L=I_{ld}

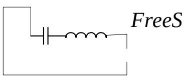
6) The event leading to a sequence transition
T_P switches 'off' before T_A

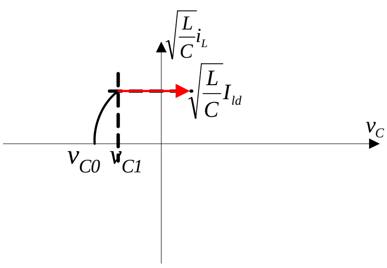
Final condition: $v_C = v_{C1}$; $i_L = I_{ld}$

Note:
In (3) the sequence exists because the voltages across all open diodes are negative and the current flowing through the closed thyristor T_P is positive. Note that no evaluation is made for T_A because it is the last switch changing state (see explanation of task 3 in Section 5.2).

Sequence 3 (T_P has switched 'off')







1) Search for the order of the system
Free-state circuit in open loop ⇒ straight line in the phase plane

2) Voltages & currents on semiconductors
$i_{TA} = i_L = I_{ld}$; $v_{DA} = -\Delta$; $v_{DP} = -E + v_C + v_L = -E + v_C$;

$v_{TP} = v_C + v_L = v_C$

3) Test of compatibility
$v_{DA} < 0$; $v_{DP} = -E + v_{C1} < 0$; $i_{TA} = i_L = I_{ld} > 0$; **Seq. OK**

4) Evolution of state variables
Straight line: $i_L = I_{ld} = Cst$; $v_C(t) = I_{ld}/C*t + v_{C1}$

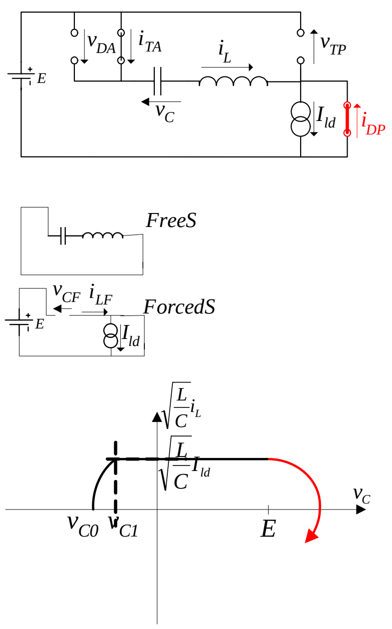
5) Events that may generate a switching
T_A keeps on; D_A keeps off; D_P switches on if $v_C > E$;
T_P switches on if $v_C > 0$ and gate signal applied

6) The event leading to a sequence transition
T_P must not switch on (lost of control); **D_P switches 'on'** when $v_C > E$

Final condition: $v_C = E$; $i_L = I_{ld}$

Note:
In (1) the capacitor is charged with constant current (load current).In (3) the sequence exists because the voltages across all open diodes are negative and the current flowing through the closed thyristor T_A is positive. No evaluation of T_P.In (6) the T_P must not be switched on at this stage to avoid loss of control. T_P can be switched on again when the transient state of the circuit is completely finished.

Sequence 4 (D_P has switched 'on')



1) *Search for the order of the system*
Free-state circuit of 2nd order ⇒ Circle in the phase plane

2) *Voltages & currents on semiconductors*
$i_{TA} = i_L\,;\quad v_{DA} = -\Delta\,;\quad i_{DP} = I_{ld} - i_L\,;\quad v_{TP} = E$

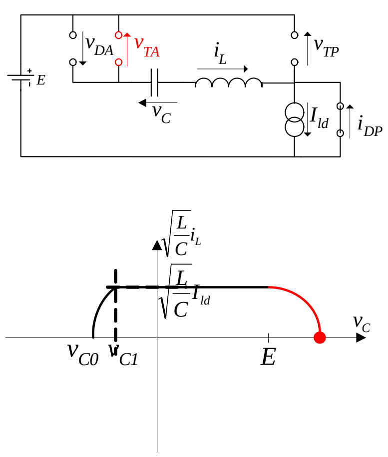
3) *Test of compatibility*
$v_{DA} < 0\,;\quad i_{TA} = i_L > 0\,;\quad v_{TP} = E > 0$ (*no gate signal*) ;
Seq. OK

4) *Evolution of state variables*
Circle: $(v_{CF}, i_{LF}) = (E, 0)$; $(v_{C0}, i_{L0}) = (E, I_{ld})$

5) *Events that may generate a switching*
T_A switches off if $i_L = 0$; D_A keeps off; D_P switches off if $i_L = I_{ld}$
T_P switches on if gating signal applied

6) *The event leading to a sequence transition*
T_P must not switch on (lost of control); **T_A switches 'off'** before D_P
Final condition: $v_C = E + I_{ld}\sqrt{L/C}\,;\ i_L = 0$

Sequence 4a (T_A has switched 'off')



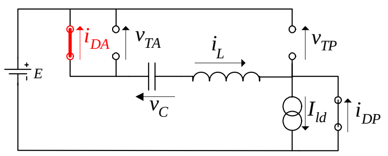
1) *Search for the order of the system*
LC circuit is open ⇒ No evolution of the state variables

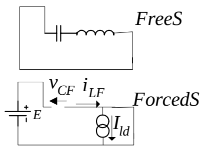
2) *Voltages & currents on semiconductors*
$v_{TA} = E - v_C\,;\quad v_{DA} = v_C - E\,;\quad i_{DP} = I_{ld}\,;\quad v_{TP} = E$

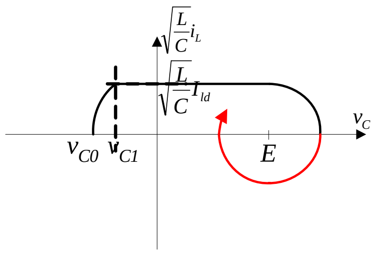
3) *Test of compatibility*
$v_{DA} = v_C - E > 0\,;\quad v_{TP} = E > 0$ (*no gate signal*) ; $i_{DP} > 0$;
Seq. not OK

The voltage across diode D_A is positive.
This sequence has no physical existence
(D_A switches on at the same time T_A switches off).

Sequence 5 (D_A has switched 'on')







1) *Search for the order of the system*
Free-state circuit of 2nd order $\Rightarrow$ Circle in the phase plane

2) *Voltages & currents on semiconductors*
$v_{\mathrm{TA}} = -\Delta\,;\quad i_{\mathrm{DA}} = -i_{\mathrm{L}}\,;\quad i_{\mathrm{DP}} = I_{\mathrm{ld}} - i_{\mathrm{L}}\,;\quad v_{\mathrm{TP}} = E$

3) *Test of compatibility*
$v_{\mathrm{TA}} < 0\,;\quad i_{\mathrm{DP}} > 0\,;\quad v_{\mathrm{TP}} > 0$ (*no gate signal*) ;
Seq. OK

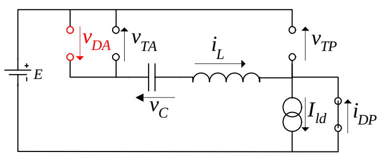
4) *Evolution of state variables*
Circle: $(v_{\mathrm{CF}}, i_{\mathrm{LF}}) = (E, 0)$; $(v_{\mathrm{C0}}; i_{\mathrm{L0}}) = (E + I_{\mathrm{ld}}\sqrt{L/C}, 0)$

5) *Events that may generate a switching*
T_A keeps off; D_A switches off if $i_L = 0$; D_P switches off if $i_L = I_{ld}$
T_P switches on if gating signal applied

6) *The event leading to a sequence transition*
T_P must not switch on (loss of control); **D_A switches 'off'** before D_P.

Final condition: $v_{\mathrm{C}} = E - I_{\mathrm{ld}}\sqrt{L/C}\,;\; i_{\mathrm{L}} = 0\,.$

Sequence 6 (D_A has switched 'off')



1) *Search for the order of the system*
LC circuit is open $\Rightarrow$ No evolutions of the state variables;

2) *Voltages & currents on semiconductors*
$v_{\mathrm{TA}} = E - v_{\mathrm{C}}\,;\quad v_{\mathrm{DA}} = v_{\mathrm{C}} - E\,;\quad i_{\mathrm{DP}} = I_{\mathrm{ld}}\,;\quad v_{\mathrm{TP}} = E$

3) *Test of compatibility*
$v_{\mathrm{TA}} > 0$ (*no gate signal*) ; $\quad i_{\mathrm{DP}} > 0\,;\quad v_{\mathrm{TP}} > 0$ (*no gate signal*) ;
Seq. OK

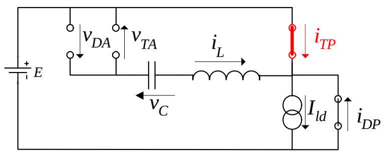
4) *Evolution of state variables*
No evolution

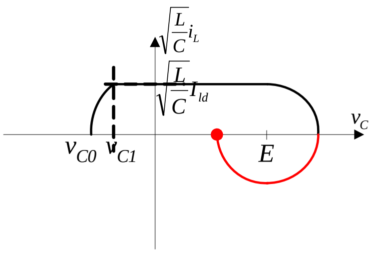
5) *Events that may generate a switching*
T_A switches on if gate signal; D_A keeps off; D_P keeps on;
T_P switches on if gate signal applied

6) *The event leading to a sequence transition*
T_A must not switch on (loss of control); **T_P switches on** with gate signal.

Final condition: $v_{\mathrm{C}} = E - I_{\mathrm{ld}}\sqrt{L/C}\,;\; i_{\mathrm{L}} = 0\,.$

Sequence 6a (T_P has switched 'on')





1) Search for the order of the system
LC circuit is open $\Rightarrow$ No evolutions of the state variables

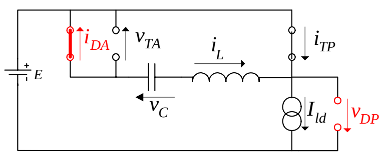
2) Voltages & currents on semiconductors
$v_{TA} = -v_C$; $v_{DA} = v_C$; $i_{DP} = I_{ld} - I_{CC}$; $i_{TP} = I_{CC}$
$I_{CC} =$ *short circuit current of* E

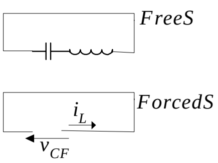
3) Test of compatibility
$v_{DA} > 0$; $v_{TA} < 0$; $i_{DP} < 0$; **Seq. not OK**

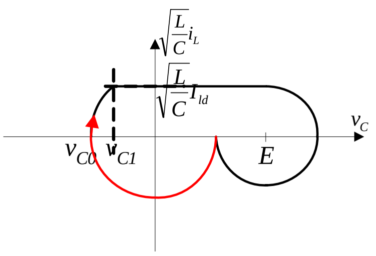
Voltage across diode D_A is positive
$\Rightarrow$ D_A switches 'on' immediately
Current on diode D_P is negative
$\Rightarrow$ D_P switches 'off' immediately

This sequence has no physical existence
(Once T_P switches 'on'; D_P switches 'off' and D_A switches on)

Sequence 7 (T_P has switched 'on'; D_P has switched 'off'; D_A has switched 'on')







1) Search for the order of the system
Free-state circuit of 2nd order $\Rightarrow$ Circle in the phase plane

2) Voltages & currents on semiconductors
$v_{TA} = -\Delta$; $i_{DA} = -i_L$; $v_{DP} = -E$; $i_{TP} = I_{ld} - i_L$

3) Test of compatibility
$v_{TA} < 0$; $v_{DP} < 0$; **Seq. OK**

4) Evolution of state variables
Circle: $(v_{CF}, i_{LF}) = (0,0)$; $(v_{C0}; i_{L0}) = (E - I_{ld}\sqrt{L/C}, 0)$

5) Events that may generate a switching
T_A keeps off; D_A switches off if $i_L = 0$; D_P keeps off
T_P switches off if $i_L = I_{ld}$

6) The event leading to a sequence transition
D_A switches 'off' *before* T_P

Final condition: $v_C = -(E - I_{ld}\sqrt{L/C})$; $i_L = 0$

The complete phase plane and time domain waveforms are shown in Fig. 14. Each part of the phase plane path is identified by the corresponding sequence number.

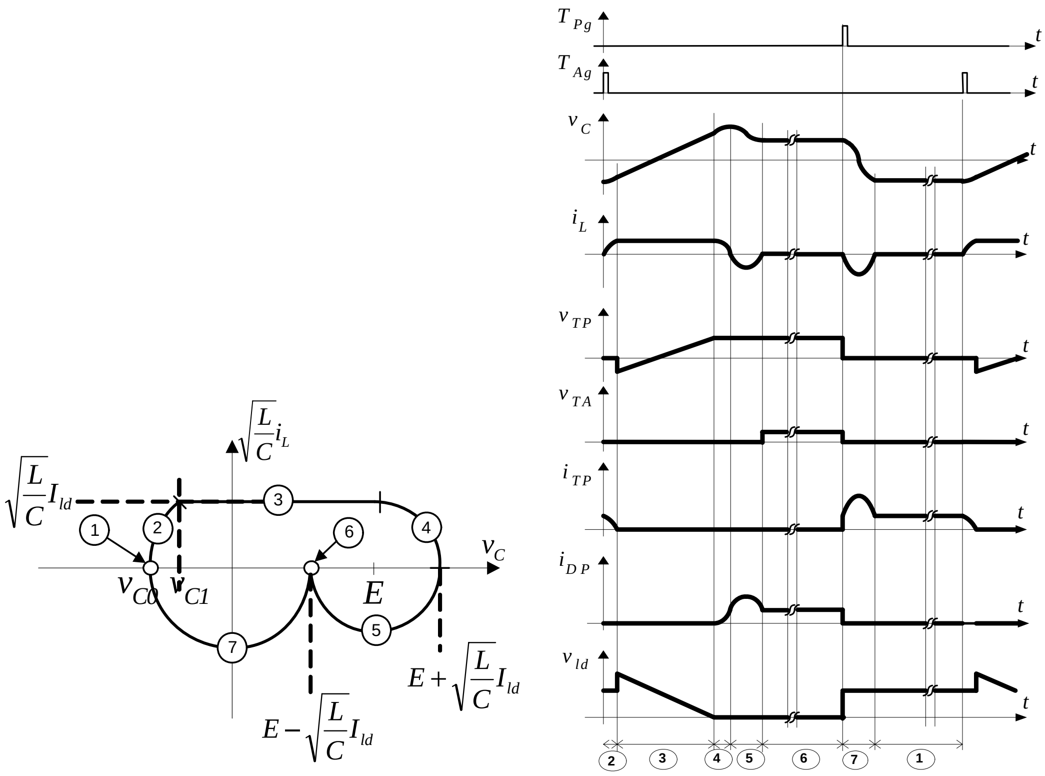


Fig. 14: Phase plane representation and time domain waveforms

As shown in Fig. 14, at the beginning of the cycle the current on the inductor is zero and the capacitor is charged at the initial negative voltage $v_{C0} = -E + \sqrt{L/C} I_{ld}$ (sequence 1). Once T_P is fired the cycle starts with sequence 2. Voltage v_C decreases in absolute value while i_L increases to reach I_{ld}, following a sine wave. In sequence 3, i_L remains constant and equal to I_{ld}, while v_C increases linearly in time to reach *E*. In sequences 4 and 5, i_L decreases down to zero following a cosine function in time. Voltage v_C increases and then decreases following a sine wave with offset *E*. Once i_L reaches zero (sequence 6), the chopper is definitely at the 'on' state. The auxiliary capacitor is charged with a positive voltage $E - \sqrt{L/C} I_{ld}$ and ready to block T_P during the second part of the cycle (sequence 6). This second part starts when the auxiliary thyristor T_A is triggered (sequence 7). The inductor current i_L increases in the negative sense following a sine wave, whereas v_C decreases following a cosine function. The sequence and cycle end when i_L reaches zero. Once the waveforms of the state variables i_L and v_C are known, it is very easy to compute all the other waveforms from the equations that have been derived within each sequence.

Part III – Modelling power converters for control loop design

In this part we overview some techniques for power converter modelling, the objective being to establish the transfer function between the input, usually a duty-cycle signal, and the output (i.e. output voltage, current, etc.). Any switch mode power converter is hybrid system: thus electronic switches with discrete time behaviour are mixed with passive devices like capacitors and inductors which have continuous time behaviour (capacitor voltages and inductor currents cannot change value instantaneously). In control theory semantics, the power converter can be classified as an actuator delivering power to a load by acting on a reference signal. The reference signal is analog, the output variable to be controlled is also analog, but the internal behaviour of the power converter as an actuator is mixed digital and analog. How to cope with this peculiarity is explained below.

6 Modelling techniques for power converters

6.1 The purpose: control oriented modelling

The power converter is considered as an actuator delivering an output quantity, $y(t)$—load voltage or current—as a function of an input signal, $u(t)$ —duty-cycle, firing angle—directly linked to the control of the electronic switches. Several external perturbations, $p(t)$ —mains disturbances, sensors noise—may have an influence on the dynamic behaviour (Fig. 15).

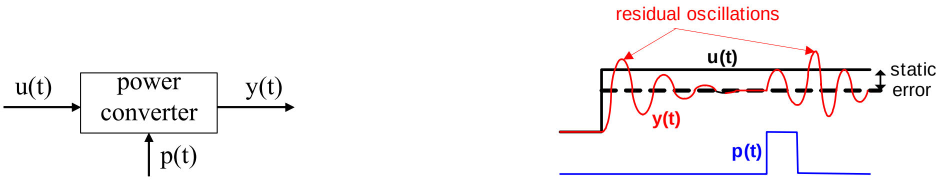


Fig. 15: Open loop system

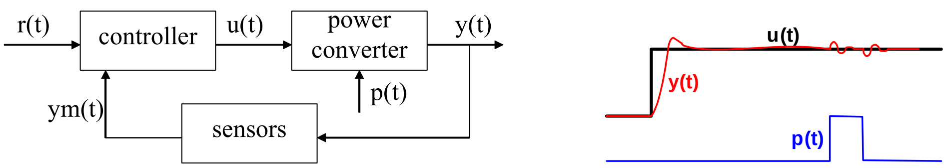


Fig. 16: Closed loop system

Power converters are usually integrated in closed loop schemes (Fig. 16) in order to improve the dynamic response of the global system to an input command, $r(t)$, to avoid or reduce residual oscillations and static errors and to improve immunity to external perturbations. Application of modelling techniques is therefore mandatory for an efficient analysis of the loop stability and calculation of the controllers' parameters, in order to tune the system to achieve the specified dynamic performances.

6.2 State space models

In the state space domain, the system is represented by the following state equations:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{25}$$

where: u is the input reference vector, x the state vector, y the output variables vector, and A,B,C,D are the constant matrices.

As an example, the boost DC/DC converter of Fig. 17 is considered. Afterwards, the method is generalized.

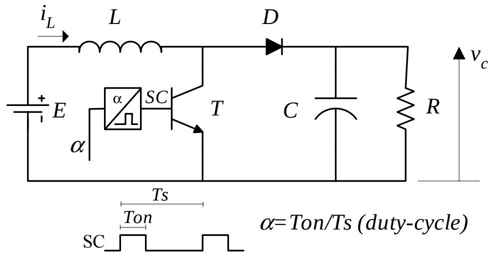
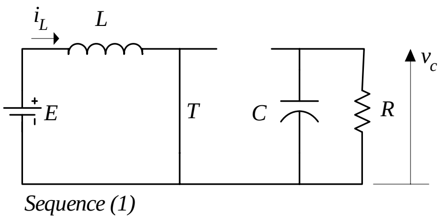


Fig. 17: Boost DC/DC converter

The objective of the present case study is to establish the transfer function between the transistor duty cycle α and the output voltage v_C in Laplace domain. For the sake of simplicity, the inductor current i_L will be considered always positive (continuous mode operation). In this case, the circuit operation can be decomposed into only two sequences: sequence 1, where transistor *T* is closed and diode *D* is open, and sequence 2, where transistor *T* is open and diode *D* is closed. The state space representation is shown below, for each sequence.

Sequence 1, duration αT_S:

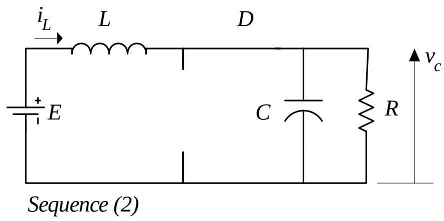


$$E = L\frac{\mathrm{d}i_\mathrm{L}}{\mathrm{d}t};\;\; v_\mathrm{C} = -RC\frac{\mathrm{d}v_\mathrm{C}}{\mathrm{d}t}$$

$$\underbrace{\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{RC} \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} i_L \\ v_C \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1/L \\ 0 \end{bmatrix}}_{B_1} \underbrace{E}_{u}$$

$$\begin{cases} \dot{x} = A_1 x + B_1 u \\ y = C_1 x + D_1 u \end{cases}$$

Sequence 2, duration $(1\text{-}\alpha)T_S$:



$$E = L\frac{\mathrm{d}i_\mathrm{L}}{\mathrm{d}t} + v_\mathrm{C};\;\; i_\mathrm{L} = C\frac{\mathrm{d}v_\mathrm{C}}{\mathrm{d}t} + \frac{v_\mathrm{C}}{R}$$

$$\underbrace{\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & -1/L \\ 1/C & -\dfrac{1}{RC} \end{bmatrix}}_{A_2} \underbrace{\begin{bmatrix} i_L \\ v_C \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1/L \\ 0 \end{bmatrix}}_{B_2} \underbrace{E}_{u}$$

$$\begin{cases} \dot{x} = A_2 x + B_2 u \\ y = C_2 x + D_2 u \end{cases}$$

If the output vector *y* is supposed to be the output voltage v_C then $y = v_\mathrm{C} \Rightarrow C_1 = C_2 = [0,\ 1];$.

The next step is to solve the differential equation system over a switching period, in order to extract the state vector *x*.

Because $u(t)$ = constant = E, the integration of the first state space equation yields:

$$\begin{aligned} \dot{x} = Ax + Bu \Rightarrow x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}B \cdot E d\tau \\ \Rightarrow\; x(t) &= e^{At}x(0) + A^{-1}(e^{At} - I)B \cdot E\,. \end{aligned} \tag{26}$$

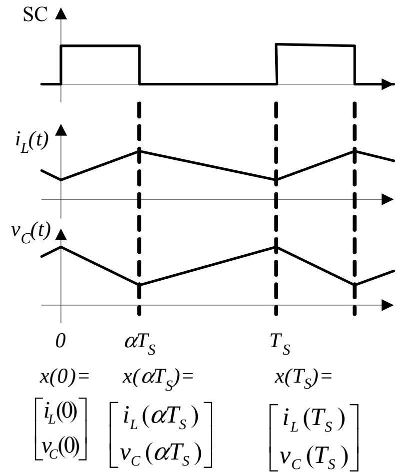


Fig. 18: State variable evolution at the switching period scale

By applying Eq. (26) to the present example, the state space vector is computed at each transition instant of one period as follows:

$$\begin{cases} x(\alpha T_S) = e^{A_1 \alpha T_S} x(0) + A_1^{-1}(e^{A_1 \alpha T_S} - I)B_1 E \\ x(T_S) = e^{A_2 (1-\alpha) T_S} x(\alpha T_S) + A_2^{-1}(e^{A_2 (1-\alpha) T_S} - I)B_2 E \end{cases},$$

or:

$$\begin{cases} x(\alpha T_S) = F_1 x(0) + G_1 E \\ x(T_S) = F_2 x(\alpha T_S) + G_2 E \end{cases} \quad \text{with} \quad \begin{matrix} F_1 = e^{A_1 \alpha T_S};\ G_1 = A_1^{-1}(e^{A_1 \alpha T_S} - I)B_1 E \\ F_2 = e^{A_2 (1-\alpha) T_S};\ G_2 = A_2^{-1}(e^{A_2 (1-\alpha) T_S} - I)B_2 E \end{matrix} \tag{27}$$

where $x(0)$, $x(\alpha T_S)$, $x(T_S)$ are the state vector, respectively, at the beginning of the switching period, at the instant of sequence transition, and at the end of the switching period (Fig. 18).

By substituting the first equation of (27) into the second one, a unique equation giving the state vector at each sampling period is obtained:

$$\begin{aligned} x(T_S) &= F_2(F_1 x(0) + G_1 E) + G_2 E \\ &= F_2 F_1 x(0) + (F_2 G_1 + G_2)E \qquad \text{with} \quad F = F_2 F_1, G = F_2 G_1 + G_2\ . \\ x(T_S) &= Fx(0) + GE \end{aligned} \tag{28}$$

Matrices F and G contain terms with exponential matrices. They can be further simplified by approximating the exponential to the first order. This approximation is valid if the state variable time constants are larger compared to the switching period (at least 3 times).

$$\begin{aligned} e^{A_1 \alpha T_S} &\approx I + A_1 \alpha T_S + \frac{(A_1 \alpha T_S)^2}{2!} + \ldots \\ e^{A_2 (1-\alpha) T_S} &\approx I + A_2 (1-\alpha) T_S + \frac{(A_2 (1-\alpha) T_S)^2}{2!} + \ldots \end{aligned}$$

By neglecting the second-order terms:

$$\begin{aligned} e^{A_1 \alpha T_S} &\approx I + A_1 \alpha T_S \\ e^{A_2 (1-\alpha) T_S} &\approx I + A_2 (1-\alpha) T_S \end{aligned}\ . \tag{29}$$

By substituting, first (29) into (27) and then (27) into (28), and neglecting second-order terms in T_S^2, the following result for matrices F and G is obtained:

$$\begin{aligned} F &\approx (I + A_2(1-\alpha)T_S)\cdot(I + A_1\alpha T_S) = I + A_1\alpha T_S + A_2(1-\alpha)T_S \\ G &\approx (I + A_2(1-\alpha)T_S)\cdot(A_1^{-1}(I + A_1\alpha T_S - I)\cdot B_1) + A_2^{-1}(I + A_2(1-\alpha)T_S - I)\cdot B_2 \\ &= \alpha T_S B_1 E + (1-\alpha)T_S B_2 E\,. \end{aligned} \tag{30}$$

By substituting (30) into (28), the discrete-time state equation giving the state variables at the end of each switching period is obtained as follows:

$$\begin{aligned} x(T_S) &= Fx(0) + GE = \\ &= (I + A_1\alpha T_S + A_2(1-\alpha)T_S)x(0) + (B_1\alpha T_S + B_2(1-\alpha)T_S)E\,. \end{aligned} \tag{31}$$

Applying a linear approximation to the state variables within a switching period (Fig. 19), the continuous-time state equation can be derived from Eq. (31) by linear interpolation:

$$\dot{x} = \frac{x(T_S) - x(0)}{T_S} = \underbrace{\left[A_1\alpha + A_2(1-\alpha)\right]}_{A} x(0) + \underbrace{\left[B_1\alpha + B_2(1-\alpha)T_S\right]}_{B} E\,. \tag{32}$$

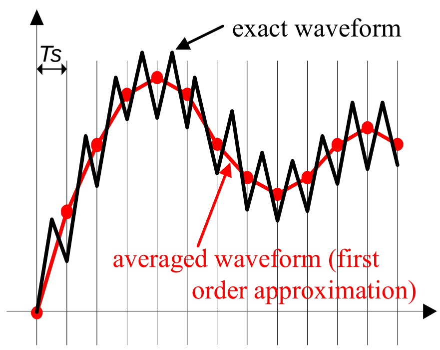


Fig. 19: State variable averaging

In general, any power converter with two sequences can be modelled in state space by the following state space equations:

Averaged state space model

$$\begin{cases} \dot{x} = Ax + B\cdot E \\ y = Cx + D\cdot E \end{cases} \text{ with } \begin{cases} A = A_1\alpha + A_2(1-\alpha) \\ B = B_1\alpha + B_2(1-\alpha) \end{cases} \text{ and } \begin{cases} C = C_1\alpha + C_2(1-\alpha) \\ D = D_1\alpha + D_2(1-\alpha) \end{cases}. \tag{33}$$

Rule for setting up the averaged state space model

The matrix A of the global circuit is the sum of the matrices of the circuit corresponding to each sequence (A_1 and A_2), weighted by their 'existence duration'. The same applies to matrices B, C, and D.

Note, however that, according to the assumptions made above, this model can be applied only in the following conditions:

a) the time constants of all state variables are significantly higher than the switching period;

b) the dynamic behaviour of all state variables is not influenced by their harmonic content at switching frequency and higher terms. This is a consequence of the linear interpolation and averaging illustrated in Fig. 19.

Coming back to the former boost DC/DC converter example, matrices A_1, B_1, A_2, B_2 of the first equation of the state space system are recalled below:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{RC} \end{bmatrix} ; \quad B_1 = \begin{bmatrix} 1/L \\ 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & -1/L \\ 1/C & -\dfrac{1}{RC} \end{bmatrix} ; \quad B_2 = \begin{bmatrix} 1/L \\ 0 \end{bmatrix} . \tag{34}$$

Using the above rule, matrices A and B become:

$$A = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{RC} \end{bmatrix}}_{A_1} \alpha + \underbrace{\begin{bmatrix} 0 & -1/L \\ 1/\mathrm{C} & -\dfrac{1}{RC} \end{bmatrix}}_{A_2} (1-\alpha) \qquad B = \underbrace{\begin{bmatrix} 1/L \\ 0 \end{bmatrix}}_{B_1 = B_2} . \tag{35}$$

By applying Eq. (33), the first equation of the state space model is

(Averaged state space model for the boost DC/DC converter)

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -\dfrac{1}{L}(1-\alpha) \\ \dfrac{1-\alpha}{C} & -\dfrac{1}{RC} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1/L \\ 0 \end{bmatrix}}_{B} E . \tag{36}$$

As can be seen, matrix A depends on α. This result, which applies to the majority of power converters, means that the system is non-linear. In order to study the system behaviour in the vicinity of a given operating point, a small-signal linear model will be derived.

Small-signal linearization

By assuming E to be constant, only x and α may vary with time:

$$\begin{aligned} &E \text{ is constant } [E(s) = E] \\ &x = x_0 + \hat{x} \quad ; \quad \alpha = \alpha_0 + \hat{\alpha} \end{aligned} \tag{37}$$

where x_0 and α_0 are, respectively, the state vector and the duty cycle in steady state, for the given operating point; $\hat{x}$ and $\hat{\alpha}$ are the small-signal quantities for the state vector and duty cycle, respectively.

Computation of the DC operating point

$$\begin{aligned} &\dot{x} = A(\alpha)x + B(\alpha)E \\ &\dot{x} = 0 \Rightarrow x_0 = -A(\alpha_0)^{-1} \cdot B(\alpha_0) \cdot E . \end{aligned} \tag{38}$$

Linearization

Because $\dot{x}$ = $f(x,\alpha,E)$, /E = const, the small signal equation can be derived from Eq. (38), by partial derivation:

$$\dot{\hat{x}} = \left. \frac{\partial f}{\partial x} \right|_{(x_0, \alpha_0)} \hat{x} + \left. \frac{\partial f}{\partial \alpha} \right|_{(x_0, \alpha_0)} \hat{\alpha} . \tag{39}$$

By applying Eq. (39) to the former averaged state space model (33), one obtains:

$$\begin{aligned}\dot{\hat{x}} &= \underbrace{A(\alpha_0)}_{A_0}\hat{x} + \underbrace{\left[(A_1 - A_2)x_0 + (B_1 - B_2)E\right]}_{B_0}\hat{\alpha} \\ \hat{y} &= \underbrace{C(\alpha_0)}_{C_0}\hat{x} + \underbrace{\left[(C_1 - C_2)x_0 + (D_1 - D_2)E\right]}_{D_0}\hat{\alpha}\end{aligned} \quad . \tag{40}$$

Note that in Eq. (39) the new input of the system becomes $\hat{\alpha}$..

Small-signal average state space model

$$\begin{cases}\dot{\hat{x}} = A_0\hat{x} + B_0\hat{\alpha} \\ \hat{y} = C_0\hat{x} + D_0\hat{\alpha}\end{cases} \quad \text{with} \quad \begin{cases}A_0 = A(\alpha_0); \quad B_0 = (A_1 - A_2)x_0 + (B_1 - B_2)E \\ C_0 = C(\alpha_0); \quad D_0 = (C_1 - C_2)x_0 + (D_1 - D_2)E\end{cases} . \tag{41}$$

By applying Laplace transformation to Eq. (41), the small signal-transfer functions are obtained as a result:

$$s\hat{x}(s) = A_0\hat{x}(s) + B_0\hat{\alpha}(s) \quad \Rightarrow \quad \begin{cases}\dfrac{\hat{x}(s)}{\hat{\alpha}(s)} = \left[sI - A_0\right]^{-1} \cdot B_0 \\ \dfrac{\hat{y}(s)}{\hat{\alpha}(s)} = C_0(sI - A_0)^{-1}B_0 + D_0\end{cases} . \tag{42}$$

Small-signal average state space model of the boost DC/DC converter

Coming back to the boost DC/DC converter, the DC operating point can be computed from Eq. (38), giving:

$$x_0 = -A(\alpha_0)^{-1} \cdot B(\alpha_0) \cdot E = \frac{LC}{(1-\alpha_0)^2}\begin{bmatrix} -\dfrac{1}{RC} & \dfrac{1}{L}(1-\alpha_0) \\ -\dfrac{1}{C}(1-\alpha_0) & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/L \\ 0 \end{bmatrix} \cdot E = E\begin{bmatrix} \dfrac{1}{R(1-\alpha_0)^2} \\ \dfrac{1}{1-\alpha_0} \end{bmatrix}$$

or:

$$\begin{aligned} i_{L0} &= \frac{E}{R}\frac{1}{(1-\alpha_0)^2} \\ v_{C0} &= \frac{1}{1-\alpha_0}E \,. \end{aligned} \tag{43}$$

Fig. 20 shows the inductor current and capacitor voltage in steady state, as a function of the duty cycle.

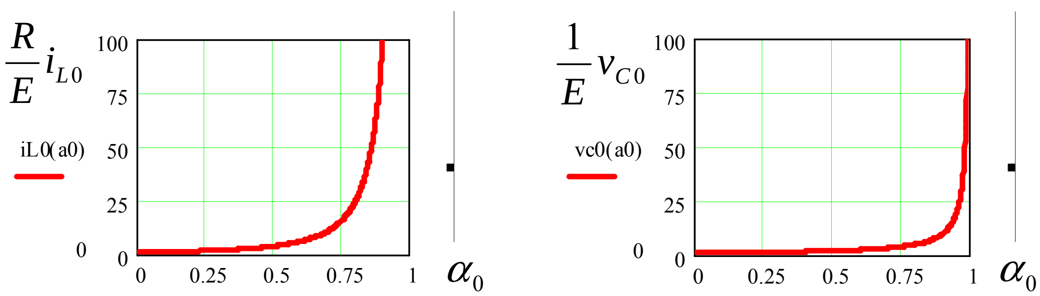


Fig. 20: State variables versus duty cycle in steady state for the boost DC/DC converter

From Eq. (41):

$$\dot{\hat{x}} = A_0\hat{x} + B_0\hat{\alpha} \quad \text{with} \quad A_0 = \begin{bmatrix} 0 & -\frac{1}{L}(1-\alpha_0) \\ \frac{1}{C}(1-\alpha_0) & -\frac{1}{RC} \end{bmatrix} \quad ; \quad B_0 = \begin{bmatrix} \frac{E}{L}\frac{1}{1-\alpha_0} \\ -\frac{E}{RC}\frac{1}{(1-\alpha_0)^2} \end{bmatrix} . \tag{44}$$

Finally, the small signal transfer function, in Laplace, are derived from the first equation of (42):

$$\frac{\hat{x}(s)}{\hat{\alpha}(s)} = \left[sI - A_0\right]^{-1} \cdot B_0 = \frac{1}{s^2 + s\frac{1}{RC} + \frac{1}{LC}(1-\alpha_0)^2} \begin{bmatrix} s\left(\frac{E}{L}\frac{1}{(1-\alpha_0)}\right) + \frac{2E}{RCL}\frac{1}{(1-\alpha_0)} \\ \frac{E}{LC}\left(1 - s\frac{L}{R}\frac{1}{(1-\alpha_0)^2}\right) \end{bmatrix}$$

which gives:

$$\frac{\hat{i}_L(s)}{\hat{\alpha}(s)} = \frac{s\left(\frac{E}{L}\frac{1}{(1-\alpha_0)}\right) + \frac{2E}{RCL}\frac{1}{(1-\alpha_0)}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}(1-\alpha_0)^2} \quad \text{and} \quad \frac{\hat{v}_C(s)}{\hat{\alpha}(s)} = \frac{\frac{E}{LC}\left(1 - s\frac{L}{R}\frac{1}{(1-\alpha_0)^2}\right)}{s^2 + s\frac{1}{RC} + \frac{1}{LC}(1-\alpha_0)^2} \tag{45}$$

6.3 Equivalent average circuit models

Here we present a more straightforward and practical method to derive the state space model. It is an alternative method which gives, in the cases where it can be applied, the same results as the full mathematical approach.

This method can be applied only if the following conditions are met:

The time constants of all the state variables in the circuit are significantly higher (approximately 3 times greater) than the switching period.

The duration of all the sequences of the circuit are known and imposed only by the control signals of the electronic switches. Note that this condition is not met in cases where discontinuous conduction exists. Indeed, in these cases the duration of the sequence corresponding to the discontinuous conduction (all switches off) also depends on the values of the state variables. To illustrate this statement, note that discontinuous conduction often occurs at low inductor current levels but not at higher levels; thus the duration of this sequence, when it exists, depends on the average current level.

As in state space models obtained by the full mathematical derivation, the dynamic behaviour of all state variables are not influenced by their harmonic contents at switching frequency and higher terms. This is a consequence of the linear interpolation and averaging illustrated in Fig. **19**, which is also fundamental in this modelling technique.

The principle of the method consists in replacing all the switches in the circuit by equivalent voltage or current sources in order to obtain an equivalent linear and time-continuous circuit. The equivalent circuit, free from all electronic switches, can therefore be modelled by any circuit theory technique and also in particular by state space.

'Recipe' to obtain the equivalent average circuit

1. Select any sequence of known duration.
2. For that sequence, compute:

– the current through all closed switches as a function of state variables and/or source values;

– the voltages across all open switches as a function of state variables and/or source values.

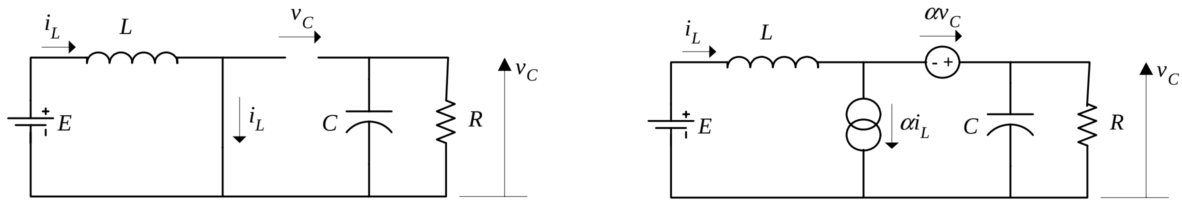
3. Draw an equivalent circuit, by replacing:

– the closed switches by their current pondered by the existence duration;

– the open switches by their voltage sources pondered by the existence duration.

Application to the BOOST converter

By applying the above described ‘recipe’ to the circuit of Fig. 17, choosing sequence 1, the following equivalent circuit is obtained (Fig. 21):



Sequence 1: - durationα *Sequence 1: - durationα*

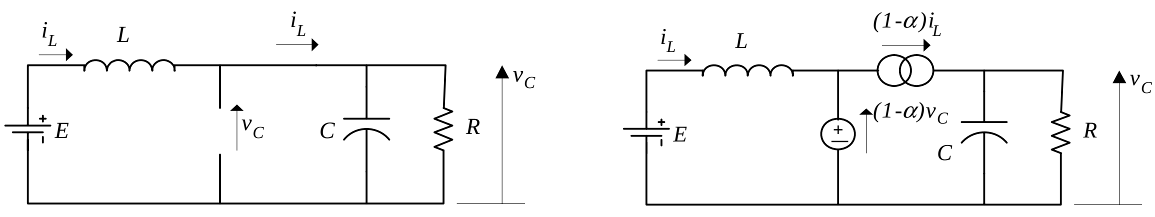
Fig. 21: BOOST DC/DC converter circuit in sequence 1 and equivalent average circuit

The transistor in closed state (see Fig. 17) was replaced by a current source whose value is the on-state transistor current (i_L) multiplied by the sequence existence duration (α). The diode in open state (Fig. 17) is replaced by a voltage source whose value is the voltage across it (v_C), again multiplied by the sequence duration.

The equivalent average circuit can now be easily modelled by the following system of equations.

$$\begin{cases} E = L\dfrac{\mathrm{d}i_\mathrm{L}}{\mathrm{d}t} - \alpha v_\mathrm{C} + v_\mathrm{C} \\ i_\mathrm{L} - \alpha i_\mathrm{L} = C\dfrac{\mathrm{d}v_\mathrm{C}}{\mathrm{d}t} + \dfrac{v_\mathrm{C}}{R} \end{cases} . \tag{46}$$

If sequence 2 were chosen instead, the equivalent average circuit would be (Fig. 22):



Sequence 2: - duration$(1-\alpha)$ *Sequence 2: - duration$(1-\alpha)$*

Fig. 22: BOOST DC/DC converter circuit in sequence 2 and equivalent average circuit

In this case, the open transistor is replaced by a voltage source with value equal to the voltage across its terminals multiplied by this sequence existence duration ($1-\alpha$). In the same way, the closed diode is replaced by a current source with value equal to the current flowing through (i_L) multiplied by the same sequence existence duration.

This equivalent average circuit can also be easily modelled by the system of equations below:

$$\begin{cases} E = L\dfrac{\mathrm{d}i_{\mathrm{L}}}{\mathrm{d}t} + (1-\alpha)v_{\mathrm{C}} \\ (1-\alpha)i_{\mathrm{L}} = C\dfrac{\mathrm{d}v_{\mathrm{C}}}{\mathrm{d}t} + \dfrac{v_{\mathrm{C}}}{R} \end{cases} . \tag{47}$$

Note that the equation systems given by (46) and (47) are identical and can be modelled in state space by the following state space equation:

$$\underbrace{\begin{bmatrix} \dot{i}_{\mathrm{L}} \\ \dot{v}_{\mathrm{C}} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & -\dfrac{1-\alpha}{L} \\ \dfrac{1-\alpha}{C} & -\dfrac{1}{RC} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i_{\mathrm{L}} \\ v_{\mathrm{C}} \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1/L \\ 0 \end{bmatrix}}_{B} \underbrace{E}_{u} . \tag{48}$$

The DC operating point can be computed from Eq. (49):

$$x_0 = -A(\alpha_0)^{-1} \cdot B(\alpha_0) \cdot E \;\Rightarrow\; x_0 = \begin{bmatrix} i_{\mathrm{L0}} \\ v_{\mathrm{C0}} \end{bmatrix} = \begin{bmatrix} \dfrac{E}{R}\dfrac{1}{(1-\alpha_0)^2} \\ \dfrac{1}{1-\alpha_0}E \end{bmatrix} . \tag{49}$$

A small signal model can also be derived by linearization of the state equation around an operating point:

$$\begin{matrix} \dot{x} = f(x,\alpha,E) \\ E = const \end{matrix}; \qquad \dot{\hat{x}} = \left.\frac{\partial f}{\partial x}\right|_{(x_0,\alpha_0)} \hat{x} + \left.\frac{\partial f}{\partial \alpha}\right|_{(x_0,\alpha_0)} \hat{\alpha}\,; \qquad \dot{\hat{x}} = \overbrace{A(\alpha_0)}^{A_0}\hat{x} + \overbrace{\left.\frac{\partial A}{\partial \alpha}\right|_{(\alpha_0)} x_0}^{B_0} \hat{\alpha}\;,$$

or:

$$\dot{\hat{x}} = \underbrace{\begin{bmatrix} 0 & -\dfrac{1}{L}(1-\alpha_0) \\ \dfrac{1}{C}(1-\alpha_0) & -\dfrac{1}{RC} \end{bmatrix}}_{A_0} \hat{x} + \underbrace{\begin{bmatrix} \dfrac{1}{(1-\alpha_0)}\dfrac{E}{L} \\ -\dfrac{E}{RC}\dfrac{1}{(1-\alpha_0)^2} \end{bmatrix}}_{B_0} \hat{\alpha} . \tag{50}$$

By applying the Laplace transformation to Eq. (50), the transfer functions between the input duty cycle and the state variables are obtained:

$$\frac{\hat{x}(s)}{\hat{\alpha}(s)} = \left[sI - A_0\right]^{-1} \cdot B_0$$

$$\frac{\hat{i}_{\mathrm{L}}(s)}{\hat{\alpha}(s)} = \frac{s\left(\dfrac{E}{L}\dfrac{1}{(1-\alpha_0)}\right) + \dfrac{2E}{RCL}\dfrac{1}{(1-\alpha_0)}}{s^2 + s\dfrac{1}{RC} + \dfrac{1}{LC}(1-\alpha_0)^2}; \qquad \frac{\hat{v}_{\mathrm{C}}(s)}{\hat{\alpha}(s)} = \frac{\dfrac{E}{LC}\left(1 - s\dfrac{L}{R}\dfrac{1}{(1-\alpha_0)^2}\right)}{s^2 + s\dfrac{1}{RC} + \dfrac{1}{LC}(1-\alpha_0)^2} . \tag{51}$$

The expressions given by (51) are the same as the ones obtained with the state modelling approach based on the mathematical developments (45).

7 Conclusion

The phase plane representation is a straightforward graphical tool for the analysis of power converters formed by sequences with up-to second-order circuits. The evolution of the state variables (inductor current and capacitor voltage) within a switching cycle can be easily drawn by following the specific set of rules described. The corresponding time domain waveforms can also be directly derived.

This graphical tool is used within a sequential analytical method of study, in which the power converter circuit is decomposed into different sequences, each one corresponding to one configuration of states of the electronic switches. In each sequence, the state variables evolution in the phase plane is derived and the conditions for transition to another sequence are evaluated. By following the specified flowchart, any power converter can be analysed in a systematic approach, avoiding dead-locks and turn-arounds in the study flow and being sure that all possible sequences are considered.

Two modelling techniques of power converters are presented: the first one, more general, is based on mathematical developments in the state space domain; the second one, more direct and practical, in which a linear and time-continuous equivalent average circuit is drawn. The validity assumptions of both techniques are presented and discussed.

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