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# E. R. LOVE TYPE LEFT FRACTIONAL INTEGRAL INEQUALITIES

**Abstract.** Here first we derive a general reverse Minkowski integral inequality. Then motivated by the work of E. R. Love [4] on integral inequalities we produce general reverse and direct integral inequalities. We apply these to ordinary and left fractional integral inequalities. The last involve ordinary derivatives, left Riemann-Liouville fractional integrals, left Caputo fractional derivatives, and left generalized fractional derivatives. These inequalities are of Opial type.

**Key words:** *Minkowski integral inequality, Opial inequality, Riemann-Liouville fractional integral, fractional derivatives.*

**2010 Mathematical Subject Classification:** *26A33, 26D10, 26D15.*

**1. Introduction.** This paper deals with ordinary and left fractional integral inequalities. We are motivated by the following results:

**Theorem 1**. *(Hardy's Inequality, integral version [3, Theorem 327]) If $f$ is a complex-valued function in $L^r(0,\infty)$, $\|\cdot\|$ is the $L^r(0,\infty)$ norm and $r>1$, then*

$$\left\| \frac{1}{x} \int_0^x f(t)\,dt \right\| \leqslant \frac{r}{r-1} \|f\|. \tag{1}$$

**Theorem 2**. *[4] If $s \geqslant r \geqslant 1$, $0 \leqslant a < b \leqslant \infty$, $\gamma$ is real, $\omega(x)$ is decreasing and positive in $(a,b)$, $f(x)$ and $H(x,y)$ are measurable and non-negative on $(a,b)$, $H(x,y)$ is homogeneous of degree $-1$,*

$$(Hf)(x) = \int_a^x H(x,y) f(y)\,dy \tag{2}$$

*and*

$$\|f\|_r = \left( \int_a^b f(x)^r x^{\gamma-1} \omega(x)\, dx \right)^{\frac{1}{r}}, \tag{3}$$

*then*

$$\|Hf\|_r \leqslant C \|f\|_s, \tag{4}$$

*where*

$$C = \int_{\frac{a}{b}}^{1} H(1,t)\, t^{-\frac{\gamma}{r}} \left( \int_a^{bt} x^{\gamma-1} \omega(x)\, dx \right)^{\frac{1}{r}-\frac{1}{s}} dt. \tag{5}$$

*Here* $\frac{a}{b}$ *is to mean* $0$ *if* $a = 0$ *or* $b = \infty$ *or both; and* $bt$ *is to mean* $\infty$ *if* $b = \infty$.

An application of Theorem 2 follows:

**Theorem 3**. *[4] If* $p > 0$, $q > 0$, $p + q = r \geqslant 1$, $0 \leqslant a < b \leqslant \infty$, $\gamma < r$, $\omega(x)$ *is decreasing and positive in* $(a, b)$, $f(x)$ *is measurable and non-negative on* $(a, b)$, $I^{\alpha}$ *is the left Riemann-Liouville operator of fractional integration defined by*

$$(I^{\alpha} f)(x) = \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\, dt \quad \text{for } \alpha > 0, \tag{6}$$

$I^0 f(x) = f(x)$, *where* $\Gamma$ *is the gamma function, and* $I^{\beta} f$ *is defined similarly for* $\beta \geqslant 0$, *then*

$$\int_a^b [(I^{\alpha} f)(x)]^p [(I^{\beta} f)(x)]^q x^{\gamma-\alpha p-\beta q-1} \omega(x) dx \leqslant C \int_a^b f(x)^r x^{\gamma-1} \omega(x) dx, \tag{7}$$

*where*

$$C = \left( \frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\alpha+1-\frac{\gamma}{r}\right)} \right)^p \left( \frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\beta+1-\frac{\gamma}{r}\right)} \right)^q. \tag{8}$$

Also Theorem 2 implies Theorem 1 (by [4]), just take $a = 0$, $b = \infty$, $\gamma = 1$, $s = r > 1$, $\omega(x) = 1$ and $H(x, y) = \frac{1}{x}$.

## 2. Main Results.

We start with a general result, see also [2].

**Theorem 4**. *(Reverse Minkowski integral inequality) Let* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *be* $\sigma$*-finite measure spaces and let* $0 < p < 1$. *Here* $f$ *is a* *nonnegative function on* $X \times Y$ *with* $f(x,y) > 0$ *for almost all* $x \in X$*, almost all* $y \in Y$ *and* $\int_Y (f(x,y))^p \, d\nu(y) < \infty$ *for almost all* $x \in X$*.*

*Then*

$$\left( \int_Y \left( \int_X f(x,y) \, d\mu(x) \right)^p d\nu(y) \right)^{\frac{1}{p}} \geqslant \int_X \left( \int_Y (f(x,y))^p \, d\nu(y) \right)^{\frac{1}{p}} d\mu(x), \tag{9}$$

*if left-hand side is finite.*

**Proof.** Notice that $\int_X f(x,y) \, d\mu(x) > 0$, for almost all $y \in Y$ and

$$\int_Y \left( \int_X f(x,y) \, d\mu(x) \right)^p d\nu(y) > 0.$$

We observe that

$$\int_Y \left( \int_X f(x,y) \, d\mu(x) \right)^p d\nu(y) =$$

$$= \int_Y \left[ \left( \int_X f(x,y) \, d\mu(x) \right) \left( \int_X f(x',y) \, d\mu(x') \right)^{p-1} \right] d\nu(y) =$$

$$= \int_Y \left[ \int_X f(x,y) \left( \int_X f(x',y) \, d\mu(x') \right)^{p-1} d\mu(x) \right] d\nu(y) =$$

(by Tonelli's theorem)

$$= \int_X \left[ \int_Y f(x,y) \left( \int_X f(x',y) \, d\mu(x') \right)^{p-1} d\nu(y) \right] d\mu(x) \geqslant$$

(by applying the reverse Hölder's inequality in the inside we get)

$$\geqslant \int_X \left[ \left( \int_Y (f(x,y))^p d\nu(y) \right)^{\frac{1}{p}} \left( \int_Y \left( \int_X f(x',y) d\mu(x') \right)^p d\nu(y) \right)^{\frac{p-1}{p}} \right] d\mu(x) \tag{10}$$

$$= \left[ \int_X \left( \int_Y (f(x,y))^p d\nu(y) \right)^{\frac{1}{p}} d\mu(x) \right] \left[ \int_Y \left( \int_X f(x,y) d\mu(x) \right)^p d\nu(y) \right]^{\frac{p-1}{p}}.$$

Finally, divide both ends of (10) by $\left[ \int_Y \left( \int_X f(x,y)\, d\mu(x) \right)^p d\nu(y) \right]^{\frac{p-1}{p}} > 0$ to obtain (9). □

We continue with a reverse analog of Theorem 2. The proof involves a special kind of variation of reverse Minkowski integral inequality which we establish completely.

We present

**Theorem 5**. *Let $0 < r < 1$, $0 < a < b < \infty$, $f(x)$ and $H(x,y)$ are measurable and non-negative on $(a,b)$, $(a,b)^2$, respectively, $H(x,y)$ is homogeneous of degree $-1$,*

$$(Hf)(x) = \int_a^x H(x,y)\, f(y)\, dy, \tag{11}$$

*and suppose that*

$$\|Hf\|_{r,[a,b]} = \left( \int_a^b (Hf)(x)^r\, dx \right)^{\frac{1}{r}} < \infty, \tag{12}$$

*and $\|f\|_{r,[a,b]}$ is defined similarly.*

*We assume that $H(1,t)\, f(x,t) > 0$, for almost all $t \in \left[\frac{a}{x}, 1\right]$, for almost all $x \in [a,b]$, and $\frac{H(1,t)}{t^{\frac{1}{r}}} \|f\|_{r,[a,bt]} < \infty$, for almost all $t \in \left[\frac{a}{b}, 1\right]$.*

*Then*

$$\|Hf\|_{r,[a,b]} \geqslant \int_{\frac{a}{b}}^1 \frac{H(1,t)}{t^{\frac{1}{r}}} \|f\|_{r,[a,bt]}\, dt. \tag{13}$$

**Proof.** For $a < x < b$ the homogeneity of degree $-1$ of $H$ gives

$$(Hf)(x) = \int_{\frac{a}{x}}^1 H(x,xt)\, f(xt)\, x dt = \int_{\frac{a}{x}}^1 H(1,t)\, f(xt)\, dt,$$

where $t = \frac{y}{x}$. As $a \leqslant y \leqslant x$, then $0 < t \leqslant 1$. We will prove first $(0 < r < 1)$

$$\|Hf\|_{r,[a,b]} = \left( \int_a^b \left( \int_{\frac{a}{x}}^1 H(1,t)\, f(xt)\, dt \right)^r dx \right)^{\frac{1}{r}} \geqslant \tag{14}$$

$$\geqslant \int_{\frac{a}{b}}^{1}\Big(\int_{\frac{a}{t}}^{b} H(1,t)^r f(xt)^r\,dx\Big)^{\frac{1}{r}} dt = (*)\,.$$

Indeed we observe that

$$\int_a^b\Big(\int_{\frac{a}{x}}^{1} H(1,t)\,f(xt)\,dt\Big)^r dx = \int_a^b\Big(\int_0^1 \chi_{[\frac{a}{x},1]}(t)\,H(1,t)\,f(xt)\,dt\Big)^r dx =$$

(where $\chi$ is the characteristic function)

$$= \int_a^b\Big\{\Big(\int_0^1 \chi_{[\frac{a}{x},1]}(t)H(1,t)f(xt)dt\Big)\Big(\int_0^1 \chi_{[\frac{a}{x},1]}(t')H(1,t')f(xt')dt'\Big)^{r-1}\Big\}dx$$

$$= \int_a^b\Big\{\int_0^1 \chi_{[\frac{a}{x},1]}(t)\,H(1,t)\,f(xt)\Big(\int_0^1 \chi_{[\frac{a}{x},1]}(t')\,H(1,t')\,f(xt')\,dt'\Big)^{r-1} dt\Big\}dx$$

$$\overset{\text{(by Tonelli's theorem)}}{=}$$

$$\int_0^1\Big\{\int_{\frac{a}{t}}^{b} \chi_{[\frac{a}{x},1]}(t)\,H(1,t)f(xt)\Big(\int_0^1 \chi_{[\frac{a}{x},1]}(t')\,H(1,t')f(xt')dt'\Big)^{r-1} dx\Big\}dt =$$

$$= \int_{\frac{a}{b}}^{1}\Big[\int_{\frac{a}{t}}^{b} H(1,t)\,f(xt)\Big(\int_{\frac{a}{x}}^{1} H(1,t')\,f(xt')\,dt'\Big)^{r-1} dx\Big]dt \geqslant$$

(by reverse Hölder's inequality applied inside)

$$\geqslant \int_{\frac{a}{b}}^{1}\Big[\Big(\int_{\frac{a}{t}}^{b} H(1,t)f(xt)^r dx\Big)^{\frac{1}{r}}\Big(\int_a^b\Big(\int_{\frac{a}{x}}^{1} H(1,t')f(xt')dt'\Big)^r dx\Big)^{\frac{r-1}{r}}\Big]dt = \quad (15)$$

$$= \Big(\int_{\frac{a}{b}}^{1}\Big(\int_{\frac{a}{t}}^{b} (H(1,t)\,f(xt))^r\,dx\Big)^{\frac{1}{r}} dt\Big)\Big(\int_a^b\Big(\int_{\frac{a}{x}}^{1} H(1,t)\,f(xt)\,dt\Big)^r dx\Big)^{\frac{r-1}{r}}.$$

Clearly here, by the assumptions, it holds $\int_a^b\Big(\int_{\frac{a}{x}}^{1} H(1,t)\,f(xt)\,dt\Big)^r dx > 0$ and all we did they make sense.

Finally, we divide both ends of (15) by $\left(\int\limits_a^b\left(\int\limits_{\frac{a}{x}}^1 H(1,t)f(xt)dt\right)^r dx\right)^{\frac{r-1}{r}} > 0$, to validate (14), which is a particular case of a reverse Minkowski type integral inequality.

By (14) we continue

$$(*) = \int\limits_{\frac{a}{b}}^{1} \frac{H(1,t)}{t^{\frac{1}{r}}} \left( \int\limits_{\frac{a}{t}}^{b} f(xt)^r \, t dx \right)^{\frac{1}{r}} dt = \int\limits_{\frac{a}{b}}^{1} \frac{H(1,t)}{t^{\frac{1}{r}}} \left( \int\limits_{a}^{bt} f(y)^r \, dy \right)^{\frac{1}{r}} dt = \tag{16}$$

$$= \int\limits_{\frac{a}{b}}^{1} \frac{H(1,t)}{t^{\frac{1}{r}}} \|f\|_{r,[a,bt]} dt,$$

proving (13). □

We give a reverse left fractional inequality.

**Corollary 1.** *(to Theorem 5) Let $0 < r < p$ with $r < 1$, $0 < a < b < \infty$, $f$ is measurable and non-negative on $(a,b)$ such that $f(x) > 0$ almost everywhere on $[a,b]$. Here $I^{\alpha}$ is the left fractional Riemann-Liouville integral operator of order $\alpha > 0$, defined by*

$$(I^{\alpha} f)(x) = \int\limits_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t) \, dt, \quad I^0 f(x) = f(x), \tag{17}$$

*$\forall\ x \in [a,b]$, and suppose that $\|x^{-\alpha} I^{\alpha} f(x)\|_{r,[a,b]} < \infty$. We assume that $\|f\|_{r,[a,bt]} < \infty$, for almost all $t \in \left[\frac{a}{b}, 1\right]$. Then*

$$\int\limits_a^b (I^{\alpha} f(x))^p (f(x))^{r-p} x^{-\alpha p} dx \geqslant \frac{1}{\Gamma(\alpha)^p} \left( \int\limits_{\frac{a}{b}}^{1} \frac{(1-t)^{\alpha-1}}{t^{\frac{1}{r}}} \|f\|_{r,[a,bt]} dt \right)^p \|f\|_r^{r-p}. \tag{18}$$

**Proof.** In Theorem 5, let $H(x,y) = \dfrac{(x-y)^{\alpha-1}}{x^{\alpha}\Gamma(\alpha)}$ for $x > y > a$ and $\alpha > 0$, which is homogeneous of degree $-1$. Then $(Hf)(x) = x^{-\alpha} I^{\alpha} f(x)$, and so by (13)

$$\|Hf\|_{r,[a,b]} \geqslant \int_{\frac{a}{b}}^{1} \frac{H(1,t)}{t^{\frac{1}{r}}} \|f\|_{r,[a,bt]}\, dt = \frac{1}{\Gamma(\alpha)} \int_{\frac{a}{b}}^{1} \frac{(1-t)^{\alpha-1}}{t^{\frac{1}{r}}} \|f\|_{r,[a,bt]}\, dt. \tag{19}$$

Here $0 < r < p$, hence $0 < \frac{r}{p} < 1$, also $\frac{r}{r-p} < 0$, and $f(x) > 0$ almost everywhere in $[a,b]$. Next we apply the reverse Hölder's inequality:

$$\int_a^b \left(x^{-\alpha}(I^\alpha f)(x)\right)^p (f(x))^{r-p}\, dx \geqslant$$

$$\geqslant \left(\int_a^b \left(x^{-\alpha}(I^\alpha f)(x)\right)^r dx\right)^{\frac{p}{r}} \left(\int_a^b (f(x))^r\, dx\right)^{\frac{r-p}{r}} =$$

$$= \|Hf\|_{r,[a,b]}^p \|f\|_r^{r-p} \overset{(19)}{\geqslant} \frac{1}{\Gamma(\alpha)^p} \left(\int_{\frac{a}{b}}^{1} \frac{(1-t)^{\alpha-1}}{t^{\frac{1}{r}}} \|f\|_{r,[a,bt]}\, dt\right)^p \|f\|_r^{r-p}, \tag{20}$$

proving the claim. □

Next we present a reverse Opial type [5] inequality.

**Corollary 2.** *(to Corollary 1) Let $0 < r < p$ with $r < 1$, $m \in \mathbb{N}$, $0 < a < b < \infty$, $f : [a,b] \to \mathbb{R}$ such that $f^{(m-1)}$ is an absolutely continuous function over $[a,b]$, where $f(a) = f'(a) = ... = f^{(m-1)}(a) = 0$, and $f^{(m)}$ is non-negative, with $f^{(m)}(x) > 0$ almost everywhere over $[a,b]$. We assume that $\|x^{-m} f(x)\|_{r,[a,b]} < \infty$, and $\|f^{(m)}\|_{r,[a,bt]} < \infty$, a.e. for $t \in \left[\frac{a}{b}, 1\right]$. Then*

$$\int_a^b (f(x))^p \left(f^{(m)}(x)\right)^{r-p} x^{-mp} dx \geqslant$$

$$\geqslant \frac{1}{((m-1)!)^p} \left(\int_{\frac{a}{b}}^{1} \frac{(1-t)^{m-1}}{t^{\frac{1}{r}}} \|f^{(m)}\|_{r,[a,bt]} dt\right)^p \|f^{(m)}\|_r^{r-p}. \tag{21}$$

**Proof.** By Taylor's formula with integral remainder we have

$$f(x) = \int_a^x \frac{(x-t)^{m-1}}{(m-1)!} f^{(m)}(t)\, dt = \left(I^m f^{(m)}\right)(x), \quad \forall\, x \in [a,b] \tag{22}$$

then apply Corollary 1 for $f^{(m)}$. □

We need

**Definition 1.** *Let $\alpha > 0$, $n = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ is the ceiling), $f \in AC^n([a,b])$ (i.e. $f^{(n-1)}$ is absolutely continuous function). The left Caputo fractional derivative is given by*

$$D_{*a}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt \tag{23}$$

*and exists almost everywhere for $x$ in $[a,b]$, $D_{*a}^{0} f = f$, see [1, p. 394].*

We mention

**Corollary 3.** *[1, p. 395] Let $\alpha > 0$, $n = \lceil \alpha \rceil$, $f \in AC^n([a,b])$, and $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$. Then*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} D_{*a}^{\alpha} f(t)\, dt = I^{\alpha} D_{*a}^{\alpha} f(x), \ \ \forall\, x \in [a,b]. \tag{24}$$

We give a reverse left fractional Opial type [1] inequality.

**Corollary 4.** *(to Corollary 1) Let $0 < r < p$ with $r < 1$, $0 < a < b < \infty$, $\alpha > 0$, $n = \lceil \alpha \rceil$, $f \in AC^n([a,b])$, and $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$. Assume here that $D_{*a}^{\alpha} f$ is non-negative over $(a,b)$ such that $D_{*a}^{\alpha} f > 0$ almost everywhere on $[a,b]$. Suppose that $\|x^{-\alpha} f(x)\|_{r,[a,b]} < \infty$ and $\|D_{*a}^{\alpha} f\|_{r,[a,bt]} < \infty$, for almost all $t \in \left[\frac{a}{b}, 1\right]$. Then*

$$\int_a^b (f(x))^p \left(D_{*a}^{\alpha} f(x)\right)^{r-p} x^{-\alpha p} dx \geqslant$$

$$\geqslant \frac{1}{\Gamma(\alpha)^p} \left( \int_{\frac{a}{b}}^1 \frac{(1-t)^{\alpha-1}}{t^{\frac{1}{r}}} \|D_{*a}^{\alpha} f\|_{r,[a,bt]}\, dt \right)^p \|D_{*a}^{\alpha} f\|_r^{r-p}. \tag{25}$$

**Proof.** By Corollaries 1, 3, see also (23). □

We need

**Definition 2.** *[1, pp. 7-8] Let $\nu > 0$, $n := [\nu]$ $[\cdot]$ the integral part, and $\alpha := \nu - n$ $(0 < \alpha < 1)$; $x$, $x_0 \in [a, b] \subset \mathbb{R}$ such that $x \geqslant x_0$, $x_0$ is fixed. Let $f \in C([a, b])$ and define*

$$\left(J_{\nu}^{x_0} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} f(t)\, dt, \quad x_0 \leqslant x \leqslant b, \tag{26}$$

*the left Riemann-Liouville integral. We define the subspace $C_{x_0}^{\nu}([a, b])$ of $C^n([a, b])$ :*

$$C_{x_0}^{\nu}([a, b]) := \left\{ f \in C^n([a, b]) : J_{1-\alpha}^{x_0} D^n f \in C^1([x_0, b]) \right\}. \tag{27}$$

*For $f \in C_{x_0}^{\nu}([a, b])$ we define the left generalized $\nu$-fractional derivative of $f$ over $[x_0, b]$ as*

$$D_{x_0+}^{\nu} f := D J_{1-\alpha}^{x_0} f^{(n)} \quad (f^{(n)} := D^n f). \tag{28}$$

*Notice $D_{x_0+}^{\nu} f \in C([x_0, b])$.*

We also need

**Theorem 6.** *[1, from Theorem 2.1, p. 8] Let $f \in C_{x_0}^{\nu}([a, b])$, $x_0 \in [a, b]$ fixed.*

1) *If $\nu \geqslant 1$, and $f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, then $f(x) = \left(J_{\nu}^{x_0} D_{x_0+}^{\nu} f\right)(x)$, all $x \in [a, b] : x \geqslant x_0$.*
2) *If $0 < \nu < 1$, then $f(x) = \left(J_{\nu}^{x_0} D_{x_0+}^{\nu} f\right)(x)$, all $x \in [a, b] : x \geqslant x_0$.*

*That is, in both cases we have*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} \left(D_{x_0+}^{\nu} f\right)(t)\, dt, \quad x_0 \leqslant x \leqslant b. \tag{29}$$

*If $x_0 = a$, we get*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{a}^{x} (x-t)^{\nu-1} \left(D_{a+}^{\nu} f\right)(t)\, dt = \left(J_{\nu}^{a} D_{a+}^{\nu} f\right)(x), \quad \text{all } a \leqslant x \leqslant b. \tag{30}$$

We give another reverse left fractional Opial type inequality.

**Corollary 1.** *(to Corollary 1) Let* $0 < r < p$ *with* $r < 1$, $0 < a < b < \infty$, $\nu > 0$, $n = [\nu]$; $f \in C_a^\nu([a,b])$, *such that* $f^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$ *for only the case of* $\nu \geqslant 1$. *Assume here that* $D_{a+}^\nu f$ *is non-negative over* $(a, b)$ *such that* $D_{a+}^\nu f > 0$ *almost everywhere on* $[a, b]$. *Suppose that* $\|x^{-\nu} f(x)\|_{r,[a,b]} < \infty$ *and* $\|D_{a+}^\nu f\|_{r,[a,bt]} < \infty$, *for almost all* $t \in \left[\frac{a}{b}, 1\right]$. *Then*

$$\int_a^b (f(x))^p \left(D_{a+}^\nu f(x)\right)^{r-p} x^{-\nu p} dx \geqslant$$

$$\geqslant \frac{1}{\Gamma(\nu)^p} \left( \int_{\frac{a}{b}}^1 \frac{(1-t)^{\nu-1}}{t^{\frac{1}{r}}} \left\|D_{a+}^\nu f\right\|_{r,[a,bt]} dt \right)^p \left\|D_{a+}^\nu f\right\|_r^{r-p}. \quad (31)$$

**Proof.** By Corollary 1, Theorem 6, see also (28). □

We need the following representation result.

**Theorem 7.** *[1, p. 395] Let* $\nu \geqslant \overline{\gamma} + 1$, $\overline{\gamma} \geqslant 0$. *Call* $n = \lceil \nu \rceil$, $m := \lceil \overline{\gamma} \rceil$. *Assume* $f \in AC^n([a, b])$, *such that* $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$, *and* $D_{*a}^\nu f \in L_\infty(a, b)$. *Then* $D_{*a}^{\overline{\gamma}} f \in C([a, b])$, $D_{*a}^{\overline{\gamma}} f(x) = I^{m-\overline{\gamma}} f^{(m)}(x)$, *and*

$$D_{*a}^{\overline{\gamma}} f(x) = \frac{1}{\Gamma(\nu - \overline{\gamma})} \int_a^x (x-t)^{\nu-\overline{\gamma}-1} D_{*a}^\nu f(t)\, dt = \left(I^{\nu-\overline{\gamma}} D_{*a}^\nu f\right)(x), \quad (32)$$

$\forall\, x \in [a, b]$.

**Remark 1.** *(to Theorem 7) By Corollary 3 we also have*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_{*a}^\nu f(t)\, dt = \left(I^\nu D_{*a}^\nu f\right)(x), \quad (33)$$

$\forall\, x \in [a, b]$.

It follows left fractional direct Opial type integral inequalities.

**Theorem 8.** *If* $p > 0$, $q > 0$, $p + q = r \geqslant 1$, $0 \leqslant a < b < \infty$, $\gamma < r$, $\omega(x)$ *is decreasing and positive in* $(a, b)$. *Let* $\nu \geqslant \overline{\gamma} + 1$, $\overline{\gamma} \geqslant 0$, *call* $n = \lceil \nu \rceil$, $f \in AC^n([a, b]) : f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$; $D_{*a}^\nu f \in L_\infty(a, b)$, *with* $D_{*a}^\nu f \geqslant 0$ *over* $(a, b)$. *Then*

$$\int_a^b (f(x))^p \left(D_{*a}^{\overline{\gamma}} f(x)\right)^q x^{\gamma-\nu p-(\nu-\overline{\gamma})q-1} \omega(x)\, dx \leqslant$$

$$\leqslant C \int_a^b \left(\left(D_{*a}^{\nu} f\right)(x)\right)^r x^{\gamma-1} \omega(x)\, dx, \tag{34}$$

where

$$C = \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu+1-\frac{\gamma}{r}\right)}\right)^p \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma}+1-\frac{\gamma}{r}\right)}\right)^q. \tag{35}$$

**Proof.** Directly from Theorem 3. Notice that

$$\int_a^b \left[\left(I^{\nu} D_{*a}^{\nu} f\right)(x)\right]^p \left[\left(I^{\nu-\overline{\gamma}} D_{*a}^{\nu} f\right)(x)\right]^q x^{\gamma-\nu p-(\nu-\overline{\gamma})q-1} \omega(x)\, dx \overset{((32),(33))}{=}$$

$$= \int_a^b (f(x))^p \left(D_{*a}^{\overline{\gamma}} f(x)\right)^q x^{\gamma-\nu p-(\nu-\overline{\gamma})q-1} \omega(x)\, dx. \tag{36}$$

So, in applying (7), now instead of $f$ we take $D_{*a}^{\nu} f$. $\square$

**Theorem 9.** *If $p>0$, $q>0$, $p+q=r\geqslant 1$, $0\leqslant a<b<\infty$, $\gamma<r$, $\omega(x)$ is decreasing and positive in $(a,b)$. Let $\nu \geqslant \overline{\gamma_i}+1$, $\overline{\gamma_i}\geqslant 0$, $i=1,2$, call $n=\lceil \nu \rceil$, $f\in AC^n([a,b])$ : $f^{(k)}(a)=0$, $k=0,1,\ldots,n-1$; $D_{*a}^{\nu} f\in L_{\infty}(a,b)$, with $D_{*a}^{\nu} f \geqslant 0$ over $(a,b)$. Then*

$$\int_a^b \left(D_{*a}^{\overline{\gamma_1}} f(x)\right)^p \left(D_{*a}^{\overline{\gamma_2}} f(x)\right)^q x^{\gamma-(\nu-\overline{\gamma_1})p-(\nu-\overline{\gamma_2})q-1} \omega(x)\, dx \leqslant$$

$$\leqslant C^* \int_a^b \left(\left(D_{*a}^{\nu} f\right)(x)\right)^r x^{\gamma-1} \omega(x)\, dx, \tag{37}$$

*where*

$$C^* = \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_1}+1-\frac{\gamma}{r}\right)}\right)^p \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_2}+1-\frac{\gamma}{r}\right)}\right)^q. \tag{38}$$

**Proof.** Use of Theorem 7 and similar to Theorem 8. $\square$

**Corollary 1.** *All as in Theorem 8. Then*

$$\int_a^b \left(D_{*a}^{\overline{\gamma}} f(x)\right)^p \left(\left(D_{*a}^{\nu} f\right)(x)\right)^q x^{\gamma-(\nu-\overline{\gamma})p-1} \omega(x)\, dx \leqslant$$

$$\leqslant \overline{C} \int_a^b \left(\left(D_{*a}^{\nu} f\right)(x)\right)^r x^{\gamma-1} \omega(x)\, dx, \tag{39}$$

*where*

$$\overline{C} = \left( \frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma} + 1 - \frac{\gamma}{r}\right)} \right)^p.$$

**Proof.** By Theorems 3, 7. □

We need

**Remark 2.** *[1, p. 26] Let* $\nu \geqslant \overline{\gamma} + 1$, $\overline{\gamma} \geqslant 0$, $n = [\nu]$, $x_0 \in [a, b]$ *fixed,* $f \in C_{x_0}^{\nu}([a, b]) : f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$. *Then*

$$\left(D_{x_0+}^{\overline{\gamma}} f\right)(x) = \frac{1}{\Gamma(\nu - \overline{\gamma})} \int_{x_0}^{x} (x - t)^{(\nu - \overline{\gamma}) - 1} \left(D_{x_0+}^{\nu} f\right)(t)\, dt, \tag{40}$$

*which is continuous in* $x$ *on* $[x_0, b]$.

We continue with

**Theorem 10.** *If* $p > 0$, $q > 0$, $p + q = r \geqslant 1$, $0 \leqslant a < b < \infty$, $\gamma < r$, $\omega(x)$ *is decreasing and positive in* (a,b). *Let* $\nu \geqslant \overline{\gamma} + 1$, $\overline{\gamma} \geqslant 0$, $n = [\nu]$, $f \in C_a^{\nu}([a, b]) : f^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$. *Assume that* $D_{a+}^{\nu} f \geqslant 0$ *on* $(a, b)$. *Then*

$$\int_a^b \left(D_{a+}^{\overline{\gamma}} f(x)\right)^p \left(D_{a+}^{\nu} f(x)\right)^q x^{\gamma - (\nu - \overline{\gamma})p - 1} \omega(x)\, dx \leqslant$$

$$\leqslant C_1 \int_a^b \left(D_{a+}^{\nu} f(x)\right)^r x^{\gamma-1} \omega(x)\, dx, \tag{41}$$

*where*

$$C_1 = \left( \frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma} + 1 - \frac{\gamma}{r}\right)} \right)^p.$$

**Proof.** By Theorem 3 and see Remark 2. □

We make

**Remark 3.** *By [4], see Theorem 2, let* $s = r \geqslant 1$, $0 \leqslant a < b \leqslant \infty$, $\gamma$ *is real,* $\omega(x)$ *is decreasing and positive in* $(a, b)$, $f(x)$ *and* $H_k(x, y)$

*$(k=1,\dots,n)$ are measurable and non-negative on $(a,b)$, $H_k(x,y)$ is homogeneous of degree $-1$,*

$$(H_k f)(x)=\int_a^b H_k(x,y)\,f(y)\,dy,\quad k=1,\dots,n, \tag{42}$$

*and*

$$\|f\|_r=\left(\int_a^b f(x)^r\,x^{\gamma-1}\omega(x)\,dx\right)^{\frac{1}{r}}, \tag{43}$$

*then*

$$\|H_k f\|_r\leqslant C_k\,\|f\|_r\,, \tag{44}$$

*where*

$$C_k=\int_{\frac{a}{b}}^{1} H_k(1,t)\,t^{-\frac{\gamma}{r}}dt,\quad k=1,\dots,n. \tag{45}$$

*Here $\frac{a}{b}$ means $0$ if $a=0$ or $b=\infty$ or both; and $bt$ means $\infty$ if $b=\infty$.*

*Let now $p_k>0$ such that $\sum_{k=1}^{n} p_k=r$.*

*We notice the following (apply generalized Hölder's inequality)*

$$\int_a^b \prod_{k=1}^{n}(H_k f(x))^{p_k}\,x^{\gamma-1}\omega(x)\,dx\leqslant\prod_{k=1}^{n}\left(\int_a^b (H_k f(x))^r\,x^{\gamma-1}\omega(x)\,dx\right)^{\frac{p_k}{r}}=$$

$$=\prod_{k=1}^{n}\|H_k f\|_r^{p_k}\overset{(44)}{\leqslant}\prod_{k=1}^{n}C_k^{p_k}\,\|f\|_r^{p_k}=\left(\prod_{k=1}^{n}C_k^{p_k}\right)\|f\|_r^{\sum_{k=1}^{n}p_k}= \tag{46}$$

$$=\left(\prod_{k=1}^{n}C_k^{p_k}\right)\|f\|_r^{r}=\widehat{C}\left(\int_a^b f(x)^r\,x^{\gamma-1}\omega(x)\,dx\right),$$

*where*

$$\widehat{C}:=\prod_{k=1}^{n}C_k^{p_k}. \tag{47}$$

We have proved

**Theorem 11**. *Let $0 \leqslant a < b \leqslant \infty$, $\gamma$ is real, $\omega(x)$ is decreasing and positive in $(a,b)$, $f(x)$ and $H_k(x,y)$ $(k = 1,\dots,n)$ are measurable and non-negative on $(a,b)$, $H_k(x,y)$ is homogeneous of degree $-1$,*

$$(H_k f)(x) = \int_a^x H_k(x,y) f(y)\, dy, \quad k = 1,\dots,n. \tag{48}$$

*Let $p_k > 0 : \sum_{k=1}^{n} p_k = r \geqslant 1$. Then*

$$\int_a^b \prod_{k=1}^{n} (H_k f(x))^{p_k} x^{\gamma-1} \omega(x)\, dx \leqslant \widehat{C} \Big( \int_a^b f(x)^r x^{\gamma-1} \omega(x)\, dx \Big), \tag{49}$$

*where*

$$\widehat{C} = \prod_{k=1}^{n} \Big( \int_{\frac{a}{b}}^{1} H_k(1,t)\, t^{-\frac{\gamma}{r}} dt \Big)^{p_k}. \tag{50}$$

Next we give an application.

**Theorem 12**. *Here $p_k > 0 : \sum_{k=1}^{n} p_k = r \geqslant 1$. Let $0 \leqslant a < b \leqslant \infty$, $\gamma < r$, $\omega(x)$ is decreasing and positive in $(a,b)$, $f(x)$ is measurable and non-negative on (a,b), $I^{\alpha_k}$ is the left Riemann-Liouville operator of fractional integration defined by*

$$(I^{\alpha_k} f)(x) = \int_a^x \frac{(x-t)^{\alpha_k - 1}}{\Gamma(\alpha_k)} f(t)\, dt, \quad \text{for } \alpha_k > 0, \tag{51}$$

*and*

$$I^{\alpha_k} f(x) = f(x), \text{ for } \alpha_k = 0;\ k = 1,\dots,n.$$

*Then*

$$\int_a^b \prod_{k=1}^{n} ((I^{\alpha_k} f)(x))^{p_k} x^{\gamma - \sum_{k=1}^{n} \alpha_k p_k - 1} \omega(x)\, dx \leqslant \widetilde{C} \Big( \int_a^b f(x)^r x^{\gamma-1} \omega(x)\, dx \Big), \tag{52}$$

*where*

$$\widetilde{C} = \prod_{k=1}^{n} \left( \frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\alpha_k + 1 - \frac{\gamma}{r}\right)} \right)^{p_k}. \tag{53}$$

**Proof.** Here we apply Theorem 11. Inequality (52) derives from (49) directly. We let

$$H_k(x,y) = \frac{(x-y)^{\alpha_k - 1}}{x^{\alpha_k}\Gamma(\alpha_k)} \quad \text{for } x > y > a \text{ and } \alpha_k > 0.$$

Then $H_k f(x) = x^{-\alpha_k} I^{\alpha_k} f(x)$, $k \in \{1, \ldots, n\}$. Notice that

$$\int_{\frac{a}{b}}^{1} H_k(1,t)\, t^{-\frac{\gamma}{r}} dt = \int_{\frac{a}{b}}^{1} \frac{(1-t)^{\alpha_k - 1}}{\Gamma(\alpha_k)} t^{-\frac{\gamma}{r}} dt \leqslant$$

$$\leqslant \int_0^1 \frac{(1-t)^{\alpha_k - 1}}{\Gamma(\alpha_k)} t^{-\frac{\gamma}{r}} dt = \frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\alpha_k + 1 - \frac{\gamma}{r}\right)}, \tag{54}$$

for $k \in \{1, \ldots, n\}$.

Therefore

$$\widetilde{C} = \prod_{k=1}^{n} \left( \frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\alpha_k + 1 - \frac{\gamma}{r}\right)} \right)^{p_k}.$$

□

Next we give general left fractional direct Opial type integral inequalities.

**Theorem 13.** *Here $p_j > 0 : \sum_{j=1}^{N} p_j = r \geqslant 1$. Let $0 \leqslant a < b < \infty$, $\gamma < r$, $\omega(x)$ is decreasing and positive in $(a,b)$. Let $\nu \geqslant \overline{\gamma_j} + 1$, $\overline{\gamma_j} \geqslant 0$, $j = 2, \ldots, N$, $n = \lceil \nu \rceil$, $f \in AC^n([a,b]) : f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$, and $D_{*a}^{\nu} f \in L_\infty(a,b)$, with $D_{*a}^{\nu} f \geqslant 0$ over $(a,b)$. Then*

$$\int_a^b (f(x))^{p_1} \prod_{j=2}^{N} \left( D_{*a}^{\overline{\gamma_j}} f(x) \right)^{p_j} x^{\gamma - \nu p_1 - \sum_{j=2}^{N} (\nu - \overline{\gamma_j}) p_j - 1} \omega(x)\, dx \leqslant$$

$$\leqslant \left[ \left( \frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu + 1 - \frac{\gamma}{r}\right)} \right)^{p_1} \prod_{j=2}^{N} \left( \frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma_j} + 1 - \frac{\gamma}{r}\right)} \right)^{p_j} \right] \times$$

$$\times \left( \int_a^b \left( D_{*a}^{\nu} f(x) \right)^r x^{\gamma - 1} \omega(x)\, dx \right). \tag{55}$$

**Proof.** By Theorem 12, use of Theorem 7 and (33). ☐

**Theorem 14.** *All as in Theorem 13. Then*

$$\int_a^b \left(D_{*a}^{\nu} f(x)\right)^{p_1} \prod_{j=2}^{N} \left(D_{*a}^{\overline{\gamma_j}} f(x)\right)^{p_j} x^{\gamma - \sum_{j=2}^{N} (\nu - \overline{\gamma_j}) p_j - 1} \omega(x)\, dx \leqslant$$

$$\leqslant \left(\prod_{j=2}^{N} \left(\frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma_j} + 1 - \frac{\gamma}{r}\right)}\right)^{p_j}\right) \left(\int_a^b \left(D_{*a}^{\nu} f(x)\right)^r x^{\gamma - 1} \omega(x)\, dx\right). \quad (56)$$

**Proof.** By Theorem 12, use of Theorem 7. ☐

**Corollary 1.** *All as in Theorem 13, and* $\overline{\gamma_2} = \overline{\gamma_3} = \ldots = \overline{\gamma_\lambda}$, $\overline{\gamma_{\lambda+1}} = \overline{\gamma_{\lambda+2}} = \ldots = \overline{\gamma_\mu}$, *and* $\overline{\gamma_{\mu+1}} = \overline{\gamma_{\mu+2}} = \ldots = \overline{\gamma_N}$. *Then*

$$\int_a^b \left(D_{*a}^{\nu} f(x)\right)^{p_1} \left(D_{*a}^{\overline{\gamma_\lambda}} f(x)\right)^{\sum_{j=2}^{\lambda} p_j} \left(D_{*a}^{\overline{\gamma_\mu}} f(x)\right)^{\sum_{j=\lambda+1}^{\mu} p_j} \left(D_{*a}^{\overline{\gamma_N}} f(x)\right)^{\sum_{j=\mu+1}^{N} p_j} \times$$

$$\times x^{\gamma - (\nu - \overline{\gamma_\lambda})\left(\sum_{j=2}^{\lambda} p_j\right) - (\nu - \overline{\gamma_\mu})\left(\sum_{j=\lambda+1}^{\mu} p_j\right) - (\nu - \overline{\gamma_N})\left(\sum_{j=\mu+1}^{N} p_j\right) - 1} \omega(x)\, dx \leqslant$$

$$\leqslant \left(\frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma_\lambda} + 1 - \frac{\gamma}{r}\right)}\right)^{\sum_{j=2}^{\lambda} p_j} \left(\frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma_\mu} + 1 - \frac{\gamma}{r}\right)}\right)^{\sum_{j=\lambda+1}^{\mu} p_j} \times$$

$$\times \left(\frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma_N} + 1 - \frac{\gamma}{r}\right)}\right)^{\sum_{j=\mu+1}^{N} p_j} \left(\int_a^b \left(D_{*a}^{\nu} f(x)\right)^r x^{\gamma - 1} \omega(x)\, dx\right). \quad (57)$$

**Proof.** By Theorem 14. ☐

We finish with

**Theorem 15.** *Here* $p_j > 0 : \sum_{j=1}^{N} p_j = r \geqslant 1$. *Let* $0 \leqslant a < b < \infty$, $\gamma < r$, $\omega(x)$ *is decreasing and positive in* $(a, b)$. *Let* $\nu \geqslant \overline{\gamma_j} + 1$, $\overline{\gamma_j} \geqslant 0$, $j = 2, \ldots, N$, $n = [\nu]$, $f \in C_a^{\nu}([a,b]) : f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$. *Assume that* $D_{a+}^{\nu} f \geqslant 0$ *on* $(a, b)$. *Then*

$$\int_a^b \left(D_{a+}^{\nu} f(x)\right)^{p_1} \prod_{j=2}^{N} \left(D_{a+}^{\overline{\gamma_j}} f(x)\right)^{p_j} x^{\gamma - \sum_{j=2}^{N} (\nu - \overline{\gamma_j}) p_j - 1} \omega(x)\, dx \leqslant$$

$$\leqslant \left( \prod_{j=2}^{N} \left( \frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_j}+1-\frac{\gamma}{r}\right)} \right)^{p_j} \right) \left( \int_a^b \left(D_{a+}^{\nu} f(x)\right)^r x^{\gamma-1} \omega(x)\, dx \right). \quad (58)$$

**Proof.** By Theorem 12, use of (40). □

**Comment.** With the exhibited methods above one can derive all kinds of variation of left fractional Opial type integral inequalities, as well as of ordinary differentiation ones, due to lack of space we omit this task.

## References

[1] Anastassiou G. A. *Fractional Differentiation Inequalities.* Springer, Heidelberg, New York, 2009.
DOI: https://doi.org/10.1007/978-0-387-98128-4

[2] Benaissa B. *On the reverse Minkowski's integral inequality.* Kragujevac Journal of Mathematics, 2022, vol. 46(3), pp. 407 – 416.

[3] Hardy G. H., Littlewood J. E., Pólya G.*Inequalities.*Cambridge, UK, 1934.

[4] Love E. R. *Inequalities like Opial's inequality.* Rocznik naukowo dydaktyczny WSP w Krakowie, Pr. Mat., 1985, vol. 97, pp. 109 – 118.

[5] Opial Z., *Sur une inégalité.* Ann. Polon. Math., 1960, vol. 8, pp. 29 – 32.
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