Essays on Dynamic Sales Mechanisms

by

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Abstract

This thesis is a collection of three essays on dynamic sales mechanisms. The first chapter analyzes the Name Your Own Price (NYOP) mechanism adopted by Priceline.com. Priceline.com, a website helping travelers obtain discount rates for travel-related items, gained prominence for its Name Your Own Price system. Under Name Your Own Price, a traveler names his price for airline tickets, hotel rooms, or car rentals. Priceline then checks if there is any seller willing to accept the offer. If no one accepts, the buyer has to wait for a certain period of time (the lockout period) before rebidding. This paper builds a one-to-many dynamic model without commitment to examine the buyer's and the sellers' equilibrium strategies. I show that without a lockout period, in equilibrium, the sellers with different costs are either almost fully discriminated or pooled in intervals except the one with the lowest possible cost. In the latter case, the buyer does not raise the bids much until the very end, so the price pattern is convexly increasing, consistent with the empirical finding, and most transactions occur just before the day of the trip, which illustrates the deadline effect that is observed in many negotiation processes. The lockout period restriction, which limits the buyer's bidding chances and seems to hurt the buyer, thus moves the transactions forward and can actually benefit a buyer in some circumstances.

The second chapter studies a one-to-many negotiation process in which a seller with an indivisible object negotiates with two asymmetric buyers to determine who gets the object and at what price. The seller repeatedly submits take-it-or-leave-it offers to the two buyers until one of them accepts. Unlike a Dutch auction, the seller has the discretion to offer two different prices to the two buyers. I show that when committing to some price paths is possible, the optimal outcome for the seller stated by Myerson (1981) is achievable. When commitment is impossible, the optimal outcome is no longer attainable. Instead, there exists an equilibrium in which the seller's equilibrium payoff is the same as that in a second-price auction, which implies that the seller's payoff might be lower than in a Dutch auction. The result thus illustrates the value of a simple institution like a Dutch auction, which seems to restrict a player's freedom but actually benefits the player by providing a commitment tool. The analysis also sheds light on the procurement literature.

The third chapter provides a rationale for why a seller may package goods in bundles that are too large for a consumer to consume all by himself. I show that selling in bulk packages is an alternative way for the seller to discriminate buyers when resale cannot be excluded among buyers. When bulk packages are offered, buyers who value the product more usually have stronger incentive to buy the package, and buyers who value the product less tend to buy from resale. Moreover, the seller can make more profit by selling bulk packages than by selling single-unit packages when the buyers' values of the product are more negatively correlated.

Thesis Supervisor: Bengt Holmstrom
Title: Paul A. Samuelson Professor of Economics

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To my parents

陳德信

黃秋艷

Chapter 1

Name Your Own Price at Priceline.com: Strategic Bidding and Lockout Periods

1.1 Introduction

Priceline.com, known for its Name Your Own Price (NYOP) system, is a website devoted to helping travelers obtain discount rates for travel-related items such as airline tickets and hotel stays. The NYOP mechanism works as follows. First, a customer enters a bid that specifies the general characteristics of what she wants to buy (travel dates, location, hotel rating, etc.) and the price that she is willing to pay. Next, Priceline.com either communicates the customer's bid to participating sellers or accesses their private database to determine whether Priceline.com can satisfy the customer's specified terms and the bid price. If a seller accepts the bid, the offer cannot be cancelled. If no seller accepts the bid, the customer can rebid either by changing the desired specifications or by waiting for a minimum period of time, the *lockout period*, before submitting a new, higher price offer. For a hotel, the lockout period is 24 hours, for rental cars it is three days and for an airline ticket seven days. Priceline says in its seller's guideline that the rule is designed to protect the sellers. Our analysis suggests that the lockout period may often benefit the buyer, because it allows the buyer to commit to fewer rounds of bidding (the bidding must end before the date of travel, of course.)

To represent the Priceline.com auction, we use a dynamic model in which a single buyer suggests prices to N potential sellers for a finite number of rounds. The number of rounds T determines the length of the lockout period. By letting T go to infinity, we can also consider the case of no lockout period. For simplicity, we assume that the buyer's valuation is known. The sellers' costs are privately known and independently drawn from a common distribution.

We first show that without a lockout period and no discounting, there are two kinds of equilibrium bidding paths. As T goes to infinity, either sellers are almost fully discriminated over time or they get pooled into a finite number of cost intervals with bids clustering at the lowest price which is accepted by the seller with the minimum cost. In the latter case, the price pattern is convexly increasing as the buyer keeps her bids close to the price accepted by the minimum-cost seller until the very end. The pattern of bidding will be convex and most of trades (if any) will be realized at the end. This is consistent with empirical evidence and similar to the deadline effect observed in many bargaining processes (see for instance Hart (1989) and Spier (1992) on strikes and pretrial negotiation.)

The buyer's bidding strategy influences the rate at which she learns about the sellers' valuations. Ideally, the buyer would like to commit to a strategy that optimally reveals this information. If she could do that, she would gradually raise the price to price discriminate among the sellers and stop at the optimal reserve price, much like a Dutch auction, but in reverse. But when commitment is impossible, as we assume, the buyer cannot help but respond to the information revealed by rejections. As a consequence, she may want to bid so that her initial bids reveal little information and only at the end will they be more informative. The last minute rush will lead to pooling and inefficient outcomes, because many sellers will accept simultaneously and the winner will be determined by lottery.

The other equilibrium where sellers are discriminated through a gradually increasing bid sequence is, on the other hand, fully efficient, since the maximum bid equals the highest seller cost and the sellers are almost fully discriminated.

We also show that without a lockout period, the expected payoff of a customer is weakly higher than that in a first-price sealed-bid reverse auction (where service providers submit their bids to a customer) without a reserve price, but lower than that in a first-price sealed-bid reverse auction with the optimal reserve price. Moreover, when the expected payoff is strictly higher than that in a first-price reverse auction without a reserve price, the equilibrium bidding path is convexly increasing.

The lockout period, by reducing the number of bidding rounds, affects the process of information revelation. It makes the buyer bid more aggressively early on, because she does not need to be as concerned about the detrimental effects of learning more about the sellers' information while still having many bidding opportunities. This can be especially valuable if the buyer moderately discounts the future, that is, she wants to learn early about bookings. Thus, the lockout period can be advantageous to the buyer, because it permits the buyer to commit to fewer rounds of bidding. However, the welfare effects are ambiguous in general. The finding that the lockout period can be valuable is in line with McAdams and Schwarz (2007)'s view that an intermediary can create value by offering a credible commitment device.

Our analysis also provides insights into the unexplained bidding paths found by Spann and Tellis (2006). They analyze buyers' bidding patterns under NYOP without the lockout period restriction and find that 36% of the patterns are concavely increasing, while 23% are convexly increasing. They argue that the concave patterns can be explained by the positive bidding cost, but the convex ones suggest irrational consumer behavior on the Internet. Our paper shows that a convex pattern where a buyer raises bids more aggressively at the end can occur in a fully rational environment.

The environment studied here is similar to a durable goods monopoly, but with the roles of buyer and seller reversed. In a durable goods monopoly, the seller makes bids. Here the buyer does it. To avoid confusion, call the side that determines the price the principal and the other side the agents. There are two differences between our setting and a durable goods monopoly. First, there is a deadline in our environment, which results in very different equilibrium paths than those of the Coase conjecture.[1] Secondly, there is competition among the agents. With competition, an agent may accept the current price even though the future price path looks attractive, because there is the risk that another agent will accept. Therefore, our model works

[1]In Stokey's discrete-time model, she also considers the case when there is a deadline and shows that the Coase conjecture still holds when the length of the period shrinks. The conclusion is different from ours because in Stocky's model, the deadline is not just the last day to trade, it is also the last day on which a buyer can enjoy the good and derive utility from it. That is, a buyer derives less utility if he gets the good on a day closer to the deadline. In our model, a buyer derives the same utility no matter when he gets the good.

even when there is no discounting.

The paper is organized as follows. Section 2 describes the model. Section 3 presents an example that motivates our research. Section 4 constructs an equilibrium. Section 5 characterizes the equilibrium bidding behavior. Section 6 analyzes a model with waiting cost to see under what conditions the lockout period rule benefits customers and Section 7 concludes.

1.2 The Model

There are $N \geq 2$ sellers and 1 buyer in the market. The buyer has one unit of demand for the good provided by the sellers. The buyer's reservation value for the good is v, which is known by everyone. Seller i privately knows his cost θ^i to provide the good. Each θ^i is independently and identically distributed on $[\underline{c}, \overline{c}]$, where $\underline{c} \geq 0$ and $\overline{c} \leq v$, according to a distribution function $\overline{F}$. $\overline{F}$ admits a continuous density f and has full support. And $x + \frac{\overline{F}(x)}{f(x)}$ strictly increases in x. A buyer's payoff is $v - b$, where b is his payment to the seller, if he gets the object, and 0 otherwise. All the players are risk neutral. The setting is common knowledge to everyone in the market.

There is one platform allowing the buyer to submit his bid price to sellers. The buyer is allowed to adjust his bids for T times. In round t, the buyer announces the bid price, and sellers decide whether to accept or not. If n sellers accept the bid, each of them gets the chance to provide the good with probability $\frac{1}{n}$, and the game stops. If no seller accepts and $t < T$, the process proceeds to the next round, and the buyer submits a new price. If $t = T$, then the market closes and no further transaction can happen.

1.2.1 Equilibrium concept

The equilibrium concept used in this paper is the perfect Bayesian equilibrium. An equilibrium consists of the buyer's strategy and belief, and the sellers' strategies and beliefs. Only symmetric pure strategy equilibria are considered. Let p_t be the price that the buyer offers the sellers in round t. Denote by $h_t = (p_1, p_2, \cdots, p_t)$ the history of the prices submitted by the buyer in the first t rounds.

Let $b_t\left(h_{t-1}\right)$ be the price that the buyer would submit in round t given the price history

h_{t-1} and the fact that no seller accepts in the first $t-1$ rounds. The buyer's strategy is a set of functions $\{b_t(h_{t-1})\}_{t=1}^T$. A seller's strategy can be summarized by functions $\{x_t(h_t)\}_{t=1}^T$. In round t, given h_t, a seller accepts the buyer's offer if and only if his cost is less than or equal to $x_t(h_t)$. The buyer's and the sellers' beliefs are summarized by a set of functions $\{y_t(h_{t-1})\}_{t=1}^T$, which specifies the greatest lower bound of a seller's cost believed by the buyer and the other sellers given history h_{t-1} and the fact that no seller accepts in the first $t-1$ rounds. Denote by $u_t^0(b, x \mid h_{t-1}, y_t(h_{t-1}))$ the buyer's expected utility given history h_{t-1} and belief $y_t(h_{t-1})$, and $u_t^i\left(b, x^{-i}, x^i \mid h_t, \theta^i, y_t(h_{t-1})\right)$ seller i's expected utility, where x^{-i} is the other sellers' strategy,[2] and x^i is seller i's strategy, given h_t, the realization θ^i of seller i's cost, and belief $y_t(h_{t-1})$.

Definition 1 *A symmetric equilibrium is a (b, y, x) that satisfies*

(a) $y_{t+1}(h_t) = \max\{x_t(h_t), x_{t-1}(h_{t-1}), \cdots, x_1(h_1)\}, \forall t, h_t,$ *and*

(b) $u_t^0(b, x \mid h_{t-1}, y_t(h_{t-1})) \geq u_t^0(b', x \mid h_{t-1}, y_t(h_{t-1}))$ *and*
$u_t^i\left(b, x, x \mid h_t, \theta^i, y_t(h_{t-1})\right) \geq u_t^i\left(b, x, x' \mid h_t, \theta^i, y_t(h_{t-1})\right), \forall b', x', t, h_t, h_{t-1}.$

Condition (a) implies that players' belief about the greatest lower bound of seller i's cost at time t is the same as the maximum of seller i's costs with which seller i would have accepted a price occurring on the history price path. Condition (b) means that players cannot do better by deviating from the equilibrium strategy.

1.3 An Example

Before proceeding to constructing an equilibrium for the general model, we show calculations for finding the equilibrium path by using the example where $N = 2, v = 1$, and $\overline{F}$ is a uniform distribution on $[0, 1]$, and highlight some interesting points.

In addition to NYOP, a reverse auction is another mechanism commonly used by a buyer to determine allocation. Thus, we are interested in comparing the performances of the two mechanisms. In this example, the reverse auction is analogous to a standard auction with one seller and two buyers whose values are uniformly distributed on $[0, 1]$. In the standard auction,

[2] x^{-i} is a tuple consisting of the other sellers' strategies. But when the other sellers use the same strategies, x^{-i} can be a single function without confusion.

a buyer with value v bids $\frac{v}{2}$ in equilibrium. Therefore, in the reverse auction, a seller with cost x analogously submits ask price $\frac{1}{2}+\frac{1}{2}x$. The buyer buys from the seller with the lowest ask price and gets expected payoff $\frac{1}{3}$. On the other hand, if the buyer is allowed to set a reserve price to commit that he buys the object only if the price is lower than the reserve price, then by setting the reserve price at $\frac{1}{2}$, the buyer gets $\frac{5}{12}$, the same as the expected payoff realized in Myerson's optimal mechanism.

T=1: Now suppose the buyer and the sellers trade under an NYOP mechanism where $T=1$. The buyer has one chance to submit his bid b. Seeing the bid, a seller whose cost is below b accepts the offer. Therefore, the buyer maximizes the expected payoff $(1-b)\left[1-(1-b)^2\right]$ by choosing $b=1-\frac{1}{\sqrt{3}}$ and gets expected payoff $\frac{2}{3\sqrt{3}}$. From the example, we see that for the buyer, NYOP outperforms a reverse auction without a reserve price even when there is only one chance to bid.

T=2: Next consider the case where $T=2$. Suppose that in round 1, the bid price is b_1 and no one buys. In round 2, the buyer believes that both sellers' costs are above $x_1(b_1)$, and each seller also believes the other one's cost is above $x_1(b_1)$. The updated belief about the distribution of a seller's cost is $U[x_1(b_1),1]$. Since it is the last round, both sellers will accept if the bid is higher than their costs. Thus $x_2(b_1,b_2)=b_2$. Given the belief, the buyer will bid at $b_2(b_1)=1-\frac{1-x_1(b_1)}{\sqrt{3}}$ to maximize his expected revenue.

In round 1, suppose the buyer has submitted a bid at b_1. A seller with cost x decides whether to accept the bid in this round or wait until the next one with the belief that the other seller would accept if his cost is below or equal to $x_1(b_1)$. If the seller accepts in this round, with probability x_1 the other accepts too, and each of them gets to sell with probability $\frac{1}{2}$; and with probability $1-x_1$, the seller gets to sell for sure, so the seller's expected payoff is $(b_1-x)\left[\frac{1}{2}x_1(b_1)+(1-x_1(b_1))\right]$. If the seller waits, with probability $1-x_1$, the game moves to the next round. In round 2, the buyer is expected to submit $b_2(b_1)$. With probability $\frac{b_2(b_1)-x_1(b_1)}{1-x_1(b_1)}$, the other seller accepts too and each of them gets to sell with probability $\frac{1}{2}$; and with probability $\frac{1-b_2(b_1)}{1-x_1(b_1)}$, the seller gets to sell for sure, so the seller's expected payoff is

$(b_2(b_1)-x)\left[\frac{1}{2}\frac{b_2(b_1)-x_1(b_1)}{1-x_1(b_1)}+\frac{1-b_2(b_1)}{1-x_1(b_1)}\right]$. The seller accepts b_1 in round 1 if

$$(b_1-x)\left[\frac{1}{2}x_1(b_1)+(1-x_1(b_1))\right]\geq \max\left\{0,(b_2(b_1)-x)\left[\frac{1}{2}\left[b_2(b_1)-x_1(b_1)\right]+\left[1-b_2(b_1)\right]\right]\right\}.$$

Note that if a seller with x accepts in round 1, then a seller with $x'<x$ would also accept. In equilibrium, a seller with $x\leq x_1(b_1)$ decides to accept, so we can get $x_1(b_1)=1-\frac{-3b_1+\sqrt{9b_1^2+12(1-b_1)}}{2}$ by solving $(b_1-x_1)\left[\frac{1}{2}x_1+(1-x_1)\right]=(b_2(b_1)-x_1)\left[\frac{1}{2}\left[b_2(b_1)-x_1\right]+\left[1-b_2(b_1)\right]\right]$.

With belief $x_1(b_1)$, the buyer chooses b_1 to maximize his total expected revenue in the two rounds

$$\max_{b_1}\left[1-b_1\right]\left[1-(1-x_1(b_1))^2\right]+\left[1-b_2\left(b_1\right)\right]\left[\left(1-x_1\left(b_1\right)\right)^2-\left(1-b_2\left(b_1\right)\right)^2\right].$$

$\{b_1,x_1(b_1),b_2(b_1),x_2(b_1,b_2)\}$ form a symmetric equilibrium. In equilibrium $b_1=0.4214$, $b_2=0.5212$, $x_1=0.1709$, $x_2=0.5212$, and the buyer's payoff is 0.40024.

Numerical results:

In the following table, we show the equilibrium paths of x_t and b_t and the expected buyer's payoffs when $T=1$, 2, 3, 4, and 5. We assume that the game begins at time 0 and ends at time 1. If the buyer's bid in the tth round is accepted, the transaction occurs at $\frac{(t-1)}{T}$. Column

$E(\tau)$ lists the expected transaction time conditional on that transaction occurs.

	Buyer's Payoff	$E(\tau)$	x_{T-4} (b_{T-4})	x_{T-3} (b_{T-3})	x_{T-2} (b_{T-2})	x_{T-1} (b_{T-1})	x_T (b_T)
$T=1$	0.38490	0					0.4225 (0.4225)
$T=2$	0.40024	0.2972				0.1709 (0.4214)	0.5212 (0.5212)
$T=3$	0.40111	0.4563			0.0597 (0.4099)	0.2165 (0.4538)	0.5475 (0.5475)
$T=4$	0.40115	0.5826		0.0154 (0.4007)	0.0597 (0.4127)	0.2165 (0.4538)	0.5475 (0.5475)
$T=5$	0.40115	0.6626	0.0070 (0.3990)	0.0154 (0.4021)	0.0597 (0.4127)	0.2165 (0.4538)	0.5475 (0.5475)

There are several points worth noticing:

1. The buyer's payoff increases in T, the number of rounds,[3] but the increment becomes smaller and smaller. Therefore, the profit of having one more bidding chance shrinks as T increases. (proved in Proposition 4)

2. The cost cutoff in round $T-t$, x_{T-t}, converges when T goes to infinity. (proved in Proposition 5)

3. The last-round bid increases in T, but the increment also shrinks as T increases. Observe that given T, the bidding path b_t is increasing. But with larger T, the increasing rate is small in the first few rounds and big jumps occur in the last few rounds. (characterized in Theorem 2)

4. The payoff for all T is lower than the payoff in a reverse auction with the optimal reserve price; and when T is large enough (in this example, when $T \geq 1$), the payoff is higher than the payoff in a reverse auction with no reserve price. (proved in Theorem 3)

[3]Note that the numbers in the table are not accurate enough to show small differences.

5. In equilibrium the buyer does not get the object only if both sellers' costs are above x_T. Therefore, we know the probability that the buyer gets the object increases in T, but the increment shrinks as T increases. From the table, we see that when T increases from 3 to 4, and to 5, neither the buyer's payoff nor the probability that the buyer gets the object increases much. However, the expected transaction time is much later. This fact suggests that if the buyer has waiting cost and prefers earlier transactions, having fewer rounds might be good for him. The analysis in Section 1.6 confirms the conjecture.

1.4 Construction of the Equilibrium

In this section, we construct an equilibrium by solving a series of programs backward and prove the existence of the equilibrium.

To construct the equilibrium, we need to introduce more notations. For convenience, define

$$F(x) = 1 - \overline{F}(x).$$

Note that $F(x)$ strictly decreases in x. Suppose only sellers with costs between x_{t-1} and x_t are willing to provide the good. Let

$$P(x_{t-1}, x_t) = \begin{cases} F(x_{t-1})^N - F(x_t)^N, & \text{if } x_{t-1} \leq x_t \\ 0, & \text{if } x_{t-1} > x_t \end{cases}$$

be the probability that the demand is fulfilled. Let

$$\begin{aligned} H(x_{t-1}, x_t) &= \begin{cases} \sum_{n=0}^{N-1} \frac{1}{n+1} \frac{(N-1)!}{n!(N-n-1)!} \left(F(x_{t-1}) - F(x_t)\right)^n \left(F(x_t)\right)^{N-n-1} / F(x_{t-1})^N, & \text{if } x_{t-1} \leq x_t \\ 1, & \text{if } x_{t-1} > x_t \end{cases} \\ &= \begin{cases} \frac{\sum_{n=0}^{N-1} F(x_t)^{N-1-n} F(x_{t-1})^n}{N F(x_{t-1})^N}, & \text{if } x_{t-1} \leq x_t \\ 1, & \text{if } x_{t-1} > x_t \end{cases} \end{aligned}$$

be the conditional probability that a seller gets to provide the good if he accepts the buyer's offer conditional on that the other sellers' costs are above x_{t-1}.[4] Define

$$G\left(x_{t-1}, x_t\right) \equiv H\left(x_{t-1}, x_t\right) NF\left(x_{t-1}\right)^N.$$

$\overline{b}_t\left(x_{t-1}\right), \widehat{x}_t\left(b_t, x_{t-1}\right), \widetilde{x}_t\left(b_t, x_{t-1}\right)$, and $V_t\left(x_{t-1}\right)$ defined below are used to characterize equilibrium strategies, beliefs, and the buyer's payoff for the continuation games starting from round t. If $t = T$, let

$$\begin{aligned} V_T\left(x_{T-1}\right) &= \max_{\{b_T, x_T\}} \left(v - b_T\right) P\left(x_{T-1}, x_T\right) \\ & \quad s.t.\ b_T = x_T, \end{aligned}$$

and

$$\begin{aligned} \left(\overline{b}_T\left(x_{T-1}\right), \overline{x}_T\left(x_{T-1}\right)\right) &\in \arg\max_{\{b_T, x_T\}} \left(v - b_T\right) P\left(x_{T-1}, x_T\right) \\ & \quad s.t.\ b_T = x_T. \end{aligned} \tag{P1.1}$$

The constraint $b_T = x_T$ of the program comes from that in the last round, a seller accepts the last-round bid b_T as long as his cost is below b_T, so the cutoff x_T equals b_T. Knowing this and given the belief that all the sellers have cost higher than x_{T-1}, the buyer chooses b_T to maximize his payoff—the objective function. Note that there might be multiple solutions to program P1.1. If there is more than one solution, only those that ensure the existence of equilibrium can be candidates for $\overline{b}_T\left(x_{T-1}\right)$ and $\overline{x}_T\left(x_{T-1}\right)$ (see the proof of Proposition 1 for more details).

[4]Conditional on that the other sellers' costs are above x_{t-1}, if a seller accepts the buyer's offer, with probability $\frac{(N-1)!}{n!(N-n-1)!}\left(F(x_{t-1}) - F(x_t)\right)^n \left(F(x_t)\right)^{N-n-1} / F\left(x_{t-1}\right)^N$, there are n other sellers accepting, and each of them gets to sell the good with probability $\frac{1}{n+1}$.

If $t < T$, let

$$V_t(x_{t-1}) = \max_{\{b_t, x_t\}} (v - b_t) P(x_{t-1}, x_t) + V_{t+1}(x_t) \tag{P1.2}$$

$$s.t.(b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t)$$

$$\text{where } C_{t+1}(x_t) = (\overline{b}_{t+1}(x_t) - x_t)G(x_t, \overline{x}_{t+1}(x_t)),$$

$$\text{where } \overline{b}_{t+1}(x_t), \overline{x}_{t+1}(x_t) \text{ are defined as below;}$$

and let

$$\left(\overline{b}_t(x_{t-1}), \overline{x}_t(x_{t-1})\right) \in \arg\max_{\{b_t, x_t\}} (v - b_t) P(x_{t-1}, x_t) + V_{t+1}(x_t) \tag{P1.3}$$

$$s.t.(b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t).$$

Note that to solve the round-t program, we must solve all the programs for later rounds first, so function $C_{t+1}(x)$ is determined before solving the program. The right-hand side of the constraint, $C_{t+1}(x_t)$, is the expected payoff of a seller with cost x_t if he waits and accepts in the next period. The left-hand side is the expected payoff of a seller with cost x_t if he accepts in period t. Given b_t, sellers with costs lower than x_t prefer to accept in period t, and sellers with costs higher than x_t prefer to accept in period $t+1$. So for each b_t, we find the sellers' equilibrium strategy x_t from the constraint. Given the sellers' strategy and the belief that all sellers' costs are above x_{t-1}, the buyer chooses b_t to maximize his payoff—the objective function. The following proposition proves that programs P1.1 and P1.3 have a solution

Proposition 1 *There exists a set of solutions* $\left\{\overline{b}_t(x_{t-1}), \overline{x}_t(x_{t-1})\right\}_t$ *that solves program P1.1 and P1.3 for all* t.

Proof. The details of the proof are in Appendix A. Here is the sketch. First, by Berge's maximum theorem, $V_T(x_{T-1})$ is continuous, and the solution set of x_T for program P1.1 is upper hemi-contiuous. Therefore, we are able to pick $\overline{x}_T(x_{T-1})$ from the solution set such that $C_T(x_{T-1})$ is lower semi-continuous. Next, substituting the constraint into the objective function in round $T-1$ in program P1.3, the objective function is graph-continuous defined in Leininger (1984), and by Leininger's generalized maximum theorem, V_{T-1} is upper semi-continuous, and the solution set of x_{T-1} exists and is upper hemi-continuous. Applying the same procedure backward, we guarantee the existence of a solution to each round-t program. ■

The following assumption is for defining $\widehat{x}_t(b_t, x_{t-1})$ and $\widetilde{x}_t(b_t, x_{t-1})$. We make the assumption to ensure the existence of pure strategy equilibrium.[5] Without the assumption, we are still able to construct an equilibrium in which mixed strategies are applied off the equilibrium path. Therefore, Assumption 1 is not necessary for an equilibrium to exist.

Assumption 1 *Given x_{t-1}, assume that there exists $\underline{b}$ such that if $b_t \in [\underline{b}, \overline{c}]$, there exists x_t such that $(b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t)$, and if $b_t < \underline{b}$, $(b_t - x_t)G(x_{t-1}, x_t) < C_{t+1}(x_t)$ for all $x_t \in [x_{t-1}, \overline{c}]$.*

Let

$$\widehat{x}_T(b_T, x_{T-1}) = \begin{cases} \overline{c} & \text{if } b_T > \overline{c} \\ b_T & \text{if } x_{T-1} \leq b_T \leq \overline{c} \\ x_{T-1} & \text{if } b_T < x_{T-1} \end{cases} \text{ and } \widetilde{x}_T(b_T, x_{T-1}) = \begin{cases} \overline{c} & \text{if } b_T > \overline{c} \\ b_T & \text{if } b_T \leq \overline{c} \end{cases}. \quad \text{(P1.4)}$$

[5] Assumption 1 implies that when the price is not too low, i.e. $b_t \in [\underline{b}, \overline{c}]$, there exists x_t such that

$$\begin{aligned} (b_t - x)\,G(x_{t-1}, x_t) &> \left(\overline{b}_{t+1}(x_t) - x\right)G(x_t, \overline{x}_{t+1}(x_t)) \text{ for } x < x_t \text{ and} \\ (b_t - x)\,G(x_{t-1}, x_t) &< \left(\overline{b}_{t+1}(x_t) - x\right)G(x_t, \overline{x}_{t+1}(x_t)) \text{ for } x > x_t. \end{aligned}$$

So a seller with cost lower than x_t accepts in round t, and a seller with cost higher than x_t accepts in later rounds. Without the assumption, since $C_{t+1}(x_t)$ might have a jump at some $c \in [\underline{c}, \overline{c}]$, there might exist b such that

$$\begin{aligned} \{x \mid (b - x)G(x_{t-1}, x) < C_{t+1}(x)\} &\neq \phi, \\ \{x \mid (b - x)G(x_{t-1}, x) > C_{t+1}(x)\} &\neq \phi, \text{ and} \\ \{x \mid (b - x)G(x_{t-1}, x) = C_{t+1}(x)\} &= \phi. \end{aligned}$$

In this case, there does not exist x_t such that

$$\begin{aligned} (b - x)\,G(x_{t-1}, x_t) &> \left(\overline{b}_{t+1}(x_t) - x\right)G(x_t, \overline{x}_{t+1}(x_t)) \text{ for } x < x_t \text{ and} \\ (b - x)\,G(x_{t-1}, x_t) &< \left(\overline{b}_{t+1}(x_t) - x\right)G(x_t, \overline{x}_{t+1}(x_t)) \text{ for } x > x_t, \end{aligned}$$

so we are not able to find a cost cutoff in round t given that the buyer submits b. One way to solve the problem is to let the buyer play mixed strategies in round $t + 1$.

For $t < T$, let

$$\widehat{x}_t(b_t, x_{t-1}) = \begin{cases} \overline{c} & \text{if } b_t \geq \overline{c} \\ x_{t-1} & \text{if } b_t < \underline{b} \text{ (defined in Assumption 1)} \end{cases}, \tag{P1.5}$$

otherwise,

$$\widehat{x}_t(b_t, x_{t-1}) \in \{x_t \mid (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t)\}. \tag{1.1}$$

and

$$\widetilde{x}_t(b_t, x_{t-1}) = \begin{cases} \overline{c} & \text{if } b_t \geq \overline{c} \\ b_t - \frac{C_{t+1}(x_{t-1})}{G(x_{t-1}, x_{t-1})} & \text{if } b_t < \underline{b} \end{cases}, \tag{P1.6}$$

otherwise,

$$\widetilde{x}_t(b_t, x_{t-1}) \in \{x_t \mid (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t)\}.$$

$\widehat{x}_t(b_t, x_{t-1})$ is for determining a player's belief about the greatest lower bound of a seller's cost, so $\widehat{x}_t(b_t, x_{t-1}) \geq x_{t-1}$; and $\widetilde{x}_t(b_t, x_{t-1})$ is for determining a seller's strategy. The difference between $\overline{x}_t(x_{t-1})$ and $\widehat{x}_t(b_t, x_{t-1})$ (or $\widetilde{x}_t(b_t, x_{t-1})$) is that $\overline{x}_t(x_{t-1})$ is determined at the same time when the buyer determines b_t, and $\widehat{x}_t(b_t, x_{t-1})$ (or $\widetilde{x}_t(b_t, x_{t-1})$) is determined after the buyer submits b_t. When deriving $\widehat{x}_t(b_t, x_{t-1})$ and $\widetilde{x}_t(b_t, x_{t-1})$, we have to take care of the cases when the buyer submits off-equilibrium bids. If an off-equilibrium bid b_t is too high, all the sellers accept and $\widehat{x}_t = \widetilde{x}_t = \overline{c}$. If b_t is too low, sellers with values higher than x_{t-1} do not accept, so the belief about the greatest lower bound of the sellers' costs after all the sellers reject b_t is still x_{t-1}, i.e. $\widehat{x}_t = x_{t-1}$. However, a seller with cost lower than $b_t - \frac{C_{t+1}(x_{t-1})}{G(x_{t-1}, x_{t-1})} < x_{t-1}$ gets higher payoff if he accepts in round t, so $\widetilde{x}_t = b_t - \frac{C_{t+1}(x_{t-1})}{G(x_{t-1}, x_{t-1})}$.

Lastly,

$$\widehat{x}_0 = \underline{c}.$$

Theorem 1 *Assume Assumption 1. Let $\overline{b}_t$ be as defined in (P1.1) and (P1.3), and $\widehat{x}_t$ be as* *defined in (P1.4), (P1.5), and (1.3). The following (b, y, x) is an equilibrium of the game.*

$$\begin{aligned} b_t(h_{t-1}) &= \overline{b}_t\left(\widehat{x}_{t-1}\left(p_{t-1}, \widehat{x}_{t-2}\left(p_{t-2}, \cdots \widehat{x}_1\left(p_1, \widehat{x}_0\right)\ldots\right)\right)\right), \\ x_t(h_t) &= \widetilde{x}_t\left(p_t, \widehat{x}_{t-1}\left(p_{t-1}, \cdots \widehat{x}_1\left(p_1, \widehat{x}_0\right)\ldots\right)\right), \\ y_{t+1}(h_t) &= \widehat{x}_t\left(p_t, \widehat{x}_{t-1}\left(p_{t-1}, \cdots \widehat{x}_1\left(p_1, \widehat{x}_0\right)\ldots\right)\right). \end{aligned}$$

Proof. See Appendix A. ∎

Corollary 1 *The equilibrium path $\{(b_1, \cdots, b_T), (x_1, \cdots, x_T)\}$ can be found by solving the recursive program*

$$V_1(\underline{c}) = \max_{\{b_1, x_1\}} (v - b_1) P(\underline{c}, x_1) + V_2(x_1) \tag{P1.7}$$

$$s.t.(b_1 - x_1) G(\underline{c}, x_1) = C_2(x_1).$$

The value of the program is the buyer's payoff in equilibrium.

The program shows that the equilibrium path $\{(b_1, \cdots, b_T), (x_1, \cdots, x_T)\}$ maximizes the buyer's payoff but is subject to two constraints. The first one is the sellers' IC constraint, which exists in every mechanism and is shown in the constraint part of the program. The second constraint comes from the recursive form of the program. In each round, the buyer makes his bidding decision based on his current information and is not able to commit to a bidding path at the beginning. The second constraint keeps the buyer from achieving the outcome derived from the optimal mechanism stated by Myerson (1981).

1.5 Equilibrium Bidding Behavior

With T chances to submit prices, the buyer is able to segment the sellers in up to T groups according to their costs. However, the buyer cannot commit to a bidding path in advance, and in each round, he will choose a price that maximizes his expected payoff based on his belief. Thus, the buyer would suffer from the inability to commit and get lower payoff than when commitment is possible. In this section, we focus on the case when there is no lockout period restriction so that the buyer can submit as many bids as he wants. We first show that when committing to a bidding path is impossible, the optimal outcome for the buyer stated by Myerson (1981) is not attainable if the optimal auction design involves setting a reserve price. Next, we characterize the equilibrium bidding behavior and show that there are two possible types of equilibrium bidding paths. One incurs constant trades over time, and the other leads to late transactions.

1.5.1 Commitment and optimality

In this section, we consider the situation when the buyer can commit to a bidding path in advance as a benchmark case. We show that with commitment, the buyer can achieve the optimal outcome realized in Myerson's optimal mechanism.

Note that in our setting, a first-price or second-price reverse auction works as follows — sellers submit their asks and the buyer chooses to buy an object from the seller with the lowest ask price. The buyer can announce a reserve price before the auction starts so that the buyer buys the object only if there is at least one ask price below the reserve price. A first-price or second-price reverse auction with a reserve price r such that $r + \frac{\overline{F}_i(r)}{f_i(r)} = v$ is an optimal mechanism prescribed by Myerson (1981). Under NYOP, it is the buyer who submits bids. When there are a large number of bidding chances, if the buyer commits to raise bids gradually and stop at r, then to the sellers, the game, like a reverse Dutch auction with a reserve price, is almost strategically equivalent to a first-price reverse auction with reserve price r, and the optimal outcome for the buyer can be approximately achieved. The following proposition elucidates this point.

Proposition 2 *Let $\pi(T)$ be the buyer's maximum payoff when there are T rounds and commitment to a path is possible. Let π^* be the buyer's payoff in Myerson's optimal mechanism. Given any $\epsilon > 0$, there exists T' such that for all $T > T'$, $\pi^* - \pi(T) < \epsilon$.*

Proof. See Appendix B. The proof shows that by committing to a path $(b_1, b_2, \cdots, b_T)$ such that in round t, sellers with cost below $x_t = \underline{c} + t\frac{r - \underline{c}}{T}$ (where r is the optimal reserve price) accept, the buyer's payoff can be arbitrarily close to π^* when T goes to infinity. ∎

However, when commitment is not possible, even though the buyer is allowed to adjust the price as many times as he wants, the maximum payoff resulting from the optimal mechanism is not approximately achievable. By corollary 1 and (P1.1), we know that on the equilibrium path, the last-round b_T and x_T can be found by solving

$$x_T = b_T = \arg\max_b (v - b) \left[F(x_{T-1})^N - F(b)^N \right].$$

A necessary condition for b_T is

$$F(x_{T-1})^N = F(b_T)^N + (v - b_T) NF(b_T)^{N-1} f(b_T). \tag{1.2}$$

Suppose the optimal auction involves setting a reserve price $r < \bar{c}$. If the optimal auction can be approximately implemented when T goes to infinity, then it must be that $\lim_{T\to\infty} b_T = \lim_{T\to\infty} x_T = r$ and $\lim_{T\to\infty} x_{T-1} = r$. But by equation (1.2), if $\lim_{T\to\infty} b_T = r$, $\lim_{T\to\infty} x_{T-1} < r$, so the optimal auction cannot be approximately implemented.

Proposition 3 *When commitment to a path is impossible, the buyer's payoff under NYOP is bounded away from the payoff in Myerson's optimal auction if the optimal auction involves setting a reserve price.*

1.5.2 Possible forms for the equilibrium paths when no lockout period restriction is imposed

In this section, we characterize the pattern of the equilibrium bidding path when $T \to \infty$ (i.e. when there is no lockout period restriction). The question is how the buyer designs a bidding path to discriminate sellers. When commitment is possible, it is optimal for the buyer to induce sellers to reveal information about their costs gradually in every round. But when commitment is impossible, acquiring new information will change the buyer's pricing strategy later on, and it is not clear whether doing so is beneficial for the buyer. In Theorem 2, we characterize the equilibrium paths. Although the equilibrium paths would be different in different environments, we show that the paths can be neatly classified into two types: either the sellers with different costs are almost fully discriminated so the sellers' private information is revealed gradually over time, or they are pooled in intervals and most information about the sellers' costs is revealed just before the deadline.

Before characterizing the equilibrium paths, we first analyze how the buyer's payoff changes when the number of rounds increases.

Proposition 4 *The buyer's payoff increases with* T*, and the payoff converges when* $T \to \infty$*.*

Proof. When the number of rounds increases from M to $M+1$, the buyer can submit price $\underline{c}$ in the first round and then in the remaining rounds, do the same thing as when there are M rounds. Following this strategy, the buyer's payoff is the same as when $T = M$, and he might be able to do better by using other strategies. Therefore, the buyer's payoff is weakly increasing with T. Moreover, the buyer's payoff is bounded by the payoff in Myerson's optimal mechanism, so the payoff converges when $T \to \infty$. ∎

Therefore, when the buyer does not have time preference, having more rounds is weakly better for him. We need the following condition for subsequent discussion.

Condition 1 *Assume that* F *is such that* $\overline{x}_t^T(x_{t-1})$ *defined in (P1.1) and (P1.3) is continuous on* $[\underline{c}, \overline{c}]$ *for all* t *and* T*.*

It can be proved that if the distribution F is uniform on $[\underline{c}, \overline{c}]$, Condition 1 holds.[6] Condition 1 ensures that the objective functions and constraints of the programs in Section 1.4 are continuous, so the generalized envelope theorem by Milgrom and Segal (2002) can be applied.

For convenience, we denote x_t and b_t on the equilibrium path when there are T rounds by x_t^T and b_t^T. The following proposition shows a convergence property of x_{T-t}^T when T goes to infinity.

Proposition 5 *Assume Condition 1.* $\lim_{T\to\infty} x_{T-t}^T$ *exists for all* $t \in \{0, 1, \cdots\}$*.*

Proof. Note that given any t and T, $\overline{x}_t^T(\cdot) = \overline{x}_{t+1}^{T+1}(\cdot)$ (defined in program P1.3 on page 19). When we increase the number of rounds from T to $T+1$, $x_1^{T+1} \geq x_0^{T+1} = x_0^T$. By Lemma 1 in Appendix B, $x_1^{T+1} \geq x_0^T$ implies $x_{T+1-t}^{T+1} \geq x_{T-t}^T$ for all t. Hence, x_{T-t}^T increases in T. Furthermore, x_{T-t}^T has an upper bound $\overline{c}$, so we conclude that $\lim_{T\to\infty} x_{T-t}^T$ exists. ∎

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[6] If

$$\begin{aligned}\phi_t^T(x_{t-1}, x_t) \equiv{}& (v - x_t)\left[F(x_{t-1})^N - F(x_t)^N\right] \\ &- C_{t+1}^T(x_t, \delta)\left[F(x_{t-1}) - F(x_t)\right] + V_{t+1}^T(x_t, \delta)\end{aligned}$$

is concave in x_t for any t and T, then Condition 1 holds.

Proposition 5 shows that x^T_{T-t} converges as $T \to \infty$ with t held fixed. Intuitively, there are two different ways in which this could happen. One is that trade could be taking place gradually so that $x^T_{T-t} \approx \bar{c}$ for T large enough. Another is that the pattern of sales could look like the one we saw in the example in Section 1.3 where most trade occurred at the end. The main result of this section shows formally that there are two different possibilities like this. To state the result, we need a preliminary definition:

Let $X^T = \{x^T_t\}^T_{t=1}$. The following defines a cluster point of the cutoff set X^T when $T \to \infty$.

Definition 2 *$z \in [\underline{c}, \bar{c}]$ is a cluster point if for any $\epsilon > 0$, there exists y such that (i) $0 < |y - z| < \epsilon$, and (ii) for any $\delta > 0$, there exists T' such that for all $T > T'$, there exists $x \in X^T$ such that $|y - x| < \delta$.*

Let B be the set of cluster points, and $[\underline{c}, \bar{c}] \setminus B$ be the complement of B.

Theorem 2 *Assume Condition 1.*

1. *The cluster point set B is either the whole interval $[\underline{c}, \bar{c}]$ or a single point $\{\underline{c}\}$, i.e. $B = [\underline{c}, \bar{c}]$ or $\{\underline{c}\}$.*

2. *The cluster point set B is a single point $\{\underline{c}\}$ if and only if the last period cutoff x^T_T is bounded away from $\bar{c}$ when $T \to \infty$, i.e. $B = \{\underline{c}\}$ if and only if $\lim_{T\to\infty} x^T_T < \bar{c}$.*

3. *If $B = [\underline{c}, \bar{c}]$, the buyer's payoff is approximately the same as that in a first-price auction without a reserve price.*

Proof. The details of the proof are in Appendix B. Here is the sketch. Lemma 3 shows that if the number of rounds left in a continuation game starting with belief x_{t-1} goes to infinity, then the difference between x_t and x_{t-1} goes to 0. So, if $a \in [\underline{c}, \bar{c}]$ is a cluster point, any point $x < a$ must be a cluster point too. However, Lemma 6 shows that it cannot be the case that $a \in (\underline{c}, \bar{c})$, $[\underline{c}, a]$ belongs to the cluster point set B, and $(a, \bar{c}]$ belongs to the complement of B because it violates the necessary condition under which the buyer chooses the optimal strategy for himself in every round. Therefore, the cluster point set is either $[\underline{c}, \bar{c}]$ or $\{\underline{c}\}$. The third statement comes from the revenue equivalence principle. ■

The first statement of the theorem implies that there are only two possible equilibrium paths: one with the cluster point set B equal to the whole interval $[\underline{c}, \overline{c}]$ and one with the cluster point set equal to a single point $\{\underline{c}\}$. If the cluster point set is $[\underline{c}, \overline{c}]$, then it is implied that the sellers are almost fully discriminated in equilibrium, and information about sellers' costs is revealed gradually over time. To discriminate the sellers, the buyer will increase the bids gradually and stop at $\overline{c}$, and transactions occur constantly along the path. If the cluster point set is $\{\underline{c}\}$, then most cutoff points x_t cluster at $\underline{c}$, and only a few cutoff points spread around other places. In this case, sellers with costs within the same cutoff interval accept the same price, so they are not fully discriminated. In addition, most information about sellers' costs is revealed just before the deadline. Since the prices in the first many rounds are accepted by sellers with costs around $\underline{c}$, and the prices in the last few rounds are accepted by sellers with costs in higher intervals, by the revenue equivalence principle, we can derive the price path and show that the prices in the first many rounds are roughly the same, and there are big jumps in prices in the last few rounds. Therefore, if the cluster point set is $\underline{c}$, the equilibrium bidding path is convex. Moreover, since the bids in the first many rounds are only accepted by sellers with costs around $\underline{c}$, transaction is more likely to occur in the last few rounds.

The second statement of the theorem says that the occurrence that $\{\underline{c}\}$ is the only cluster point occurs if and only if $\lim_{T\to\infty} x_T^T$ is strictly lower than $\overline{c}$. In other words, late transaction and information revelation coincide with the possibility that the buyer's demand is not fulfilled.

The result could explain the puzzle proposed by Spann and Tellis. Spann and Tellis (2006) employ the data of a NYOP retailer in Germany that sells airline tickets for various airlines and allows multiple bidding to analyze buyers' bidding patterns. They argue that with positive bidding cost, the pattern should be concavely increasing because at the beginning, consumers try to increase the probability of successful bidding by bidding higher, but when the bids are closer to their reservation value, the increasing rate slows down; and with zero bidding cost, the pattern should reflect linearly increasing bids. However, the result shows that only 36% of the data fit the first pattern and 5% fit the second pattern. 23% of the data fit the pattern which is convexly increasing, so they conclude that consumer behavior on the internet is not so rational. Nevertheless, a convexly increasing pattern corresponds to the case $B = \{\underline{c}\}$ in Theorem 2.

Thus, a convex path can actually occur in a fully rational environment.[7] In addition to the convex bidding path, the case $B = \{\underline{c}\}$ also implies that most transactions occur near the end. This is related to the deadline effect that has been observed in many negotiation processes such as bargaining during strikes and pretrial negotiation. Our model thus provides insight into this phenomenon.

1.5.3 Factors that affect the type of the equilibrium path

What would happen on the equilibrium path depends on the distribution of sellers' cost F, the buyer's value v, and the number of sellers N. Under NYOP, the buyer is allowed to set up a price path so that $\lim_{T\to\infty} b_T^T = \lim_{T\to\infty} x_T^T < \bar{c}$, which functions as a reserve price. But since there is no commitment, to sustain $\lim_{T\to\infty} b_T^T < \bar{c}$, the buyer must have $\lim_{T\to\infty} x_T^T > \lim_{T\to\infty} x_{T-1}^T > \lim_{T\to\infty} x_{T-2}^T > \cdots > \lim_{T\to\infty} x_1^T$ (by Lemma 2 in Appendix B), and this requirement incurs some costs. First, the buyer has to charge the same price for sellers between x_t and x_{t-1}, and hence sellers receive more information rent than when fully discriminated. Furthermore, sellers in $[x_t, x_{t-1})$ get to sell the good with the same probability. Hence, the allocation is not efficient under NYOP. If the benefit dominates the loss of having $\lim_{T\to\infty} b_T^T < \bar{c}$, the equilibrium path will lead to $\lim_{T\to\infty} b_T^T < \bar{c}$.

Figure 1-1 shows the path of x_t when a seller's cost is uniformly distributed on $[0, 1]$ and $T = 20, N = 2$. When $v = 1$, $v = 1.2$, and $v = 1.4$, the optimal reserve prices are 0.5, 0.6, and 0.7 respectively. So when $v = 1$, the buyer is more inclined to have x_{20} much lower than $\bar{c} = 1$, and in equilibrium, a seller with cost higher than 0.1 would not sell the good until the last two periods, which implies transactions are much more likely to occur in the last two periods. On the other hand, when $v = 1.4$, the loss of having x_{20} much lower than $\bar{c} = 1$ dominates the benefit, so in equilibrium, the buyer raises bids gradually to a price close to 1, and transactions occur constantly in every period.

One thing that deserves mention is that when the benefit of having a reserve price is large enough, in order to attain $\lim_{T\to\infty} b_T^T < \bar{c}$, the buyer has to restrict himself from getting too much information about sellers' costs. Supposing he raises bids early so that sellers with higher

[7] After the buyer's waiting cost is incorporated in the next section, all the three patterns can occur in our model with different parameters.

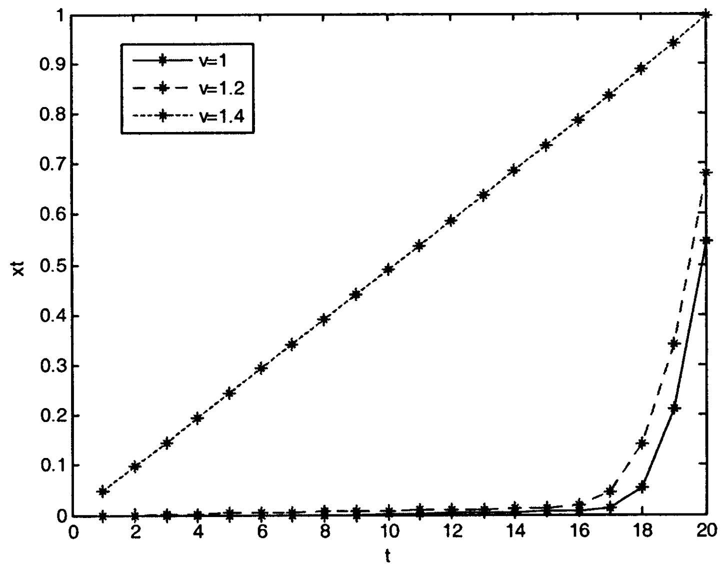

Figure 1-1: Path of x_t

cost also accept, once the bid is rejected, he believes that sellers' costs are above a higher threshold and will raise bids further in the next rounds. In the end, $\lim_{T\to\infty} b_T^T = \bar{c}$. Therefore, he has to keep the bids low most of the time and his belief about sellers' costs does not change much until the last few rounds; and since he only has a few chances left, he cannot raise bids to $\bar{c}$, so $\lim_{T\to\infty} b_T^T < \bar{c}$. The following proposition provides some means to check whether $\lim_{T\to\infty} b_T^T < \bar{c}$ or $\lim_{T\to\infty} b_T^T = \bar{c}$.

Proposition 6 *If* $\lim_{T\to\infty} x_T^T = \bar{c}$, *there does not exist a finite number* M *such that the buyer's expected payoff when there are* M *rounds is higher than that in a first-price reverse auction without a reserve price.*

Proof. If the buyer's payoff when $T = M$ is higher than that in a first-price reverse auction without a reserve price, by Proposition 4, the buyer's payoff when $T \to \infty$ is weakly higher than when $T = M$. Hence, by the third statement of Theorem 2, $\lim_{T\to\infty} b_T^T = \bar{c}$ would not happen. ■

For example, when $N = 2$, $\overline{F}(x) = x$ on $[0,1]$ and $v = 1$, the expected payoff of the buyer is $\frac{1}{3}$ in a first-price reverse auction. But if the buyer is allowed to submit the price once, and he chooses $b = 0.4225$, then the expected payoff is 0.3849. Thus, we know that x_T is bounded away from $\bar{c}$ when $T \to \infty$.

1.5.4 Payoff comparison among different mechanisms

The proposition and the theorem give insights into why Priceline.com has to limit bidding chances within a period of time. Suppose travelers realize their demand for a hotel room M days in advance. If allowed to submit bids many times a day, under some circumstances, travelers would not submit serious bids until the last day, and so successful transactions only occur on the day just before the trip. This would somewhat inconvenience the hotels and travelers. If only one bid is allowed a day, then transactions will occur much earlier, but the negative impact on travelers' payoff is infinitesimal. This intuition is formalized and analyzed in the next section.

Based on the analysis above, we can also characterize the buyer's payoff with different equilibrium paths and obtain an upper bound and a lower bound for the buyer's expected payoff under NYOP.

Theorem 3 *When $T \to \infty$, if on the equilibrium path, $\lim_{T\to\infty} b_T^T = \lim_{T\to\infty} x_T^T < \bar{c}$, the buyer's expected payoff is strictly greater than that in a reverse auction without a reserve price. Thus, when $T \to \infty$, the buyer's expected payoff is between the payoff in a reverse auction without a reserve price and the payoff in a reverse auction with the optimal reserve price.*

Proof. Note that when $T \to \infty$, a path that almost fully discriminates sellers and satisfies sellers' IC constraint is a feasible solution candidate to program P1.7 (it is the stationary solution to program P1.7 when $T = \infty$, see Appendix B, Proposition 9) and it brings the buyer almost the same expected payoff as in a reverse auction with no reserve price. Therefore, if the solution to program P1.7 is the path with $\lim_{T\to\infty} b_T^T = \lim_{T\to\infty} x_T^T < \bar{c}$, it must yield a higher value for the program than in a reverse auction with no reserve price. This proves the first statement. The second statement follows from Theorem 2, Proposition 3, and the first statement. ■

We can consider the mechanism used in Hotwire.com as a first-price reverse auction without a reserve price. Hotels submit their prices to Hotwire.com, and Hotwire.com picks the lowest one and announces it on the website. Customers see the price and decide whether to buy or not. Therefore, we should expect that customers get higher expected savings under NYOP.

1.6 Model with Buyer's Waiting Cost

At Priceline, when a bid is rejected, a customer has to wait for a period of time to submit another bid, but some other NYOP websites in Europe allow customers to rebid immediately once their bids are rejected. In this section, we examine the conditions under which having the lockout period restriction benefits customers.

1.6.1 The model and an example

We modify the model in Section 1.2 to fit the real environment better. In reality, buyers would like to pin down their travel plans as early as possible, so late transactions actually incur some waiting costs. Therefore, we incorporate buyers' waiting cost and show that setting an appropriate lockout period rule may benefit the buyer. However, we assume that sellers have no preference for early or late transactions.

The model is modified as follows. The buyer realizes his demand for the good at time 0 and tries to fulfill the demand in time period $[0, M]$. After time M, the buyer no longer needs the good. If the buyer gets the good at price B at time t, his utility is $\delta^{\frac{t}{M}}(v - B)$, where $\delta \in (0, 1)$. The platform sets a lockout period rule which regulates how frequently the buyer can submit a bid. If the lockout period is s, the buyer can submit bids for $\lfloor \frac{M}{s} \rfloor$ times, that is, $T = \lfloor \frac{M}{s} \rfloor$.

After incorporating waiting cost, let us revisit the example in Section 1.3 and confirm our conjecture about how the lockout period improves the buyer's payoff. The following table is for the case when $\delta = 0.9$.

	Buyer's Payoff	x_{T-4} (b_{T-4})	x_{T-3} (b_{T-3})	x_{T-2} (b_{T-2})	x_{T-1} (b_{T-1})	x_T (b_T)
$T = 1$	0.3849					0.4225 (0.4225)
$T = 2$	0.3897				0.2066 (0.4405)	0.5418 (0.5418)
$T = 3$	0.3844			0.1204 (0.4422)	0.2885 (0.5006)	0.5891 (0.5891)
$T = 4$	0.3802		0.0959 (0.4479)	0.1840 (0.4855)	0.3356 (0.5343)	0.6163 (0.6163)
$T = 5$	0.3773	0.0579 (0.4391)	0.1086 (0.4617)	0.1875 (0.4885)	0.3356 (0.5348)	0.6163 (0.6163)

Compared to the result when $\delta = 1$, we see that a buyer with waiting cost trades more eagerly. However, he would still like to have x_5 much lower than $\bar{c}$ to serve as a reserve price, so he has to suppress his intention to induce early transaction and cannot raise bids too fast. With the conflict, the table shows that allowing two bidding chances yields the highest payoff for the buyer. Having more rounds causes delay, which is costly to the buyer. The example illustrates that the lockout period rule which puts restriction on the buyer's bidding chances might actually help the buyer.

1.6.2 Equilibrium bidding path with no lockout period and $\delta < 1$

When $\delta = 1$, we show in Section 1.5.2 that when there is no lockout period, there are two possible equilibrium paths – either sellers are almost fully discriminated over time or they get pooled into some cost intervals. In the latter case, the price pattern is convexly increasing, and most of trades will be realized at the end. In this section, we show that with $\delta < 1$, there is one more possible path along which sellers with costs below some level are almost fully discriminated and sellers with costs above the level are pooled in intervals.

First, let

$$\left(\overline{b}_T^T(x_{T-1},\delta),\overline{x}_T^T(x_{T-1},\delta)\right) \in \arg\max_{\{b_T,x_T\}}(v-b_T)P(x_{T-1},x_T) \quad \text{s.t. } b_T = x_T, \tag{P1.8}$$

and

$$\left(\overline{b}_t^T(x_{t-1},\delta),\overline{x}_t^T(x_{t-1},\delta)\right) \in \arg\max_{\{b_t,x_t\}}(v-b_t)P(x_{t-1},x_t)+\sqrt[T]{\delta}V_{t+1}(x_t,\delta) \quad \text{s.t.} (b_t-x_t)G(x_{t-1},x_t)=C_{t+1}(x_t,\delta). \tag{P1.9}$$

for $t<T$. We need Condition 2 and Condition 3 for subsequent discussion.

Condition 2 *Assume that F is such that $\overline{x}_t^T(x_{t-1},\delta)$ is continuous in x_{t-1} on $[\underline{c},\overline{c}]$ for all t and T.*

Condition 3 *Assume that F is such that for any T, k, and $x\in[\underline{c},\overline{c}]$, $\overline{x}_{T-k}^T(x,\delta)$ converges to $\overline{x}_{T-k}^T(x,1)$ when T goes to infinity.*

Note that $\overline{x}_{T-k}^T(\cdot,1)$ is independent of T. If the distribution F is uniform on $[\underline{c},\overline{c}]$, it can be proved that Condition 2 and Condition 3 hold.[8]

Proposition 7 *Assume Conditions 2 and 3. Given δ, $\lim_{T\to\infty}x_{T-k}^T$ exists for all $k\in\{0,1,\cdots\}$.*

Proof. See Appendix B ■

The following theorem is a companion of Theorem 2, which characterizes the equilibrium path given $\delta\in(0,1]$ when there is no lockout period. The cluster point set B is defined on page 26.

[8] Given δ, if

$$\begin{aligned}\phi_t^T(x_{t-1},x_t) \equiv\ & (v-x_t)\left[F(x_{t-1})^N-F(x_t)^N\right] \\ & -C_{t+1}^T(x_t,\delta)\left[F(x_{t-1})-F(x_t)\right]+\sqrt[T]{\delta}V_{t+1}^T(x_t,\delta)\end{aligned}$$

is concave in x_t for any t and T, then Condition 2 holds. If

$$\begin{aligned}\phi_t^T(x_{t-1},x_t) \equiv\ & (v-x_t)\left[F(x_{t-1})^N-F(x_t)^N\right] \\ & -C_{t+1}^T(x_t,\delta)\left[F(x_{t-1})-F(x_t)\right]+V_{t+1}^T(x_t,\delta)\end{aligned}$$

is concave in x_t for any t and T, then Condition 3 holds.

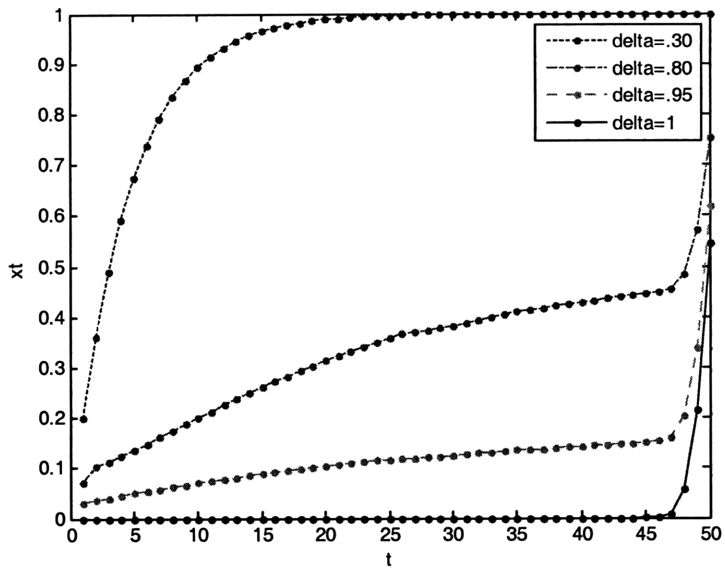

Figure 1-2: Path of x_t with different values of δ

Theorem 4 *Assume Conditions 2 and 3. Given* $\delta \in (0, 1]$,

1. *The cluster point set* B *is* $[\underline{c}, a]$, *where* $a \in [\underline{c}, \overline{c}]$.

2. *The cluster point set is not the whole interval* $[\underline{c}, \overline{c}]$ *if and only if the last period cutoff* x_T^T *is bounded away from* $\overline{c}$ *when* $T \to \infty$, *i.e.* $a < \overline{c}$ *if and only if* $\lim_{T\to\infty} x_T^T < \overline{c}$.

Proof. See Appendix B. ■

Figure 1-2 illustrates the points made in Theorem 4. It shows the paths of x_t for different values of δ when $v = 1$, $T = 50$, $N = 2$, and a seller's cost is uniformly distributed on $[0, 1]$. The paths with $\delta = 1$, 0.95, and 0.8 are consistent with the case of $a < \overline{c}$, and the path with $\delta = 0.3$ is consistent with the case of $a = \overline{c}$. When $\delta = 0.3$, the differences between adjacent x_t's in the first few rounds are relatively large. However, they shrink as the number of rounds increases, as shown in Figure 1-3. Figure 1-3 depicts the paths of x_t in the first 30 rounds given $\delta = 0.3$ with $T = 50, 100$, and 150. When $T \to \infty$, the differences between adjacent x_t's go to 0. When $\delta = 1$, 0.95, and 0.8, the differences between adjacent x_t's in the last three rounds are

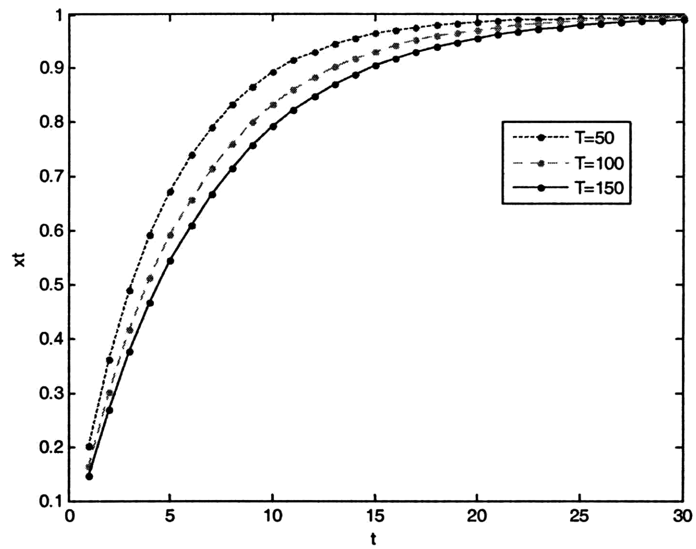

Figure 1-3: Path of x_t in the first 30 rounds with different numbers of total rounds, $\delta = 0.3$.

large. However, they do not shrink when the number of rounds increases, as shown in Figure 1-4. Figure 1-4 depicts the paths of x_t given $\delta = 0.95$ with $T = 50, 100$, and 150.

Theorem 2 is a special case of Theorem 4. When $\delta = 1$, $a = \underline{c}$ or $\overline{c}$; and when $\delta \in (0, 1)$, a can be anything in $[\underline{c}, \overline{c}]$, and Figure 1-2 shows that a decreases in δ. The difference comes from the fact that with $\delta < 1$, after several rounds, some waiting cost has been sunk and the remaining time left before the deadline is shorter. It is as if the buyer now has a higher discount factor, so the buyer's bidding behavior changes accordingly. We can see from Figure 1-2 that with lower δ, the path is more concave at the beginning since the buyer is more eager to get the good. As time passes by and less time is left, the path turns convex.

1.6.3 Optimal lockout period

In this section, we use some numerical examples to study the pros and cons of the lockout period rule and characterize the circumstances under which setting an appropriate lockout period increases the buyer's payoff.

With a discount factor δ lower than 1, the example in Section 1.6.1 shows that the buyer's payoff does not monotonically increase with the number of rounds, which contrasts to the result

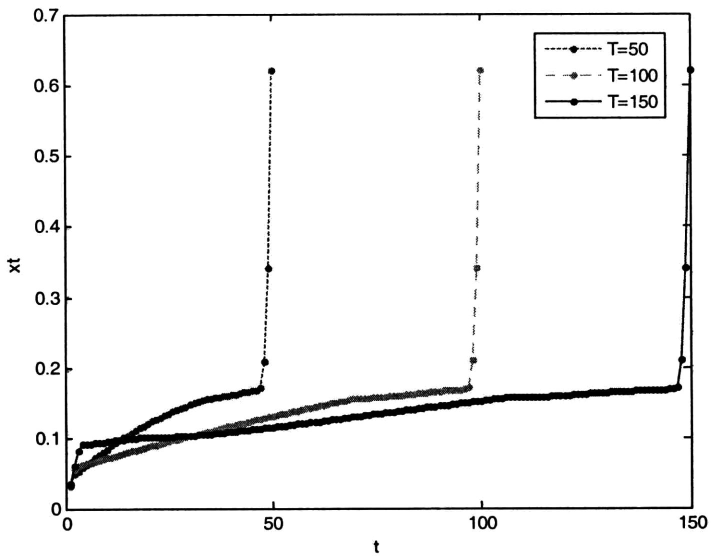

Figure 1-4: Path of x_t with different numbers of total rounds, $\delta = 0.95$.

in Proposition 4.

Proposition 8 *With $\delta \in (0, 1)$, the buyer's payoff might not monotonically increase with the number of rounds T.*

The following discusses how a lockout period rule affects the buyer's payoff given different values of δ. We focus on the settings in which Myerson's optimal mechanism involves setting a reserve price. If setting a reserve price is unnecessary, having more rounds always benefits the buyer because it helps the buyer discriminate the sellers better and be able to close the transaction earlier.

With high discount factor When the discount factor is high but lower than 1, if there is no lockout period, in equilibrium, the path of x_t is convex (see Figure 1-2), the last-round price is lower than $\bar{c}$, and most transactions occur late. If there is a lockout period, the buyer has fewer bidding chances and will bid seriously from the beginning, so transactions occur earlier. However, the buyer also loses chances to discriminate sellers with cost around $\underline{c}$.

With low discount factor When the discount factor is low, if there is no lockout period, the buyer raises the bid aggressively, and the bidding path is concave. With a lockout period, the buyer cannot raise the bid all the way up to $\bar{c}$, so there is a reserve-price-like effect. But the lockout period limits the buyer's bidding chances so that the buyer cannot discriminate the sellers well, and it also prevents the buyer from bidding aggressively and getting the good early.

We consider the example where $N = 2$, $v = 1.2$, and a seller's cost is uniformly distributed on $[0, 1]$. The following table summarizes the number of rounds T^* that maximizes the buyer's payoff and the corresponding buyer's payoff $\pi(T^*)$, given different values of δ.

δ	$\lim_{T\to\infty} x_T$	T^*	$\pi(T^*)$
1.00	0.668	∞	0.5559
0.95	0.774	2	0.5461
0.90	1	2	0.5399
0.85	1	∞	0.5333

The result shows that setting a lockout period so that the buyer has two bidding chances maximizes the buyer's payoff when $\delta = 0.95$ and 0.9. With $\delta = 0.95$, when there is no lockout period, $\lim_{T\to\infty} x_T < 1$, so the equilibrium path of x_t is mostly convex, and transaction is very likely to occur late. By setting a lockout period, the buyer benefits from having early transactions but suffers from not being able to discriminate sellers with costs around $\underline{c}$. With $\delta = 0.9$, when there is no lockout period, $\lim_{T\to\infty} x_T = 1$, so the equilibrium path of x_t is concave, and transactions occur early. By setting a lockout period, the buyer benefits from having a last-round price lower than $\bar{c}$, which functions like a reserve price, but suffers from not being able to close transaction early and discriminate sellers finely. In these two cases, the benefit of having a lockout period dominates the loss. However, with δ very close to 1 and δ lower than 0.85, the loss dominates the benefit, so setting a lockout period hurts the buyer.

In addition, setting a lockout period can be valuable for the buyer when having a reserve price benefits the buyer a lot. Consider another example where $v = 1$ and the other parameters are the same as before. The optimal reserve price is 0.5. In this case, if δ is lower than 0.62, $\lim_{T\to\infty} x_T = 1$, so the buyer's payoff when there is no lockout period is at most $\frac{1}{3}$, the payoff in a reverse auction with no reserve price. On the other hand, the buyer's payoff when only one bidding chance is allowed is 0.3849 for all δ. Therefore, setting a lockout period benefits the buyer if $\delta < 0.62$ (it also benefits the buyer for higher values of δ.)

From the discussion above, we see that NYOP websites with different designs of rebidding rules are preferred by different kinds of customers. Priceline's lockout period rule seems to hurt customers by restricting their rebidding opportunities, but in fact, a customer with waiting cost might find it beneficial.

1.7 Conclusion

This paper analyzes the Name Your Own Price (NYOP) mechanism adopted by Priceline.com. We characterize the buyer's and the sellers' equilibrium strategies and show that Priceline.com's lockout period restriction, a design to protect sellers that seems to hurt customers, can actually benefit a customer with moderate discount factor.

We show that when there is no lockout period and no waiting cost, the equilibrium paths can be categorized into two classes. In the first class, the cluster point set of the sellers' cost cutoffs in all rounds is the whole cost interval $[\underline{c}, \overline{c}]$, which implies that sellers with different costs are almost fully discriminated and information about sellers' cost is revealed gradually over time. In this case, the buyer raises bids constantly, the ending price is the highest possible cost $\overline{c}$, and the buyer's payoff is approximately the same as the payoff in a reverse auction without a reserve price. In the second class, the cluster point set is a single point $\{\underline{c}\}$, which implies that sellers with different costs are pooled in intervals except the one with the lowest possible cost, and information about the sellers' cost is barely revealed in the first many rounds. In this case, the buyer does not raise the bid much until the very end, the ending price is lower than $\overline{c}$, and the buyer's payoff is greater than the payoff in a reverse auction without a reserve price. In the second type of equilibrium paths, most transactions occur just before the deadline. The delay of transactions incurs waiting cost if the buyer has time preference. Therefore, setting a lockout period might actually benefit a buyer by moving transactions forward.

This paper also indicates some interesting extensions for future research. Based on our analysis, one might be curious about whether Priceline can do better by adopting other measures, such as restricting the number of bidding chances instead of the frequency of bidding.

Moreover, one can extend the model to consider the case when there are multiple buyers with private information about their own valuations, which better characterizes the situation of high travel season.

1.8 Appendix A

Proof of Proposition 1. There exists a set of solutions $\left\{\overline{b}_t(x_{t-1}), \overline{x}_t(x_{t-1})\right\}_t$ that solves program P1.1 and P1.3 for all t. In the last period, recall that

$$\begin{aligned}
V_T(x_{T-1}) &= \max_{x_T \in [x_{T-1}, \overline{c}]} (v - x_T) P(x_{T-1}, x_T), \\
\overline{x}_T(x_{T-1}) &\in X_T(x_{T-1}) = \arg \max_{x_T \in [x_{T-1}, \overline{c}]} (v - x_T) P(x_{T-1}, x_T), \\
\text{and } C_T(x_{T-1}) &= (\overline{x}_T(x_{T-1}) - x_{T-1}) G(x_{T-1}, \overline{x}_T(x_{T-1})).
\end{aligned}$$

By Berge's maximum theorem, we know $V_T(x_{T-1})$ is continuous and $X_T(x_{T-1})$ is upper hemi-continuous. In period $t, t < T$, let

$$\begin{aligned}
\phi_t(x_{t-1}, x_t) &= (v - x_t)\left[F(x_{t-1})^N - F(x_t)^N\right] \\
&\quad - C_{t+1}(x_t)\left[F(x_{t-1}) - F(x_t)\right] + V_{t+1}(x_t), \\
\alpha(x_{t-1}) &= [x_{t-1}, \overline{c}].
\end{aligned}$$

Then

$$\begin{aligned}
V_t(x_{t-1}) &= \max_{x_t \in \alpha(x_{t-1})} \phi_t(x_{t-1}, x_t) \\
\overline{x}_t(x_{t-1}) \in X_t(x_{t-1}) &= \arg \max_{x_t \in \alpha(x_{t-1})} \phi_t(x_{t-1}, x_t).
\end{aligned}$$

We show that by picking a proper $\overline{x}_t(x_{t-1})$ from $X_t(x_{t-1}), t \leq T$, each round-t program has a solution.

First observe that for upper hemi-continuous correspondence X_T, we are able to find n_T closed intervals $[a_k, a_{k+1}], k = 1, \cdots, n_T$, such that $\cup_k [a_k, a_{k+1}] = [\underline{c}, \overline{c}]$, and n_T continuous functions $\overline{x}_{T,k} : [a_k, a_{k+1}] \longrightarrow [a_k, \overline{c}]$ such that $\overline{x}_{T,k}(x) \in X_T(x), \forall x \in [a_k, a_{k+1}]$. Let

$$\begin{aligned}
C_T(x_{T-1}) &= \begin{cases} (\overline{x}_{T,k}(x_{T-1}) - x_{T-1})G(x_{T-1}, \overline{x}_{T,k}(x_{T-1})), \text{ if } x_{T-1} \in (a_k, a_{k+1}) \\ \min \begin{Bmatrix} (\overline{x}_{T,k}(x_{T-1}) - x_{T-1})G(x_{T-1}, \overline{x}_{T,k}(x_{T-1})), \\ (\overline{x}_{T,k+1}(x_{T-1}) - x_{T-1})G(x_{T-1}, \overline{x}_{T,k+1}(x_{T-1})) \end{Bmatrix}, \text{ if } x_{T-1} = a_{k+1}, k < n_T \end{cases}, \\
\overline{x}_T(x_{T-1}) &= \begin{cases} \overline{x}_{T,k}(x_{T-1}), \text{ if } x_{T-1} \in (a_k, a_{k+1}) \\ \arg\min_{x \in \{\overline{x}_{T,k}(x_{T-1}), \overline{x}_{T,k+1}(x_{T-1})\}} (x - x_{T-1})G(x_{T-1}, x), \text{ if } x_{T-1} = a_{k+1}, k < n_T \end{cases}, \\
\overline{b}_T(x_{T-1}) &= \overline{x}_T(x_{T-1}).
\end{aligned}$$

C_T is lower semi-continuous and V_T is continuous, so ϕ_{T-1} is upper semi-continuous. Note that ϕ_{T-1} is graph-continuous with respect to α, which is defined in Leininger (1984). So by Leininger's generalized maximum theorem, V_{T-1} is upper semi-continuous, and X_{T-1} is upper hemi-continuous.

Similarly, since X_{T-1} is upper hemi-continuous, we are able to find n_{T-1} closed intervals $[a'_k, a'_{k+1}], k = 1, \cdots, n_{T-1}$, such that $\cup_k [a'_k, a'_{k+1}] = [\underline{c}, \overline{c}]$, and n_{T-1} continuous functions $\overline{x}_{T-1,k} : [a'_k, a'_{k+1}] \longrightarrow [a'_k, \overline{c}]$ such that $\overline{x}_{T-1,k}(x) \in X_{T-1}(x), \forall x \in [a'_k, a'_{k+1}]$. Let

$$C_{T-1}(x_{T-2}) =$$
$$\begin{cases} (\overline{x}_{T-1,k}(x_{T-2}) - x_{T-2})G(x_{T-2}, \overline{x}_{T-1,k}(x_{T-2})) + C_T(x_{T-1}), \text{ if } x_{T-2} \in (a'_k, a'_{k+1}) \\ \min \begin{Bmatrix} (\overline{x}_{T-1,k}(x_{T-2}) - x_{T-2})G(x_{T-2}, \overline{x}_{T-1,k}(x_{T-2})) + C_T(x_{T-1}), \\ (\overline{x}_{T-1,k+1}(x_{T-2}) - x_{T-2})G(x_{T-2}, \overline{x}_{T-1,k+1}(x_{T-2})) + C_T(x_{T-1}) \end{Bmatrix}, \text{ if } x_{T-2} = a'_{k+1} \end{cases},$$

$$\overline{x}_{T-1}(x_{T-2}) =$$
$$\begin{cases} \overline{x}_{T-1,k}(x_{T-2}), \text{ if } x_{T-2} \in (a'_k, a'_{k+1}) \\ \arg\min_{x \in \{\overline{x}_{T-1,k}(x_{T-2}), \overline{x}_{T-1,k+1}(x_{T-2})\}} (x - x_{T-2})G(x_{T-2}, x) + C_T(x), \text{ if } x_{T-2} = a'_{k+1} \end{cases},$$
$$\overline{b}_{T-1}(x_{T-2}) = \overline{x}_{T-1}(x_{T-2}) + \frac{C_T(\overline{x}_{T-1}(x_{T-2}))}{G(\overline{x}_{T-2}, \overline{x}_{T-1}(x_{T-2}))}.$$

C_{T-1} is lower semi-continuous and V_{T-1} is upper semi-continuous, so ϕ_{T-2} is upper semi-continuous. Check that ϕ_{T-2} is graph-continuous with respect to α. Applying the same procedure, we conclude that there exists a set of solutions $\{\overline{b}_t(x_{t-1}), \overline{x}_t(x_{t-1})\}_t$ that solves program

P1.1 and P1.3 for all t. ■

Proof of Theorem 1. First we show that $u_t^i\left(b, x, x \mid h_t, \theta^i, y_t\left(h_{t-1}\right)\right) \geq u_t^i\left(b, x, x' \mid h_t, \theta^i, y_t\left(h_{t-1}\right)\right)$. If $t = T$, $x_T\left(h_T\right) = p_T$. Seller i with cost $\theta^i \leq p_T$ gets positive expected payoff if accepting and 0 if not, so he should accept. Seller i with cost $\theta^i > p_T$ gets negative expected payoff if accepting and 0 if not, so he would not accept. Therefore, he should follow x. For $t < T$, let $x_{t-1} = x_{t-1}\left(h_{t-1}\right)$. In the continuation game, the price path $\left(b_{t+1}, b_{t+2}, \cdots, b_T\right)$ and the belief path $\left(y_{t+1}, y_{t+2}, \cdots, y_T\right) = \left(x_t, x_{t+1}, \cdots, x_{T-1}\right)$ can be found by solving the recursive program

$$\begin{aligned} &\max_{x_t}\left(v - b_t\right) P\left(x_{t-1}, x_t\right) + V_{t+1}\left(x_t\right) \\ &s.t. \text{if } \left(b_t - x_{t-1}\right) < C_{t+1}\left(x_{t-1}\right), x_t = x_{t-1}, \\ &\text{otherwise, } \left(b_t - x_t\right) G\left(x_{t-1}, x_t\right) = C_{t+1}\left(x_t\right). \end{aligned}$$

Seller i's deviation does not affect $\left(b_{t+1}, \cdots, b_T\right)$ and $\left(y_{t+1}, \cdots, y_T\right)$. Suppose seller i's cost θ^i is in $\left(x_{s-1}, x_s\right], s \geq t$, so he should buy in round s. If he accepts in round $s' \neq s$, $u_t^i\left(b, x, x' \mid h_t, \theta^i, y_t\left(h_{t-1}\right)\right) = \left(b_{s'} - \theta^i\right) \frac{G\left(x_{s'-1}, x_{s'}\right)}{NF\left(x_{t-1}\right)^{N-1}}$. If he sticks to x (accepts in round s), $u_t^i\left(b, x, x \mid h_t, \theta^i, y_t\left(h_{t-1}\right)\right) = \left(b_s - \theta^i\right) \frac{G\left(x_{s-1}, x_s\right)}{NF\left(x_{t-1}\right)^{N-1}}$. If $s' > s$, we know that

$$\begin{aligned} \left(b_s - x_s\right) G\left(x_{s-1}, x_s\right) &= \left(b_{s+1} - x_s\right) G\left(x_s, x_{s+1}\right) \\ &\vdots \\ \left(b_{s'-1} - x_{s'-1}\right) G\left(x_{s'-2}, x_{s'-1}\right) &= \left(b_{s'} - x_{s'-1}\right) G\left(x_{s'-1}, x_{s'}\right) \end{aligned}$$

Since $G\left(x_{s'-1}, x_{s'}\right) < NF\left(x_{s'-1}\right)^{N-1} < G\left(x_{s'-2}, x_{s'-1}\right)$, for any $x < x_{s'-1}$, $\left(b_{s'-1} - x\right) G\left(x_{s'-2}, x_{s'-1}\right) > \left(b_{s'} - x\right) G\left(x_{s'-1}, x_{s'}\right)$. Applying the same argument, since $\theta^i < x_s \leq \cdots \leq x_{s'-1}$, $\left(b_s - \theta^i\right) G\left(x_{s-1}, x_s\right) > \left(b_{s+1} - \theta^i\right) G\left(x_s, x_{s+1}\right) \geq \cdots \geq \left(b_{s'} - \theta^i\right) G\left(x_{s'-1}, x_{s'}\right)$. On the other hand, if $t \leq s' < s$, applying similar arguments, since $\theta^i > x_{s-1} \geq \cdots \geq x_{s'}$, $\left(b_s - \theta^i\right) G\left(x_{s-1}, x_s\right) > \left(b_{s-1} - \theta^i\right) G\left(x_{s-2}, x_{s-1}\right) \geq \cdots \geq \left(\theta^i - b_{s'}\right) G\left(x_{s'-1}, x_{s'}\right)$. Therefore, $\left(b_s - \theta^i\right) \frac{G\left(x_{s-1}, x_s\right)}{NF\left(x_{t-1}\right)^{N-1}} > \left(b_{s'} - \theta^i\right) \frac{G\left(x_{s'-1}, x_{s'}\right)}{NF\left(x_{t-1}\right)^{N-1}}$.

Next, we show that $u_t^0\left(b, x \mid h_{t-1}, y_t\left(h_{t-1}\right)\right) \geq u_t^0\left(b', x \mid h_{t-1}, y_t\left(h_{t-1}\right)\right)$. For any t and any h_{t-1}, given x, the buyer's optimal strategy must generate the path that maximizes his conditional utility

$$\max_{p_t} \frac{(v - p_t) P(x_{t-1}(h_{t-}), x_t((h_{t-}, p_t))) + V_{t+1}(x_t((h_{t-}, p_t)))}{F(x_{t-1}(h_{t-}))^N}. \tag{P1.10}$$

That is, the strategy b is consistent with the solution $(p_t, \cdots, p_T)$ derived from (P1.10) in the sense that $b_t(h_{t-1}) = p_t, b_{t+1}(h_{t-1}, p_t) = p_{t+1}, \cdots$. Under our construction of $x_t(h_t)$, the solution to (P1.10) is the same as $(b_t, \cdots, b_T)$ derived from (P1.2). Hence the strategy b constructed from (P1.2) is consistent with $(p_t, \cdots, p_T)$ and is optimal. ■

1.9 Appendix B

Proof of Proposition 2. Let r be the optimal reserve price. Submitting a path $(b_1, b_2, \cdots, b_T)$ so that in round t, sellers with cost below $x_t = \underline{c} + t\frac{r-\underline{c}}{T}$ accept, is a feasible choice. We show that the buyer's payoff with the path can be arbitrarily close to π^* when T goes to infinity.

Given the path, in the last round, sellers with cost below r accept, so $b_T = x_T = r$. In round t, $t < T$, a seller with cost x_t feels indifferent between accepting now or accepting in the next round, so the following constraint holds:

$$\begin{aligned}
&(b_t - x_t)G(x_{t-1}, x_t) = (b_{t+1} - x_t)G(x_t, x_{t+1}) \\
\Rightarrow &(b_t - x_t)\left(F(x_t)^{N-1} + F(x_t)^{N-2}F(x_{t-1}) + \cdots + F(x_{t-1})^{N-1}\right) \\
&= (b_{t+1} - x_t)\left(F(x_{t+1})^{N-1} + F(x_{t+1})^{N-2}F(x_t) + \cdots + F(x_t)^{N-1}\right).
\end{aligned}$$

If $\delta \equiv x_{t+1} - x_t$ and $\Delta \equiv b_{t+1} - b_t$ are small, an approximation of the equation is

$$\begin{aligned}
&(b_t - x_t)\left(NF(x_t)^{N-1} + \frac{(N-1)N}{2}F(x_t)^{N-2}f(x_t)\delta\right) \\
&\approx (b_t + \Delta - x_t)\left(NF(x_t)^{N-1} - \frac{(N-1)N}{2}F(x_t)^{N-2}f(x_t)\delta\right) \\
\Rightarrow &\frac{\Delta}{\delta} \approx \frac{(b_t - x_t)(N-1)F(x_t)^{N-2}f(x_t)}{F(x_t)^{N-1}} = \frac{(N-1)f(x_t)(b_t - x_t)}{F(x_t)}.
\end{aligned}$$

In a reverse Dutch auction with reserve price r, a seller with cost x accepts at price

$$b(x) = r\frac{F(r)^{N-1}}{F(x)^{N-1}} + \frac{1}{F(x)^{N-1}}\int_x^r y(N-1)F(y)^{N-2}f(y)\,dy,$$

which is also the price submitted by a seller with cost x in a first-price reverse auction with reserve price r.

$$\begin{aligned} b'(x) &= (N-1)f(x)\left[\frac{xF(x)^{N-1}+\int_x^r F(y)^{N-1}dy}{F(x)^N} - \frac{xF(x)^{N-2}}{F(x)^{N-1}}\right] \\ &= \frac{(N-1)f(x)(b(x)-x)}{F(x)}. \end{aligned}$$

Since $b(x_T) = b_T = r$ and $b'(x) = \frac{\Delta}{\delta}$, $b(x_t) - b_t = O(\delta^2)\cdot T(\delta) = O(\delta)$, where $T(\delta) = \frac{r-\underline{c}}{\delta}$. The optimal payoff is $\pi^* = \int_{b=\underline{c}}^r (v-b(x))\,dF(x)^N$. By the Riemann-Stieltjes integral, for all $\epsilon > 0$, there exists $\delta' > 0$ such that for all $\delta < \delta'$,

$$\left|\sum_{t=1}^{T(\delta)}(v-b(x_t))\left(F(x_t)^N - F(x_{t-1})^N\right) - \pi^*\right| < \frac{\epsilon}{2}.$$

Since $b(x_t) - b_t = O(\delta)$ and $\sum_{t=1}^{T(\delta)}\left(F(x_t)^N - F(x_{t-1})^N\right) = 1$, there exists $\delta'' > 0$ such that for all $\delta < \delta''$,

$$\left|\sum_{t=1}^{T(\delta)}[(v-b_t)-(v-b(x_t))]\left(F(x_t)^N - F(x_{t-1})^N\right)\right| < \frac{\epsilon}{2}.$$

Therefore, for any $\delta \le \min\{\delta', \delta''\}$, i.e. for any $T \ge \frac{r-\underline{c}}{\min\{\delta',\delta''\}}$,

$$\left|\pi^* - \sum_{t=1}^{T(\delta)}(v-b_t)\left(F(x_t)^N - F(x_{t-1})^N\right)\right| < \epsilon.$$

The buyer can do weakly better by choosing a better path, so $\pi^* - \pi(T(\delta)) < \epsilon$. ■

Lemma 1 *Assume Condition 1.* $\overline{x}_t^T(x_{t-1})$ *(defined in program P1.3) increases in* x_{t-1}.

Proof. It is easy to check that $\overline{x}_T^T(x_{T-1})$ defined in program P1.1 increases in x_{T-1}. With

$t < T$, $\overline{x}_t^T(x_{t-1})$ is derived from program P1.3. Let

$$\begin{aligned}\varphi(x_t, x_{t-1}) &\equiv (v - b_t(x_t; x_{t-1}))\left[F(x_{t-1})^N - F(x_t)^N\right] + V_{t+1}^T(x_t), \\ \text{where } b_t(x_t; x_{t-1}) &= \frac{C_{t+1}^T(x_t)}{F(x_t)^{N-1} + F(x_t)^{N-2}F(x_{t-1}) + \cdots + F(x_{t-1})^{N-1}} + x_t.\end{aligned}$$

$$\begin{aligned}\frac{\partial\varphi(x_t, x_{t-1})}{\partial x_t} &= -\left[F(x_{t-1}) - F(x_t)\right]\left[\left(F(x_{t-1})^{N-1} + \cdots + F(x_t)^{N-1}\right) + \frac{dC_{t+1}^T(x_t)}{dx_t}\right] \\ &\quad + f(x_t)\left(\overline{x}_{t+1}(x_t) - x_t\right)\left[NF(x_t)^{N-1} - \left(F(\overline{x}_{t+1}(x_t))^{N-1} + \cdots + F(x_t)^{N-1}\right)\right]\end{aligned}$$

and

$$\frac{\partial^2\varphi(x_t, x_{t-1})}{\partial x_t \partial x_{t-1}} = \frac{1}{\overline{c} - \underline{c}}\left[NF(x_{t-1})^{N-1} + \frac{dC_{t+1}^T(x_t)}{dx_t}\right].$$

For any x_t, x_{t-1} and $x_{t-1}' \in [x_{t-1}, x_t]$, if $\left(F(x_{t-1})^{N-1} + \cdots + F(x_t)^{N-1}\right) + \frac{dC_{t+1}^T(x_t)}{dx_t} < 0$, $\frac{\partial\varphi(x_t, x_{t-1})}{\partial x_t} > 0$ and $\frac{\partial\varphi(x_t, x_{t-1}')}{\partial x_t} > 0$. If $\left(F(x_{t-1})^{N-1} + \cdots + F(x_t)^{N-1}\right) + \frac{dC_{t+1}^T(x_t)}{dx_t} > 0$, $\frac{\partial^2\varphi(x_t, x_{t-1})}{\partial x_t \partial x_{t-1}} > 0$, so $\frac{\partial\varphi}{\partial x_t}(x_t, x_{t-1}) > 0$ implies $\frac{\partial\varphi}{\partial x_t}\left(x_t, x_{t-1}'\right) > 0$. Therefore, $\varphi(x_t, x_{t-1})$ satisfies single crossing property of marginal returns. By Milgrom-Shannon theorem, $\overline{x}_t^T(x_{t-1})$ increases in x_{t-1}. ■

We use Lemmas 2, 3, and 4 to prove Lemma 5 and Lemma 6, and use Lemmas 5, 6, and 4 to prove Theorem 2. We sometimes add superscript T to $V_t(x)$ and $C_t(x)$ (defined in (P1.2)) for clarification. Note that for two sets (t, T) and (t', T'), if $T - t = T' - t'$, then $V_t^T(x) = V_{t'}^{T'}(x)$ and $C_t^T(x) = C_{t'}^{T'}(x)$. So, we let $c_k(x) = C_{T-k}^T(x)$.

Proof of Theorem 2. By Lemma 6, if $B \neq \{\underline{c}\}$, there does not exist $a \in (\underline{c}, \overline{c})$ such that $(a, \overline{c}] \subset [\underline{c}, \overline{c}] \setminus B$. Then by Lemma 5, $B = [\underline{c}, \overline{c}]$. So the first statement is proved. The third statement follows from the revenue equivalence principle. For the second statement, if $\lim_{T\to\infty} x_T^T < \overline{c}$, it must be that $B = \{\underline{c}\}$. On the other hand, if $B = \{\underline{c}\}$, there exists $t < \infty$ such that $\overline{c} - \lim_{T\to\infty} x_{T-t}^T > 0$. By Lemma 4, $\lim_{T\to\infty} x_T^T < \overline{c}$. ■

Lemma 2 *Assume Condition 1. Given $k > 0$, there exists $\delta_k(\epsilon, x) > 0$ such that for any t and T where $T - t = k$, if $x_t^T = x_{t+1}^T - \epsilon$, $\epsilon > 0$, then $x_{t-1}^T \leq x_t^T - \delta_k\left(\epsilon, x_t^T\right)$. $\delta_k(\epsilon, x)$ does not depend on T.*

Proof. Given any t, T such that where $T-t=k$ and given belief x_{t-1}, the continuation equilibrium x_t^* and b_t^* are derived from

$$V_t^T(x_{t-1}) = \max_{\{b_t, x_t\}} (v-b_t)\left[F(x_{t-1})^N - F(x_t)^N\right] + V_{t+1}^T(x_t) \tag{P1.11}$$

$$s.t.(b_t - x_t)(F(x_t)^{N-1} + F(x_t)^{N-2}F(x_{t-1}) + \cdots + F(x_{t-1})^{N-1}) = C_{t+1}^T(x_t). \tag{1.2}$$

From (1.2),

$$b_t = \frac{C_{t+1}^T(x_t)}{F(x_t)^{N-1} + F(x_t)^{N-2}F(x_{t-1}) + \cdots + F(x_{t-1})^{N-1}} + x_t,$$

$$\begin{aligned}\frac{db_t}{dx_t} &= \frac{C_{t+1}^{T\prime}(x_t)}{F(x_t)^{N-1} + F(x_t)^{N-2}F(x_{t-1}) + \cdots + F(x_{t-1})^{N-1}} + 1 \\ &+ \frac{(b_t - x_t)\left[(N-1)F(x_t)^{N-2} + (N-2)F(x_t)^{N-3}F(x_{t-1}) + \cdots + F(x_{t-1})^{N-2}\right]f(x_t)}{F(x_t)^{N-1} + F(x_t)^{N-2}F(x_{t-1}) + \cdots + F(x_{t-1})^{N-1}}\end{aligned}$$

The solution $\{x_t^*, b_t^*\}$ must satisfy the first order condition

$$\begin{aligned} 0 &= -\frac{db_t(x_t^*)}{dx_t}\left[F(x_{t-1})^N - F(x_t^*)^N\right] + (v - b_t(x_t^*))\left(NF(x_t^*)^{N-1}f(x_t^*)\right) + V_{t+1}^{T\prime}(x_t^*) \\ \Rightarrow\ 0 &= \left[F(x_{t-1}) - F(x_t^*)\right]\left[-\left(F(x_{t-1})^{N-1} + \cdots + F(x_t^*)^{N-1}\right) - C_{t+1}^{T\prime}(x_t^*)\right] \\ &\quad - C_{t+1}^T(x_t^*)f(x_t^*) + N(v - x_t^*)F(x_t^*)^{N-1}f(x_t^*) + V_{t+1}^{T\prime}(x_t^*). \end{aligned} \tag{1.3}$$

Note that

$$\begin{aligned} V_{t+1}^T(x_t) &= \max_{\{b_{t+1}, x_{t+1}\}} (v - b_{t+1}(x_{t+1}; x_t))\left[F(x_t)^N - F(x_{t+1})^N\right] + V_{t+2}^T(x_{t+1}) \\ \text{where } b_{t+1}(x_{t+1}; x_t) &= \frac{C_{t+2}^T(x_{t+1})}{F(x_{t+1})^{N-1} + F(x_{t+1})^{N-2}F(x_t) + \cdots + F(x_t)^{N-1}} + x_{t+1}. \end{aligned} \tag{1.4}$$

Let $\{x_{t+1}^*(x_t), b_{t+1}^*(x_t)\}$ be the solution to program (1.4). By (1.2), $C_{t+2}^T(x_{t+1}^*) = (b_{t+1}^* - x_{t+1}^*)(F(x_{t+1}^*)^{N-1} + F(x_{t+1}^*)^{N-2}F(x_t) + \cdots + F(x_t)^{N-1})$. By the envelope theorem,

$$V_{t+1}^{T\prime}(x_t) = -NF(x_t)^{N-1}f(x_t)\left(v - x_{t+1}^*\right) + f(x_t)C_{t+2}^T\left(x_{t+1}^*\right). \tag{1.5}$$

Plugging into (1.3), we get

$$
\begin{aligned}
0 &= [F(x_{t-1}) - F(x_t^*)]\left[-C_{t+1}^{T\prime}(x_t^*) - \left(F(x_{t-1})^{N-1} + \cdots + F(x_t^*)^{N-1}\right)\right] \\
&\quad - \left[C_{t+1}^T(x_t^*) - C_{t+2}^T(x_{t+1}^*)\right] f(x_t^*) - NF(x_t^*)^{N-1} f(x_t^*)\left[x_t^* - x_{t+1}^*\right] \\
&= -[F(x_{t-1}) - F(x_t^*)]\left[\left(F(x_{t-1})^{N-1} + \cdots + F(x_t^*)^{N-1}\right) + c_k'(x_t^*)\right] \\
&\quad + f(x_t^*)\left(x_{t+1}^* - x_t^*\right)\left[NF(x_t^*)^{N-1} - \left(F(x_{t+1}^*)^{N-1} + \cdots + F(x_t^*)^{N-1}\right)\right]. \qquad (1.6)
\end{aligned}
$$

If $x_t^* = x_{t+1}^* - \epsilon$,
$\left(x_{t+1}^* - x_t^*\right)\left[NF(x_t^*)^{N-1} - \left(F(x_{t+1}^*)^{N-1} + \cdots + F(x_t^*)^{N-1}\right)\right]$ is strictly positive, and so by (1.6), $[F(x_{t-1}) - F(x_t^*)]\left[\left(F(x_{t-1})^{N-1} + \cdots + F(x_t^*)^{N-1}\right) + c_k'(x_t^*)\right]$ is strictly positive. Condition 1 implies that $c_k(x)$ is continuous on $[\underline{c}, \overline{c}]$, and $c_k'(x)$ exists almost everywhere and is bounded. Therefore, by (1.6), if $x_{t+1}^* - x_t^* > 0$, $x_t^* - x_{t-1} > 0$. Moreover, the difference between x_{t-1} and x_t^* only depends on x_t^*, ϵ ($\epsilon = x_{t+1}^* - x_t^*$), and k. ■

Lemma 3 *Assume Condition 1. In a continuation game starting from round t ($t < T-1$) with the belief that the greatest lower bound of a seller's cost is x_{t-1}, when the number of rounds left in the continuation game goes to infinity, $x_t \to x_{t-1}$ on the continuation equilibrium path but $x_t \neq x_{t-1}$.*

Proof. In the continuation game, the equilibrium path $\{x_\tau\}_{t\leq\tau\leq T}$ and $\{b_\tau\}_{t\leq\tau\leq T}$ are derived from program P1.11. As $T-t \to \infty$, the value of the program converges, so the additional payoff a buyer can get by adding one more round goes to 0. In the following proof, we show that given any $T-t$, when one more round is added to the continuation game, the additional payoff the buyer can get is strictly positive if $x_t > x_{t-1} + \epsilon, \epsilon > 0$. However, the buyer's payoff is bounded by the payoff in Myerson's optimal mechanism, so when $T-t \to \infty$, $x_t \to x_{t-1}$.

Let $\{x_\tau^*, b_\tau^*\}_{t\leq\tau\leq T}$ be the equilibrium path when there are $T-t$ rounds left, which can be derived from program P1.11. If we add a constraint $x_t = x_{t-1}$ to P1.11 and let $\{x_\tau', b_\tau'\}_{t\leq\tau\leq T}$ be the solution to the program, then the buyer's payoff and $\{x_\tau', b_\tau'\}_{t+1\leq\tau\leq T}$ would be the same as those in the continuation game with $T-t-1$ rounds. The value of the program is

$$V_t(x_{t-1}) = \left(v - b_t'\right)\left[F(x_{t-1})^N - F(x_t')^N\right] + V_{t+1}\left(x_t'\right), \text{ where } x_t' = x_{t-1}.$$

Without the constraint, x_t' can be increased by ε, and $V_t(x_{t-1})$ increases approximately by

$$\begin{aligned}
& \left[\left(v-b_t'\right) NF\left(x_t'\right)^{N-1} f\left(x_t'\right)+V_{t+1}'\left(x_t'\right)\right]\varepsilon \\
= & \ [\left(v-b_t'\right) NF\left(x_t'\right)^{N-1} f\left(x_t'\right)-NF\left(x_t'\right)^{N-1}\left(v-x_{t+1}'\right) f\left(x_t'\right) \\
& +(b_{t+1}'-x_{t+1}')(F(x_{t+1}')^{N-1}+F(x_{t+1}')^{N-2}F(x_t')+\cdots+F(x_t')^{N-1})f\left(x_t'\right)]\varepsilon \\
= & \left(x_{t+1}'-x_t'\right)\left[NF\left(x_t'\right)^{N-1}-\left(F(x_{t+1}')^{N-1}+F(x_{t+1}')^{N-2}F(x_t')+\cdots+F(x_t')^{N-1}\right)\right] f\left(x_t'\right)\varepsilon.
\end{aligned}$$

The second equation comes from $\left(b_t'-x_t'\right) NF\left(x_t'\right)^{N-1}=\left(b_{t+1}'-x_t'\right)\left(F\left(x_t'\right)^{N-1}+\cdots+F\left(x_{t+1}'\right)^{N-1}\right)$. Therefore, if $x_{t+1}'>x_t'+\epsilon$, the value is positive and increasing the number of rounds from $T-t-1$ to $T-t$ strictly increases the buyer's payoff. ■

Lemma 4 *Given any T and $t<\infty$, if $x_{T-t}^T<\overline{c}$, then $x_T^T<\overline{c}$ and $x_T^T-x_{T-1}^T>0$.*

Proof. When $t=1$, by (1.1), $x_T^T<\overline{c}$ and $x_T^T-x_{T-1}^T>0$. When $t=2$, if $x_{T-1}^T=\overline{c}$, then $x_T^T=\overline{c}$ and the buyer pays for the good at a price higher than or equal to $\overline{c}$, which cannot happen in equilibrium. Therefore, $x_{T-1}^T<\overline{c}$, and we can apply the result we get in the case when $t=1$.

Applying the same argument to the case where $t=3,4,\cdots$, we can conclude that, for any t, if $x_{T-t}^T<\overline{c}$, then $x_T^T<\overline{c}$ and $x_T^T-x_{T-1}^T>0$. ■

Recall that B is the set of cluster points, and $[\underline{c},\overline{c}]\setminus B$ is the complement of B.

Lemma 5 *Assume Condition 1. If $a\in B$, $[\underline{c},a]\subset B$.*

Proof. By Lemma 3, $\underline{c}\in B$. If not the whole interval $[\underline{c},a]$ belongs to B, there must exist $[b,c]\subset[\underline{c},a]$ such that $(b,c)\subset[\underline{c},\overline{c}]\setminus B$ and $b,c\in B$. Since $(b,c)\subset[\underline{c},\overline{c}]\setminus B$, there exist functions $t(T)$ and $t'(T)$, $t(T)<t'(T)$, such that $\left\{x_{t(T)}^T\right\}_T$ and $\left\{x_{t'(T)}^T\right\}_T$ converge, $b\le\lim_{T\to\infty}x_{t(T)}^T<\lim_{T\to\infty}x_{t'(T)}^T\le c$, and no other sequences $\left\{x_k^T\right\}_T$, $x_k^T\in\left\{x_t^T\right\}_{t(T)<t<t'(T)}$, converge. However, since $a\in B$, i.e. $\lim_{T\to\infty}T-t(T)=\infty$, Lemma 3 implies $x_{t(T)+1}^T$ is arbitrarily close to $x_{t(T)}^T$ when T is large. Therefore, $\lim_{T\to\infty}x_{t(T)+1}^T-\lim_{T\to\infty}x_{t(T)}^T=0$, a contradiction. ■

Lemma 6 *Assume Condition 1. If there exists $a\in(\underline{c},\overline{c})$ such that $(a,\overline{c}]\subset[\underline{c},\overline{c}]\setminus B$, then $[\underline{c},\overline{c}]\setminus B=(\underline{c},\overline{c}]$.*

Proof. By Lemma 5, we only need to show that it cannot be the case that $(a, \overline{c}] \subset [\underline{c}, \overline{c}] \setminus B$ and $[\underline{c}, a] \subset B$. Suppose $[\underline{c}, a) \subset B$. First, we show that when $T \to \infty$, there exists $x \in X^T$, $x > a$, that is arbitrarily close to a. Since $(a, \overline{c}] \subset [\underline{c}, \overline{c}] \setminus B$, there exists $t < \infty$ such that $\overline{c} - \lim_{T\to\infty} x^T_{T-t} > 0$. By Lemma 4, $\lim_{T\to\infty} x^T_T < \overline{c}$ and $\lim_{T\to\infty} x^T_T - \lim_{T\to\infty} x^T_{T-1} > 0$. By Lemma 2, $\lim_{T\to\infty} x^T_{T-s+1} - \lim_{T\to\infty} x^T_{T-s} > 0$, for all $s < \infty$. Since $(a, \overline{c}] \subset [\underline{c}, \overline{c}] \setminus B$, it must be the case that $\lim_{t\to\infty} \lim_{T\to\infty} x^T_{T-t} = a$, which also implies $a \in B$.

Since $[\underline{c}, a] \subset B$ and $\lim_{t\to\infty} \lim_{T\to\infty} x^T_{T-t} = a$, we can rewrite the necessary condition (1.6) for the optimality problem as

$$\begin{aligned} 0 &= \left[F\left(x - dx^-\right) - F\left(x\right)\right]\left[-F\left(x - dx^-\right)^{N-1} - \cdots - F\left(x\right)^{N-1} - C'\left(x\right)\right] \\ &\quad -f\left(x\right)dx^+\left[F\left(x + dx^+\right)^{N-1} + \cdots + F\left(x\right)^{N-1} - NF\left(x\right)^{N-1}\right] \end{aligned} \tag{1.7}$$

where $C(x) = \lim_{k\to\infty} c_k(x)$, $x \in [\underline{c}, a]$, and dx^- and dx^+ are two positive numbers which can be arbitrarily small. For $x \in [\underline{c}, a)$, $dx^- \in O(dx^+)$ but $dx^- \notin o(dx^+)$, and an approximation of equation (1.7) is

$$\begin{aligned} \Rightarrow \quad 0 &= f\left(x\right)dx^-\left[-NF\left(x\right)^{N-1} - \frac{\left(N-1\right)N}{2}F\left(x\right)^{N-2}f\left(x\right)dx^- - C'\left(x\right)\right] \\ &\quad -f\left(x\right)dx^+\left[NF\left(x\right)^{N-1} - \frac{\left(N-1\right)N}{2}F\left(x\right)^{N-2}f\left(x\right)dx^+ - NF\left(x\right)^{N-1}\right] \end{aligned}$$

Since dx^- and dx^+ are arbitrarily small, the equation implies $C'(x) = -NF(x)^{N-1}$ for $x \in [\underline{c}, a)$.

However, $\lim_{x\to a+} C'(x) \neq -NF(x)^{N-1}$. If $\lim_{x\to a+} C'(x) = -NF(x)^{N-1}$, in order to satisfy equation (1.7), there exists $\epsilon > 0$ such that for $x \in (a, a+\epsilon)$, $dx^- \in O(dx^+)$ but $dx^- \notin o(dx^+)$.[9] So $(a, a+\epsilon) \subset B$, a contradiction. Since $C'(x)$ is not continuous at a, and dx^- and dx^+ can be arbitrarily small, the necessary condition (1.7) does not hold around a. Therefore, a path that $[\underline{c}, a] \subset B$ and $(a, \overline{c}] \subset [\underline{c}, \overline{c}] \setminus B$ cannot occur in equilibrium. Note that there must be at least one cluster point in $[\underline{c}, \overline{c}]$. Since only $\underline{c}$ can be in B, $[\underline{c}, \overline{c}] \setminus B = (\underline{c}, \overline{c}]$. ■

[9] Suppose $dx^- \in o\left(dx^+\right)$. Since dx^- is derived from equation (1.7), $dx^- \in o\left(dx^+\right)$ implies $dx^- \in O\left(\left(dx^+\right)^2\right)$ and $C'(x) = -NF(x)^{N-1} + \frac{(N-1)N}{2}F(x)^{N-2}f(x) \neq -NF(x)^{N-1}$.

Proposition 9 *A path that fully discriminates sellers is a stationary solution to program P1.7 when $T = \infty$.*

Proof. If the buyer fully discriminates sellers, we can rewrite the necessary condition (1.6) for a stationary solution as

$$
\begin{aligned}
& 0 = [F(x-dx) - F(x)]\left[-F(x-dx)^{N-1} - \cdots - F(x)^{N-1} - C'(x)\right] \\
& -f(x)\,dx\left[F(x+dx)^{N-1} + \cdots + F(x)^{N-1} - NF(x)^{N-1}\right] \\
\Rightarrow\quad & 0 = f(x)\,dx\left[-NF(x)^{N-1} - \frac{(N-1)N}{2}F(x)^{N-2}f(x)\,dx - C'(x)\right] \\
& -f(x)\,dx\left[NF(x)^{N-1} - \frac{(N-1)N}{2}F(x)^{N-2}f(x)\,dx - NF(x)^{N-1}\right],
\end{aligned}
$$

where dx is a positive number which can be arbitrarily small. Note that $\frac{C(x)}{N}$ can be considered as the information rent given to a seller with cost x. In our setting, in an incentive compatible mechanism that fully discriminates sellers with different costs, the information rent $R(x)$ has the property that $R'(x) = -F(x)^{N-1}$, so $\frac{C'(x)}{N} = -F(x)^{N-1}$. Therefore, the necessary condition holds. Given x_{t-1}, supposing $x_{t+1} - x_t$ is arbitrarily small, one can check that the objective function of (P1.11) is concave in x_t. Therefore, a path that fully discriminates sellers is a stationary solution. ■

Proof of Proposition 7. Since $\overline{x}_{T-k}^{T}(x_{T-k-1}, \delta)$ is continuous in x_{T-k-1} and $\lim_{T\to\infty} \overline{x}_{T-k}^{T}(x_{T-k-1}, \delta) = \overline{x}_{T-k}^{T}(x_{T-k-1}, 1)$, given any x, $\lim_{T\to\infty} \overline{x}_{T-k}^{T}\left(\overline{x}_{T-k-1}^{T}\left(\cdots \overline{x}_{T-k-t}^{T}(x,\delta)\cdots\right),\delta\right) = \overline{x}_{T-k}^{T}\left(\overline{x}_{T-k-1}^{T}\left(\cdots \overline{x}_{T-k-t}^{T}(x,1)\cdots\right),1\right)$ for $t \in \{0, 1, \cdots\}$. Since the sequence $\{\lim_{T\to\infty} \overline{x}_{T-k}^{T}\left(\overline{x}_{T-k-1}^{T}\left(\cdots x_{T-k-t}^{T}(x,\delta)\cdots\right),\delta\right),$ $\lim_{T\to\infty} \overline{x}_{T-k-1}^{T}\left(\cdots x_{T-k-t}^{T}(x,\delta)\cdots\right), \cdots, \lim_{T\to\infty} x_{T-k-t}^{T}(x,\delta)\}$ decreases and is bounded below by $\underline{c}$, there exists a limit of the sequence $\{\lim_{T\to\infty} \overline{x}_{T-k}^{T}\left(\overline{x}_{T-k-1}^{T}\left(\cdots x_{T-k-t}^{T}(x,\delta)\cdots\right),\delta\right), \lim_{T\to\infty} \overline{x}_{T-k-1}^{T}\left(\cdots x_{T-k-t}^{T}(x,\delta)\cdots\right),$ $\cdots, \lim_{T\to\infty} x_{T-k-t}^{T}(x,\delta)\}$ when $t \to \infty$. Let χ be the supremum of the cluster point set B given δ. Then $\lim_{T\to\infty} x_{T-k}^{T} = \lim_{t\to\infty} \overline{x}_{T-k}^{T}\left(\overline{x}_{T-k-1}^{T}\left(\cdots x_{T-k-t}^{T}(\chi,1)\cdots\right),1\right)$. ■

Proof of Theorem 4. The model in Section 1.2 is a special case when $\delta = 1$. Given $\delta < 1$, since $\lim_{T\to\infty} \sqrt[T]{\delta} = 1$, Lemmas 3 and 4 hold, so Lemma 5 holds for $\delta < 1$. However, Lemma 6 might not hold with $\delta < 1$. By Lemma 5, the first statement of Proposition 4 is proved. For the second statement, if $\lim_{T\to\infty} x_T^T < \overline{c}$, it must be that $a < \overline{c}$. On the other hand, if $a < \overline{c}$, there exists $t < \infty$ such that $\overline{c} - \lim_{T\to\infty} x_{T-t}^T > 0$. By Lemma 4, $\lim_{T\to\infty} x_T^T < \overline{c}$. ■

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Chapter 2

One-to-Many Negotiation Between a Seller and Asymmetric Buyers

2.1 Introduction

The Dutch auction is considered strategically equivalent to the first-price sealed-bid auction. In a Dutch auction, the auctioneer, who is usually the seller, plays a passive role and non-strategically lowers the price until the object is sold. However, if the seller is allowed to submit different prices to different buyers at the same time, will the seller be able to attain a higher profit than when such discretion is not permitted?

In this paper, we consider a two-buyer auction where the buyers are ex ante asymmetric: the buyers' private values of the object are drawn independently from different distributions. When buyers are asymmetric, the seller's payoffs in a first-price and a second-price auction are not optimal. An optimal auction stated by Myerson (1981) requires the seller to treat buyers asymmetrically,[1] that is, the seller might not sell the object to the buyer with the highest value. Therefore, one may imagine that if the seller does not have to submit the same price to the buyers as in a Dutch auction, the seller might be able to do better. We show that if the seller is allowed to submit different prices to different buyers, when commitment to some

[1] In a second-price auction, buyers bid their true value, so it is always the case that the buyer with the highest value gets the good, and hence it is obviously not optimal. In a first-price auction, a buyer whose value distribution is first-order stochastically dominated will bid more aggressively, but still, the result is not optimal for the seller.

price paths is possible, Myerson's optimal outcome for the seller is achievable; however, when commitment is impossible, the optimal outcome is no longer attainable, but instead, there exists an equilibrium such that the seller's equilibrium payoff is the same as that in a second-price auction. As the auction literature (Vickrey (1961) and Maskin and Riley (2000)) shows, with asymmetric bidders, the seller's payoff in a first-price auction (or a Dutch auction) might be greater than that in a second-price auction. Therefore, when allowed to submit different prices, the seller's payoff might be lower than the payoff when he must submit the same price. The result thus suggests that the benefit brought by the discretion to determine the price paths might be outweighed by the loss caused by not being able to commit.

A example of the trading process we consider is the Name-Your-Own-Price (NYOP) auctions conducted at Priceline.com. An NYOP auction is a procurement auction (also called reverse auction) where the roles of a buyer and a seller are reversed from an ordinary auction. In an NYOP auction, when a customer tries to get a travel-related item such as an air ticket or a hotel room, he can submit different prices to different groups of sellers. For instance, when bidding for a hotel room, a buyer can specify the area to stay and the rating of hotels, and submit different bids to hotels with different characteristics. The customer adjusts his bids over time until one of the sellers accepts. The analysis in this paper helps us get a better understanding of the buyer's payoff and bidding behavior under the NYOP mechanism.

This paper is also related to the procurement literature that compares the performances of two different trading institutions—auctions versus negotiations. Manelli and Vincent (1995) model negotiations as sequential offers and find that under certain conditions, negotiations outperform auctions from the buyer's perspective. The sequential-offer process they consider for negotiation is such that the order of sellers that the buyer gets to negotiate and the prices offered are determined in advance, and the buyer only gets at most one chance to negotiate with each seller. In this paper, we generalize the negotiation process, considering the case when the number of chances to negotiate is not limited. We show that if the seller can commit to certain price paths in advance, the optimal outcome stated by Myerson (1981) can always be achieved, so negotiations always perform better. We further consider the case when the price paths are determined as the process goes along, which is closer to the real-world negotiation process. In this case, whether negotiations or auctions do better depends on the environment and is not conclusive.

This paper builds a bridge between the auction and the bargaining literature. The model differs from a Dutch auction environment by allowing the auctioneer to set different prices for different buyers, and differs from a bargaining setting by considering a one-to-many negotiation process in which a individual party chooses his partner from a group of candidates.[2]

The paper is organized as follows. Section 2 presents an example illustrating the main points of this paper. Section 3 describes the model. Section 4 characterizes the optimal outcome that the seller can achieve when commitment is possible. Section 5 characterizes the equilibrium without commitment, and Section 6 concludes the paper.

2.2 An Example

In this section, we use a simple example to illustrate some of the main points of this paper. Consider the case when a seller tries to sell a single object to two buyers. The buyers have private values for the object. Buyer 1's value is drawn from the set $\{0, 3\}$ with respective probabilities $\{\frac{1}{2}, \frac{1}{2}\}$; buyer 2's value is drawn from the set $\{1, 3.5, 6\}$ with respective probabilities $\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\}$. The seller maximizes the expected amount of money he collects from the buyers. A buyer's payoff if he gets the object is the difference between his value and the payment. The seller makes offers to the two buyers until one of them accepts. If the two buyers accept at the same time, each of them gets the object with probability $\frac{1}{2}$. For simplicity, we assume that the prices offered must be integers. We consider three different sales mechanisms: (i) the seller offers the same price sequence $\{5, 4, 3, 2, 1\}$ to the two buyers, (ii) the seller determines two price paths in advance and makes offers according to the paths, and (iii) the seller makes offers without making commitment in advance.

2.2.1 Offer price sequence $\{5, 4, 3, 2, 1\}$

In this case, the mechanism works like a Dutch auction. The seller lowers the price gradually until a buyer accepts. Given the price path, the following table summarizes the prices accepted

[2]The existing one-to-many negotiation literature considers the case where the individual party has to work with every member in the other party instead of choosing one partner from all of them.

by different types of buyers: buyer 2 with value 6 accepts \$3,[3] buyer 1 with value 3 accepts \$2, and buyer 2 with value 1 and 3.5 accepts \$1. The seller's expected payoff is $\frac{9}{4}$.

	Price pairs offered to buyers				
	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
Buyer 1	5	4	3	2	1
Buyer 2	5	4	3	2	1
accepted by			B2 with 6	B1 with 3	B2 with 1, 3.5

Note that buyer 1 is stronger than buyer 2 in the sense that the distribution function of buyer 1's value, F_1, stochastically dominates the distribution function of buyer 2's value, F_2. That is, $F_1(x) \leq F_2(x)$. We can observe from the table that buyer 1 with value 3 accepts at a higher price than buyer 2 with value 3.5. This is analogous to the well-known fact that in a first-price auction (which is strategically equivalent to a Dutch auction), the weaker bidder bids more aggressively than the strong bidder.

2.2.2 Commit to certain price paths determined in advance

Now we allow the seller to determine the price paths in advance, and the prices offered to the two buyers can be different. Suppose the seller offers the following price sequences:

	Price pairs offered to buyers			
	$t=1$	$t=2$	$t=3$	$t=4$
Buyer 1	2	1	1	0
Buyer 2	4	4	3	3
accepted by	B2 with 6	B1 with 3	B2 with 3.5	B1 with 0

The last line shows that at $t=1$, buyer 2 with value 6 accepts \$4, at $t=2$, buyer 1 with value

[3]To see why buyer 2 with value 6 accepts at \$3, note that if he waits and accepts \$2, with probability $\frac{1}{2}$, buyer 1 also accepts, and each of them gets the object with probability $\frac{1}{2}$, so the probability that he gets the object is $\frac{3}{4}$. Since the buyer's expected payoff if accepting at \$2 is $\frac{3}{4}(6-2)=3$, the same as his payoff if he accepts at \$3, he is willing to accept at \$3. Similarly, buyers with other values will find that accepting at the price specified in the table is the optimal strategy.

3 accepts \$1, at $t = 3$, buyer 2 with value 3.5 accepts \$3, and at $t = 4$, buyer 1 with value 0 accepts \$0. The seller's expected payoff is $\frac{21}{8}$, greater than the payoff in the first mechanism.

Notice that given the paths, buyer 2 with value 1 never gets the good. Therefore, by committing to the price paths, the seller loses the chance to sell the object to a buyer with positive value. Furthermore, in each period, the price offered to buyer 2 is higher than the price offered to buyer 1. This design raises competition between the buyers, so the seller is able to extract more surplus from buyer 2.

2.2.3 Make offers without commitment

If the seller makes offers without committing to certain paths in advance, the price paths characterized in the first two mechanisms cannot be implemented. First consider the paths implemented in the second mechanism. In the last period $t = 4$, the seller believes that buyer 1 has value 0 and buyer 2 has value 1. Given the belief, the seller can offer \$1 to buyer 2 and a price higher than 0 to buyer 1, and gets a higher payoff than if he follows the paths in the second mechanism at $t = 4$.

Next consider the paths implemented in the first mechanism. After $t = 3$, if no buyer accepts, the seller believes that buyer 1's value is either 0 or 3, and buyer 2's value is either 1 or 3. Given the belief, the optimal paths for the seller in the continuation game are the following:

	Price pairs offered to buyers		
	$t = 4$	$t = 5$	$t = 6$
Buyer 1	3	2	2
Buyer 2	2	2	1
accepted by	B2 with 3.5	B1 with 3	B2 with 1

.

With the paths, buyer 2 with value 3.5 accepts \$2 at $t = 4$ and buyer 1 with value 3 accepts \$2 at $t = 5$, and buyer 1 with value 1 accepts \$1 at $t = 6$, so the seller gets a higher payoff than if he follows the paths in the first mechanism. Given the optimal paths in the continuation game, in equilibrium, buyer 2 with value 6 would not accept a price higher than 2. If in equilibrium, the seller believes that buyer 2 with value 6 accepts \$3, then in the next period, he follows the optimal paths and offers the price pair $(3, 2)$. Given the pair $(3, 2)$, buyer 1 will not accept, so buyer 2 with value 6 gets a higher payoff if he rejects price \$3 and waits for one more period to accept \$2. The following table characterizes the equilibrium price paths and the types of buyers accepting in each period. The seller's expected payoff is $\frac{15}{8}$.

	Price pairs offered to buyers		
	$t = 1$	$t = 2$	$t = 3$
Buyer 1	3	2	2
Buyer 2	2	2	1
accepted by	B2 with 3.5, 6	B1 with 3	B2 with 1

.

2.2.4 Comparison

From the above discussion, we can compare the seller's expected payoffs and the equilibrium allocations in the three mechanisms.

1. The seller can do better in the second mechanism than in the first mechanism. In both mechanisms, the price paths are determined in advance, but in the second mechanism, the seller has the freedom to offer different prices to different buyers, and hence, he can get a higher payoff. However, the fact that the seller's payoff in the third mechanism is less than the payoff in the first mechanism shows that if the seller cannot commit to certain price paths in advance, even with the freedom to offer different prices to different buyers, the seller might do worse.

2. In the first and the second mechanisms, the final allocation might be inefficient: if buyer 1 and buyer 2 have values 3 and 3.5 respectively, buyer 1 gets the object although his value is lower than buyer 2's. On the other hand, in the third mechanism where the seller cannot commit, the seller tends to induce the buyer with the higher value to buy first in every period. Therefore, the allocation is efficient — the buyer with the higher value always gets the object.

In the remaining sections, we analyze a model in which buyers' values are drawn from continuous distribution functions and the seller adjusts prices continuously. We show that the conclusions made above for the simple example hold in the general model.

2.3 The Model

A seller has one indivisible object for sale and faces two risk-neutral buyers. The value of the object to the seller is 0, which is publicly known. Buyers have private values for the object, and the values are independently distributed. Buyer i's value X^i is distributed over the interval $[\underline{w}_i, \overline{w}_i], \underline{w}_i \geq 0$, according to distribution function F_i with associated density function f_i. For $i = 1, 2$, $\psi_i(x) \equiv x - \frac{1-F_i(x)}{f_i(x)}$ strictly increases in x and $f_i(x) > 0, \forall x \in [\underline{w}_i, \overline{w}_i]$. The seller maximizes the expected amount of money he collects from the buyers. Buyer i's utility is $z_i x_i - M$, where M is his payment to the seller, and $z_i = 1$ if buyer i gets the object, $z_i = 0$ otherwise. The setting is common knowledge to everyone in the market.

The seller is allowed to conduct a negotiation process. From the beginning, the seller makes simultaneous offers to the two buyers and adjusts the prices continuously until some buyer indicates his interest. Without loss of generality, we assume that the prices offered have to be weakly decreasing. The two simultaneous offers to the two buyers can be different and are observable by both buyers. Once a buyer accepts, the object is sold to the buyer at the price offered. There is no discounting. The seller is unable to commit some price paths in advance.

2.4 Optimal Outcome with Commitment

In this section, we consider the case when the seller can commit to some price paths before the negotiation begins. Recall that when deriving the optimal mechanism, we compare the virtual valuation of buyers. The virtual valuation of a buyer with value x_i is defined as

$$\psi_i(x_i) \equiv x_i - \frac{1 - F_i(x_i)}{f_i(x_i)},$$

which is assumed strictly increasing in our model. The optimal allocation rule characterized by Myerson (1981) is that the good goes to the buyers whose virtual valuations are the greatest and positive. Therefore, the seller should design paths such that, in the following negotiation process, a buyer whose virtual valuation is higher will accept earlier.

Without loss of generality, assume that $\psi_1(\underline{w}_1) \geq \psi_2(\underline{w}_2)$, where $\underline{w}_i$ is the lowest possible value of buyer i. If $\psi_1(\underline{w}_1) < 0$, let $\underline{x}_1 = \psi_1^{-1}(0)$; otherwise, $\underline{x}_1 = \underline{w}_1$. Let $\widehat{x}_2(x_1) =$

$\psi_2^{-1}(\psi_1(x_1))$ so that $\psi_1(x_1) = \psi_2(\widehat{x}_2(x_1))$, and let $\widehat{x}_1(x) \equiv \widehat{x}_2^{-1}(x)$. Furthermore, let

$$b_1(x_1) = x_1 - \frac{\int_{\underline{x}_1}^{x_1} F_2(\widehat{x}_2(x))\,dx}{F_2(\widehat{x}_2(x_1))},$$

and

$$b_2(x_2) = x_2 - \frac{\int_{\widehat{x}_2(\underline{x}_1)}^{x_2} F_1(\widehat{x}_1(x))\,dx}{F_1(\widehat{x}_1(x_2))}.$$

Note that both $b_1(x)$ and $b_2(x)$ are strictly increasing.

Theorem 5 *If the seller commits to a path on which the prices to buyer 1 and buyer 2 have the relation $p_2(p_1) = b_2\left(\widehat{x}_2\left(b_1^{-1}(p_1)\right)\right)$ and stop at $\underline{x}_1$ and $\widehat{x}_2(\underline{x}_1)$ respectively, then there exists an equilibrium such that buyer 1 with value x_1 accepts at price $b_1(x_1)$ and buyer 2 with value x_2 accepts at price $b_2(x_2)$. The seller's payoff is the same as that in an optimal mechanism.*

Proof. Given that buyer 2 with value $x_2 > \widehat{x}_2(\underline{x}_1)$ accepts at $b_2(x_2)$, we show that accepting at $b_1(x_1)$ is the best strategy for buyer 1 with value x_1. Suppose buyer 1 has value x_1. If the current price for buyer 1 is $p_1 > b_1(x_1)$, accepting now gives buyer 1 payoff

$$x_1 - p_1 = x_1 - b_1(x_1'), \text{ where } x_1' = b_1^{-1}(p_1) > x_1.$$

Accepting later at $b_1(x_1)$ gives expected payoff

$$\frac{F_2(\widehat{x}_2(x_1))}{F_2(\widehat{x}_2(x_1'))}(x_1 - b_1(x_1)).$$

$$\begin{aligned} F_2(\widehat{x}_2(x_1'))(x_1 - b_1(x_1')) &= (x_1 - x_1')F_2(\widehat{x}_2(x_1')) + \int_{\underline{x}_1}^{x_1'} F_2(\widehat{x}_2(x))\,dx \\ &< \int_{\underline{x}_1}^{x_1} F_2(\widehat{x}_2(x))\,dx = F_2(\widehat{x}_2(x_1))(x_1 - b_1(x_1)). \end{aligned}$$

Therefore waiting and accepting later is better for buyer 1. If the current price is $b_1(x_1)$, accepting now gives payoff $x_1 - b_1(x_1)$. Accepting later at $b_1(x_1''), x_1'' < x_1$, gives expected payoff $\frac{F_2(\widehat{x}_2(x_1''))}{F_2(\widehat{x}_2(x_1))}(x_1 - b_1(x_1))$.

$$\begin{aligned} F_2(\widehat{x}_2(x_1''))(x_1 - b_1(x_1'')) &= (x_1 - x_1'')F_2(\widehat{x}_2(x_1'')) + \int_{\underline{x}_1}^{x_1''} F_2(\widehat{x}_2(x))\,dx \\ &< \int_{\underline{x}_1}^{x_1} F_2(\widehat{x}_2(x))\,dx = F_2(\widehat{x}_2(x_1))(x_1 - b_1(x_1)). \end{aligned}$$

Therefore, accepting now at $b_1(x_1)$ is better. The same argument applies to buyer 2. Therefore, that buyer 1 with value x_1 accepts at price $b_1(x_1)$ and buyer 2 with value x_2 accepts at price $b_2(x_2)$ is an equilibrium.

To see that the seller gets the optimal payoff, notice that buyer i gets the object if and only if his virtual valuation is positive and at least as high as buyer j, i.e. $\psi_i(x_i) \geq \psi_j(x_j)$, so the allocation is the same as the optimal allocation rule. In addition, buyer 1 with value $\underline{w}_1$ and buyer 2 with value $\widehat{x}_2(\underline{w}_1)$ get zero utility. Thus, by the revenue equivalence principle, the seller gets the optimal payoff. ■

Manelli and Vincent (1995) compare the performance of negotiations and auctions in procurement and conclude that under certain conditions, negotiations are outperformed by auctions from the buyer's perspective (seller in our model). The sequential-offer process they consider for negotiation requires that the order of sellers (buyers in our model) the buyer gets to negotiate and the prices offered are both determined in advance, and the buyer only gets at most one chance to negotiate with each seller. However, we show that if the number of chances to negotiate is not limited and the price paths are determined in advance, Myerson's optimal outcome can always be achieved, so negotiations always outperform auctions. Therefore, Manelli and Vincent's result is contingent on the limited number of chances to negotiate.

2.5 Equilibrium without Commitment

In this section, we consider the case when the seller is not able to commit to any price paths. First we describe the equilibrium concept. Next we restrict our attention to a particular class of equilibria and show that without commitment, Myerson's optimal outcome, which can be attained when commitment is possible, is no longer achievable.

2.5.1 Equilibrium Concept

Let $p_i(\tau)$ be the price submitted by the seller to buyer i at time τ. At time t, $t \geq 0$, denote by $h^t = \{(p_1(\tau), p_2(\tau)); 0 \leq \tau \leq t\}$ the price history submitted by the seller to the two buyers until time t. Without loss of generality, let the prices offered to the two buyers start at $\overline{w}_1$ and $\overline{w}_2$, that is, $p_1(0) = \overline{w}_1$ and $p_2(0) = \overline{w}_2$. Then at time t, given history h^t and the fact that no buyer accepts, the seller determines the price for the next instant. Therefore, the seller's strategy can be characterized by $\left(\frac{dp_1(t;h^t)}{dt}, \frac{dp_2(t;h^t)}{dt}\right)$, which determines the increments of p_1 and p_2. Because the prices are weakly decreasing, $\frac{dp_1(t;h^t)}{dt} \leq 0$ and $\frac{dp_2(t;h^t)}{dt} \leq 0$. Since there is no discounting, speeding up or slowing down the price paths should not affect the players' payoffs. What really matters is how the two prices are paired along the path. Therefore, we restrict the seller's strategy space to $\mathcal{S} = \left\{\left(\frac{dp_1(t;h^t)}{dt}, \frac{dp_2(t;h^t)}{dt}\right) \mid \frac{dp_1}{dt} + \frac{dp_2}{dt} = -1, \frac{dp_1}{dt} \leq 0, \text{ and } \frac{dp_2}{dt} \leq 0\right\}$, and denote the seller's strategy by $\mu(t; h^t) = \left(\frac{dp_1(t;h^t)}{dt}, \frac{dp_2(t;h^t)}{dt}\right)$.

At time t, given the current price (p_1, p_2) offered by the seller and the price history until t, h^t, buyer i determines whether to accept p_i. We use $P_1(x_1; h^t)$ and $P_2(x_2; h^t)$ to characterize buyer 1's and buyer 2's strategies: buyer i with value x_i accepts the current price $p_i(t)$ if and only if $p_i(t) \leq P_i(x_i; h^t)$. Given history h^t and an equilibrium in which $P_i(x_i; h^t)$ is monotone in x_i, the seller's and buyer j's belief about buyer i's value is summarized by function $y_i(h^t)$, which specifies the lowest upper bound of buyer i's value believed by the seller and buyer j given history h^t and the fact that no buyer accepts.

Denote by $v_0(\mu, P_1, P_2 \mid h^t, y_1(h^t), y_2(h^t))$ the seller's expected utility given h^t and belief y, and $v_i(\mu, P_1, P_2 \mid h^t, \theta_i, y_j(h^t))$ buyer i's expected utility, given h^t, the realization θ_i of buyer i's value, and belief $y_j(h^t), j \neq i$.

Definition 3 *A pure strategy perfect Bayesian equilibrium is a* $(\mu, y_1, y_2, P_1, P_2)$ *that satisfies*

(a) $y_i(h^t) = \inf\{x \mid P_i(x; h^\tau) \geq p_i(\tau), \textit{ for any } \tau \in [0, t]\}, \forall t, h^t.$

(b) $v_0(\mu, P_1, P_2 \mid h^t, y_1(h^t), y_2(h^t)) \geq v_0(\mu', P_1, P_2 \mid h^t, y_1(h^t), y_2(h^t))$ *and*
$v_i(\mu, P_i, P_j \mid h^t, \theta_i, y_j(h^t)) \geq v_i(\mu, P_i', P_j \mid h^t, \theta_i, y_j(h^t)), \textit{for } i = 1, 2 \textit{ and } \forall t, \theta_i, \mu', P_i', h^t.$

Condition (a) implies that players' belief about the lowest upper bound of buyer i's value at time t is the same as the infimum of buyer i's values with which buyer i would have accepted a price occurring on the history price path. Condition (b) implies that players cannot do better by deviating from the equilibrium strategy.

2.5.2 The class of equilibria considered

We focus on pure strategy perfect Bayesian equilibria with the property that the seller's strategy depends on history only through his belief about the buyers' values, $y_1\left(h^t\right)$ and $y_2\left(h^t\right)$, and the current prices, $p_1(t)$ and $p_2(t)$; and buyer i's strategy depends on history only through his belief about buyer j's value, $y_j\left(h^t\right)$, if his value x_i is less than or equal to $\lim_{\tau\uparrow t} y_i\left(h^\tau\right)$, the lowest upper bound of his value believed by other players given that he did not accept any price before t. Buyer i's strategy if $x_i > \lim_{\tau\uparrow t} y_i\left(h^\tau\right)$, which implies that he has deviated, is characterized in Proposition 12. Therefore, when there is no deviation, $P_i\left(x_i; h^t\right)$ can be expressed as $P_i\left(x_i, x_j\right)$ where $x_j = y_j\left(h^t\right)$, and $\mu\left(t; h^t\right) = \left(\frac{dp_1\left(t;h^t\right)}{dt}, \frac{dp_2\left(t;h^t\right)}{dt}\right)$ can be expressed as $\mu\left(t; p_1, p_2, x_1, x_2\right)$ where $p_i = p_i(t)$ and $x_i = y_i\left(h^t\right)$. We restrict our attention to the equilibria in which $\frac{\partial P_i(x_i,x_j)}{\partial x_i} \geq 0$, $\frac{\partial P_i(x_i,x_j)}{\partial x_j} \geq 0$, and $\frac{\frac{\partial P_1}{\partial x_1}}{\frac{\partial P_1}{\partial x_2}} \neq \frac{\frac{\partial P_2}{\partial x_1}}{\frac{\partial P_2}{\partial x_2}}$. $\frac{\partial P_i(x_i,x_j)}{\partial x_i} \geq 0$ is natural because given the other conditions the same, a buyer with higher value should be willing to accept a higher price.

Condition 4 *We consider equilibria with the property that* $\frac{\partial P_i(x_i,x_j)}{\partial x_i} \geq 0$, $\frac{\partial P_i(x_i,x_j)}{\partial x_j} \geq 0$, *and* $\frac{\frac{\partial P_1}{\partial x_1}}{\frac{\partial P_1}{\partial x_2}} \neq \frac{\frac{\partial P_2}{\partial x_1}}{\frac{\partial P_2}{\partial x_2}}$ *for all* $x_i \in [\underline{w}_i, \overline{w}_i]$, $x_j \in [\underline{w}_j, \overline{w}_j]$.

2.5.3 Seller's strategy

In this section, we characterize the seller's equilibrium strategies. First consider a continuation game at time t after the buyers have rejected prices $p_1(t)$ and $p_2(t)$. Let the beliefs about the lowest upper bounds of the buyers' values be x_1 and x_2. The first condition in Definition 3 implies that $p_1(t) \geq P_1(x_1, x_2)$ and $p_2(t) \geq P_2(x_1, x_2)$.

Equilibrium strategy when $p_i(t) > P_i(x_1, x_2)$ for some i

In this section, we first characterize the set of price paths that result in the same expected payoff of the seller in Lemma 7 and Lemma 8. Then by the two lemmas, we conclude the seller's strategy when $p_i(t) > P_i(x_1, x_2)$ for some i in Proposition 10.

Lemma 7 *Given belief* (x_1, x_2), *for any price path* $\{(p_1(\tau), p_2(\tau)) \mid t \leq \tau \leq t'\}$ *such that* $p_1(\tau) \geq P_1(x_1, x_2)$ *and* $p_2(\tau) \geq P_2(x_1, x_2)$ *for all* $\tau \in [t, t']$, *the seller's expected payoff received along the path is* 0.

Proof. Given the belief that buyer 1's and buyer 2's values are below x_1 and x_2 respectively, any price that is higher than or equal to $P_i(x_1, x_2)$ will not be accepted by buyer i with a value below x_i. Therefore, the seller's expected payoff along the path is 0. ∎

Lemma 8 *Given belief* (x_1, x_2), *consider two price paths*

1. $\{(p_1(\tau), p_2(\tau)) \mid t \leq \tau \leq t'\}$ *such that for all* $\tau \in [t, t']$, $p_i(\tau) = P_i(x, x_j)$ *and* $p_j(\tau) \geq P_j(x, x_j)$ *for some* $x \leq x_i$;

2. $\{(\bar{p}_1(\tau), \bar{p}_2(\tau)) \mid t \leq \tau \leq t''\}$ *such that for all* $\tau \in [t, t'']$, $\bar{p}_i(\tau) = P_i(x, x_j)$ *and* $\bar{p}_j(\tau) = P_j(x, x_j)$ *for some* $x \leq x_i$,

such that $p_i(t) = \bar{p}_i(t)$ *and* $p_i(t') = \bar{p}_i(t'')$. *The seller's expected payoffs received along the two price paths are the same.*

Proof. Given the belief that the lowest upper bounds of the buyers' values are x_1 and x_2 at t, along the first and the second paths, $p_i(\tau)$ is accepted by buyer i with value x such that $p_i(\tau) = P_i(x, x_j)$, $\bar{p}_i(\tau)$ is accepted by buyer i with value y such that $\bar{p}_i(\tau) = P_i(y, x_j)$, and $p_j(\tau)$ and $\bar{p}_j(\tau)$ are not accepted by buyer j with a value lower than x_j. Therefore, the seller receives no expected payoff from buyer j along the two paths, and the belief about the lowest upper bound of buyer j's value stays at x_j. Given that the belief about the lowest upper bound of buyer j's value stays at x_j on both paths, the prices accepted by buyer i with some value x are the same on both paths. Since both paths start at $p_i(t)$ and end at $p_i(t')$, the expected payoffs received from buyer i are the same along the two paths. ∎

Proposition 10 *At* t, *given belief* $(x_1, x_2) = (y_1(h^t), y_2(h^t))$ *and the current prices* $(p_1(t), p_2(t))$, *if* $p_i(t) > P_i(x_1, x_2)$ *and* $p_j(t) = P_j(x_1, x_2)$, *the seller's strategy is* $\frac{dp_i}{dt} = 1$ *and* $\frac{dp_j}{dt} = 0$; *if* $p_i(t) > P_i(x_1, x_2)$ *for both* $i = 1, 2$, *then any* $\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right) \in \mathcal{S}$ *can be the seller's strategy.*

Proof. Lemmas 7 and 8 imply that given belief (x_1, x_2), for any path $\mathcal{P}$ starting with prices (p_1, p_2) such that $p_1 \geq P_1(x_1, x_2)$ and $p_2 \geq P_2(x_1, x_2)$, there exists a path starting with prices $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$ that yields the same seller's payoff as $\mathcal{P}$, that is, a path starting with prices $p_1 \geq P_1(x_1, x_2)$ and $p_2 \geq P_2(x_1, x_2)$ is weakly dominated by some path starting with prices $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$. Therefore, at t, if $p_i(t) > P_i(x_1, x_2)$ and $p_j(t) = P_j(x_1, x_2)$, it does not hurt the seller to first lower p_i to $P_i(x_1, x_2)$; and if $p_i(t) > P_i(x_1, x_2)$ for both $i = 1, 2$, it does not hurt the seller to first lower p_i and p_j to $P_i(x_1, x_2)$ and $P_j(x_1, x_2)$ respectively. (Note that all the paths that lower p_i and p_j to $P_i(x_1, x_2)$ and $P_j(x_1, x_2)$ yield the same seller's payoff, 0.) ■

Equilibrium strategy when $p_i(t) = P_i(x_1, x_2)$ for both i

Proposition 10 characterizes the seller's strategy when $p_i(t) > P_i(x_1, x_2)$ for some i. In this section, we consider the case when $p_1(t) = P_1(x_1, x_2)$ and $p_2(t) = P_2(x_1, x_2)$. We show that without loss of generality, the seller's strategy of choosing $\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right)$ is equivalent to choosing $\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}\right)$, where x_i is the infimum of buyer i's values with which buyer i would have accepted a price occurring on the history price path.

If the seller chooses strategy $\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right) \in \mathcal{S}$, we can derive how the beliefs x_1 and x_2 change accordingly by solving

$$\begin{cases} \frac{dp_1}{dt} = \frac{\partial P_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial P_1}{\partial x_2}\frac{dx_2}{dt} \\ \frac{dp_2}{dt} = \frac{\partial P_2}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial P_2}{\partial x_2}\frac{dx_2}{dt} \end{cases}, \tag{2.1}$$

where $\frac{dp_1}{dt} + \frac{dp_2}{dt} = -1$. Note that since x_i is the belief about the lowest upper bound of a buyer's value, x_i can never go up after the belief is updated, so $\frac{dx_1}{dt} \leq 0$ and $\frac{dx_2}{dt} \leq 0$. Because $\frac{\partial P_i(x_i, x_j)}{\partial x_i} \geq 0$, $\frac{\partial P_i(x_i, x_j)}{\partial x_j} \geq 0$, $\frac{dp_1}{dt} \leq 0$, and $\frac{dp_2}{dt} \leq 0$, given (x_1, x_2), there exist $a_{x_1x_2}, b_{x_1x_2} \in [-1, 0]$ such that if $\frac{dp_1}{dt} \in [a_{x_1x_2}, b_{x_1x_2}]$ and $\frac{dp_2}{dt} = 1 - \frac{dp_1}{dt}$, then $\frac{dx_1}{dt} \leq 0$ and $\frac{dx_2}{dt} \leq 0$.[4] Hence, when $\frac{dp_1}{dt} \in [a_{x_1x_2}, b_{x_1x_2}]$, choosing $\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right)$ is equivalent to choosing $\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}\right)$ derived from (2.1). Moreover, Lemma 8 implies that without loss of generality, for each (x_1, x_2), we can focus on a smaller strategy space, $\hat{\mathcal{S}}_{x_1x_2} = \left\{\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right) \in \mathcal{S} \mid \frac{dp_1}{dt} \in [a_{x_1x_2}, b_{x_1x_2}], \frac{dp_2}{dt} = -1 - dp_1\right\}$, because for any path derived from strategies in $\mathcal{S}$, there exists a path derived from strategies in $\left\{\hat{\mathcal{S}}_{x_1x_2}\right\}_{x_1,x_2}$ that yields the same seller's expected payoff. Therefore, when $p_1(t) = P_1(x_1, x_2)$ and $p_2(t) = P_2(x_1, x_2)$, we can focus on the smaller strategy space $\hat{\mathcal{S}}_{x_1x_2}$ and redefine the seller's strategy as $u(t; x_1, x_2) = \left(\frac{dx_1(t; x_1, x_2)}{dt}, \frac{dx_2(t; x_1, x_2)}{dt}\right)$. As we discussed before, since there

[4] $a_{x_1x_2} = \min\left\{\frac{\frac{\partial P_1}{\partial x_1}}{\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_1}}, \frac{\frac{\partial P_1}{\partial x_2}}{\frac{\partial P_1}{\partial x_2} + \frac{\partial P_2}{\partial x_2}}\right\}$ and $b_{x_1x_2} = \max\left\{\frac{\frac{\partial P_1}{\partial x_1}}{\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_1}}, \frac{\frac{\partial P_1}{\partial x_2}}{\frac{\partial P_1}{\partial x_2} + \frac{\partial P_2}{\partial x_2}}\right\}$.

is no discounting, what really matters is how the two paths $x_1(t)$ and $x_2(t)$ are paired, not the exact values of $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$. Therefore, we let $\frac{dx_1}{dt} = 1$, and the seller determines $u(x_1, x_2) \equiv \frac{dx_2(x_1;x_1,x_2)}{dx_1} = \frac{dx_2(t;x_1,x_2)}{dt} \in [0, \infty]$. Note that for each $\frac{dx_2(x_1;x_1,x_2)}{dx_1} \in [0, \infty]$, there is a corresponding strategy $\left(\frac{dp_1}{dt}, \frac{dp_2}{dt}\right) \in \hat{\mathcal{S}}_{x_1x_2}$. After the transformation, the seller's strategy $\frac{dx_2(x_1;x_1,x_2)}{dx_1}$ also depends on history only through the belief about the buyers' values, $x_1 = y_1(h^t)$ and $x_2 = y_2(h^t)$.

The seller's strategy profile is $\left\{u(x_1, x_2) \equiv \frac{dx_2(x_1;x_1,x_2)}{dx_1} \mid x_1 \in [\underline{w}_1, \overline{w}_1], x_2 \in [\underline{w}_2, \overline{w}_2]\right\}$. In a continuation game starting with belief (w_1, w_2), we can derive the equilibrium path $x_{2,w_1w_2}(x)$ for $x \in [\underline{w}_1, w_1]$ from the strategy profile $\{u(x_1, x_2)\}$, and that path maximizes the seller's expected payoff in the continuation game given the buyers' strategies $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$. Therefore, for any (w_1, w_2), the seller's strategy $u(x_1, x_2)$ along the equilibrium path $x_{2,w_1w_2}(x)$ is such that

$$
\begin{aligned}
& u(x_1, x_2, _{w_1w_2}(x_1)) \qquad\qquad\qquad \text{(P2.1)} \\
\in\ & \arg \max_{u(x_1, x_{2,w_1w_2}(x_1))} \int_{\underline{w}_1}^{w_1} \left\{ \begin{array}{c} P_1(x_1, x_2) F_2(x_2) f_1(x_1) \\ +P_2(x_1, x_2) F_1(x_1) f_2(x_2) u \end{array} \right\} dx_1 \\
& +P_2(x_1, x_{2,w_1w_2}(x_1)) [F_2(w_2) - F_2(x_{2,w_1w_2}(x_1))] \\
s.t.\quad & x_2(w_1) \leq w_2, \\
& \frac{dx_2}{dx_1} = u, \\
& 0 \leq u < \infty.
\end{aligned}
$$

The first part of the integrand represents the seller's payoff increment from buyer 1 when he decreases x_1 by dx_1. The good is sold to buyer 1 at price $P_1(x_1, x_2)$ if buyer 2's value is below x_2 and buyer 1's value is x_1, which occurs with probability $F_2(x_2) f_1(x_1) dx_1$. The second part represents the payoff increment from buyer 2. When $x_{2,w_1w_2}(w_1) \neq w_2$, $u(w_1, w_2) = \infty$, and the second line is the seller's payoff from buyer 2 with values between $x_2(w_1)$ and w_2, who would accept $P_2(w_1, x_{2,w_1w_2}(w_1))$. To apply standard dynamic programming techniques, we allow the seller to have jumps on $x_{2,w_1w_2}(x_1)$ path only at the beginning when he chooses the initial value $x_2(w_1)$. With the restriction on the seller's strategy, we derive an equilibrium, and will show that even jumps are allowed along the path, the seller would not deviate.

By considering program (P2.1) for all (w_1, w_2) in $[\underline{w}_1, \overline{w}_1] \times [\underline{w}_2, \overline{w}_2]$, we derive the seller's strategy profile $\{u(x_1, x_2) \mid x_1 \in [\underline{w}_1, \overline{w}_1], x_2 \in [\underline{w}_2, \overline{w}_2]\}$. Note that the principal of optimality ensures that if $x_2(x_1)$ is the optimal path for a game starting with (w_1, w_2), then for a continuation game starting with $(x_1', x_2(x_1'))$ where $x_1' \in (\underline{w}_1, w_1)$, $x_2(x_1)$ for $x_1 \leq x_1'$ is also the optimal path. Therefore, $u(x_1, x_2)$ derived from (P2.1) with different (w_1, w_2) must be consistent. By Lemma 8 and the discussion in this section, we have the following proposition.

Proposition 11 *At t, given belief $(w_1, w_2) = (y_1(h^t), y_2(h^t))$ and the current prices $(p_1(t), p_2(t))$, if $p_1(t) = P_1(w_1, w_2)$ and $p_2(t) = P_2(w_1, w_2)$, the seller's strategy is $\left(\frac{dp_1(t;b_1,b_2,w_1,w_2)}{dt}, \frac{dp_2(t;b_1,b_2,w_1,w_2)}{dt}\right) \in \mathcal{S}$ such that*

$$\frac{\frac{dp_2}{dt}}{\frac{dp_1}{dt}} = \left\{ \begin{array}{ll} \frac{dP_2(x_1,x_2,_{w_1w_2}(x_1))}{dx_1} / \frac{dP_1(x_1,x_2,_{w_1w_2}(x_1))}{dx_1} \,|_{x_1=w_1} & \text{if } x_{2,w_1w_2}(w_1) = w_2 \\ \frac{\partial P_2(x_1,x_2)}{\partial x_2} / \frac{\partial P_1(x_1,x_2)}{\partial x_2} \,|_{x_1=w_1,x_2=w_2} & \text{if } x_{2,w_1w_2}(w_1) < w_2 \end{array} \right\}, \text{ where } x_{2,w_1w_2}(w_1)$$

is derived from (P2.1).

Differences between the commitment and non-commitment cases

When commitment is possible, the buyers form their beliefs and strategies after seeing the pre-committed price paths. The seller's optimal strategy is to choose the path with the induced buyers' beliefs and strategies that maximizes his payoff. Without commitment, the buyers form their beliefs and strategies based on the price history. The seller determines the prices at every instant based on his expectation of how the buyers will react. In short, with commitment, the buyer's strategies are formed corresponding to the pre-committed price paths; and without commitment, the buyers' strategies are formed corresponding to each realized history. Therefore, the seller's payoff-maximizing problems are different under the two circumstances.

In addition, buyers' reactions to prices are different in the two situations. Consider the case when the optimal mechanism requires setting reserve prices $r_1 > \underline{w}_1$ and $r_2 > \underline{w}_2$ for buyers 1 and 2 respectively. If the buyers expect the seller to submit the path derived in Theorem 5 and stop at r_1 and r_2, then buyer 1 with value r_1 and buyer 2 with value r_2 will accept at prices r_1 and r_2 respectively, that is, $P_1(r_1, r_2) = r_1$ and $P_2(r_1, r_2) = r_2$. When there is no commitment, since $P_1(x_1, x_2) < r_1$ and $P_2(x_1, x_2) < r_2$ for all $x_1 < r_1$ and $x_2 < r_2$, once the seller submits prices (r_1, r_2) and no buyer accepts, the seller believes that the buyers' values are below r_1 and r_2 respectively and will lower the prices further. Hence, $P_1(r_1, r_2) = r_1$ and $P_2(r_1, r_2) = r_2$

cannot be sustained in equilibrium.[5] The differences in buyers' reactions to prices prevent the seller from achieving the optimal outcome even though he maximizes his payoff in response to the buyers' strategies.

2.5.4 Buyers' strategy

$P_1(x_1, x_2)$ and $P_2(x_1, x_2)$ characterize the buyers' strategies, and they are the prices that the seller needs to offer in order to induce buyer 1 and buyer 2 whose values are above x_1 and x_2 respectively to accept. We can represent $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$ in the following forms

$$\begin{aligned} P_1(x_1, x_2) &= x_1 - \frac{C_1(x_1, x_2)}{F_2(x_2)} \text{ if } x_2 > \underline{w}_2, \\ P_2(x_1, x_2) &= x_2 - \frac{C_2(x_1, x_2)}{F_1(x_1)} \text{ if } x_2 > \underline{w}_1. \end{aligned} \tag{B1}$$

$C_i(x_1, x_2) = F_j(x_j)(x_i - P_i(x_1, x_2))$ can be regarded as the information rent asked by buyer i with value x_i when he believes that if buyer j does not accept the current price, the lowest upper bound of buyer j's value is x_2. Buyer i with value x_i accepts a price smaller than or equal to $P_i(x_1, x_2)$, and he gets the good if buyer j's value is below x_j. How much a buyer asks for as information rent depends on his expectation of the prices that the seller will offer later and his belief about how the other buyer behaves. Therefore, $C_i(x_1, x_2)$ also characterizes buyer i's strategy, and must be the best response to the other players' strategies. Given the current beliefs (w_1, w_2), if the seller's strategy is to implement a path $x_{2,w_1w_2}(x_1)$ in the future, then to satisfy incentive compatibility so that buyer 1 and buyer 2 will not deviate, we must have

$$\begin{aligned} C_1(w_1, w_2) &= \int_{\underline{w}_1}^{w_1} F_2(x_{2,w_1w_2}(x))\, dx, \\ C_2(w_1, w_2) &= \int_{\underline{w}_2}^{w_2} F_1\left(x_{2,w_1w_2}^{-1}(x)\right) dx \end{aligned} \tag{B2}$$

in equilibrium, where

$$x_{2,w_1w_2}^{-1}(x_2) \equiv \sup\{x_1 \mid x_{2,w_1w_2}(x_1) \le x_2\} \tag{2.2}$$

which shows the same path as $x_{2,w_1w_2}(x_1)$ but represents x_1 as a function of x_2.

[5] In equilibrium, P_1 and P_2 satisfy (B1) and (B2) defined in the next section.

Proposition 12 formalizes the arguments above and characterizes the buyers' strategies.

Proposition 12 *After the seller submits $p_1(t)$ and $p_2(t)$ at time t, with the belief that the lowest upper bound of buyer j's value is $x_j = y_j(h^t)$ provided that buyer j rejects all the prices on the history path until t, if buyer i's value $x_i \leq y_i \equiv \lim_{\tau \uparrow t} y_i(h^\tau)$, then the maximum price he accepts is $P_i(x_i, x_j)$ defined in (B1) and (B2); if $x_i > y_i$, which implies buyer i has deviated, then the maximum price he accepts is $P_i(y_i, x_j)$.*

Proof. We show that buyer i maximizes his payoff by following the strategy specified in the proposition. Let $p_1(\tau)$ and $p_2(\tau)$ be the price paths submitted by the seller. Consider a continuation game starting at t with belief $(y_1, y_2) = (\lim_{\tau \uparrow t} y_1(h^\tau), \lim_{\tau \uparrow t} y_2(h^\tau))$ and prices $(p_1(t), p_2(t))$.

First consider the case when there exist $w_1 \leq y_1$ and $w_2 \leq y_2$ such that $p_1(t) = P_1(w_1, w_2)$ and $p_2(t) = P_2(w_1, w_2)$. Given the seller's strategy profile characterized in Proposition 11, we can derive the price paths after t as well as the other two paths $x_1(\tau)$ and $x_2(\tau)$ by solving $\begin{cases} p_1(\tau) = P_1(x_1(\tau), x_2(\tau)) \\ p_2(\tau) = P_2(x_1(\tau), x_2(\tau)) \end{cases}$. $x_i(\tau)$ is the seller's and buyer j's beliefs about the greatest lower bound of buyer i's values with which buyer i would have accepted $p_i(\tau)$. In addition, $(x_1(\tau), x_2(\tau))$, $\tau \geq t$, constitute the graph of $x_{2,w_1w_2}(x_1)$, the solution path to program (P2.1). Note that if $x_{2,w_1w_2}(w_1) < w_2$, then there exists $s > t$ such that $x_1(\tau) = w_1$, and $x_2(\tau)$ decreases from w_2 to $x_{2,w_1w_2}(w_1)$ for $\tau \in [t, s]$. Because $\frac{\partial P_i(x_i, x_j)}{\partial x_i} \geq 0$, to prove the proposition, it is enough to show that given the price paths $(p_1(\tau), p_2(t))$ derived from the seller's strategy profile, at any time $t' \geq t$, buyer i with a value higher than or equal to $x_i(t')$ should accept $p_i(t')$, and buyer i with a value lower than $x_i(t')$ should reject. At t', buyer 1 with value x_1 gets payoff

$$x_1 - p_1(t') = (x_1 - x_1(t')) + \int_{\underline{w}_1}^{x_1(t')} F_2(x_{2,w_1w_2}(x))\, dx \text{ [6]}$$

if he accepts now, and gets expected payoff

$$\frac{F_2(x_2(t''))}{F_2(x_2(t'))}(x_1 - p_1(t'')) = \frac{F_2(x_2(t''))}{F_2(x_2(t'))}\left((x_1 - x_1(t'')) + \int_{\underline{w}_1}^{x_1(t'')} F_2(x_{2,w_1w_2}(x))\, dx\right)$$

[6] $x_{2,w_1w_2}(x) = x_{2,x_1(t')x_2(t')}(x)$ for $x \in [\underline{w}_1, x_1(t')]$.

if he accepts later at t''. Because $x_2(t'') \leq x_2(t')$, similar to the argument in Theorem 5, buyer 1 with $x_1 \geq x_1(t')$ gets weakly higher payoff by accepting at t' and buyer 1 with $x_1 = x_1(t'')$ gets weakly higher payoff by accepting at $t'' > t'$. The same argument applies to buyer 2. Therefore, buyer i with a value higher than or equal to $x_i(t')$ should accept $p_i(t')$, and buyer i with a value lower than $x_i(t')$ should reject.

Next consider the situation when there does not exist $w_1 \leq y_1$ and $w_2 \leq y_2$ such that $p_1(t) = P_1(w_1, w_2)$ and $p_2(t) = P_2(w_1, w_2)$. This implies either (i) $p_i(t) > P_i(y_i, y_j)$ for both $i = 1, 2$, or (ii) there exists $w_j \leq y_j$ such that $p_i(t) > P_i(y_i, w_j)$ and $p_j(t) = P_j(y_i, w_j)$ for some $i \neq j$. Given the seller's strategy in Proposition 11 and buyer j's strategy $P_j(x_1, x_2)$, in case (i), buyer i expects that buyer j would not accept before p_i falls to $P_i(y_i, y_j)$ so buyer i rejects all the prices $p_i > P_i(y_i, y_j)$; and in case (ii), buyer i expects that buyer j would not accept before p_i falls to $P_i(y_i, w_j)$ so buyer i rejects all the prices $p_i > P_i(y_i, w_j)$. ■

Substituting $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$ in the seller's problem (P2.1), we get

$$\begin{aligned}
&\max_{u(x_1, x_{2,w_1w_2}(x_1))} \int_{\underline{w}_1}^{w_1} \left\{ \begin{array}{c} [x_1 F_2(x_2) - C_1(x_1, x_2)] f_1(x_1) \\ + [x_2 F_1(x_1) - C_2(x_1, x_2)] f_2(x_2) u \end{array} \right\} dx_1 \\
&+ [x_{2,w_1w_2}(w_1) F_1(w_1) - C_2(w_1, x_{2,w_1w_2}(w_1))] [F_2(w_2) - F_2(x_{2,w_1w_2}(w_1))]
\end{aligned} \tag{P2.2}$$

$$\begin{aligned}
s.t. \quad & x_{2,w_1w_2}(w_1) \leq w_2, \\
& \frac{dx_2}{dx_1} = u, \\
& 0 \leq u < \infty.
\end{aligned}$$

2.5.5 An equilibrium

In this subsection, we first characterize the conditions for an equilibrium in Proposition 13. Then we show that there exists an equilibrium in which the equilibrium allocation and payoffs of all the players are the same as those in a second-price auction in Theorem 6.

Proposition 13 *Suppose there exist $x_{2,w_1w_2}(x_1), C_1(w_1, w_2), C_2(w_1, w_2)$ such that for any $(w_1, w_2) \in [\underline{w}_1, \overline{w}_1] \times [\underline{w}_2, \overline{w}_2]$,*

1. $x_{2,w_1w_2}(x_1), x_1 \in [\underline{w}_1, w_1]$, *is the path derived from (P2.2);*

2. $C_1(w_1,w_2)=\int_{\underline{w}_1}^{w_1}F_2(x_{2,w_1w_2}(x))\,dx$;

3. $C_2(w_1,w_2)=\int_{\underline{w}_2}^{w_2}F_1\left(x_{2,w_1w_2}^{-1}(x)\right)dx$.[7]

If the vector-valued function $P(w_1,w_2)=(P_1(w_1,w_2),P_2(w_1,w_2))$ *defined in (B1) satisfies Condition 4, then there exists a perfect Bayesian equilibrium as follows:*

Let $X_{w_1w_2}(p_1,p_2)=(X_{1,w_1w_2}(p_1,p_2),X_{2,w_1w_2}(p_1,p_2))=P^{-1}(p_1,p_2),(p_1,p_2)\in\mathcal{B}_{w_1w_2}$, *where* $\mathcal{B}_{w_1w_2}=\{P(y_1,y_2)\mid 0<y_1\le w_1, 0<y_2\le w_2\}$.

- *The belief about the lowest upper bound of buyer i's value is formed as follows:*
  At t, let $(w_1,w_2)=\begin{cases}(\overline{w}_1,\overline{w}_2), & \text{if } t=0\\ (\lim_{\tau\uparrow t}y_1(h^\tau),\lim_{\tau\uparrow t}y_2(h^\tau)), & \text{if } t>0\end{cases}$,
  - *If the prices* (p_1,p_2) *offered by the seller is in* $\mathcal{B}_{w_1w_2}$, $\left(y_1\left(h^t\right),y_2\left(h^t\right)\right)=(X_{1,w_1w_2}(p_1,p_2),X_{2,w_1w_2}(p_1,p_2))$.
  - *If* $p_1\le P_1(w_1,w_2)$ *and* (p_1,p_2) *is above* $\mathcal{B}_{w_1w_2}$, $y_1\left(h^t\right)=x_1$ *such that* $P_1(x_1,w_2)=p_1$, *and* $y_2\left(h^t\right)=w_2$.
  - *If* $p_2\le P_2(w_1,w_2)$ *and* (p_1,p_2) *is on the right of* $\mathcal{B}_{w_1w_2}$, $y_1\left(h^t\right)=w_1$, *and* $y_2\left(h^t\right)=x_2$ *such that* $P_1(w_1,x_2)=p_2$.
  - *If* $p_1>P_1(w_1,w_2)$ *and* $p_2>P_2(w_1,w_2)$, $\left(y_1\left(h^t\right),y_2\left(h^t\right)\right)=(w_1,w_2)$.

- *At t, given belief* $(w_1,w_2)=\left(y_1\left(h^t\right),y_2\left(h^t\right)\right)$ *and the current prices* (b_1,b_2) *(note that* $P_i(w_1,w_2)\le b_i$ *for* $i=1,2$[8]*), if* $b_1=P_1(w_1,w_2)$ *and* $b_2=P_2(w_1,w_2)$, *the seller's strategy is*
  $\left(\frac{dp_1(t;b_1,b_2,w_1,w_2)}{dt},\frac{dp_2(t;b_1,b_2,w_1,w_2)}{dt}\right)\in\mathcal{S}$ *such that*
  $\frac{\frac{dp_2}{dt}}{\frac{dp_1}{dt}}=\left\{\begin{array}{ll}\frac{dP_2(x_1,x_{2,w_1w_2}(x_1))}{dx_1}\Big/\frac{dP_1(x_1,x_{2,w_1w_2}(x_1))}{dx_1}\Big|_{x_1=w_1} & \text{if } x_{2,w_1w_2}(w_1)=w_2\\ \frac{\partial P_2(x_1,x_2)}{\partial x_2}\Big/\frac{\partial P_1(x_1,x_2)}{\partial x_2}\Big|_{x_1=w_1,x_2=w_2} & \text{if } x_{2,w_1w_2}(w_1)<w_2\end{array}\right\}$; *if* $b_i>P_i(x_1,x_2)$ *and* $b_j=P_j(x_1,x_2)$, *the seller's strategy is* $\frac{dp_i}{dt}=1$ *and* $\frac{dp_j}{dt}=0$; *if* $b_i>P_i(x_1,x_2)$ *for both* $i=1,2$, *then any* $\left(\frac{dp_1}{dt},\frac{dp_2}{dt}\right)\in\mathcal{S}$ *can be the seller's strategy.*

---

[7] $x_{2,w_1w_2}^{-1}(x)$ *is defined in (2.2).*

[8] *When the seller determines* $\left(\frac{dp_1}{dt},\frac{dp_2}{dt}\right)$, *the buyers have rejected* (b_1,b_2), *so the belief* $(w_1,w_2)=\left(y_1\left(h^t\right),y_2\left(h^t\right)\right)$, *and* $P_i(w_1,w_2)\le b_i$ *for* $i=1,2$.

- *At t, given history h^t, if buyer i's value $x_i \leq \lim_{\tau \uparrow t} y_i(h^\tau)$, the maximum price he accepts is $P_i(x_i, y_j(h^t))$; if $x_i > \lim_{\tau \uparrow t} y_i(h^\tau)$, the maximum price he accepts is $P_i(\lim_{\tau \uparrow t} y_i(h^\tau), y_j(h^t))$.*

Proof. The result comes directly from Propositions 10, 11, and 12. ∎

Example 1 *When the two buyers' values are uniformly distributed on $[0, \overline{w}_1]$ and $[0, \overline{w}_2]$ respectively, there exists an equilibrium as follows:*

In this example, the function that characterizes buyer i's strategy is $P_i(w_i, w_j) = \begin{cases} w_i - \frac{w_i^2}{2w_j}, \text{ if } w_j \geq w_i \\ \frac{w_j}{2}, \text{ if } w_j < w_i \end{cases}$.

Let $w_1(t)$ and $w_2(t)$ be the beliefs about buyer 1's and buyer 2s values at time t. $w_i(0) = \overline{w}_i$, and the value of $w_i(t), t > 0$, is defined recursively below.

- *At time $t \geq 0$, suppose the current posted prices are (p_1, p_2) and beliefs are $(w_1, w_2) = (w_1(t), w_2(t))$ Without loss of generality, suppose $w_i \leq w_j$. Note that $p_i \geq w_i - \frac{w_i^2}{2w_j}$ and $p_j \geq \frac{w_i}{2}$.[9] the seller's strategy is*

$$\left(\frac{dp_i(t; p_i, p_j, w_i, w_j)}{dt}, \frac{dp_j(t; b_i, b_j, w_i, w_j)}{dt}\right)$$
$$= \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right), \text{ if } p_i = p_j = \frac{1}{2}w_i = \frac{1}{2}w_j \\ (0, 1), \text{ if } p_i \geq w_i - \frac{w_i^2}{2w_j} \text{ and } p_j > \frac{w_i}{2} \\ (1, 0), \text{ if } p_j = \frac{w_i}{2} \text{ and } p_i > \frac{w_i}{2} \end{cases}.$$

- *At time t, let* $(w_1, w_2) = \begin{cases} (\overline{w}_1, \overline{w}_2), \text{ if } t = 0 \\ (\lim_{\tau \uparrow t} w_1(\tau), \lim_{\tau \uparrow t} w_2(\tau)), \text{ if } t > 0 \end{cases}$ *and (p_1, p_2) be the prices submitted by the seller (without loss of generality, suppose $p_i \geq p_j$), the buyers' strategies and the values of $(w_1(t), w_2(t))$ are as follows:*

  - *If $2p_j \leq w_i$ and $\frac{2p_j^2}{2p_j - p_i} \leq w_j$, buyer i accepts p_i if his value is above $2p_j$, and buyer j accepts p_j if his value is above $\frac{2p_j^2}{2p_j - p_i}$. $w_i(t) = 2p_j$ and $w_j(t) = \frac{2p_j^2}{2p_j - p_i}$.*
  - *If $2p_j \leq w_i$ and $\frac{2p_j^2}{2p_j - p_i} > w_j$, buyer i does not accept p_i, and buyer j accepts p_j if and only if $w_i - \sqrt{w_i^2 - 2w_i p_j} \leq w_j$ and his value is above $w_i - \sqrt{w_i^2 - 2w_i p_j}$. $w_i(t) = w_i$ and $w_j(t) = \min\left\{w_j, w_i - \sqrt{w_i^2 - 2w_i p_j}\right\}$.*

[9] *This is because in this example, if $w_j > w_i$, $P_i(w_i, w_j) = w_i - \frac{w_i^2}{2w_j}$, and $P_j(w_i, w_j) = \frac{w_i}{2}$. It must be that $P_i(w_i, w_j) \leq p_i$ and $P_j(w_i, w_j) \leq p_j$ so that the belief is updated consistently (see Proposition 13).*

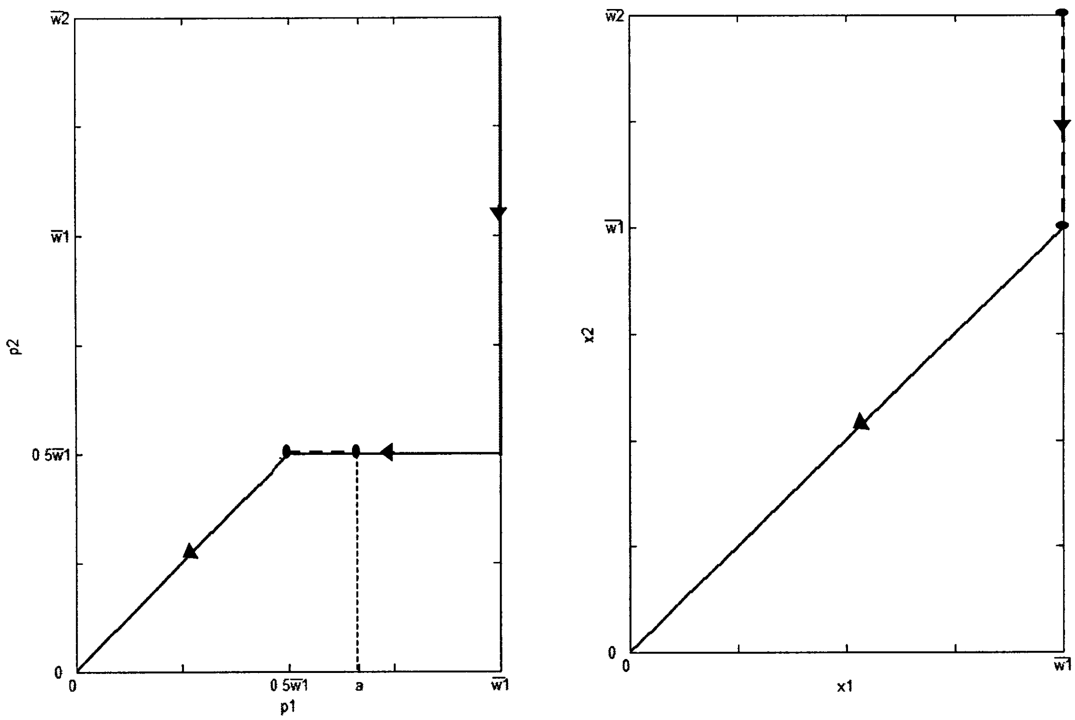

Figure 2-1: Equilibrium $p_1 - p_2$ and $x_1 - x_2$ paths

- *If* $2p_j > w_i$ *and* $\frac{2p_j^2}{2p_j - p_i} \leq w_j$, *buyer* j *does not accept* p_j, *and buyer* i *accepts* p_i *if and only if* $w_j - \sqrt{w_j^2 - 2w_j p_i} \leq w_i$ *and his value is above* $w_j - \sqrt{w_j^2 - 2w_j p_i}$. $w_j(t) = w_j$ *and* $w_i(t) = \min\left\{w_i, w_j - \sqrt{w_j^2 - 2w_j p_i}\right\}$.
- *If* $2p_j > w_i$ *and* $\frac{2p_j^2}{2p_j - p_i} > w_j$, *both buyer* i *and buyer* j *do not accept.* $w_i(t) = w_i$ *and* $w_j(t) = w_j$.

Figure 2-1 shows the equilibrium price path submitted by the seller and the corresponding $x_1 - x_2$ path in Example 1 when $\overline{w}_2 > \overline{w}_1$. The seller first keeps p_1 at $\overline{w}_1$ and lowers p_2 to $\frac{1}{2}\overline{w}_1$. Then he keeps p_2 at $\frac{1}{2}\overline{w}_1$ and lowers p_1 to $\frac{1}{2}\overline{w}_1$. Finally he lowers both p_1 and p_2 to 0 at the same speed. The buyers correctly expect the path, so buyer 2 will not accept any price along the vertical part of the path. When the seller starts lowering p_1 (the horizontal part), buyer 2 starts accepting $p_2 = \frac{1}{2}\overline{w}_1$. To be more precise, buyer 2 starts accepting after p_1 drops to

$a = \overline{w}_1 - \frac{\overline{w}_1^2}{2\overline{w}_2}$. Buyer 1 then has to consider whether to accept a higher price now or to accept a lower price later at the risk of buyer 2 accepting before him. In the equilibrium characterized in Example 1, buyer 2 accepts p_2 at a speed such that buyer 1 is better to wait until p_1 drops to $\frac{1}{2}\overline{w}_1$. When p_1 reaches $\frac{1}{2}\overline{w}_1$, buyer 2 with a value higher than $\overline{w}_1$ would have accepted. Therefore, the dash-line part of the $p_1 - p_2$ path is mapped to the dash-line part of the $x_1 - x_2$ path. After that, the seller lowers p_1 and p_2 at the same speed, and buyer i with a value equal to $2p_i$ accepts p_i. The seller can choose other paths. However, given the buyers' reactions and expectations of the price path after deviation, which is consistent with the seller's strategy in the continuation game after deviation, the seller cannot do better than following the path in Example 1.

In this example, the seller's and the buyers' expected payoffs are the same as in a second-price auction, which is lower than the payoff in a Dutch auction. The lower payoff results from the seller's lack of commitment power. In equilibrium, the seller lowers the two prices simultaneously only when $p_1 = p_2 = \frac{1}{2}\overline{w}_1 = \frac{1}{2}\overline{w}_2$. Otherwise, he only lowers one of the prices. Given the seller's strategy, a buyer does not feel competition from the other buyer as much as in a Dutch auction, so he is reluctant to accept a higher price. And given the buyers' strategies, the seller can do no better by deviating. On the other hand, in a Dutch auction, the two prices are lowered simultaneously. This causes competition between the buyers, so the buyers are willing to accept higher prices. The example thus shows that the inability to make a commitment might result in an equilibrium unfavorable to the seller. The advantage of being able to determine the price paths at will is dominated by the loss caused by not being able to commit. In the next theorem, we show that existing an equilibrium in which the buyers' and the seller's expected payoffs are the same as in a second-price auction is generally true, not just for the uniform distribution case. And in Section 2.5.6, we further prove the uniqueness of the equilibrium in the uniform distribution case.

Theorem 6 $x_{2,w_1w_2}(x_1) = \begin{cases} \underline{w}_2, \ \textit{for } \underline{w}_1 \leq x_1 < \max\{\underline{w}_1, \underline{w}_2\} \\ x_1, \ \textit{for } \max\{\underline{w}_1, \underline{w}_2\} \leq x_1 \leq \min\{w_1, w_2\} \\ w_2, \ \textit{for } \min\{w_1, w_2\} < x_1 \leq w_1 \end{cases}$,

$C_1(w_1,w_2) = \int_{\underline{w}_1}^{w_1} F_2(x_{2,w_1w_2}(x_1))\,dx_1$, *and* $C_2(w_1,w_2) = \int_{\underline{w}_2}^{w_2} F_1\left(x_{2,w_1w_2}^{-1}(x_2)\right) dx_2$[10] *are a set of functions satisfying the conditions in Proposition 13. Therefore, there exists an equilibrium such that the allocation rule, the buyers', and the seller's expected payoffs are the same as in a second-price auction.*

Proof. To show that $x_{2,w_1w_2}(x_1)$ is the path derived from (P2.2), we check the sufficient condition that value function $V(w_1,w_2)$ satisfies the HJB equation

$$V_{x_1}(x_1,x_2) = \max_u \{G(x_1,x_2,u) - V_{x_2}(x_1,x_2)\,g(x_1,x_2,u) \mid 0 < u < \infty\},^{11} \tag{2.3}$$

where

$$\begin{aligned} G(x_1,x_2,u) &= [x_1F_2(x_2) - C_1(x_1,x_2)]\,f_1(x_1) + [x_2F_1(x_1) - C_2(x_1,x_2)]\,f_2(x_2)\,u, \\ g(x_1,x_2,u) &= u. \end{aligned}$$

Without loss of generality, assume that $\underline{w}_1 \geq \underline{w}_2$. The form of $x_{2,w_1w_2}(x_1)$ implies that $u(x_1,x_2(x_1)) = 1$. Therefore, when $x_2 \geq x_1$,

$$\begin{aligned} V(x_1,x_2) &= \int_{\underline{w}_1}^{x_1} \left[xF_2(x) - \int_{\underline{w}_1}^{x} F_2(t)\,dt\right] f_1(x) + \left[xF_1(x) - \int_{\underline{w}_1}^{x} F_1(t)\,dt\right] f_2(x)\,dx \\ &\quad + \left[x_1F_1(x_1) - \int_{\underline{w}_1}^{x_1} F_1(x)\,dx\right] [F_2(x_2) - F_2(x_1)]\,. \end{aligned}$$

Then the left-hand side of equation (2.3) is $\left[x_1F_2(x_2) - \int_{\underline{w}_1}^{x_1} F_2(x)\,dx\right] f_1(x_1)$, and so is the right-hand side. Similarly, equation 2.3 holds when $x_1 \geq x_2$.

One can check that $P(w_1,w_2) = (P_1(w_1,w_2), P_2(w_1,w_2))$ defined in (B1) satisfies Condition 4. Thus, there exists an equilibrium by Proposition 13. The allocation rule is that the buyer with the higher value gets the good, and a buyer whose value is smaller than or equal to $\max\{\underline{w}_1, \underline{w}_2\}$ gets zero payoff. Therefore, by the revenue equivalence principle, all the players' payoffs are the same as in the second-price auction. ■

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[10] *If* $\underline{w}_2 < \underline{w}_1$, $x_2^{-1}(x_2) = \underline{w}_1$ *for* $x_2 < \underline{w}_1$. *If* $w_2 > w_1$, $x_2^{-1}(x_2) = w_1$ *for* $x_2 > w_1$.

[11] The HJB equation here is different from the version generally used. This is because the index x_1 here is going down instead of going up as the commonly-used index t is.

A Dutch auction requires the seller to call out a single price for all the buyers even though they are asymmetric. In the previous section, we show that if different prices for different buyers are permitted and the seller is able to commit to a price path in advance, then the seller can achieve the optimal outcome. However, if different prices are allowed but the seller is not able to commit, there exists an equilibrium in which the seller's payoff is the same as in a second-price auction. Vickrey (1961) and Maskin and Riley (2000) show that, with asymmetric bidders, the seller's payoff in a first-price auction (or a Dutch auction) might be greater than that in a second-price auction; this is always true when the two buyers' values are both uniformly distributed. Our result together with the conclusion from the literature suggests that the benefit brought by the discretion to determine the price paths might be outweighed by the loss caused by not being able to commit.

In reality, an extremely sophisticated institution is difficult to implement, so the mechanisms adopted are usually simple. At times, people complain that simple mechanisms hinder their ability and freedom to do what is best for them. However, our result shows that even though people lose their discretion with a simple institution, that institution still retains its value because it helps people make commitment. In a Dutch auction, the seller is forced to commit to a price path and cannot charge different buyers different prices. It seems that the seller might get a better payoff without the restriction and benefit from having the discretion to design price paths. However, sometimes the advantage of having the discretion might be sabotaged by not being able to commit, and the seller's payoff turns out to be lower when he can negotiate with buyers at will.

2.5.6 Uniqueness of the equilibrium

It might be the concern that there exist multiple equilibria for the negotiation game, so the comparison between the seller's payoffs in different institutions is not conclusive. In this subsection, we show that under some circumstances, there is actually a unique equilibrium, so our result is robust. We consider the case in Example 1 when the two buyers' values are uniformly distributed on $[0, \overline{w}_1]$ and $[0, \overline{w}_2]$ respectively. To see why the number of equilibria is limited, note that each off-equilibrium history h^t is reachable by deviations of the seller. Since the deviations are made by a player who does not have private information, the beliefs about the buyers' values cannot be arbitrary and must be consistent with the buyers' strategies at each h^t. In addition, the buyers' expectations of future paths are not arbitrary either. They must be consistent with the seller's strategy, which is the best response to the buyers' strategies.

The following analysis focuses on the equilibrium in which given any (w_1, w_2), the seller's strategy $u(x_1, x_{2,w_1w_2}(x_1))$ (defined in Section 2.5.3 and derived from program (P2.2)) on the continuation equilibrium path starting with belief (w_1, w_2) is in $(0, \infty)$ for all $x_1 \in (0, \overline{x}_{1,w_1w_2})$ for some $\overline{x}_{1,w_1w_2} \leq w_1$. With the restriction, program (P2.2) becomes

$$\max_{\overline{x}_{1,w_1w_2}, u(x_1, x_{2,w_1w_2}(x_1))} \int_{\underline{w}_1}^{\overline{x}_1} \left\{ \begin{array}{c} [x_1 F_2(x_2) - C_1(x_1, x_2)] f_1(x_1) \\ + [x_2 F_1(x_1) - C_2(x_1, x_2)] f_2(x_2) u \end{array} \right\} dx_1 \tag{P2.3}$$
$$+\mathbf{1}_{(\overline{x}_1 < w_1)} [\overline{x}_1 F_2(x_{2,w_1w_2}(\overline{x}_1)) - C_1(\overline{x}_1, x_{2,w_1w_2}(\overline{x}_1))] [F_1(w_1) - F_1(\overline{x}_1)]$$
$$+\mathbf{1}_{(x_2(\overline{x}_1) < w_2)} [x_{2,w_1w_2}(\overline{x}_1) F_1(\overline{x}_1) - C_2(\overline{x}_1, x_{2,w_1w_2}(\overline{x}_1))] [F_2(w_2) - F_2(x_{2,w_1w_2}(\overline{x}_1))]$$

$$\begin{aligned} s.t. \quad & \overline{x}_1 = w_1 \text{ and } x_{2,w_1w_2}(w_1) \leq w_2, \text{ or } \overline{x}_1 \leq w_1 \text{ and } x_{2,w_1w_2}(\overline{x}_1) = w_2, \\ & \frac{dx_2}{dx_1} = u, \\ & 0 < u < \infty, \end{aligned}$$

where $\mathbf{1}_{(\cdot)}$ is the indicator function. Note that we do not restrict the seller's strategy space. The seller can still adopt a strategy such that $u(x_1, x_2) = 0$ or ∞, but he will find that doing so does not make him better.

In addition, we require $C_1(x_1, x_2)$ and $C_2(x_1, x_2)$, the functions that characterize buyers' information rent, to be smooth, i.e. $\frac{\partial C_i}{\partial x_j}, i, j = 1, 2$, is continuous. If $\frac{\partial C_1}{\partial x_2}$ is not continuous at (x_1, x_2), it is implied that the equilibrium strategies and beliefs are quite different in the two continuation games with beliefs (x_1, x_2) and $(x_1, x_2 - \epsilon)$, where ϵ is small. Moreover, we require $\frac{\partial C_i}{\partial x_j} \geq 0$.

Definition 4 *An equilibrium is smooth if the seller's equilibrium strategy $u(x_1, x_2(x_1))$ on the continuation equilibrium path starting with any belief (w_1, w_2) is in $(0, \infty)$ for all $x_1 \in (0, \overline{x}_{1,w_1w_2})$, where $u(x_1, x_2(x_1))$ and $\overline{x}_{1,w_1w_2}$ are derived from program (P2.3).. Moreover, $\frac{\partial C_i}{\partial x_j}$, $i, j = 1, 2$, is continuous and $\frac{\partial C_i}{\partial x_j} \geq 0$.*

First, we use Pontryagin's maximum principle to derive necessary conditions for the seller's equilibrium strategy. Along with the condition implied by the revenue equivalence principle, we show that there is only one set of strategies and beliefs satisfying all the conditions and the requirements for a smooth equilibrium. Therefore, the equilibrium derived in Theorem 6 is the unique smooth equilibrium. The details of the proof can be found in Appendix.

Proposition 14 *When the two buyers' values are uniformly distributed on* $[0, \overline{w}_1]$ *and* $[0, \overline{w}_2]$ *respectively, the equilibrium characterized in Example 1 is the unique smooth perfect Bayesian equilibrium.*

Proof. The proposition comes from lemma 16 in Appendix. ■

2.6 Conclusion

Our paper studies the case when a seller with an indivisible object negotiates with two asymmetric buyers to determine who gets the object and at what price. The seller repeatedly submits take-it-or-leave-it offers to the two buyers until one of them accepts. Unlike a Dutch auction, the two prices offered to the two buyers do not have to be the same. We show that if the seller can commit to some price paths, the payoff realized in Myerson's optimal mechanism is achievable. However, if commitment is not possible, the seller's equilibrium payoff is the same as that in a second-price auction, which might be lower than the payoff in a Dutch auction. Therefore, although a simple institution, like a Dutch auction, restricts a player's freedom, it might actually benefit the player by providing a commitment tool. Our analysis also sheds light on the procurement literature and gives insights into the performance of atypic auctions conducted at Priceline.com.

The paper builds a bridge between the auction and the bargaining literature. The model differs from an auction environment by allowing the auctioneer to set different prices for different buyers, and differs from a bargaining setting by considering a one-to-many negotiation process in which one party chooses his partner from a group of people. Although we only consider a two-buyer case, the analysis and methodology can be applied to a more complex environment, and the conclusion can be generalized to a n-buyer case.

2.7 Appendix

Consider the seller's optimal control problem in (P2.3). Define the initial value function as

$$I\left(\overline{x}_1, x_2\left(\overline{x}_1\right)\right) = \begin{cases} \left[\overline{x}_1 F_2\left(x_2\left(\overline{x}_1\right)\right) - C_1\left(\overline{x}_1, x_2\left(\overline{x}_1\right)\right)\right]\left[F_1\left(w_1\right) - F_1\left(\overline{x}_1\right)\right] & \text{if } \overline{x}_1 < w_1, x_2\left(\overline{x}_1\right) = w_2 \\ \left[x_2\left(\overline{x}_1\right) F_1\left(\overline{x}_1\right) - C_2\left(\overline{x}_1, x_2\left(\overline{x}_1\right)\right)\right]\left[F_2\left(w_2\right) - F_2\left(x_2\left(\overline{x}_1\right)\right)\right] & \text{if } \overline{x}_1 = w_1, x_2\left(\overline{x}_1\right) < w_2 \end{cases}.$$

Define the Hamiltonian function H as

$$H\left(x_1, x_2, u, \lambda\right) = G\left(x_1, x_2, u\right) + \lambda g\left(x_1, x_2, u\right),$$

where $G\left(x_1, x_2, u\right)$ and $g\left(x_1, x_2, u\right)$ are defined in equation (2.3). The following theorem is a restatement of Pontryagin's maximum principle.

Theorem 7 *If a control $u\left(\cdot\right)$ with a corresponding state trajectory $x\left(\cdot\right)$ is optimal, there exists an absolutely continuous function $\lambda : [0, w_1] \mapsto \mathbb{R}$ such that the maximum condition*

$$H\left(x_1, x_2\left(x_1\right), u\left(x_1\right), \lambda\left(x_1\right)\right) = \max\left\{H\left(x_1, x_2\left(x_1\right), u, \lambda\left(x_1\right)\right) \mid 0 \leq u \leq \infty\right\},$$

the adjoint equation

$$\lambda'\left(x_1\right) = -\frac{\partial H\left(x_1, x_2\left(x_1\right), u\left(x_1\right), \lambda\left(x_1\right)\right)}{\partial x_2},$$

and the transversality conditions

$$\begin{aligned} \lambda(0) &= 0; \\ \textit{if } \overline{x}_{1,w_1w_2} &= w_1 \textit{ and } x_2\left(w_1\right) < w_2,\ \lambda\left(w_1\right) = \frac{\partial I}{\partial x_2}; \quad (2.4) \\ \textit{if } \overline{x}_{1,w_1w_2} &< w_1 \textit{ and } x_2\left(\overline{x}_1\right) = w_2,\ H\left(\overline{x}_1\right) + \frac{\partial I}{\partial \overline{x}_1} = 0. \quad (2.5) \end{aligned}$$

are satisfied.

For later convenience, let $c_1\left(x_1, x_2\right) = \overline{w}_2 C_1\left(x_1, x_2\right)$ and $c_2\left(x_1, x_2\right) = \overline{w}_1 C_2\left(x_1, x_2\right)$.

Lemma 9 $c_1\left(x_1, x_2\right) = x_1 x_2 - c_2\left(x_1, x_2\right)$.

Proof. Recall that $C_i\left(w_1, w_2\right)$ can be regarded as information rent to buyer i with value w_i while at some stage of the negotiation game, it is the belief that w_1 and w_2 are the lowest upper bounds of the buyers' values. Given w_1 and w_2, suppose $x_{2,w_1w_2}(x_1) : [0, \overline{x}_{1,w_1w_2}] \mapsto [0, w_2]$, is the equilibrium path of the continuation game. In equilibrium, incentive compatibility is satisfied, so the information rent to buyer 1 with value w_1 must be

$$C_1(w_1, w_2) = \int_0^{\overline{x}_{1,w_1w_2}} F_2(x_{2,w_1w_2}(x))\,dx + F_2(x_{2,w_1w_2}(\overline{x}_{1,w_1w_2}))[w_1 - \overline{x}_{1,w_1w_2}],$$

and the information rent to buyer 2 with value w_2 must be

$$C_2(w_1, w_2) = \int_0^{x_2(\overline{x}_{1,w_1w_2})} F_1\left(x_{2,w_1w_2}^{-1}(x)\right)dx + F_1(\overline{x}_{1,w_1w_2})[w_2 - x_{2,w_1w_2}(\overline{x}_{1,w_1w_2})].$$

Multiply both sides of the two equations by $\overline{w}_2$ and $\overline{w}_1$ respectively, we get

$$\begin{aligned} c_1(w_1, w_2) &= \int_0^{\overline{x}_{1,w_1w_2}} x_{2,w_1w_2}(x)\,dx + x_{2,w_1w_2}(\overline{x}_{1,w_1w_2})[w_1 - \overline{x}_{1,w_1w_2}], \\ c_2(w_1, w_2) &= \int_0^{x_{2,w_1w_2}(\overline{x}_{1,w_1w_2})} x_{2,w_1w_2}^{-1}(x)\,dx + \overline{x}_{1,w_1w_2}[w_2 - x_{2,w_1w_2}(\overline{x}_{1,w_1w_2})]. \end{aligned} \tag{2.6}$$

Note that either $\overline{x}_{1,w_1w_2} = w_1$ or $x_{2,w_1w_2}(\overline{x}_{1,w_1w_2}) = w_2$. Therefore, $c_1(w_1, w_2) + c_2(w_1, w_2) = w_1w_2$. ■

Lemma 10 $c_i(0, x_2) = 0$ *and* $c_i(x_1, 0) = 0$.

Proof. By lemma 9, $c_1(x_1, 0) + c_2(x_1, 0) = 0$. Since $c_i(\cdot) \geq 0$, $c_1(x_1, 0) = 0$ and $c_2(x_1, 0) = 0$. ■

Lemma 11 *Let* $x_{2,w_1w_2}(x_1)$ *be the equilibrium path of the continuation game starting with belief* w_1, w_2. *Then*

$$c_1(x_1, x_{2,w_1w_2}(x_1)) - \int_0^{x_1} x - \frac{\partial c_1(x, x_{2,w_1w_2}(x))}{\partial x_2} + \frac{\partial c_1(x, x_{2,w_1w_2}(x))}{\partial x_2}\frac{dx_{2,w_1w_2}(x)}{dx_1}dx = 0 \tag{2.7}$$

for all $x_1 \in [0, \overline{x}_{1,w_1w_2}]$.

Proof. By the maximum condition,

$$u \in (0, \infty) \text{ if } \frac{\partial H}{\partial u} = [x_2F_1(x_1) - C_2(x_1, x_2)]\,f_2(x_2) + \lambda = 0. \tag{2.8}$$

By the adjoint equation,

$$\lambda'(x_1) = -\left\{\left[x_1 f_2(x_{2,w_1w_2}(x_1)) - \frac{\partial C_1}{\partial x_2}\right] f_1(x_1) + \left[F_1(x_1) - \frac{\partial C_2}{\partial x_2}\right] f_2(x_{2,w_1w_2}(x_1))\, u\right\},$$

and $\lambda(0) = 0$ by the transversality condition, so $\lambda(x_1) = -\int_0^{x_1}\left[x f_2(x_{2,w_1w_2}(x)) - \frac{\partial C_1}{\partial x_2}\right] f_1(x) + \left[F_1(x) - \frac{\partial C_2}{\partial x_2}\right] f_2(x_{2,w_1w_2}(x))\, u(x)\, dx$. Plugging into equation (2.8),

$$\begin{aligned} & [x_{2,w_1w_2}(x_1) F_1(x_1) - C_2(x_1, x_2)] f_2(x_2) \\ - & \int_0^{x_1}\left[x f_2(x_{2,w_1w_2}(x)) - \frac{\partial C_1}{\partial x_2}\right] f_1(x) + \left[F_1(x) - \frac{\partial C_2}{\partial x_2}\right] f_2(x_2(x))\, u(x)\, dx = 0. \end{aligned}$$

Multiplying both sides by $\overline{w}_1\overline{w}_2 = \frac{1}{f_1(x) f_2(y)}$,

$$[x_1 x_{2,w_1w_2}(x_1) - c_2(x_1, x_{2,w_1w_2}(x_1))] - \int_0^{x_1}\left[x - \frac{\partial c_1}{\partial x_2}\right] + \left[x - \frac{\partial c_2}{\partial x_2}\right] u(x)\, dx = 0.$$

By lemma 9, we get $c_1(x_1, x_{2,w_1w_2}(x_1)) - \int_0^{x_1} x_1 - \frac{\partial c_1}{\partial x_2} + \frac{\partial c_1}{\partial x_2}\frac{dx_{2,w_1w_2}}{dx_1} dx_1 = 0$ for all $x_1 \in [0, \overline{x}_{1,w_1w_2}]$. ■

Lemma 12 *Letting $x_{2,w_1w_2}(x_1)$ be the equilibrium path of the continuation game starting with belief w_1, w_2,*

$$\frac{\partial c_1(x_1, x_{2,w_1w_2}(x_1))}{\partial x_1} + \frac{\partial c_1(x_1, x_{2,w_1w_2}(x_1))}{\partial x_2} = x_1 \text{ for all } x_1 \in [0, \overline{x}_{1,w_1w_2}]. \tag{2.9}$$

Proof. Since equation (2.7) holds for all $x_1 \in [0, \overline{x}_{1,w_1w_2}]$, taking derivative with respect to x_1 on both sides, we get

$$\frac{\partial c_1}{\partial x_1} + \frac{\partial c_1}{\partial x_2} u = x_1 - \frac{\partial c_1}{\partial x_2} + \frac{\partial c_1}{\partial x_2} u.$$

Therefore, $\frac{\partial c_1(x_1, x_{2,w_1w_2}(x_1))}{\partial x_1} + \frac{\partial c_1(x_1, x_{2,w_1w_2}(x_1))}{\partial x_2} = x_1$. ■

Definition 5 *Let $A = \{(x_1, x_{2,w_1w_2}(x_1)) \mid x_1 \in [0, \overline{x}_{1,w_1w_2}], 0 \le w_1 \le \overline{w}_1, 0 \le w_2 \le \overline{w}_2\}$, and $\overline{A}$ is the closure of A.*

A contains all the points on the equilibrium paths of all the continuation games. Note that $(x, x) \in A$, where $x \in [0, \min\{\overline{w}_1, \overline{w}_2\}]$ by Theorem 6.

Lemma 13 *For all* $(x_1, x_2) \in \overline{A}$, $c_1(x_1, x_2)$ *is of the form* $\frac{x_1^2}{2} + \phi(x_2 - x_1)$.

Proof. If (x_1, x_2) is on the equilibrium path of a continuation game so that it is in A, then the differential equation (2.9) has to hold. The general solution to the differential equation is $c_1(x_1, x_2) = \frac{x_1^2}{2} + \phi(x_2 - x_1)$. Since $c_1(x_1, x_2)$ is totally differentiable on $[0, \overline{w}_1] \times [0, \overline{w}_2]$, $c_1(x_1, x_2) = \frac{x_1^2}{2} + \phi(x_2 - x_1)$ for all $(x_1, x_2) \in \overline{A}$. ■

Let $\overline{d} = \max\left\{x_2 - x_1 \mid (x_1, x_2) \in \overline{A}\right\}$ and $\underline{d} = \min\left\{x_2 - x_1 \mid (x_1, x_2) \in \overline{A}\right\}$.

Lemma 14 $\phi(d) = 0$ *if* $d \in \left[0, \overline{d}\right]$.

Proof. Let $UL(x_2) = \min\left\{x_1 \in \overline{A}\right\}$ for $x_2 \in [0, \overline{w}_2]$. UL describes the upper-left boundary of $\overline{A}$ and provides information about how $\phi(d)$ is like when $d \in \left[0, \overline{d}\right]$. Let $B_u = \{(x_1, x_2) \mid 0 \leq x_2 \leq \overline{w}_2, 0 \leq x_1 \leq UL(x_2)\}$ be the set above the upper-left boundary. If there exists (a_1, a_2) in the interior of B_u, it is implied that in a continuation game with belief (a_1, a_2), the continuation equilibrium path is $\left(x_1, UL^{-1}(x_1)\right), 0 \leq x_1 \leq a_1$, which means buyer 1 with value a_1 is paired with buyer 2 with value $UL^{-1}(a_1)$, so $c_1(a_1, a_2) = c_1\left(a_1, UL^{-1}(a_1)\right)$. Since we focus on the set of c_i such that $\frac{\partial c_i}{\partial x_j} \geq 0, i, j = 1, 2$, $c_1(a_1, x_2) = c_1(a_1, a_2)$ for all $x_2 \in \left[UL^{-1}(a_1), a_2\right]$, i.e. c_1 is independent of x_2. Besides, $c_1(0, x_2) = 0$, so $c_1(x_1, x_2)$ is of the form $\psi(x_1)$.

For any $d \in \left[0, \overline{d}\right]$, there exists (x_1, x_2) such that $x_2 - x_1 = d$, $(x_1, x_2) \in B_u \cap \overline{A}$, that is, (x_1, x_2) is on the upper-left boundary of $\overline{A}$. Therefore, $c_1(x_1, x_2)$ is of the form $\psi(x_1)$ as well as $\frac{x_1^2}{2} + \phi(x_2 - x_1)$. Since we require $\frac{\partial c_1}{\partial x_2}$ to be continuous, $\frac{\partial c_1(x_1, x_2)}{\partial x_2} = \phi'(x_2 - x_1) = \phi'(d) = \frac{d\psi(x_1)}{dx_2} = 0$. Since $c_1(0, 0) = 0$, $\phi(0) = 0$. So $\phi(d) = 0$ for all $d \in \left[0, \overline{d}\right]$, and $\psi(x_1) = \frac{x_1^2}{2}$ ■

Lemma 15 $\phi(d) = -\frac{(x_2 - x_1)^2}{2}$ *if* $d \in [\underline{d}, 0]$.

Proof. Let $LR(x_1) = \min\left\{x_2 \in \overline{A}\right\}$ for $x_1 \in [0, \overline{w}_1]$. LR describes the lower-right boundary of $\overline{A}$ and provides information about how $\phi(d)$ is like when $d \in [\underline{d}, 0]$. Let $B_l = \{(x_1, x_2) \mid 0 \leq x_1 \leq \overline{w}_1, 0 \leq x_2 \leq LR(x_1)\}$ be the set below the lower-right boundary. If there exists (a_1, a_2) in the interior of B_l, it is implied that in a continuation game with belief (a_1, a_2), the continuation equilibrium path is $\left(LR^{-1}(x_2), x_2\right), 0 \leq x_2 \leq a_2$, which means buyer 2 with value a_2 is paired with buyer 1 with value $LR^{-1}(a_2)$, so $c_2(a_1, a_2) = c_2\left(LR^{-1}(a_2), a_2\right)$. Since we focus on the set of c_i such that $\frac{\partial c_i}{\partial x_j} \geq 0, i, j = 1, 2$, $c_2(x_1, a_2) = c_2(a_1, a_2)$ for all $x_1 \in \left[LR^{-1}(a_2), a_1\right]$. Besides, $c_2(x_1, 0) = 0$, so $c_2(x_1, x_2)$ is of the form $\varphi(x_2)$.

For any $d \in [\underline{d}, 0]$, there exists (x_1, x_2) such that $x_2 - x_1 = d$, $(x_1, x_2) \in B_l$, and $(x_1, x_2) \in \overline{A}$. Therefore, $c_2(x_1, x_2)$ is of the form $\varphi(x_2)$ as well as $x_1 x_2 - \frac{x_1^2}{2} - \phi(x_2 - x_1)$. Since we require $\frac{\partial c_1}{\partial x_1}$ and $\frac{\partial c_1}{\partial x_2}$ to be continuous, $\frac{\partial c_2(x_1,x_2)}{\partial x_1} = x_2 - x_1 + \phi'(x_2 - x_1) = d + \phi'(d) = 0$. Since $c_2(0,0) = 0$, $\phi(0) = 0$. So $\phi(d) = -\frac{(x_2 - x_1)^2}{2}$ for all $d \in [\underline{d}, 0]$, and $\varphi(x_2) = \frac{x_2^2}{2}$. ■

Lemma 16 *In equilibrium,* $C_1(x_1, x_2) = \begin{cases} \frac{1}{\overline{w}_2} \frac{x_1^2}{2}, & \text{if } x_1 \leq x_2 \\ \frac{1}{\overline{w}_2}\left(x_1 x_2 - \frac{x_2^2}{2}\right), & \text{if } x_1 \geq x_2 \end{cases}$,

$C_2(x_1, x_2) = \begin{cases} \frac{1}{\overline{w}_1}\left(x_1 x_2 - \frac{x_1^2}{2}\right), & \text{if } x_1 \leq x_2 \\ \frac{1}{\overline{w}_1} \frac{x_2^2}{2}, & \text{if } x_1 \geq x_2 \end{cases}$, *and* $A = \{(x_1, x_2) \mid x_1 = x_2, 0 \leq x_1 \leq \min\{\overline{w}_1, \overline{w}_2\}\}$.

Proof. From lemmas 14, 15 and the discussion in their proofs, we know that if the equilibrium is smooth, the only possible c_1 is that

$$c_1(x_1, x_2) = \begin{cases} \frac{x_1^2}{2}, & \text{if } x_1 \leq x_2 \\ \left(x_1 x_2 - \frac{x_2^2}{2}\right), & \text{if } x_1 \geq x_2 \end{cases},$$

which implies that the seller's strategy leads to the path $x_{2,w_1 w_2}(x_1) = \begin{cases} x_1, x_1 \in [0, \min\{w_1, w_2\}] \\ w_2, x_1 \in (\min\{w_1, w_2\}, w_1] \end{cases}$,
given the belief (w_1, w_2) about the buyers' values at the beginning of a continuation game. ■

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Chapter 3

Screening with Resale: A Rationale for Selling in Bulk Packages

3.1 Introduction

Retail stores, referred to as warehouse stores and warehouse clubs such as Food 4 Less, Super Saver Foods, Costco, and Sam's Club, sell their goods in bulk packages at discount. The package is usually too large for a consumer to consume all by himself. A buyer must take the package all together and try to resell or share the product with his friends. So, what is the rationale behind the bulk package strategy and how does a seller determine the optimal quantity of goods in a package?

The literature on nonlinear pricing pioneered by Mussa and Rosen (1978) and Maskin and Riley (1984) addresses a problem of how a seller discriminates buyers by offering a quantity-price menu. However, nonlinear pricing works well only when the seller can exclude the possibility of resale among buyers, which is not easy to achieve in reality. Many papers (e.g. Laffont and Martimort (1997, 2000), Jeon and Menicucci (2005), Che and Kim (2006)) thus have been devoted to the mechanism design issue when buyers or agents have costless communication and can form coalitions easily. Che and Kim (2006) show that collusion among agents actually imposes no cost in a large class of circumstances, including both uncorrelated and correlated types of agents. However, the implementation of the mechanism is complex. The final allocation and payment depend on all the agents' reports, and it is not implementable in reality. In reality, firms first determine the quantity of goods per package and post the price. Buyers arrive at the market, see the price, and make their purchase decisions. So under the circumstance of costless resale and limited contractibility between a seller and buyers, how can a seller discriminate buyers?

This paper connects this question with the previous question of finding the rationale behind the bulk package strategy, showing that each of them is the answer to the other. In short, selling in big packages is a way to help a seller discriminate buyers when resale among buyers cannot be prohibited and communication between a seller and buyers is limited. To be more specific, consider the situation when a seller offers big packages to buyers who might have high or low valuations of the product. A buyer buying the big package can resell some units to the other buyers. Therefore, a buyer buying the package directly from the seller and a buyer buying the product from resale may pay different unit prices. We show that in equilibrium, if big packages are offered, it cannot be the case that only buyers with a low value of the product take the package. Among those who take the packages, some must have a high value of the product. We further show that if the buyers do not value the product too differently, buyers with a high value of the product have stronger incentive to buy the package directly from the seller, and buyers with a low value tend to wait and buy from resale. Therefore, through the process of resale, the seller is able to screen the buyers. Moreover, when buyers' valuations are correlated, the seller can incorporate buyers' information and understanding of the other buyers through the process of resale.

In our model, we consider the case when there are two buyers with different values of a product, and each of them demands only one unit of the product. Our packaging problem can be regarded as a variation of Stigler (1963)'s and Adams and Yellen (1976)'s bundling problems. As Adams and Yellen point out, their model can be applied to the situation when a bundle consists of multiple units of the same commodity. In their case, a buyer has different values for different units of the product he consumes, and the seller chooses the size of the bundle sold to a buyer. In our model, different buyers have different valuations for the product, and each of them needs only one unit. The seller chooses the size of the package that can be shared between buyers. However, our problem is much more complex since the purchase decision and resale process involve more than one buyer who has private information. Therefore, when a buyer decides whether to take the package, he has to consider the possibility of selling the extra unit to the other buyer and make an estimate of the expected revenue from resale. With the additional concern, we show that selling a two-unit package can dominate selling single-unit packages, but only when the buyers' valuations are negatively correlated. Hence, the result suggests that a seller might be able to make more profit by selling in big packages when a consumer is likely to know someone valuing the product differently from him and share the big package with the person.

The paper is organized as follows. Section 2 describes the model. Section 3 characterizes the equilibrium and finds conditions for bulk packages to do better than single-unit packages. Section 4 concludes.

3.2 The Model

A monopolist has two units of a product and tries to sell them to two buyers. The monopolist maximizes the total payment he receives from the buyers. Each buyer has a single-unit demand for the product and has utility function $u(x, T; \theta) = \theta x - T$, where θ is the buyer's reservation value for the product, $x \in \{0, 1\}$ is the quantity he consumes, and T is his net payment for the product. θ equals θ_H with probability p_H and equals θ_L with probability p_L, $p_H + p_L = 1$. There is correlation between the two buyers' reservation values, which can be measured by ρ. The probability that both buyers have high value is p_{HH}, the probability that both buyers have low value is p_{LL}, and the probability that buyer 1 has high (respectively low) value and buyer 2 has low (respectively high) value is p_{HL} (respectively p_{LH}). $p_{HL} = p_{LH}$. We can represent p_{HH}, p_{HL}, p_{LL} in the form of

$$\begin{aligned} p_{HH} &= p_H p_H (1 + \frac{p_L}{p_H} \rho), \\ p_{LL} &= p_L p_L (1 + \frac{p_H}{p_L} \rho), \\ p_{HL} &= p_{LH} = p_H p_L (1 - \rho). \end{aligned}$$

We assume that $1 - \frac{1}{\max(p_H, p_L)} < \rho < 1$ so that all the probabilities are well defined.

Assumption 2 $1 - \frac{1}{\max(p_H, p_L)} < \rho < 1$.

The setting above is publicly known. Buyers have private information about their own reservation value. Denote by $p_{t|k}$ the probability that a buyer believes that the other buyer has value θ_t conditional on his own value θ_k.

$$\begin{aligned} p_{H|H} &= p_H + p_L\rho,\ p_{L|H} = p_L(1-\rho), \\ p_{L|L} &= p_L + p_H\rho,\ p_{H|L} = p_H(1-\rho). \end{aligned}$$

The monopolist first commits to some package size and price[1] and announces it to the buyers. The monopolist can choose to offer a two-unit package or two single-unit packages. Buyers then decide whether to buy the package or not. There is no resale cost between the buyers. Without loss of generality, we assume that the package price of a two-unit package has to be higher than $2\theta_L$ because selling two single-unit packages at price θ_L can always do weakly better than selling a two-unit package at a price lower than or equal to $2\theta_L$.

Assumption 3 *If the seller sells a two-unit package, the package price is higher than* $2\theta_L$.

If single-unit packages are offered, resale cannot do a buyer any good since the resale price cannot be higher than the price set by the monopolist. Therefore, each buyer buys the product directly from the seller.

If a two-unit package is offered, buyers who decide to buy go to the seller. If no one goes to the seller, the game ends. If only one buyer goes, he gets the package. If both buyers go, the seller sells the package to the buyer who arrives first. Both buyers have the same probability to be the first buyer. The second buyer arrives after the first buyer leaves and knows that the other buyer has taken the package, but the first buyer does not know whether there will be a second buyer coming. After taking the package, the first buyer can sell the extra unit to the other buyer. We assume that the buyer with the package gets the whole bargaining power and makes a take-it-or-leave-it offer to the other buyer. If the other buyer does not accept, the product perishes, and no one can consume it anymore. (The first buyer can consume his unit before resale.)

[1] If the monopolist cannot commit to a package plan, he might lower the price if he finds no one is taking the package.

3.3 Results

3.3.1 Offering single-unit packages

First we derive the optimal package price and the seller's revenue if the seller decides to offer two single-unit packages. If the price is θ_H, only θ_H type buyers will buy, so the revenue is $2p_H\theta_H$. If the price is θ_L, both buyers will buy, and the revenue is $2\theta_L$. Therefore, we have the following proposition.

Proposition 15 *Given that the monopolist decides to offer single-unit packages, he sets unit price at θ_L if $\theta_L - p_H\theta_H \geq 0$ and sets unit price at θ_H if $\theta_L - p_H\theta_H < 0$.*

3.3.2 Offering a two-unit package

The following example helps us get a better understanding of how offering a two-unit package may benefit the seller.

An example Consider the case where $\rho = -1$. It is known that one buyer has value θ_H, and the other has θ_L. By offering single-unit packages, the seller's profit is $\max\{2\theta_L, \theta_H\}$. If the seller offers a two-unit package with price $\theta_H + \theta_L$, both buyers will be willing to buy the package and resell the extra unit to the other buyer at a price equal to the other buyer's value. Therefore, offering a two-unit package might be more profitable than offering single-unit packages.

The packaging problem in the example is similar to the bundling problem in Stigler (1963) and Adams and Yellen (1976). It thus suggests how selling in big packages might help the seller. However, when ρ is not equal to 1 or -1, a buyer is uncertain about the other buyer's value, so the purchase decision and resale process would involve more complex interactions.

Characterize continuation equilibria If the seller offers a two-unit package, buyers have to determine whether to buy directly from the seller or wait and buy from resale. In this section, we characterize the possible continuation equilibria given that the seller has chosen to sell a two-unit package. We analyze which type of the buyers is more likely to get the package from the seller. Note that we only characterize the continuation equilibria that will occur on the equilibrium path and do not include the continuation equilibria off the equilibrium path.

We first discuss whether a θ_H type or a θ_L type buyer has more incentive to take the package. If a θ_H type buyer gets the package, he can set the resale price at θ_L and the extra unit is sold without fail. Therefore, the package is worth at least $\theta_H + \theta_L$ to a θ_H type buyer. Since the package is worth less than $\theta_L + \theta_H$ to a θ_L type buyer, a θ_H type buyer values the package more than a θ_L type buyer. On the other hand, when a θ_H type buyer expects the resale price to be θ_L with positive probability, he finds buying from resale profitable too, while a θ_L type buyer never gets a positive profit from resale since the resale price is never lower than θ_L. Therefore, a θ_H type buyer can benefit more than a θ_L type buyer both from taking the package and from resale, so it is unclear whether a θ_H type or a θ_L type buyer has more incentive to take the package. Nevertheless, Proposition 16 shows that it cannot occur in equilibrium that only θ_L type buyers are willing to take the package.

Lemma 17 *In equilibrium, if a θ_L type buyer gets the package, the resale price is θ_H.*

Proof. If a θ_L type buyer gets the package and sells the extra unit at θ_L, the package price T is at most $2\theta_L$ so that a θ_L type buyer is willing to take the package. Therefore, with Assumption 3, θ_L cannot be the resale price in equilibrium. ■

Proposition 16 *In equilibrium, if the seller provides a two-unit package, it cannot happen that only θ_L type buyers go to the seller.*

Proof. Suppose that in equilibrium, a θ_L type buyer goes to the seller with probability u_L and waits to buy from resale with probability $(1 - u_L)$, and a θ_H type buyer always buys from resale. By Lemma 17, the resale price is θ_H. For the equilibrium to exist, there are two incentive compatibility constraints that need to be satisfied:

$$(1 - \frac{p_{L|L}u_L}{2})[\theta_L + \frac{p_{H|L}}{1 - \frac{p_{L|L}u_L}{2}}\theta_H - T] = 0, \tag{3.1}$$

and

$$(1 - \frac{p_{L|H}u_L}{2})[\theta_H + \max\left\{\frac{p_{H|H}}{1 - \frac{p_{L|H}u_L}{2}}\theta_H, \theta_L\right\} - T] \leq 0. \tag{3.2}$$

Constraints (3.1) and (3.2) are for θ_L type and θ_H type buyers respectively. The left-hand side

	(a)	(b)	(c)	(d)	(e)	(f)
Go	H	H	HL	HL	HL	HL
Wait	L	HL	L		H	HL

Table 3.1:

is a buyer's expected payoff if he decides to go to the seller, and the right-hand side is a buyer's expected payoff if he waits to buy from resale. If a θ_L type buyer goes to the seller, he gets the package with probability $(1 - \frac{p_{L|L}u_L}{2})$. Given that the resale price is θ_H, the probability that the other buyer buys the extra unit is $\frac{p_{H|L}}{1-\frac{p_{L|L}u_L}{2}}$,[2] so the highest utility he can derive if he gets the package is $\theta_L + \frac{p_{H|L}}{1-\frac{p_{L|L}}{2}}\theta_H - T$. On the other hand, if he does not get the package or if he does not go to the seller, he gets 0 no matter whether the other one takes the package or not. Constraint (3.1) ensures that a θ_L type buyer feels indifferent between going to the seller and waiting to buy from resale. For a θ_H type buyer, if he goes to the seller, he gets the package with probability $(1 - \frac{p_{L|H}u_L}{2})$. After he gets the package, he will set the resale price at θ_H if $\frac{p_{H|H}}{1-\frac{p_{L|H}}{2}}\theta_H \geq \theta_L$ and θ_L if $\frac{p_{H|H}}{1-\frac{p_{L|H}}{2}}\theta_H < \theta_L$. If he does not get the product or if he does not go to the seller, he gets 0 because if the package is taken by the other buyer who is of θ_L type, the resale price is θ_H. Constraint (3.2) ensures that a θ_H type buyer prefers waiting to buy from resale to going to the seller. However, (3.1) and (3.2) cannot be satisfied at the same time, so in equilibrium, it cannot happen that only θ_L type buyers go to the seller. ■

The proposition implies that in equilibrium, θ_H type buyers weakly prefer buying the package directly from the seller to buying from resale, while θ_L type buyers might find buying from resale more favorable. Table 3.1 summarizes all the possible scenarios that occur in equilibrium in which the seller offers a two-unit package.

Proposition 17 characterizes the condition under which scenarios (e) and (f) will not occur in equilibrium. So in equilibrium, a θ_H type buyer is more inclined to take the package than a θ_L type buyer.

[2] From the buyer's perspective, the probability that the other buyer has value θ_L and goes to the seller is $p_{L|L}u_L$. If the buyer decides to go to the seller, with probability $\frac{1}{2}p_{L|L}u_L$, the other buyer arrives before him. Therefore, the ex ante probability that he gets the package is $1 - \frac{p_{L|L}u_L}{2}$, and the probability that the other buyer has value θ_H conditional on that he gets the package is $\frac{p_{H|L}}{1-\frac{p_{L|L}u_L}{2}}$.

Lemma 18 *If scenarios (b), (e), and (f) occur in equilibrium, i.e. a θ_H type buyer chooses not to go to the seller with positive probability, then if a θ_H type buyer gets the package, the resale price is θ_L.*

Proof. If scenario (b) occurs in equilibrium and a θ_H type buyer sets resale price at θ_H, the seller's expected revenue is lower than the expected revenue if he sells two single-unit packages at price θ_H, because only θ_H type buyers consume the product, and the probability that a θ_H type buyer consume the product is less than 1. So the resale price cannot be θ_H in equilibrium.

If scenarios (e) and (f) occur in equilibrium, a θ_L type buyer is willing to take the package. Since a θ_H type buyer values the package more than a θ_L type buyer, his expected payoff must be larger than 0 if he takes the package. Since a θ_H type buyer is also willing to wait to buy from resale in scenarios (e) and (f), he must also get positive expected payoff if buying from resale, so the resale price must be θ_L with positive probability. By Lemma 17, the resale price is θ_H if a θ_L type buyer takes the package in equilibrium. Therefore, the resale price if a θ_H type buyer gets the package is θ_L. ■

Proposition 17 *If $\frac{\frac{1}{2}p_{H|H}u_H}{1-\frac{p_{L|H}+p_{H|H}u_H}{2}}(\theta_H-\theta_L) < \frac{\frac{1}{2}p_{L|L}}{1-\frac{p_{L|L}+p_{H|L}u_H}{2}}\theta_H$ for all $u_H \in (0,1)$, only scenarios (a), (b), (c), and (d) can occur in equilibrium.*

Proof. Suppose scenarios (e) and (f) occur in equilibrium so that a θ_L type buyer goes to the seller with probability $u_L \in (0,1]$, and a θ_H type buyer goes to the seller with probability $u_H \in (0,1)$. By Lemma 17 and Lemma 18, the resale price is θ_L if a θ_H type gets the package and θ_H if a θ_L type gets the package. For the scenarios to occur in equilibrium, the following two incentive compatibility constraints must be satisfied:

$$(1-\frac{p_{L|L}u_L+p_{H|L}u_H}{2})[\theta_L+\frac{p_{H|L}\left(1-\frac{u_H}{2}\right)}{1-\frac{p_{L|L}u_L+p_{H|L}u_H}{2}}\theta_H-T] \geq 0, \tag{3.3}$$

and

$$(1-\frac{p_{L|H}u_L+p_{H|H}u_H}{2})[\theta_H+\theta_L-T]+\frac{p_{H|H}u_H}{2}\left[\theta_H-\theta_L\right] = p_{H|H}u_H\left[\theta_H-\theta_L\right]. \tag{3.4}$$

For scenario (f), the equality holds in (3.3). Constraints (3.3) and (3.4) are for θ_L type and θ_H type buyers respectively. The left-hand side is a buyer's expected payoff if he goes to the seller, and the right-hand side is a buyer's expected payoff if he waits to buy from resale. On the left-hand side of equation (3.4), $(1-\frac{p_{L|H}u_L+p_{H|H}u_H}{2})$ is the probability that a θ_H type buyer gets the package if he goes to the buyer, and $\frac{p_{H|H}u_H}{2}$ is the probability that the other buyer takes the package and sells the extra unit at θ_L. On the right-hand side, $p_{H|H}u_H$ is the probability that the other buyer takes the package and sells the extra unit at θ_L conditional on that the buyer does not go to the seller.

For (3.3) and (3.4) to hold at the same time,

$$\frac{\frac{1}{2}p_{H|H}u_H}{1-\frac{p_{L|H}u_L+p_{H|H}u_H}{2}}(\theta_H-\theta_L) \geq \frac{\left(1-\frac{u_L}{2}\right)p_{L|L}}{1-\frac{p_{L|L}u_L+p_{H|L}u_H}{2}}\theta_H.$$

Since

$$\begin{aligned}
\frac{\frac{1}{2}p_{H|H}u_H}{1-\frac{p_{L|H}u_L+p_{H|H}u_H}{2}}(\theta_H-\theta_L) &\leq \frac{\frac{1}{2}p_{H|H}u_H}{1-\frac{p_{L|H}+p_{H|H}u_H}{2}}(\theta_H-\theta_L) \text{ for } u_L\in(0,1]\,, \\
\frac{\left(1-\frac{u_L}{2}\right)p_{L|L}}{1-\frac{p_{L|L}u_L+p_{H|L}u_H}{2}}\theta_H &\geq \frac{\frac{1}{2}p_{L|L}}{1-\frac{p_{L|L}+p_{H|L}u_H}{2}}\theta_H \text{ for } u_L\in(0,1]\,,
\end{aligned}$$

and

$$\frac{\frac{1}{2}p_{H|H}u_H}{1-\frac{p_{L|H}+p_{H|H}u_H}{2}}(\theta_H-\theta_L) < \frac{\frac{1}{2}p_{L|L}}{1-\frac{p_{L|L}+p_{H|L}u_H}{2}}\theta_H,$$

$\frac{\frac{1}{2}p_{H|H}u_H}{1-\frac{p_{L|H}u_L+p_{H|H}u_H}{2}}(\theta_H-\theta_L) < \frac{(1-\frac{u_L}{2})p_{L|L}}{1-\frac{p_{L|L}u_L+p_{H|L}u_H}{2}}\theta_H$. So (3.3) and (3.4) cannot hold at the same time, and scenarios (e) and (f) cannot occur in equilibrium. ■

Proposition 17 shows that if θ_L is close to θ_H, and when $p_{L|L}$ is larger, in equilibrium, a θ_H type buyer is more willing to take the package than a θ_L type buyer. This is because when θ_L is closer to θ_H, a θ_H type buyer gets less benefit by buying from resale, and when $p_{L|L}$ is larger, a θ_L type buyer's expected revenue from selling the extra unit to the other buyer declines. Therefore, a θ_H type buyer is more willing to take the package than a θ_L type buyer under the two circumstances. Furthermore, one can also check that if $p_H = p_L = \frac{1}{2}$, $\frac{\frac{1}{2}p_{H|H}u_H}{1-\frac{p_{L|H}+p_{H|H}u_H}{2}} < \frac{\frac{1}{2}p_{L|L}}{1-\frac{p_{L|L}+p_{H|L}u_H}{2}}$ for all $u_H\in(0,1)$, so we have the following corollary.

Corollary 2 *If $p_H = p_L = \frac{1}{2}$, only scenarios (a), (b), (c), and (d) can occur in equilibrium.*

3.3.3 Conditions when selling a two-unit package is better

In this subsection, we provide sufficient and necessary conditions for a two-unit package to do better than two single-unit packages. In the following proposition, we show that $\rho < 0$ is a necessary condition for selling a two-unit package to do better than selling two single-unit packages.

Proposition 18 *If selling a two-unit package does better than selling two single-unit packages, then $\rho < 0$ and the seller's revenue decreases in ρ.*

Proof. We consider the six possible scenarios listed in table 3.1. In scenario (a), if a θ_H type buyer gets the package and the resale price is θ_H, the seller's revenue cannot be greater than $2p_H\theta_H$ since only θ_H type buyers get the product and they pay at most θ_H. Therefore, for a two-unit package to do better, the resale price is θ_L. With resale price θ_L, the package price T is $\frac{p_L(1-\rho)}{1-\frac{1}{2}(p_H+p_L\rho)}\theta_H + \frac{1}{1-\frac{1}{2}(p_H+p_L\rho)}\theta_L$, and the seller's expected revenue is $2p_H\left[p_L(1-\rho)\theta_H + \theta_L\right]$, which decreases in ρ. To get $2p_H\left[p_L(1-\rho)\theta_H + \theta_L\right]$ greater than $\max\{2p_H\theta_H, 2\theta_L\}$, the revenue if the seller sells two single-unit packages, we must have $\theta_H(p_H + p_L\rho) < \theta_L < \theta_H p_H(1-\rho)$. $(p_H + p_L\rho) < p_H(1-\rho)$ can only happen when $\rho < 0$.

Next consider scenario (b) in table 3.1. By Lemma 18, the resale price is θ_L, so the package price T is $\frac{1-(p_H+p_L\rho)u_H}{1-\frac{1}{2}(p_H+p_L\rho)u_H}\theta_H + \frac{1}{1-\frac{1}{2}(p_H+p_L\rho)u_H}\theta_L$, where u_H is the probability that a θ_H type buyer goes to the seller in equilibrium, and the seller's expected revenue is $2p_Hu_H\left[(1-(p_H+p_L\rho)u_H)\theta_H + \theta_L\right]$, which decreases in ρ. To get $2p_Hu_H\left[(1-(p_H+p_L\rho)u_H)\theta_H + \theta_L\right] > \max\{2p_H\theta_H, 2\theta_L\}$, we must have

$$p_Hu_H(1-(p_H+p_L\rho)u_H)\theta_H > (1-p_Hu_H)\theta_L$$

and

$$u_H\left[(1-(p_H+p_L\rho)u_H)\theta_H + \theta_L\right] > \theta_H,$$

which together imply

$$1 > \frac{1}{u_H} + (p_H+p_L\rho)u_H - p_H.$$

The right-hand side is minimized when $u_H = 1$ because $p_H + p_L\rho = p_{H|H} < 1$. Assuming $u_H = 1$, then the right hand side, $1 + p_L\rho$, is less than 1 only when $\rho < 0$.

Then consider scenario (c) in table 3.1. A θ_L type buyer feels indifferent between showing willingness and waiting. If he waits, he gets zero payoff, so his expected payoff when going to the seller is also 0. Therefore, the package price is the sum of θ_L and the expected revenue from resale. By Lemma 17, the resale price is θ_H, so the package price T is $\left[\frac{\frac{p_H(1-\rho)}{2}}{1-\frac{p_H(1-\rho)+(p_L+p_H\rho)u_L}{2}}\theta_H + \theta_L\right]$, and the seller's expected revenue is $T\left[1-p_L\left(p_L+p_H\rho\right)\left(1-u_L\right)^2\right]$. Note that T and $\left[1-p_L\left(p_L+p_H\rho\right)\left(1-u_L\right)^2\right]$ both decreases in ρ, so the expected revenue decreases in ρ. To have $T\left[1-p_L\left(p_L+p_H\rho\right)\left(1-u_L\right)^2\right] > \max\left\{2p_H\theta_H, 2\theta_L\right\}$, one necessary condition is

$$\begin{aligned}
&\left(1-p_L\left(p_L+p_H\rho\right)\left(1-u_L\right)^2\right)(1-\rho)\\
> &\left[1+p_L\left(p_L+p_H\rho\right)\left(1-u_L\right)^2\right]\left[2-p_H\left(1-\rho\right)-\left(p_L+p_H\rho\right)u_L\right].
\end{aligned}$$

$\left[2-p_H\left(1-\rho\right)-\left(p_L+p_H\rho\right)u_L\right]$ is minimized when $u_L=1$, so
$\left[2-p_H\left(1-\rho\right)-\left(p_L+p_H\rho\right)u_L\right]\geq 1$, and

$$\left(1-p_L\left(p_L+p_H\rho\right)\left(1-u_L\right)^2\right)(1-\rho) > \left[1+p_L\left(p_L+p_H\rho\right)\left(1-u_L\right)^2\right]. \tag{3.5}$$

Since $(p_L+p_H\rho)>0$, inequality holds only when $\rho<0$. The same argument can be applied to scenario (d).

For scenarios (e) and (f), by Lemmas 17 and 18, if a θ_L type buyer gets the package, the resale price is θ_H, and if a θ_H type buyer gets the package, the resale price is θ_L. This requires

$$\begin{aligned}
&\frac{p_{H|H}\left(1-\frac{u_H}{2}\right)}{1-\frac{p_{L|H}u_L+p_{H|H}u_H}{2}}\theta_H \leq \theta_L \leq \frac{p_{H|L}\left(1-\frac{u_H}{2}\right)}{1-\frac{p_{L|L}u_L+p_{H|L}u_H}{2}}\theta_H\\
\Rightarrow\ &\frac{p_{H|H}}{1-\frac{p_{L|H}u_L+p_{H|H}u_H}{2}} \leq \frac{p_{H|L}}{1-\frac{p_{L|L}u_L+p_{H|L}u_H}{2}},
\end{aligned}$$

The inequality holds only if $p_{H|H}\leq p_{H|L}$, which implies $\rho\leq 0$. By equation (3.4),

$$T=\theta_H+\theta_L-\left(\theta_H-\theta_L\right)\frac{\frac{1}{2}\left(p_H+p_L\rho\right)u_H}{1-\frac{1}{2}\left[\left(p_H+p_L\rho\right)u_H+p_L\left(1-\rho\right)u_L\right]},$$

and the probability that the package is sold is

$$\left[1-p_H\left(p_H+p_L\rho\right)\left(1-u_H\right)^2-p_L\left(p_L+p_H\rho\right)\left(1-u_L\right)^2-2p_Hp_L\left(1-\rho\right)\left(1-u_H\right)\left(1-u_L\right)\right].$$

Both T and the probability decrease in ρ, so the expected revenue also decreases in ρ. ■

The proposition shows that selling a two-unit package does better than selling two single-unit packages only when $\rho < 0$, and the seller can do better by selling a two-unit package when ρ is more negative. To see why this is so, first consider scenarios (a), (b), (e), and (f). In those scenarios, if a θ_H type buyer gets the package, the resale price is θ_L. In scenarios (e) and (f), a θ_L type buyer might also take the package, and the resale price is θ_H. Therefore, a θ_H type buyer can get positive surplus by buying from resale if the other buyer is of θ_H type and gets the package. The package prices in those scenarios are $\theta_H + \theta_L$ minus some positive rent R so that a θ_H type buyer feels indifferent between going to the seller to buy the package and waiting to buy from resale. When ρ is more negative, a θ_H type buyer expects the other buyer to be of θ_H type with lower probability, so the resale price is less likely to be θ_L, the rent R can be lower, and the package price can be higher. Next consider scenarios (c) and (d). We know that if a θ_L type buyer gets the package, the resale price is θ_H in equilibrium, and the extra unit is sold only when the other buyer is of θ_H type. Hence, the package price is at most θ_L plus the expected revenue from selling the extra unit. When ρ is more negative, a θ_L type buyer expects the other buyer to be of θ_H type with higher probability, so the expected revenue from the extra unit is higher, and the seller can set a higher package price. Therefore, with more negative ρ, the seller gets higher revenue by selling a two-unit package.

In the following proposition, we characterize the necessary and sufficient condition for a two-unit package to do better than two single-unit packages given $p_H = p_L = \frac{1}{2}$, and $\theta_H = \theta_L + 1$.

Proposition 19 *Given $p_H = p_L = \frac{1}{2}$, $\theta_L = \theta$, and $\theta_H = \theta + 1$, selling a two-unit package can do better than selling two single-unit packages if and only if $\frac{1+\rho}{1-\rho} < \theta < \frac{1-\rho}{1+\rho}$.*

Proof. We first prove the "if" part. Suppose $\frac{1+\rho}{1-\rho} < \theta < \frac{1-\rho}{1+\rho}$. If the seller offers a two-unit package and sets the package price at $\frac{\frac{1}{2}(1-\rho)}{1-\frac{1}{4}(1+\rho)} + \frac{1+\frac{1}{2}(1-\rho)}{1-\frac{1}{4}(1+\rho)}\theta$, the unique continuation equilibrium is that a θ_H type buyer always goes to the seller, a θ_L type buyer always buys from resale, and a θ_H type buyer sets resale price at θ_L. The probability that the package is taken is $\left(1 - \frac{1}{4}(1+\rho)\right)$, so the seller's revenue is $\theta + \frac{1}{2}(1-\rho)(1+\theta)$, which is greater than $\max\{2\theta, \theta + 1\}$, the seller's revenue if he sells two single-unit packages.

Then we prove the "only if" part. By corollary 2, if selling a two-unit package does better than selling two single-unit packages, only scenarios (a), (b), (c), and (d) can occur. The seller can set different package prices to induce different continuation equilibria. Among all the prices, $\frac{\frac{1}{2}(1-\rho)}{1-\frac{1}{4}(1+\rho)}+\frac{1+\frac{1}{2}(1-\rho)}{1-\frac{1}{4}(1+\rho)}\theta$ and $\theta+\frac{1}{2}(1-\rho)(1+\theta)$ give the seller the maximum revenue, $\theta+\frac{1}{2}(1-\rho)(1+\theta)$.[3] So if selling a two-unit package does better than selling two single-unit packages, $\theta+\frac{1}{2}(1-\rho)(1+\theta)>\max\{2\theta,\theta+1\}$, and this implies $\frac{1+\rho}{1-\rho}<\theta<\frac{1-\rho}{1+\rho}$. ■

Proposition 19 shows that a two-unit package can do better if ρ is close to -1 and if θ is in the middle range. As we discussed before, if ρ is more negative, a θ_H type buyer gets less expected surplus by buying from resale, and a θ_L type buyer gets higher expected revenue from selling the extra unit to the other buyer. Therefore, the buyers are willing to pay a higher package price, and the seller's revenue increases. Proposition 19 also shows that a two-unit package does better if θ is in the middle range. When a two-unit package is offered, there is probability that a θ_L type buyer does not buy the product in equilibrium. If θ is large, the loss caused by θ_L type buyers not buying the product is large, so selling a two-unit package cannot do better than selling two single-unit packages at price θ. On the other hand, when θ is small, a θ_H type buyer gets large expected surplus by buying from resale, so the seller has to set a lower package price such that a θ_H type buyer is willing to take the package. In this case, the seller can just sell two single-unit packages at price $\theta_H=\theta+1$. This will increase the seller's revenue collected from θ_H type buyers. Although the seller also loses the chance to sell the product to θ_L type buyers, the loss is small because θ is small. Therefore, a two-unit package cannot do well when θ is very large or very small.

The propositions suggest that a seller might be able to make more profit by selling in big packages when (i) a consumer is likely to know someone valuing the product differently from him and share a big package together, and (ii) the difference between consumers' valuations of the product is neither too large nor too small.

[3] With package price $\frac{\frac{1}{2}(1-\rho)}{1-\frac{1}{4}(1+\rho)}+\frac{1+\frac{1}{2}(1-\rho)}{1-\frac{1}{4}(1+\rho)}\theta$, in the continuation equilibrium, a θ_H type buyer always goes to the seller and sets resale price at θ_L, and a θ_L type buyer always buys from resale. With package price $\frac{1}{2}(1-\rho)+1+\frac{1}{2}(1-\rho)\theta$, in the continuation equilibrium, both θ_H type and θ_L type buyers go to the seller with probability 1, a θ_H type buyer sets resale price at θ_L, and a θ_L type buyer sets resale price at θ_H.

3.4 Conclusion

This paper provides a rationale for why a seller may package goods in bundles that are too large for a consumer to consume all by himself. We show that selling in bulk packages is an alternative way for the seller to discriminate buyers when resale cannot be excluded among buyers. When bulk packages are offered, in many circumstances, (for instance, when the buyers do not value the product too differently, or when a buyer is equally likely to have a high value or a low value of the product,) buyers with a high value of the product usually have stronger incentive to buy the package directly from the seller, and buyers with a low value tend to wait and buy from resale. Therefore, through the process of resale, the seller is able to screen the buyers. Moreover, when buyers' valuations are correlated, the seller can incorporate buyers' information and understanding of the other buyers through the process of resale. We thus show that the seller can make more profit by selling bulk packages when the buyers' values are more negatively correlated.

The paper takes the first step toward understanding the sales model of warehouse stores and warehouse clubs such as Food 4 Less, Super Saver Foods, Costco, and Sam's Club. Although their original intention of selling in bulk packages is to get further reduction in price and to save marketing and packing costs, we believe that there are other advantages brought through different venues that wait for us to explore.

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