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FRACTIONAL DERIVATIVE AND ITS APPLICATION IN MATHEMATICS AND PHYSICS

Kh. Namsrai[1]

Institute of Physics and Technology, Mongolian Academy of Sciences,
Ulaanbaatar 51, Mongolia
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We propose fractional derivatives and to study those mathematical and physical consequences. It is shown that fractional derivatives possess noncommutative and nonassociative properties and within which motion of a particle, differential and integral calculuses are investigated.

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[1] n_namsrai@hotmail.com

1 Introduction

Recently, fractional derivatives have played an important role in mathematical methods and their physical and chemical applications (for example, see: Dzrbashan, 1966; Samko, Kilbas and Marichev, 1993 and Hilfer ed. 2000). In this paper, we consider mathematical operations similar to the star product in the noncommutative field theory (see, e.g., Douglas and Nekrasov, 2001; Szabo, 2001; Namsrai, 2003) and study those applications in mathematics and physics. These operations are related to fractional derivatives and fractional integrals. Many attempts (Zabodal, Vilhena and Livotto, 2001; Dattoli, Quattromini and Torre, 1999; Turmetov and Umarov, 1993) (where earlier references concerning this problem are cited) have been devoted to this problem.

The most usual definition for fractional derivatives consists in a natural extension of the integer - order derivative operators (Dattoli, Quattromini and Torre, 1999) .

Let us consider the following polynomial expansion for an arbitrary function

$$f = \sum_{k=0}^{\infty} a_k x^k \tag{1}$$

The m-th derivative of $f(x)$ is given by

$$\frac{d^m f}{dx^m} = \sum_{k=0}^{\infty} a_k \frac{k!}{(k-m)!} x^{k-m}, \quad 0 \leq m \leq k \tag{2}$$

Then its extension is

$$\frac{d^\alpha f}{dx^\alpha} = \sum_{k=0}^{\infty} a_k \frac{k!}{\Gamma(k-\alpha+1)} (x-x_0)^{k-\alpha}, \quad 0 \leq \alpha \leq k \tag{3}$$

where x_0 is called the lower differintegration limit, plays a same role of the lower limit of integration. The fractional derivative of a function does not depend upon the lower limit only when α is a nonnegative integer.

The Riemann - Liouville definition for fractional derivatives instead of (3) is

$$\frac{d^\alpha f}{dx^\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-s)^{-\alpha-1} f(s) ds \tag{4}$$

$Re\, \alpha < 0$, where the lower differintegration limit was taken as zero.

2 Noncommutative and nonassociative properties of the fractional derivatives

We propose here a more simpler mathematical definition for fractional derivatives based on their noncommutative characters:

$$\left(\frac{d^\gamma}{dx^\gamma}\right) \cdot \frac{d^\beta}{dx^\beta} f(x) \neq \left(\frac{d^\beta}{dx^\beta}\right) \cdot \frac{d^\gamma}{dx^\gamma} f(x), \tag{5}$$

$$\frac{d^\gamma}{dx^\gamma}[g(x)+f(x)] \neq \frac{d^\gamma}{dx^\gamma}g(x) + \frac{d^\gamma}{dx^\gamma}f(x), \tag{6}$$

$$\frac{d^\alpha}{dx^\alpha} \cdot \frac{d^\beta}{dx^\beta} g(x) \neq \left(\frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}}\right) g(x) \tag{7}$$

and

$$\frac{d^\gamma}{dx^\gamma}[g(x)f(x)] \neq \left(\frac{d^\gamma}{dx^\gamma} \cdot g(x)\right) \cdot f(x) + g(x)\left(\frac{d^\gamma}{dx^\gamma} \cdot f(x)\right) \tag{8}$$

and so on, for any fractional numbers α, β, γ and any functions $g(x)$ and $f(x)$. The limit α, γ, $\beta \to 1$ gives usual differentiation rule.

To find an explicit form of the fractional derivatives satisfying the properties (5) - (8) one can assume that a function

$$F(\alpha; x) = \frac{d^\alpha}{dx^\alpha} f(x) \tag{9}$$

is an analytic function over the variable α, where $|\alpha| \leq 1$ and depends on the function $f(x)$ and its derivative $df(x) = f'(x)dx$. It is natural to propose boundary conditions:

$$F(0; x) = f(x)$$

and

$$F(1; x) = f'(x) = \frac{d}{dx} f(x) \tag{10}$$

Moreover, use the formal identity

$$\frac{d}{dx} f(x) = \lim_{\alpha \to 1} \exp\left[\ln \frac{d^\alpha}{dx^\alpha} f(x)\right] =$$

$$= \lim_{\alpha \to 1} \exp\left\{\alpha \ln\left[\frac{d}{dx} f^{1/\alpha}(x)\right]\right\} \tag{11}$$

Theorem.

The fractional derivative (9) satisfying the boundary conditions (10) is given by

$$\frac{d^\rho}{dx^\rho} f(x) = \left(\frac{1}{\rho}\right)^\rho [f(x)]^{1-\rho} \left[\frac{d}{dx} f(x)\right]^\rho \tag{12}$$

for any fractional numbers $0 \leq |\rho| \leq 1$.

Proof. Let us consider the differential product

$$\Lambda = \frac{d}{dx} f(x) \tag{13}$$

and use a formal identity

$$\Lambda = \exp\left\{\ln\left[\frac{d}{dx} f(x)\right]\right\} = \exp\left\{\ln \frac{d}{dx} + \ln f(x)\right\} \tag{14}$$

Notice that identity (14) is not unique. Since due to noncommutative properties

$$f(x)\frac{d}{dx} - \frac{d}{dx} f(x) \neq 0,$$

another identity should be used:

$$\Lambda_c = \exp\left\{\ln f(x) + \ln\frac{d}{dx}\right\} \tag{15}$$

First, let us study this identity by using the standard commutator in the quantum theory:

$$\left[x_i, \frac{1}{i}\frac{\partial}{\partial x_j}\right] = i\delta_{ij} \tag{16}$$

and the Baker-Hausdorf formula. Decomposing the function $f(x)$ into the power series

$$f(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}f^{(n)}(0)$$

and making use of the commutator

$$\left[x^n, \quad \frac{1}{i}\frac{d}{dx}\right] = inx^{n-1}$$

arisen from (16), one gets

$$\left[f(x), \quad \frac{1}{i}\frac{d}{dx}\right] = if'(x) \tag{17}$$

Further, to include unit operator one can rewrite formula (15) in the form

$$\Lambda_c = \exp\left\{ln\frac{d}{dx} - \ln\frac{d}{dx} + \ln f(x) + \ln\frac{d}{dx}\right\}$$

and consider expression

$$E = \widehat{A}^{-1}B\widehat{A}$$

where

$$\widehat{A} = \exp\left[\ln\frac{d}{dx}\right] = \frac{d}{dx}$$

and

$$B(x) = \exp\left[\ln f(x)\right] = f(x)$$

Then, due to the commutator (17) we have

$$E = B(x) - \widehat{A}^{-1}B'(x)$$

Finally, we obtain zero identity

$$\Lambda_c = f'(x) - f'(x) \equiv 0,$$

as it should.

Now we turn to the proof of the above theorem. One can divide the interval $[0, 1]$ by any number of subintervals N with widths $\frac{1}{N}$. Then

$$1 \equiv \sum_{j=1}^{N}[1/N]_j$$

For example, $N=11$, we have 11-sums:

$$1=\frac{1}{11}+\frac{1}{11}+\ldots+\frac{1}{11}$$

We would like to rewrite the identity (14) in the form

$$\Lambda=\exp\left\{\sum_{j=1}^{N}\ln\frac{d^{[\frac{1}{N}]_j}}{dx^{[\frac{1}{N}]_j}}+\sum_{j=1}^{N}\left[\frac{1}{N}\right]_j\ln f(x)\right\}$$

Again collecting corresponding terms step by step one gets

$$\Lambda=\prod_{j=1}^{N}\frac{d^{[\frac{1}{N}]_j}}{dx^{[\frac{1}{N}]_j}}\left[f(x)^{[\frac{1}{N}]_j}\right] \tag{18}$$

It is obvious that this expression takes the form

$$\Lambda=\prod_{j=1}^{N}\left[\frac{d}{dx}f(x)\right]^{[\frac{1}{N}]_j}\equiv\frac{d}{dx}f(x)$$

if and only if the fractional derivative in (18) is given by equation (12), where $\rho=\frac{1}{N}$. Here, we have used the identity

$$\frac{d^\rho}{dx^\rho}f^\rho(x)=\left(\frac{1}{\rho}\right)^\rho[f^\rho]^{1-\rho}\cdot\left[\rho f^{\rho-1}\cdot f'\right]^\rho\equiv\left[f'\right]^\rho$$

Thus, we have proven this theorem.

It can easily be seen that definition (12) satisfies nonassociative and noncommutative relations (5) - (8). For example,

$$\frac{d^{1/4}}{dx^{1/4}}\cdot x^2=(4)^{1/4}(x^2)^{1-1/4}[2x]^{1/4}=2^{3/4}\cdot x^{7/4} \tag{19}$$

Then

$$\frac{d^{1/2}}{dx^{1/2}}\cdot\left(\frac{d^{1/4}}{dx^{1/4}}x^2\right)=2^{1/4}\sqrt{7}x^{5/4} \tag{20}$$

and

$$\frac{d^{1/4}}{dx^{1/4}}\left(\frac{d^{1/2}}{dx^{1/2}}x^2\right)=2^{5/4}3^{1/4}x^{5/4} \tag{21}$$

Therefore

$$\frac{d^{1/2}}{dx^{1/2}}\left(\frac{d^{1/4}}{dx^{1/4}}x^2\right)-\frac{d^{1/4}}{dx^{1/4}}\left(\frac{d^{1/2}}{dx^{1/2}}x^2\right)=2^{1/4}x^{5/4}[7^{1/2}-2\cdot 3^{1/4}] \tag{22}$$

Thus, for the particular case (21) we obtain the noncommutative relation (5). In accordance with the definition (12) one gets by simple differentiation:

$$\frac{d^{1/2}}{dx^{1/2}}[\cos^2 x+\sin^2 x]=\sqrt{2}(\sin^2 x+\cos^2 x)^{1/2}\times$$

$$[-2\sin x\cos x+2\sin x\cos x]^{1/2}=0 \tag{23}$$

On the other hand,

$$\frac{d^{1/2}}{dx^{1/2}}\sin^2 x = 2\sin x\sqrt{\sin x\cos x}$$

and

$$\frac{d^{1/2}}{dx^{1/2}}\cos^2 x = 2i\cos x\sqrt{\sin x\cos x} \tag{24}$$

From Eqs. (23) and (24) we see inequality (6). Direct calculations:

$$\frac{d^{1/2}}{dx^{1/2}}x^n = (2n)^{1/2}x^{n-1/2}$$

and

$$\frac{d^{1/4}}{dx^{1/4}}\frac{d^{1/4}}{dx^{1/4}}x^n = 2n^{1/4}\left(n-\frac{1}{4}\right)^{1/4}x^{n-1/2}$$

show inequality (7) for the given case

$$\frac{d^{1/2}}{dx^{1/2}}x^n \neq \frac{d^{1/4}}{dx^{1/4}}\frac{d^{1/4}}{dx^{1/4}}x^n \tag{25}$$

However, notice that for the function x^n with the power $n = 1/3$ inequality (25) becomes equality. It means that for some cases fractional derivatives obey the usual rule of integer - order differentiation. This situation is called degeneration.

3 Main specific properties of the fractional derivatives

By using the basic rule of the fractional derivative (12) one can generalize differential and integral calculuses for any fractional numbers $|\rho| \leq 1$. Let C be constant then

$$\frac{d^\alpha}{dx^\alpha}CF(x) = \left(\frac{1}{\alpha}\right)^\alpha (CF(x))^{1-\alpha}\left[\frac{d}{dx}(CF(x))\right]^\alpha = C\frac{d^\alpha}{dx^\alpha}F(x) \tag{26}$$

For the fractional derivative case the usual rule of differentiation takes the forms:

$$\frac{d^\alpha}{dx^\alpha}[f(x)\cdot g(x)] = \left(\frac{1}{\alpha}\right)^\alpha [f(x)g(x)]^{1-\alpha}[f'g+fg']^\alpha \tag{27}$$

$$\frac{d^\alpha}{dx^\alpha}\left[\frac{f(x)}{g(x)}\right] = \left(\frac{1}{\alpha}\right)^\alpha f^{1-\alpha}(x)g^{\alpha-1}(x)\left[\frac{f'g-fg'}{g^2}\right]^\alpha \tag{28}$$

$$\frac{d^\alpha}{dx^\alpha}F(\varphi(x)) = \left(\frac{1}{\alpha}\right)^\alpha F^{1-\alpha}(\varphi(x))\left[F'(\varphi(x))\varphi'(x)\right]^\alpha . \tag{29}$$

etc. Here $f' = df/dx$ and $F'(\varphi(x)) = dF(\varphi(x))/d\varphi(x)$.

By definition, multiple fractional differentiation

$$\frac{d^{\alpha_n}}{dx^{\alpha_n}}\cdots\frac{d^{\alpha_2}}{dx^{\alpha_2}}\frac{d^{\alpha_1}}{dx^{\alpha_1}}f(x)g(x) =$$

$$= \left(\frac{1}{\alpha_n}\right)^{\alpha_n}[F_{n-1}(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x)]^{1-\alpha_n}\left[\frac{d}{dx}F_{n-1}(\alpha_1,\alpha_2,\ldots,\alpha_{n-1},x)\right]^{\alpha_n} \tag{30}$$

is taken by the chain rule. Here

$$F_0(x) = f(x)g(x),$$

$$F_1(\alpha_1, x) = \left(\frac{1}{\alpha_1}\right)^{\alpha_1} [f(x)g(x)]^{1-\alpha_1} \left[\frac{d}{dx}(f(x)g(x))\right]^{\alpha_1} =$$

$$= \left(\frac{1}{\alpha_1}\right)^{\alpha_1} [f(x)g(x)]^{1-\alpha_1} \left[f'g + fg'\right]^{\alpha_1},$$

$$F_2(\alpha_1, \alpha_2, x) = \left(\frac{1}{\alpha_2}\right)^{\alpha_2} [F_1(\alpha_1, x)]^{1-\alpha_2} \left[\frac{d}{dx}F_1(\alpha_1, x)\right]^{\alpha_2} \tag{31}$$

and so on. For example,

$$f(x) = x^n,$$

$$\frac{d^\rho}{dx^\rho} f(x) = \left(\frac{1}{\rho}\right)^\rho [x^n]^{1-\rho}[nx^{n-1}]^\rho \equiv F_1(x, \rho),$$

$$\frac{d^\gamma}{dx^\gamma} \cdot \frac{d^\rho}{dx^\rho} f(x) = \left(\frac{1}{\gamma}\right)^\gamma [F_1(x, \rho)]^{1-\gamma} \left[\frac{d}{dx}F_1(x, \rho)\right]^\gamma =$$

$$= \left(\frac{1}{\gamma}\right)^\gamma \left(\frac{n}{\rho}\right)^\rho (n-\rho)^\gamma x^{n-\rho-\gamma} \tag{32}$$

We see that, as it is expected, the case $\gamma = \rho = 1$ in (32) gives the usual result

$$\frac{d^\gamma}{dx^\gamma} \frac{d^\rho}{dx^\rho}|_{\gamma=\rho=1} x^n = n(n-1)x^{n-2}$$

This is the correspondence principle in the fractional derivative prescription.

In conclusion, notice that by using the rule (12) one can form a table of fractional derivatives for elementary functions (Section 5) and moreover from definition (12) it follows

$$\lim_{\varepsilon \to 0} \frac{d^\alpha}{dx^\alpha} f(x)|_{\alpha=1+\varepsilon} = [1 - \varepsilon \ln f(x)] \frac{d}{dx} f(x) \tag{33}$$

where we have used the identity:

$$a^{-\varepsilon} = e^{-\varepsilon \ln a} = 1 - \varepsilon \ln a \tag{34}$$

for any small quantity ε.

4 Study of motion of a particle by using the fractional derivatives

Now let us consider physical consequences of the fractional derivatives and study the motion of the particle in one dimensional space. The Lagrangian of the particle is given by

$$L_\rho = L(v^\rho, x) = \frac{m \cdot (v^\rho)^2}{2} - U(x) \tag{35}$$

where velocity of the particle is defined by means of the fractional derivative with respect to the time variable

$$v^\rho = \frac{d^\rho x}{dt^\rho} = \left(\frac{1}{\rho}\right)^\rho \cdot x^{1-\rho} \cdot \dot{x}^\rho \tag{36}$$

Here $\dot{x}^\rho = [\frac{dx}{dt}]^\rho$. The action is

$$S = \int_{t_1}^{t_2} dt^\rho \cdot L(v^\rho, x) \tag{37}$$

From the action principle one can get the Euler-Lagrange equation:

$$\frac{d^\rho}{dt^\rho}\frac{\partial L}{\partial v^\rho} - \frac{\partial L}{\partial x} = 0 \tag{38}$$

where $\frac{\partial L_\rho}{\partial v^\rho} = m \cdot v^\rho$.

It should be noted that for the general case, instead of (38) one can get a more generalized equation of the type

$$\frac{d^{\rho'}}{dt^{\rho'}}\frac{\partial L}{\partial v^\rho} - \frac{\partial L}{\partial x} = 0 \tag{39}$$

with different fractional powers ρ and ρ'. Then the Newtonian equation acquires the form

$$\left(\frac{1}{\rho'}\right)^{\rho'} (mv^\rho)^{1-\rho'} [m\dot{v}^\rho]^{\rho'} = -\frac{\partial U}{\partial x} \tag{40}$$

where $U(x)$ is some external potential field.

Further, we are interested in cases of $\rho = 1+\varepsilon$, $\rho' = 1-\varepsilon$ or $\rho = 1-\varepsilon$, $\rho' = 1+\varepsilon$, respectively. While other two cases: $\rho = 1-\varepsilon$ and $\rho' = 1-\varepsilon$ or $\rho = 1+\varepsilon$ and $\rho' = 1+\varepsilon$ lead to a wrong dimensional expression like $\varepsilon \ln(\ddot{x}/x)$ in the definition of acceleration $w^{\rho'\rho}$ in the motion of equation obtained by using the fractional derivatives with $\varepsilon \neq 0$. Here ε-is small quantity. It turns out that in the fractional derivative formalism some fundamental constants of dimensions of length and time appear in the theory. The assumption that $\varepsilon \neq 0$ gives rise introducing universal length L and time T:

$$\frac{d^{1-\varepsilon}x}{dx^{1-\varepsilon}} = \left(\frac{1}{1-\varepsilon}\right)^{1-\varepsilon} \cdot x^\varepsilon \Rightarrow \left(\frac{1}{1-\varepsilon}\right)^{1-\varepsilon} \left[1 + \varepsilon \ln \frac{x}{L}\right]$$

or

$$\frac{d^{1-\varepsilon}t}{dt^{1-\varepsilon}} = \left(\frac{1}{1-\varepsilon}\right)^{1-\varepsilon} t^\varepsilon \Rightarrow \left(\frac{1}{1-\varepsilon}\right)^{1-\varepsilon} \left[1 + \varepsilon \cdot \ln \frac{t}{T}\right]$$

These formulas have arisen from pure dimensional argument. Moreover, in our formalism there exist two kinds of velocity of the particle:

1) Current velocity is

$$V = \frac{1}{2}\left[v^{1+\varepsilon} + v^{1-\varepsilon}\right] = v$$

2) Longitudinal velocity is

$$U = \frac{1}{2}\left[v^{1+\varepsilon} - v^{1-\varepsilon}\right] = \varepsilon v \left(\ln \frac{vT}{x} - 1\right)$$

Now we turn to study the equation of motion (40), where the acceleration of the particle is given by

$$w^{\rho'} = \frac{d^{\rho'}}{dt^{\rho'}} v^\rho \equiv \left(\frac{1}{\rho'}\right)^{\rho'} \left(\frac{1}{\rho}\right)^{\rho} x^{1-\rho-\rho'} \dot{x}^\rho \left[(1-\rho)\dot{x} + \rho \frac{x}{\dot{x}} \ddot{x}\right]^{\rho'}$$

Let ρ and ρ' be $\rho = 1 + \varepsilon$ and $\rho' = 1 - \varepsilon$, then in the ε-approximation one gets

$$m\left[\ddot{x} + \varepsilon\left(\ddot{x} - \frac{\dot{x}^2}{x} + \ddot{x}\ln\frac{\dot{x}^2}{x\ddot{x}}\right)\right] = F_x \tag{41}$$

The case $\rho = 1 - \varepsilon$ and $\rho' = 1 + \varepsilon$ gives

$$m\left[\ddot{x} - \varepsilon\left(\ddot{x} - \frac{\dot{x}^2}{x} + \ddot{x}\ln\frac{\dot{x}^2}{x\ddot{x}}\right)\right] = F_x \tag{42}$$

Equation (41) for the case when an external force is zero $F_x = 0$ and $\ddot{x} \neq 0$ has the approximate solution

$$\mathcal{X}_1(t) = X_0 + \frac{\varepsilon}{2}\left[-v_0^x t \ln\frac{X_0}{x_0} + \frac{1}{2}X_0 \ln^2\frac{X_0}{x_0}\right] \tag{43}$$

where $X_0 = x_0 + v_0^x t$. It is natural that this equation satisfies initial conditions:

$$\mathcal{X}_1(t)|_{t=0} = x_0, \quad \frac{d\mathcal{X}_1(t)}{dt}|_{t=0} = v_0^x$$

While the solution of equation (42) with $F_x = 0$ possesses oscillating trajectory

$$\mathcal{X}_2(t) = De^{\sqrt{\frac{\varepsilon}{2}}B}\cos\left(A\sqrt{\frac{\varepsilon}{2}}\right) + Ne^{\sqrt{\frac{\varepsilon}{2}}B}\sin\left(A\sqrt{\frac{\varepsilon}{2}}\right) \tag{44}$$

where

$$D = X_0(t) - \frac{\varepsilon}{2}v_0^x t, \quad N = \sqrt{\frac{\varepsilon}{2}}v_0^x t$$

$$A = \ln\frac{X_0}{x_0} + \frac{\varepsilon}{2}\cdot\frac{v_0^x t}{X_0}\left(-1 + \frac{1}{2}\frac{v_0^x t}{X_0}\right), \quad B = \sqrt{\frac{\varepsilon}{2}}\frac{v_0^x t}{X_0}$$

From the three dimensional version of equation (44) one can observe that in the general case when vectors $\mathbf{R_0} = \{x_0, y_0, z_0\}$ and $\mathbf{V_0} = \{v_0^x, v_0^y, v_0^z\}$ are not collinear, the equation for $\mathbf{R_2} = \{\mathcal{X}_2(t), \mathcal{Y}_2(t), \mathcal{Z}_2(t)\}$ is the equation of ellipse. Indeed, if we introduce skew-angle coordinates and axes x and y which are directed along vectors $\mathbf{R_0} = \{x_0, y_0, z_0\}$ and $\mathbf{V_0}$, respectively, then in the Cartesian coordinates equation of trajectory is defined from the equations

$$x = a(t)\cos(\alpha(t)), \quad y = b(t)\sin(\alpha(t))$$

in the form

$$\frac{x^2}{a^2(t)} + \frac{y^2}{b^2(t)} = 1 \tag{45}$$

where

$$a^2(t) = \lambda(x)X_0^2 + \lambda(y)Y_0^2 + \lambda(z)Z_0^2$$

$$b^2(t) = \frac{\varepsilon}{2}t^2\left[\lambda(x)(v_0^x)^2 + \lambda(y)(v_0^y)^2 + \lambda(z)(v_0^z)^2\right]$$

Here

$$\lambda(x) = \exp\left[2\sqrt{\frac{\varepsilon}{2}}\frac{v_0^x t}{X_0}\right]$$

and so on.

Notice that solution (43) tells us that due to a small fractional power of ε in the differential equation of motion of the particle its trajectory may take some small deviation from the rectilinear one while solution (44) is absolutely new in the free particle motion within the classical mechanics and therefore it may shed light on the origin of very nature of oscillating motion taking place in the inertial reference frames. According to Newton's first law: Consider a body on which no net force acts. If the body is at rest, it will remain at rest. If the body is moving with a constant velocity, it will continue to move along the **almost** rectilinear trajectory (i.e., on a nearly straight line) or it will continue to **oscillate** with almost classical trajectory's amplitudes.

Application of fractional derivatives in relativistic mechanics and quantum physics will be given elsewhere.

Finally, we would like to attempt to find numerical value of the parameter ε in the fractional derivative. Let us consider the problem of the motion of a system consisting of two interacting particles. Potential energy of two interacting particles depends only upon the distance between them, i.e., on an absolute value of difference of their radius vectors. Therefore the Lagrangian function of such system is

$$L = \frac{1}{2}m\dot{\mathbf{r}}_1^2 + \frac{1}{2}M\dot{\mathbf{r}}_2^2 - U(|\mathbf{r_1} - \mathbf{r_2}|)$$

Introduce a vector of mutual distance of both points

$$\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$$

and take an origin of coordinates as the center of inertia, that gives

$$m\mathbf{r_1} + M\mathbf{r_2} = 0, \quad or \quad m\mathbf{v_1} + M\mathbf{v_2} = 0 \tag{46}$$

Taking the fractional derivative from the second equation (46) with respect to time with $\rho = 1-\varepsilon$, one finds

$$\frac{d}{dt}m\mathbf{v_1} = \left(1 - \varepsilon \ln |\frac{m\mathbf{v_1}}{M\mathbf{v_2}}|\right)\mathbf{F} \tag{47}$$

in the approximation of $o(\varepsilon^2)$. Here

$$\mathbf{F} \approx M\dot{\mathbf{v}}_2 \left[\varepsilon \ln\left(\frac{M}{m}|\frac{\dot{\mathbf{v}}_2}{\dot{\mathbf{v}}_1}|\right) - 1\right] \tag{48}$$

is some almost constant force which, it seems, is responsible for inertia in a two particles system.

Thus, one can rewrite Eq.(47) in the given direction for the general case:

$$\frac{d}{dt}mv = F\left[1 - \varepsilon \ln\left(\frac{mv}{Mv_0}\right)\right] \tag{49}$$

where as seen below Mv_0 is a typical constant having common character for given physical systems under consideration. This differential equation possesses simple integration:

$$m\int \frac{dv}{1 - \varepsilon \ln\left(\frac{mv}{Mv_0}\right)} = \int F dt$$

or

$$mv\left\{1+\varepsilon\left[\ln\left(\frac{mv}{Mv_0}\right)-1\right]\right\}=F\cdot t+C \tag{50}$$

Assume $C = 0$, the external force F to be approximately zero and obtain two remarkable solutions of Eq. (50) in the form:

1)

$$v=0 \tag{51}$$

2)

$$v=\left(\frac{Mv_0}{m}\right)\exp\left(\frac{\varepsilon-1}{\varepsilon}\right) \tag{52}$$

The first solution is the well-known Newtonian case, while the second solution is responsible for inertia exactly. The fact that Eq. (49) contains these two phenomena simultaneously is very natural. Our next goal is to find the parameter ε of the fractional derivative with respect to time. On the Earth orbit its centrifugal force is compensated to the sun attraction and therefore one can identify quantities m and v with the Earth mass $m = m_\oplus = 5.98\cdot 10^{27}g$ and its orbital velocity $v = v_\oplus = 29.76\ kms^{-1}$ and M and v_0 to the solar mass $m_\odot = 1.991\cdot 10^{33}g$ and its velocity $v_0 = v_\odot = 220\ kms^{-1}$ around the center of Galaxy, respectively. This case yields the number:

$$\varepsilon_\oplus = 0.0636 \tag{53}$$

Similar calculations for other Planets of the solar system give:

	m_i (in $m_\oplus$)	v_i (kms^{-1})	ε_i	ε_i^1
Mercury	0.053	47.84	0.0551	0.0535
Venus	0.8149	35	0.0635	0.0614
Earth	1	29.76	0.0636	0.0616
Mars	0.107	24.11	0.0551	0.0535
Jupiter	318	13.06	0.0928	0.0885
Saturn	95.22	9.64	0.0814	0.0781
Uranus	15.55	6.80	0.0689	0.0665
Neptun	17.23	5.43	0.0686	0.0663
Pluto	0.9	4.73	~ 0.0498	~ 0.0485
average			0.0665	0.0642

(54)

Here quantities ε_i^1 are obtained by using the assumption that solar velocity with respect to the cosmic background radiation (CBR) is $369.3\ kms^{-1}$. On the other hand, assuming $M = M_G = 2\cdot 10^{44}g$ and $v_0 = v_G = 600\ kms^{-1}$ with respect to CBR and $v = v_\odot$, $m = m_\odot$ one gets

$$\varepsilon_\odot = 0.0366 \quad and \quad \varepsilon_\odot^1 = 0.0373 \tag{55}$$

For the motion of the moon: $M = M_\oplus$, $v_0 = v_\oplus$ and $m = m_m = 7.23\cdot 10^{25}g$, $v = 1.02\ kms^{-1}$ we have

$$\varepsilon_{moon} = 0.114 \tag{56}$$

All the averaging over Galaxy (55), the solar system (54) and moon (56) becomes

$$\varepsilon_{mean} = 0.0633 \tag{57}$$

It means that time has the fractional derivative of the power $1 - \varepsilon = 0.937$. However from (52) it follows that for our bodies moving on the Earth surface quantity ε is not negligible small. For instance, for our car weight $m \sim 10^6 g$ and its velocity $v \sim 0.03\, kms^{-1}$, the parameter $\varepsilon_{homo} = 0.0172$. Therefore we feel the inertial force in our daily life.

Finally, notice that for astronauts travelling in Russian - USA cosmic stanza around the Earth orbit: approximately $m \sim 5 \cdot 10^7 g$, $v \sim 8\, kms^{-1}$ and therefore

$$\varepsilon_{as} \sim 0.02.$$

Moreover, an enormous momentum force due to parameter defect $\varepsilon_{loc} = \varepsilon_{main} - \left(\frac{\varepsilon_{homo}+\varepsilon_{as}}{2}\right)$, which would be acquired by bodies moving near or on the surface of the Earth is compensated by its gravity.

In the sense of an explanation of inertia by means of the fractional derivative of time, Newton and Mach were right in their ideas of the origin of inertia. Newton assumed to link inertial forces with absolute space and time properties, while Mach suggested that inertial forces are more probably generated by the general mass of heavenly bodies. Both these ideas are presented in our approach if the word "absolute" be changed by "fractional derivative".

Now let us turn to calculate fractional derivatives for elementary functions by using the main formula (12).

5 Table of fractional derivatives of elementary functions

No	$f(x)$	$F_\rho(x) = \frac{d^\rho}{dx^\rho} f(x)$
1.	x^n	$\left(\frac{n}{\rho}\right)^\rho x^{n-\rho}$
2.	e^x	$\left(\frac{1}{\rho}\right)^\rho e^x$
3.	$\sin x$	$\left(\frac{1}{\rho}\right)^\rho \sin^{1-\rho} x \cos^\rho x$
4.	$\cos x$	$\left(\frac{-1}{\rho}\right)^\rho \cos^{1-\rho} x \sin^\rho x$
5.	$\tan x$	$\left(\frac{1}{\rho}\right)^\rho \tan^{1-\rho} x \cos^{-2\rho} x$
6.	$\cot x$	$\left(\frac{-1}{\rho}\right)^\rho \cot^{1-\rho} x \sin^{-2\rho} x$
7.	$\sec x$	$\left(\frac{1}{\rho}\right)^\rho \sec^{1-\rho} x \left[\frac{\sin x}{\cos^2 x}\right]^\rho$
8.	$\csc x$	$\left(\frac{1}{\rho}\right)^\rho \csc^{1-\rho} x \left[-\frac{\cos x}{\sin^2 x}\right]^\rho$
9.	a^x	$\left(\frac{1}{\rho}\right)^\rho (a^x)^{1-\rho}(a^x \ln a)^\rho$
10.	$\ln x$	$\left(\frac{1}{\rho}\right)^\rho \ln^{1-\rho} x \cdot x^{-\rho}$
11.	$\log_a x$	$\left(\frac{1}{\rho}\right)^\rho (\log_a x)^{1-\rho}(x \ln a)^{-\rho}$
12.	$\lg x$	$\left(\frac{1}{\rho}\right)^\rho (\lg x)^{1-\rho} \left(\frac{\lg e}{x}\right)^\rho$
13.	$\arcsin x$	$\left(\frac{1}{\rho}\right)^\rho \arcsin^{1-\rho} x(1-x^2)^{-\rho/2}$
14.	$\arccos x$	$\left(\frac{-1}{\rho}\right)^\rho \arccos^{1-\rho} x(1-x^2)^{-\rho/2}$
15.	$\arctan x$	$\left(\frac{1}{\rho}\right)^\rho (\arctan^{1-\rho} x)(1+x^2)^{-\rho}$
16.	$arccotx$	$\left(\frac{-1}{\rho}\right)^\rho arccot^{1-\rho}x(1+x^2)^{-\rho}$
17.	$arcsecx$	$\left(\frac{1}{\rho}\right)^\rho arcsec^{1-\rho}x(x^2-1)^{-\rho/2} \cdot x^{-\rho}$
18.	$arccscx$	$\left(\frac{-1}{\rho}\right)^\rho arccsc^{1-\rho}x(x^2-1)^{-\rho/2} \cdot x^{-\rho}$
19.	$\sinh x$	$\left(\frac{1}{\rho}\right)^\rho \sinh^{1-\rho} x \cosh^\rho x$
20.	$\cosh x$	$\left(\frac{1}{\rho}\right)^\rho \cosh^{1-\rho} x \sinh^\rho x$
21.	$\tanh x$	$\left(\frac{1}{\rho}\right)^\rho \tanh^{1-\rho} x \cosh^{-2\rho} x$
22.	$\coth x$	$\left(\frac{-1}{\rho}\right)^\rho \coth^{1-\rho} x \sinh^{-2\rho} x$
23.	$Arcsinhx$	$\left(\frac{1}{\rho}\right)^\rho Arcsinh^{1-\rho}x(1+x^2)^{-\rho/2}$
24.	$Arccoshx$	$\left(\frac{1}{\rho}\right)^\rho Arccosh^{1-\rho}x(x^2-1)^{-\rho/2}$
25.	$Arctanhx$	$\left(\frac{1}{\rho}\right)^\rho Arctanh^{1-\rho}x(1-x^2)^{-\rho}$
26.	$Arccothx$	$\left(\frac{-1}{\rho}\right)^\rho Arccoth^{1-\rho}x(1-x^2)^{-\rho}$

This table may be extended for any complicated and special functions like the Gamma function $\Gamma(x)$ and cylindrical ones:

$$\frac{d^\alpha}{dx^\alpha}\Gamma(x) = \left(\frac{1}{\alpha}\right)^\alpha \Gamma^{1-\alpha}(x)\left[\frac{d}{dx}\Gamma(x)\right]^\alpha \tag{58}$$

and

$$\frac{d^\alpha}{dx^\alpha}J_\nu(x) = \left(\frac{1}{\alpha}\right)^\alpha J_\nu^{1-\alpha}(x)\left[\frac{d}{dx}J_\nu(x)\right]^\alpha . \tag{59}$$

6 Fractional integration

Fractional integration, as an inverse operation with respect to the fractional differentiation, can be defined as:

$$\int dx^{\alpha} F(\alpha, x) = f(x) + C \tag{60}$$

where C is constant and

$$F(\alpha, x) = \frac{d^{\alpha}}{dx^{\alpha}} f(x). \tag{61}$$

For instance,

$$\int dx^{\rho} \left[\left(\frac{n}{\rho} \right)^{\rho} \cdot x^{n-\rho} \right] = x^n + C. \tag{62}$$

It is easy to form indefinite integrals from elementary functions.

Table of the fractional integrals

No	functions	fractional integrals
1.	x^n	$\int dx^{\rho} \left(\frac{n}{\rho}\right)^{\rho} x^{n-\rho}$
2.	e^x	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} e^x$
3.	$\sin x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} \sin^{1-\rho} x \cos^{\rho} x$
4.	$\cos x$	$\int dx^{\rho} \left(\frac{-1}{\rho}\right)^{\rho} \cos^{1-\rho} x \sin^{\rho} x$
5.	$\tan x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} \tan^{1-\rho} x \cos^{-2\rho} x$
6.	$\cot x$	$\int dx^{\rho} \left(\frac{-1}{\rho}\right)^{\rho} \cot^{1-\rho} x \sin^{-2\rho} x$
7.	$\sec x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} \sec^{1-\rho} x \left[\frac{\sin x}{\cos^2 x}\right]^{\rho}$
8.	$\csc x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} \csc^{1-\rho} x \left[-\frac{\cos x}{\sin^2 x}\right]^{\rho}$
9.	a^x	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} (a^x)^{1-\rho} (a^x \ln a)^{\rho}$
10.	$\ln x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} \ln^{1-\rho} x \cdot x^{-\rho}$
11.	$\log_a x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} (\log_a x)^{1-\rho} (x \ln a)^{-\rho}$
12.	$\lg x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} (\lg x)^{1-\rho} \left(\frac{\lg e}{x}\right)^{\rho}$
13.	$\arcsin x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} \arcsin^{1-\rho} x (1-x^2)^{-\rho/2}$
14.	$\arccos x$	$\int dx^{\rho} \left(\frac{-1}{\rho}\right)^{\rho} \arccos^{1-\rho} x (1-x^2)^{-\rho/2}$
15.	$\arctan x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} (\arctan^{1-\rho} x)(1+x^2)^{-\rho}$
16.	$arccotx$	$\int dx^{\rho} \left(\frac{-1}{\rho}\right)^{\rho} arccot^{1-\rho} x (1+x^2)^{-\rho}$
17.	$arcsecx$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} arcsec^{1-\rho} x (x^2-1)^{-\rho/2} \cdot x^{-\rho}$
18.	$arccscx$	$\int dx^{\rho} \left(\frac{-1}{\rho}\right)^{\rho} arccsc^{1-\rho} x (x^2-1)^{-\rho/2} \cdot x^{-\rho}$
19.	$\sinh x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} \sinh^{1-\rho} x \cosh^{\rho} x$
20.	$\cosh x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} \cosh^{1-\rho} x \sinh^{\rho} x$
21.	$\tanh x$	$\int dx^{\rho} \left(\frac{1}{\rho}\right)^{\rho} \tanh^{1-\rho} x \cosh^{-2\rho} x$
22.	$\coth x$	$\int dx^{\rho} \left(\frac{-1}{\rho}\right)^{\rho} \coth^{1-\rho} x \sinh^{-2\rho} x$

23.	$Arcsinhx$	$\int dx^\rho \left(\frac{1}{\rho}\right)^\rho Arcsinh^{1-\rho}x(1+x^2)^{-\rho/2}$
24.	$Arccoshx$	$\int dx^\rho \left(\frac{1}{\rho}\right)^\rho Arccosh^{1-\rho}x(x^2-1)^{-\rho/2}$
25.	$Arctanhx$	$\int dx^\rho \left(\frac{1}{\rho}\right)^\rho Arctanh^{1-\rho}x(1-x^2)^{-\rho}$
26.	$Arccothx$	$\int dx^\rho \left(\frac{-1}{\rho}\right)^\rho Arccoth^{1-\rho}x(1-x^2)^{-\rho}$

These fractional integrals can be generalized for any functions depending on the fractional parameters ρ.

It is readily seen that the fractional differential dx^ρ satisfies condition

$$\int dx^\rho = x^\rho + C \tag{63}$$

since

$$\frac{d^\rho}{dx^\rho}x^\rho = 1.$$

This is exactly the expected formula which arose from definition (62).

We point out that the integration rule by parts

$$\int du \cdot v = u \cdot v - \int udv$$

is not valid for the fractional integral due to the nonassociative character of the fractional differential:

$$d^\gamma u(x)v(x) \neq d^\gamma u(x) \cdot v(x) + u(x) \cdot d^\gamma v(x)$$

that arose from inequality (8).

Finally, we would like to present basic fractional integrals in correspondence with the integer - order derivative ones for the case $\rho = 1$. These are:

Table 4

No	fractional integrals	functions
1.	$\int dx^\rho \cdot 1$	x^ρ
2.	$\int dx^\rho \cdot x^n$	$\left(\frac{\rho}{n+\rho}\right)^\rho x^{n+\rho}$
3.	$\int dx^\rho e^x$	$\rho^\rho e^x$
4.	$\int dx^\rho \sin^\rho x \cos^{2-2\rho} x$	$(-1)^\rho N \cos^{2-\rho} x$
5.	$\int dx^\rho \cos^\rho x \sin^{2-2\rho} x$	$N \sin^{2-\rho} x$
6.	$\int dx^\rho \tan^\rho x \ln^{2-2\rho} \lvert\cos x\rvert$ $[x \neq (2n+1)\pi/2]$	$(-1)^\rho N \ln^{2-\rho} \lvert\cos x\rvert$
7.	$\int dx^\rho \cot^\rho x \ln^{2-2\rho} \lvert\sin x\rvert$	$N \ln^{2-\rho} \lvert\sin x\rvert,\ [x \neq 2\pi n]$
8.	$\int dx^\rho \cos^{-2\rho} x \tan^{2-2\rho} x$	$N \tan^{2-\rho} x,\ [x \neq (2n+1)\frac{\pi}{2}]$

9. $\int dx^\rho \sin^{-2\rho} x \cot^{2-2\rho} x$ — $(-1)^\rho N cot^{2-\rho} x,\ (x \neq n\pi)$

10. $\int dx^\rho a^x$ — $\rho^\rho \ln^{-\rho} a \cdot a^x,\ (a \neq 1)$

11. $\int dx^\rho \cdot x^{-\rho} \ln^{2-2\rho} |x|$ — $N \ln^{2-\rho} |x|,\ (x \neq 0)$

12. $\int dx^\rho \sinh^\rho x \cosh^{2-2\rho} x$ — $N \cosh^{2-\rho} x$

13. $\int dx^\rho \cosh^\rho x \sin^{2-2\rho} x$ — $N \sinh^{2-\rho} x$

14. $\int dx^\rho \tanh^\rho x \ln^{2-2\rho} (\cosh x)$ — $N \ln^{2-\rho} (\cosh x)$

15. $\int dx^\rho \coth^\rho x \ln^{2-2\rho} |\sinh x|$ $(x \neq 0)$ — $N \ln^{2-\rho} |\sinh x|$

16. $\int dx^\rho \cosh^{-2\rho} x \tanh^{2-2\rho} x$ — $N \tanh^{2-\rho} x$

17. $\int dx^\rho \sinh^{-2\rho} x \coth^{2-2\rho} x$ $(x \neq 0)$ — $(-1)^\rho N \coth^{2-\rho} x$

18. $\int dx^\rho (a^2 + x^2)^{-\rho} \left(\arctan \frac{x}{a}\right)^{2-2\rho}$ $(a \neq 0)$ — $N a^{-\rho} \left(\arctan \frac{x}{a}\right)^{2-\rho}$

19. $\int dx^\rho (a^2 - x^2)^{-\rho} \left(arctanh \frac{x}{a}\right)^{2-2\rho}$ — $N a^{-\rho} \left(arctanh \frac{x}{a}\right)^{2-\rho}$ $(|\ x\ |< a)$

20. $\int dx^\rho (a^2 - x^2)^{-\rho} \ln^{2-2\rho} \left|\frac{a+x}{a-x}\right|$ — $N (2a)^{-\rho} \ln^{2-\rho} \left|\frac{a+x}{a-x}\right|$ $(a \neq 0)$

21. $\int dx^\rho (x^2 - a^2)^{-\rho} \left(arccoth \frac{x}{a}\right)^{2-2\rho}$ — $(-1)^\rho N a^{-\rho} \left(arccoth \frac{x}{a}\right)^{2-\rho}$ $(|\ x\ |> a)$

22. $\int dx^\rho (x^2 - a^2)^{-\rho} \ln^{2-2\rho} \left|\frac{x-a}{x+a}\right|$ — $N (2a)^{-\rho} \ln^{2-\rho} \left|\frac{x-a}{x+a}\right|$ $(a \neq 0)$

23. $\int dx^\rho (a^2 - x^2)^{-\rho/2} \left(\arcsin \frac{x}{a}\right)^{2-2\rho}$ — $N \left(\arcsin \frac{x}{a}\right)^{2-\rho},\ (|\ x\ |< a)$

24. $\int dx^\rho (a^2 + x^2)^{-\rho/2} \left(arcsinh \frac{x}{a}\right)^{2-2\rho}$ — $N \left(arcsinh \frac{x}{a}\right)^{2-\rho}$

25. $\int dx^\rho (a^2 + x^2)^{-\rho/2} \ln^{2-2\rho} (x + \sqrt{a^2 + x^2})$ — $N \ln^{2-\rho} (x + \sqrt{a^2 + x^2})$

26. $\int dx^\rho (x^2 - a^2)^{-\rho/2} \left(arccosh \frac{x}{a}\right)^{2-2\rho}$ — $N \left(arccosh \frac{x}{a}\right)^{2-\rho}$

27. $\int dx^\rho (x^2 - a^2)^{-\rho/2} \ln^{2-2\rho} |x + \sqrt{x^2 - a^2}|$ — $N \ln^{2-\rho} |x + \sqrt{x^2 - a^2}|$ $(|\ x\ |> a)$

28. *where* $N = \left(\frac{\rho}{2-\rho}\right)^\rho$

7 Fractional vector analysis

At the same time, with the definition of the fractional differentiation and fractional integration, it is useful to work with formal "fractional " vector analysis. We hope that this formal prescription may shed light on many physical and mathematical problems of modern science. We know that a vector $\mathbf{a}$ is defined by three numbers $\{a_x, a_y, a_z\}$ with respect to the coordinate system. In our scheme, we can define a fractional vector field $\mathbf{a}^\rho$ as follows:
1.

$$\mathbf{a}^\rho = \{a_x^\rho, a_y^\rho, a_z^\rho\}$$

2. If some of the components of the vector $\mathbf{a}$ is a negative number, say $a_z^0 = -2$, then we arranged ourselves by assumption

$$\mathbf{a}^\rho = \{a_x^\rho, a_y^\rho, -a_z^\rho\} \quad or \quad \mathbf{a}_0^\rho = \{a_x^\rho, a_y^\rho, -2^\rho\}$$

3. The scalar and vector products of two fractional vectors are defined by the usual rule:

$$\mathbf{a}^\rho \cdot \mathbf{b}^\rho = a_x^\rho \cdot b_x^\rho + a_y^\rho \cdot b_y^\rho + a_z^\rho \cdot b_z^\rho \tag{64}$$

and

$$\mathbf{a}^\rho \times \mathbf{b}^\rho = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x^\rho & a_y^\rho & a_z^\rho \\ b_x^\rho & b_y^\rho & b_z^\rho \end{pmatrix} = \mathbf{i}(a_y^\rho b_z^\rho - a_z^\rho b_y^\rho) +$$

$$+\mathbf{j}(a_z^\rho b_x^\rho - a_x^\rho b_z^\rho) + \mathbf{k}(a_x^\rho b_y^\rho - a_y^\rho b_x^\rho) \tag{65}$$

In particular, the fractional radius vector $\mathbf{r}^\rho$ has the components:

$$\mathbf{r}^\rho = (x^\rho, y^\rho, z^\rho)$$

and its length is

$$|\mathbf{r}^\rho| = \sqrt{x^{2\rho} + y^{2\rho} + z^{2\rho}} \tag{66}$$

The above mentioned formal definitions allow us to generalize fractional differential and fractional integral calculuses for vector fields. For instance, linear fractional integral of the vector $\mathbf{a}^\rho$ along a curve L is given by

$$\int\limits_L \mathbf{a}^\rho \cdot d\mathbf{r}^\rho \tag{67}$$

Since

$$\mathbf{a}^\rho \cdot d\mathbf{r}^\rho = a_x^\rho dx^\rho + a_y^\rho dy^\rho + a_z^\rho dz^\rho$$

and therefore

$$\int\limits_L \mathbf{a}^\rho \cdot d\mathbf{r}^\rho = \int\limits_L (a_x^\rho dx^\rho + a_y^\rho dy^\rho + a_z^\rho dz^\rho) \tag{68}$$

Let us calculate the integral

$$\int\limits_L (x^\rho dy^\rho - y^\rho dx^\rho)$$

taken along the reduced circle:

$$x^{2\rho} + y^{2\rho} = R^{2\rho}$$

Coordinate points of this reduced circle can be expressed by the functions of one parameter θ:

$$x^\rho = R^\rho \cos\theta \qquad y^\rho = R^\rho \sin\theta$$

where $0 \leqslant \theta \leqslant 2\pi$. Further, we have for the case $\rho = 1+\varepsilon$, ε is small quantity:

$$dx^\rho = R^\rho \left(\frac{1}{\rho}\right)^\rho \cos^{1-\rho}\theta[-\sin\theta \cdot d\theta]^\rho \approx -R\sin\theta d\theta + \varepsilon R \sin\theta \ln\cos\theta \cdot d\theta$$

and

$$dy^\rho = R^\rho \left(\frac{1}{\rho}\right)^\rho \sin^{1-\rho}\theta[\cos\theta \cdot d\theta]^\rho \approx R\cos\theta d\theta - \varepsilon R \cos\theta \ln\sin\theta \cdot d\theta$$

Here, we have used the formula (33).

Thus,

$$x^\rho dy^\rho - y^\rho dx^\rho|_{\rho=1+\varepsilon} \approx R\cos\theta[R\cos\theta \cdot d\theta - \varepsilon R\cos\theta \ln\sin\theta \cdot d\theta] +$$

$$R\sin\theta[R\sin\theta \cdot d\theta - \varepsilon R \sin\theta \ln\cos\theta \cdot d\theta] =$$

$$= R^2 d\theta - \varepsilon R^2 d\theta(\cos^2\theta \ln\sin\theta + \sin^2\theta \ln\cos\theta),$$

$$\int\limits_L (x^\rho dy^\rho - y^\rho dx^\rho)|_{\rho=1+\varepsilon} = 2\pi R^2 - \varepsilon R^2 I \tag{69}$$

where integral I is given by the formula

$$I = \int\limits_0^{2\pi} d\theta(\cos^2\theta \ln \mid \sin\theta \mid + \sin^2\theta \ln \mid \cos\theta \mid).$$

Linear fractional integral of the fractional vector $\mathbf{a}^\rho = grad^\rho\varphi$ along any curve L connecting two points $M_0(\mathbf{r}_0^\rho)$ and $M_1(\mathbf{a}_1^\rho)$ is equal to difference in values of the function at the points M_1 and M_0.

Indeed,

$$\int\limits_L grad^\rho\varphi \cdot d\mathbf{r}^\rho = \int\limits_L d^\rho\varphi = \varphi^\rho(\mathbf{r}_1^\rho) - \varphi^\rho(\mathbf{r}_0^\rho) =$$

$$= \varphi^\rho(x_1^\rho, y_1^\rho, z_1^\rho) - \varphi^\rho(x_0^\rho, y_0^\rho, z_0^\rho) \tag{70}$$

where the definition of the fractional $grad^\rho$ is

$$grad^\rho\varphi = \mathbf{i}\frac{\partial^\rho}{\partial x^\rho}\varphi + \mathbf{j}\frac{\partial^\rho}{\partial y^\rho}\varphi + \mathbf{k}\frac{\partial^\rho}{\partial z^\rho}\varphi \tag{71}$$

Eq. (70) is based on the remarkable property of the fractional differential

$$\int d^\rho\varphi = \int d\varphi^\rho = \varphi^\rho + C \tag{72}$$

Since

$$\frac{d^\rho \varphi^\rho}{d\varphi^\rho} = \left(\frac{1}{\rho}\right)^\rho (\varphi^\rho)^{1-\rho}[\rho\varphi^{\rho-1}]^\rho = 1 \tag{73}$$

i.e.,

$$d^\rho \varphi = d\varphi^\rho.$$

From Eq. (70) it follows that if φ is a synonymous function then the quantity of the linear fractional integral of $grad^\rho\varphi$ does not depend on the integration path but only on its finite points. In particular, the fractional linear integral over closed curve is equal to zero, since final and initial points of the path coincide here. The case $\rho = 1$ is well known as our usual one.

For instance, a conservative force $\mathbf{F}^\rho$ is gradient of some function U:

$$\mathbf{F}^\rho = grad^\rho U$$

where U is called a force function or potential energy (or simply potential). Thus a "fractional" work A^ρ defines as

$$A^\rho = \int\limits_L \mathbf{F}^\rho \cdot d\mathbf{r}^\rho = U^\rho(\mathbf{r}_1^\rho) - U^\rho(\mathbf{r}_0^\rho) \tag{74}$$

In order to study dynamics of physical systems, by using fractional derivative, it is necessary to define fractional derivative with respect to time. Thus, by definition of fractional derivatives:

$$\frac{d^\rho}{dt^\rho} f(t) = \left(\frac{1}{\rho}\right)^\rho f^{1-\rho}(t)[\dot{f}]^\rho,$$

$$\frac{d^\rho}{dt^\rho}\mathbf{a}(t) = \left(\frac{1}{\rho}\right)^\rho [\mathbf{a}(t)]^{1-\rho}[\dot{\mathbf{a}}(t)]^\rho$$

$$= \left\{\left(\frac{1}{\rho}\right)^\rho a_x^{1-\rho}\dot{a}_x^\rho,\ \left(\frac{1}{\rho}\right)^\rho a_y^{1-\rho}\dot{a}_y^\rho,\ \left(\frac{1}{\rho}\right)^\rho a_z^{1-\rho}\dot{a}_z^\rho\right\}, \tag{75}$$

$$\frac{d^\rho}{dt^\rho}(\mathbf{a}\cdot\mathbf{b}) = \left(\frac{1}{\rho}\right)^\rho (\mathbf{a}\cdot\mathbf{b})^{1-\rho}[\dot{\mathbf{a}}\mathbf{b} + \mathbf{a}\dot{\mathbf{b}}]^\rho \tag{76}$$

$$\mathbf{Q}^\rho = \frac{d^\rho}{dt^\rho}[\mathbf{a}(t)\times\mathbf{c}] = \left(\frac{1}{\rho}\right)^\rho [\mathbf{a}(t)\times\mathbf{c}]^{1-\rho}[\dot{\mathbf{a}}\times\mathbf{c}]^\rho \tag{77}$$

where $\mathbf{c}$ is a constant vector. How to understand the last Eq. (77). Of course, this equation tends to the equation

$$\frac{d}{dt}[\mathbf{a}\times\mathbf{c}] = [\dot{\mathbf{a}}(t)\times\mathbf{c}] \tag{78}$$

for the case $\rho = 1$.

It is easy to verify that Eq. (77) can be written in the correspondence form

$$\mathbf{Q}^\rho = \left(\frac{1}{\rho}\right)^\rho (\mathbf{A})^{1-\rho} \times \mathbf{B}^\rho \tag{79}$$

where

$$(\mathbf{A})^{1-\rho} = \left\{(a_x c_x)^{1-\rho}\dot{a}_x^\rho,\ (a_y c_y)^{1-\rho}\dot{a}_y^\rho,\ (a_z c_z)^{1-\rho}\dot{a}_z^\rho\right\}$$

and

$$\mathbf{B}^\rho = \mathbf{c}^\rho = \{c_x^\rho, c_y^\rho, c_z^\rho\}$$

Thus, in terms of the coordinate system equation (79) takes the form

$$\mathbf{Q}^\rho = \left(\frac{1}{\rho}\right)^\rho \left\{\mathbf{i}[c_z^\rho (a_y c_y)^{1-\rho} \dot{a}_y^\rho - c_y^\rho (a_z c_z)^{1-\rho} \dot{a}_z^\rho] + \right.$$
$$+\mathbf{j}[c_x^\rho (a_z c_z)^{1-\rho} \dot{a}_z^\rho - c_z^\rho (a_x c_x)^{1-\rho} \dot{a}_x^\rho] +$$
$$\left. +\mathbf{k}[c_y^\rho (a_x c_x)^{1-\rho} \dot{a}_x^\rho - c_x^\rho (a_y c_y)^{1-\rho} \dot{a}_y^\rho]\right\} \tag{80}$$

At the limit $\rho = 1$ this equation goes to Eq. (78), as it will be expected. Eqs. (79) and (80) can be used for a more complicated case:

$$\frac{d^\rho}{dt^\rho}[\mathbf{a}(t) \times \mathbf{b}(t)] = \left(\frac{1}{\rho}\right)^\rho [\mathbf{a} \times \mathbf{b}]^{1-\rho}[\dot{\mathbf{a}} \times \mathbf{b} + \mathbf{a} \times \dot{\mathbf{b}}]^\rho \tag{81}$$

Applying Eqs. (79) and (80) for each term of the binomial expansion

$$[\dot{\mathbf{a}} \times \mathbf{b} + \mathbf{a} \times \dot{\mathbf{b}}]^\rho = [\dot{\mathbf{a}} \times \mathbf{b}]^\rho + \rho[\dot{\mathbf{a}} \times \mathbf{b}]^{\rho-1}[\mathbf{a} \times \dot{\mathbf{b}}] + \ldots$$

one can realize the meaning of the expression (81).

1. As an example, consider the dynamics of a material point

$$\frac{d}{dt} m\mathbf{v} = \mathbf{F}$$

By using fractional derivative with respect to time a formal generalization of this equation acquires the form

$$\frac{d^\rho}{dt^\rho} m\mathbf{v} = \left(\frac{1}{\rho}\right)^\rho [m\mathbf{v}]^{1-\rho}[m\dot{\mathbf{v}}]^\rho =$$
$$= \left(\frac{1}{\rho}\right)^\rho [m\mathbf{v}]^{1-\rho}[\mathbf{F}]^\rho \tag{82}$$

Take a short notation

$$\mathbf{N}_F^\rho = \left(\frac{1}{\rho}\right)^\rho [m\mathbf{v}]^{1-\rho}[\mathbf{F}]^\rho$$

and form the vectorial product with the radius vector $\mathbf{r}$:

$$\mathbf{r} \times \mathbf{N}_F^\rho = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ \left(\frac{1}{\rho}\right)^\rho [mv_x]^{1-\rho} F_x^\rho & \left(\frac{1}{\rho}\right)^\rho [mv_y]^{1-\rho} F_y^\rho & \left(\frac{1}{\rho}\right)^\rho [mv_z]^{1-\rho} F_z^\rho \end{pmatrix} =$$
$$= \mathbf{i}T_x + \mathbf{j}T_y + \mathbf{k}T_z \tag{83}$$

where

$$T_x = y\left(\frac{1}{\rho}\right)^\rho [mv_z]^{1-\rho} F_z^\rho - z\left(\frac{1}{\rho}\right)^\rho [mv_y]^{1-\rho} F_y^\rho,$$
$$T_y = z\left(\frac{1}{\rho}\right)^\rho [mv_x]^{1-\rho} F_x^\rho - x\left(\frac{1}{\rho}\right)^\rho [mv_z]^{1-\rho} F_z^\rho,$$

$$T_z = x\left(\frac{1}{\rho}\right)^{\rho}[mv_y]^{1-\rho}F_y^{\rho} - y\left(\frac{1}{\rho}\right)^{\rho}[mv_x]^{1-\rho}F_x^{\rho} \tag{84}$$

These quantities can be written in the equivalent form, for example

$$T_x = \left(\frac{1}{\rho}\right)^{\rho}[mv_z y]^{1-\rho}(yF_z)^{\rho} - \left(\frac{1}{\rho}\right)^{\rho}[mzv_y]^{1-\rho}(zF_y)^{\rho}$$

etc. On the other hand, we use another fractional derivative:

$$\begin{aligned}\frac{d^{\rho}}{dt^{\rho}}[\mathbf{r}\times m\dot{\mathbf{r}}] &= \left(\frac{1}{\rho}\right)^{\rho}[\mathbf{r}\times m\dot{\mathbf{r}}]^{1-\rho}\left[\frac{d}{dt}(\mathbf{r}\times m\dot{\mathbf{r}})\right]^{\rho} \\ &= \left(\frac{1}{\rho}\right)^{\rho}[\mathbf{r}\times m\dot{\mathbf{r}}]^{1-\rho}[\mathbf{r}\times m\ddot{\mathbf{r}}]^{\rho} \qquad (85)\end{aligned}$$

where we have used the identity

$$\dot{\mathbf{r}}\times m\dot{\mathbf{r}} = 0$$

due to their collinearity. Now in accordance with Eqs. (77) and (79) we can rewrite Eq. (85) in the form

$$\frac{d^{\rho}}{dt^{\rho}}[\mathbf{r}\times m\dot{\mathbf{r}}] = \left(\frac{1}{\rho}\right)^{\rho}(\mathbf{A}_1)^{1-\rho}\times\mathbf{B}_1^{\rho} \tag{86}$$

Here

$$\mathbf{A}_1^{1-\rho} = \{(x\cdot m\dot{x})^{1-\rho}x^{\rho},\ (y\cdot m\dot{y})^{1-\rho}y^{\rho},\ (z\cdot m\dot{z})^{1-\rho}z^{\rho}\} \tag{87}$$

and

$$\mathbf{B}_1^{\rho} = \{(m\ddot{x})^{\rho},\ (m\ddot{y})^{\rho},\ (m\ddot{z})^{\rho}\} \tag{88}$$

Thus, Eq. (86) in components becomes

$$\frac{d^{\rho}}{dt^{\rho}}[\mathbf{r}\times m\dot{\mathbf{r}}] = \mathbf{i}R_x + \mathbf{j}R_y + \mathbf{k}R_z \tag{89}$$

where

$$\begin{aligned}R_x &= (y\cdot m\dot{y})^{1-\rho}(my\ddot{z})^{\rho} - (z\cdot m\dot{z})^{1-\rho}(mz\ddot{y})^{\rho}, \\ R_y &= (z\cdot m\dot{z})^{1-\rho}(mz\ddot{x})^{\rho} - (x\cdot m\dot{x})^{1-\rho}(mx\ddot{z})^{\rho}, \\ R_z &= (x\cdot m\dot{x})^{1-\rho}(mx\ddot{y})^{\rho} - (y\cdot m\dot{y})^{1-\rho}(my\ddot{x})^{\rho} \qquad (90)\end{aligned}$$

Comparing Eqs. (83) and (86) we observe that these equations in the limit $\rho = 1$ become the well-known classical law of moment of momentum:

$$\frac{d}{dt}[\mathbf{r}\times m\dot{\mathbf{r}}] = \mathbf{r}\times\mathbf{F}$$

as it should.

From an explicit form of T_i $(i = x, y, z)$ in (84) it follows that in the case of the central force $\mathbf{F} = \gamma\mathbf{r}$, quantity

$$\frac{d^\rho}{dt^\rho}[\mathbf{r} \times m\dot{\mathbf{r}}]$$

is not equal to zero for $\rho \neq 1$ and therefore in the fractional derivative dynamics of the material point conservation of areas is not valid.

2. The second example is to study the motion of a material point under an attracting force $\mathbf{F} = -\alpha\mathbf{r}$ by using fractional derivative method. The equation of motion becomes the fractional derivative

$$m\frac{d^\rho}{dt^\rho} \cdot \frac{d^\rho}{dt^\rho}\mathbf{r} = -\alpha\mathbf{r} \tag{91}$$

or

$$m\left(\frac{1}{\rho}\right)^{2\rho} \mathbf{r}^{(1-\rho)^2}\dot{\mathbf{r}}^{\rho-\rho^2}\left[\frac{d}{dt}(\mathbf{r}^{1-\rho}\dot{\mathbf{r}}^\rho)\right]^\rho = -\alpha\mathbf{r} \tag{92}$$

In the limit $\rho = 1 + \varepsilon$ the last equation reads

$$m\ddot{\mathbf{r}} - \varepsilon m\left[\ddot{\mathbf{r}}\ln(\mathbf{r}\dot{\mathbf{r}}) + \frac{\dot{\mathbf{r}}^2}{\mathbf{r}}\right] = -\alpha\mathbf{r}$$

A more simple equation is obtained by acting fractional derivative on the velocity:

$$m\left(\frac{1}{\rho}\right)^\rho (\mathbf{v})^{1-\rho}[\dot{\mathbf{v}}]^\rho = -\alpha\mathbf{r} \tag{93}$$

or

$$m\frac{d^2\mathbf{r}}{dt^2} = -\alpha\mathbf{r}(\mathbf{v}/\mathbf{v}_0)^\varepsilon \tag{94}$$

We seek its solution in the form

$$\mathbf{r} = \mathbf{A}\sin kt$$

After simple calculations, we have

$$\mathbf{r} = \sqrt{\frac{m}{\alpha}}\mathbf{v}_0\sin\left(\sqrt{\frac{\alpha}{m}}\left(1 - \frac{\alpha\varepsilon}{4m}t^2\right)t\right) + \mathbf{r}_0\times$$
$$\cos\left(\sqrt{\frac{\alpha}{m}}\left(1 - \frac{\alpha\varepsilon}{4m}t^2\right)t\right) \tag{95}$$

This is the desired form of solution that tends to be the usual one at the limit $\varepsilon \to 0$.

8 Fractional differential operations

We call operations of $grad^\rho$, div^ρ, rot^ρ and $\mathbf{v} \cdot \vec{\nabla}^\rho$ fractional differential operations of the first order. There exists fractional differential operations of the second order:

$$div^\rho grad^\rho\varphi,\ rot^\rho grad^\rho\varphi,\ div^\rho rot^\rho\mathbf{a},\ rot^\rho rot^\rho\mathbf{a},\ grad^\rho div^\rho\mathbf{a}.$$

More general fractional differential operators are also possible:

$$div^{\rho} grad^{\alpha}\varphi,\ rot^{\alpha} grad^{\beta}\varphi,\ div^{\alpha} rot^{\beta}\mathbf{a},\ rot^{\alpha} rot^{\beta}\mathbf{a} grad^{\alpha} div^{\beta}\mathbf{a}.$$

Here noncommutative and nonassociative characters are also preserved. For instance,

$$div^{\alpha} grad^{\beta}\varphi \neq div^{\beta} grad^{\alpha}\varphi,$$

$$rot^{\rho} grad^{\alpha}\varphi \neq rot^{\alpha} grad^{\rho}\varphi$$

and so on. This section is devoted to the definition of these fractional differential operations.

8.1 Gradient (grad)

By definition fractional gradient is given by the formula

$$grad^{\rho}\varphi = \mathbf{i}\frac{\partial^{\rho}}{\partial x^{\rho}}\varphi + \mathbf{j}\frac{\partial^{\rho}}{\partial y^{\rho}}\varphi + \mathbf{k}\frac{\partial^{\rho}}{\partial z^{\rho}}\varphi \tag{96}$$

a. Let r be the length of the radius vector $\mathbf{r}$:

$$r = \sqrt{x^2 + y^2 + z^2}$$

then

$$grad^{\rho} r = \left(\frac{1}{\rho}\right)^{\rho} r^{1-2\rho}(\mathbf{i}x^{\rho} + \mathbf{j}y^{\rho} + \mathbf{k}z^{\rho}) = \left(\frac{1}{\rho}\right)^{\rho} r^{1-2\rho}\mathbf{r}^{\rho} \tag{97}$$

where $\mathbf{r} = \{x^{\rho},\ y^{\rho},\ z^{\rho}\}$.

b. Let $\frac{-e}{4\pi r}$ be the Coulomb potential of the electric charge e:

$$\varphi_c = \frac{-e}{4\pi r}$$

This case gives

$$\mathbf{a}_c^{\rho} = grad^{\rho}\left(\frac{-e}{4\pi r}\right) = -\frac{e}{4\pi}\left(\frac{-1}{\rho}\right)^{\rho}\frac{\mathbf{r}^{\rho}}{r^{1+2\rho}} \tag{98}$$

where

$$a_{cx}^{\rho} = \frac{-e}{4\pi}\left(\frac{-1}{\rho}\right)^{\rho}\frac{x^{\rho}}{r^{1+2\rho}}$$

and so on.

8.2 Divergent (div)

The fractional divergent takes the form

$$div^{\alpha}\mathbf{a} = \frac{\partial^{\alpha}}{\partial x^{\alpha}}a_x + \frac{\partial^{\alpha}}{\partial y^{\alpha}}a_y + \frac{\partial^{\alpha}}{\partial z^{\alpha}}a_z \tag{99}$$

Let $\mathbf{a} = \mathbf{a}_c^{\rho}$, $\mathbf{a}_c^{\rho}$ be given by (98) then

$$div^{\rho}\cdot grad^{\rho}\left(\frac{-e}{4\pi r}\right) = \frac{e}{4\pi}(-1)^{1+\rho}\left(\frac{1}{\rho}\right)^{\rho}\left(\frac{1}{r}\right)^{1+4\rho}\times$$

$$\left[(r^2-\frac{1+2\rho}{\rho}x^2)^\rho+(r^2-\frac{1+2\rho}{\rho}y^2)^\rho+(r^2-\frac{1+2\rho}{\rho}z^2)^\rho\right] \tag{100}$$

It can easily be seen that only the case $\rho = 1$ gives

$$divgrad\left(\frac{-e}{4\pi r}\right)=0 \tag{101}$$

8.3 Whirl or rotor of a vector

Fractional rotor defines as

$$rot^\rho \mathbf{a} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial^\rho}{\partial x^\rho} & \frac{\partial^\rho}{\partial y^\rho} & \frac{\partial^\rho}{\partial z^\rho} \\ a_x & a_y & a_z \end{pmatrix}$$

$$= \mathbf{i}\left(\frac{\partial^\rho}{\partial y^\rho}a_z-\frac{\partial^\rho}{\partial z^\rho}a_y\right)+\mathbf{j}\left(\frac{\partial^\rho}{\partial z^\rho}a_x-\frac{\partial^\rho}{\partial x^\rho}a_z\right)+\mathbf{k}\left(\frac{\partial^\rho}{\partial x^\rho}a_y-\frac{\partial^\rho}{\partial y^\rho}a_x\right) \tag{102}$$

It is easy to verify that the Coulomb potential possesses a remarkable property, namely its fractional rotor of the fractional gradient is equal to zero

$$rot^\rho grad^\rho\left(-\frac{e}{4\pi r}\right)=0 \tag{103}$$

Indeed, for example,

$$\frac{\partial^\rho}{\partial y^\rho}a^\rho_{cz}=\frac{e}{4\pi}(-1)^{1+\rho}\left(\frac{1}{\rho}\right)^\rho\cdot z^\rho\left(\frac{1}{r^{1+2\rho}}\right)^{1-\rho}\left(\frac{1}{\rho}\right)^\rho\left[-(1+2\rho)r^{-2-2\rho}\cdot\frac{y}{r}\right]^\rho$$

and

$$\frac{\partial^\rho}{\partial z^\rho}a^\rho_{cy}=\frac{e}{4\pi}(-1)^{1+\rho}\left(\frac{1}{\rho}\right)^\rho\cdot y^\rho\left(\frac{1}{r^{1+2\rho}}\right)^{1-\rho}\left(\frac{1}{\rho}\right)^\rho\left[-(1+2\rho)r^{-2-2\rho}\cdot\frac{z}{r}\right]^\rho$$

and therefore we obtain Eq. (103) for any fractional number ρ. This fact is an essential difference with respect to the star product operation

$$rot\star grad\star\varphi\neq 0$$

Let us consider some examples of the fractional rotor.

1.

$$rot^\gamma(\varphi\mathbf{a})\neq\varphi rot^\gamma\mathbf{a}+grad^\gamma\varphi\times\mathbf{a}$$

Indeed, by definition

$$rot^\gamma\varphi\mathbf{a} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial^\gamma}{\partial x^\gamma} & \frac{\partial^\gamma}{\partial y^\gamma} & \frac{\partial^\gamma}{\partial z^\gamma} \\ \varphi a_x & \varphi a_y & \varphi a_z \end{pmatrix} =$$

$$= \mathbf{i}\left(\frac{\partial^\gamma}{\partial y^\gamma}\varphi a_z-\frac{\partial^\gamma}{\partial z^\gamma}\varphi a_y\right)+\mathbf{j}\left(\frac{\partial^\gamma}{\partial z^\gamma}\varphi a_x-\frac{\partial^\gamma}{\partial x^\gamma}\varphi a_z\right)+\mathbf{k}\left(\frac{\partial^\gamma}{\partial x^\gamma}\varphi a_y-\frac{\partial^\gamma}{\partial y^\gamma}\varphi a_x\right)$$

where

$$\frac{\partial^\gamma}{\partial y^\gamma}\varphi a_z=\left(\frac{1}{\gamma}\right)^\gamma(\varphi a_z)^{1-\gamma}\left[\frac{\partial}{\partial y}\varphi a_z\right]^\gamma$$

However, in the limit $\gamma = 1 + \varepsilon$ we have

$$rot^{\gamma}(\varphi\mathbf{a}) = (1+\varepsilon)[\varphi rot\mathbf{a} + grad\varphi \times \mathbf{a}] - \varepsilon rot\mathbf{A} \tag{104}$$

Here

$$\mathbf{A} = \{\varphi a_x \ln(\varphi a_x),\ \varphi a_y \ln(\varphi a_y),\ \varphi a_z \ln(\varphi a_z)\}$$

may be understood as a vector.

2. For the usual integer - order derivative case

$$rotF(r)\mathbf{r} = 0$$

But in the fractional derivative scheme it takes the form

$$rot^{\gamma}F(r)\mathbf{r}|_{\gamma=1+\varepsilon} = -\varepsilon \frac{1}{r}\frac{\partial F(r)}{\partial r}[\mathbf{r} \times \mathbf{R}] \tag{105}$$

where

$$\mathbf{R} = \mathbf{r}\ln\mathbf{r} = \{x\ln x,\ y\ln y,\ z\ln z\}$$

3. Let

$$\mathbf{v} = \mathbf{v}_0 + \vec{\omega} \times \mathbf{r}$$

be field of velocities of a rigid body at some time moment. Then,

$$rot^{\gamma}\mathbf{v}|_{\gamma=1+\varepsilon} = 2\vec{\omega} - \varepsilon(\mathbf{i}\omega_x \ln v_z v_y + \mathbf{j}\omega_y \ln v_x v_z + \mathbf{k}\omega_z \ln v_x v_y)$$

8.4 Fractional differential operation $\mathbf{v} \cdot \vec{\nabla}^{\rho}$

This differential operation has a natural generalization:

$$\mathbf{v} \cdot \vec{\nabla}^{\rho} = v_x \frac{\partial^{\rho}}{\partial x^{\rho}} + v_y \frac{\partial^{\rho}}{\partial y^{\rho}} + v_z \frac{\partial^{\rho}}{\partial z^{\rho}} \tag{106}$$

8.5 Operation of $div^{\rho} grad^{\rho}\varphi$

This second order operation using fractional derivatives is defined as

$$div^{\rho} grad^{\rho}\varphi = \left(\frac{1}{\rho}\right)^{2\rho} \varphi^{(1-\rho)^2} \left\{ \left(\frac{\partial}{\partial x}\varphi\right)^{\rho(1-\rho)} \left[\frac{\partial}{\partial x}\left(\varphi^{1-\rho}\left(\frac{\partial\varphi}{\partial x}\right)^{\rho}\right)\right]^{\rho} + \right.$$

$$\left. + \left(\frac{\partial}{\partial y}\varphi\right)^{\rho(1-\rho)} \left[\frac{\partial}{\partial y}\left(\varphi^{1-\rho}\left(\frac{\partial\varphi}{\partial x}\right)^{\rho}\right)\right]^{\rho} + \left(\frac{\partial}{\partial z}\varphi\right)^{\rho(1-\rho)} \left[\frac{\partial}{\partial z}\left(\varphi^{1-\rho}\left(\frac{\partial\varphi}{\partial x}\right)^{\rho}\right)\right]^{\rho} \right\} \tag{107}$$

This expression generalizes the Laplacian operator:

$$divgrad\varphi = \vec{\nabla} \cdot \vec{\nabla}\varphi = (\vec{\nabla} \cdot \vec{\nabla})\varphi = \nabla^2\varphi = \triangle\ \varphi \tag{108}$$

As mentioned above $div^{\rho} grad^{\rho}\left(\frac{1}{r}\right)$ is not equal to zero for any parameter ρ except $\rho = 1$.

8.6 Operation of $rot^\rho grad^\rho \varphi$

Due to the noncommutative character of the fractional derivatives

$$\frac{\partial^\rho}{\partial x^\rho}\frac{\partial^\rho}{\partial y^\rho} \neq \frac{\partial^\rho}{\partial y^\rho}\frac{\partial^\rho}{\partial x^\rho} \tag{109}$$

this second order operation is not zero in the general case. However direct calculation shows that

$$rot^\rho grad^\rho \left(\frac{1}{r}\right) \equiv 0$$

for any number ρ. In the general case, we have

$$rot^\rho grad^\rho \varphi = \mathbf{i}\left[\frac{\partial^\rho}{\partial y^\rho}\frac{\partial^\rho}{\partial z^\rho}\varphi - \frac{\partial^\rho}{\partial z^\rho}\frac{\partial^\rho}{\partial y^\rho}\varphi\right] +$$

$$+\mathbf{j}\left[\frac{\partial^\rho}{\partial z^\rho}\frac{\partial^\rho}{\partial x^\rho}\varphi - \frac{\partial^\rho}{\partial x^\rho}\frac{\partial^\rho}{\partial z^\rho}\varphi\right] + \mathbf{k}\left[\frac{\partial^\rho}{\partial x^\rho}\frac{\partial^\rho}{\partial y^\rho}\varphi - \frac{\partial^\rho}{\partial y^\rho}\frac{\partial^\rho}{\partial x^\rho}\varphi\right] \tag{110}$$

For instance, where

$$Q_x = \frac{\partial^\rho}{\partial y^\rho}\frac{\partial^\rho}{\partial z^\rho}\varphi - \frac{\partial^\rho}{\partial z^\rho}\frac{\partial^\rho}{\partial y^\rho}\varphi =$$

$$= \left(\frac{1}{\rho}\right)^{2\rho} \varphi^{(1-\rho)^2} \left\{ \left(\frac{\partial}{\partial z}\varphi\right)^{\rho-\rho^2} \left[\frac{\partial}{\partial y}\left(\varphi^{1-\rho}\left(\frac{\partial \varphi}{\partial z}\right)^\rho\right)\right]^\rho - \right.$$

$$\left. - \left(\frac{\partial}{\partial y}\varphi\right)^{\rho-\rho^2} \left[\frac{\partial}{\partial z}\left(\varphi^{1-\rho}\left(\frac{\partial \varphi}{\partial y}\right)^\rho\right)\right]^\rho \right\}$$

Other components Q_y and Q_z are obtained by cyclic permutations from Q_x.

8.7 Operation of $div^\rho rot^\rho \mathbf{a}$

In the general case, this operation is also not zero due to noncommutativity of the fractional partial derivatives (109). Its explicit form is

$$div^\rho rot^\rho \mathbf{a} = \mathbf{i}\left(\frac{1}{\rho}\right)^{2\rho} a_x^{(1-\rho)^2} \left\{ \left(\frac{\partial}{\partial z}a_x\right)^{\rho-\rho^2} \left[\frac{\partial}{\partial y}\left(a_x^{1-\rho}\left(\frac{\partial}{\partial z}a_x\right)^\rho\right)\right]^\rho - \right.$$

$$\left. - \left(\frac{\partial}{\partial y}a_x\right)^{\rho-\rho^2} \left[\frac{\partial}{\partial z}\left(a_x^{1-\rho}\left(\frac{\partial}{\partial y}a_x\right)^\rho\right)\right]^\rho \right\} +$$

$$+\mathbf{j}\left(\frac{1}{\rho}\right)^{2\rho} a_y^{(1-\rho)^2} \left\{ \left(\frac{\partial}{\partial x}a_y\right)^{\rho-\rho^2} \left[\frac{\partial}{\partial z}\left(a_y^{1-\rho}\left(\frac{\partial}{\partial x}a_y\right)^\rho\right)\right]^\rho - \right.$$

$$\left. - \left(\frac{\partial}{\partial z}a_y\right)^{\rho-\rho^2} \left[\frac{\partial}{\partial x}\left(a_y^{1-\rho}\left(\frac{\partial}{\partial z}a_y\right)^\rho\right)\right]^\rho \right\} +$$

$$+\mathbf{k}\left(\frac{1}{\rho}\right)^{2\rho} a_z^{(1-\rho)^2} \left\{ \left(\frac{\partial}{\partial y}a_z\right)^{\rho-\rho^2} \left[\frac{\partial}{\partial x}\left(a_z^{1-\rho}\left(\frac{\partial a_z}{\partial y}\right)^\rho\right)\right]^\rho - \right.$$

$$\left. - \left(\frac{\partial}{\partial x}a_z\right)^{\rho-\rho^2} \left[\frac{\partial}{\partial y}\left(a_z^{1-\rho}\left(\frac{\partial a_z}{\partial x}\right)^\rho\right)\right]^\rho \right\} \tag{111}$$

In the particular case, when

$$\mathbf{a} = \mathbf{a}_c^\rho = grad\left(\frac{-e}{4\pi r}\right)$$

the fractional operator $div^\rho rot^\rho \mathbf{a}_c^\rho = 0$ for any fractional number ρ. This is an important consequence of the fractional derivative operation. In the limit $\rho = 1+\varepsilon$, $\varepsilon \to 0$ we obtain

$$div^\rho rot^\rho \mathbf{a} = -\varepsilon\left\{\mathbf{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}a_x \cdot \ln\left(a_x\frac{\partial}{\partial z}a_x\right)\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}a_x \cdot \ln\left(a_x\frac{\partial}{\partial y}a_x\right)\right)\right] + \right.$$
$$+\mathbf{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}a_y \cdot \ln\left(a_y\frac{\partial}{\partial x}a_y\right)\right) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}a_y \cdot \ln\left(a_y\frac{\partial}{\partial z}a_y\right)\right)\right] +$$
$$\left.+\mathbf{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}a_z \cdot \ln\left(a_z\frac{\partial}{\partial y}a_z\right)\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}a_z \cdot \ln\left(a_z\frac{\partial}{\partial x}a_z\right)\right)\right]\right\}$$

This quantity is also of small order as in the star - product scheme in calculations of $div\star rot\star\mathbf{a}$. In both fractional derivative and star product cases, when $\varepsilon \to 0$ and $\theta \to 0$, we obtain the exact result

$$div rot \mathbf{a} = 0$$

as it should.

9 Multiple fractional integrals

Formal generalization of the multiple integer - order derivative integrals in the case of fractional integrals can also be done. For instance:

1.
$$\int dy^\beta \int dx^\alpha = x^\alpha y^\beta$$

2.
$$\int dy^\beta \int dx^\alpha x^\alpha y^\beta \left(\frac{1}{\alpha}\right)^\alpha \left(\frac{1}{\beta}\right)^\beta (1-2\alpha)^\beta (x^2+y^2)^{1/2-\alpha-\beta} = \sqrt{x^2+y^2}$$

3.
$$\int dx^\alpha \int dy^\beta x^\alpha y^\beta \left(\frac{1}{\alpha}\right)^\alpha \left(\frac{1}{\beta}\right)^\beta (1-2\beta)^\alpha (x^2+y^2)^{1/2-\alpha-\beta} = \sqrt{x^2+y^2}$$

4.
$$\int dy^\beta \int dx^\alpha \sin^{1-\alpha}(xy)[y\cos(xy)]^\alpha \times$$
$$\left[\frac{x(1-\alpha)\cos(xy)}{\sin(xy)} + \frac{\alpha}{y\cos(xy)}(\cos(xy) - xy\cdot\sin(xy))\right]^\beta = \sin(xy)$$

etc.

We see that integration order of multiple fractional integrals is important. This reflects nonassociative and noncommutative properties of fractional derivatives.

Two and three multiple fractional integrals correspond to the generalization of the surface and volume integrals in the usual case. However, it is difficult to understand fractional surface element $dx^\alpha dy^\alpha = dS^\alpha$ and fractional volume element $dx^\alpha dy^\alpha dz^\alpha = dV^\alpha$ in fractional integral calculuses.

10 Relativistic extension of the fractional derivative method

Relativistic dynamics of particles can also be studied by means of the fractional derivative techniques. In the relativistic theory the proper time interval is given by

$$d\tau^2 = dt^2 - d\mathbf{x}^2 = dt^2(1 - \mathbf{v}^2) \tag{112}$$

Then four velocity is

$$u^\mu = dx^\mu / d\tau, \quad u_\mu u^\mu = 1 \tag{113}$$

We know the motion of a nonrelativistic particle with the charge Ze in the electromagnetic field $\mathbf{E}(t, \mathbf{x})$ is given by the equation:

$$m\frac{d}{dt}\mathbf{v} = Ze\mathbf{E} \tag{114}$$

The natural covariant generalization of this equation has the form

$$\frac{d}{d\tau}u^\mu(t) = ZeF^{\mu\nu}u_\nu(t)/m \tag{115}$$

where $F^{\mu\nu}$ is the tensor of the electromagnetic field $\mathbf{E}$ and $\mathbf{H}$.

In the relativistic case four - differential operators are

$$\begin{aligned} \partial_\mu^\alpha &= \frac{\partial^\alpha}{\partial x^{\mu\alpha}} = (\partial_0^\alpha,\ \partial_1^\alpha,\ \partial_2^\alpha,\ \partial_3^\alpha) = \\ &= \left(\frac{1}{c}\frac{\partial^\alpha}{\partial t^\alpha},\ \frac{\partial^\alpha}{\partial x^\alpha},\ \frac{\partial^\alpha}{\partial y^\alpha},\ \frac{\partial^\alpha}{\partial z^\alpha}\right) = \\ &= \left(\frac{1}{c}\frac{\partial^\alpha}{\partial t^\alpha},\ \vec{\nabla}^\alpha\right) \end{aligned} \tag{116}$$

and

$$\partial^{\mu\alpha} = g^{\mu\nu}\partial_\nu^\alpha = \left(\frac{1}{c}\frac{\partial^\alpha}{\partial t^\alpha},\ -\vec{\nabla}^\alpha\right) \tag{117}$$

The formal D'Alembertian operator becomes

$$\Box^\alpha = \partial^{\mu\alpha}\partial_\mu^\alpha = \frac{1}{c^2}\frac{\partial^\alpha}{\partial t^\alpha}\cdot\frac{\partial^\alpha}{\partial t^\alpha} - \left(\frac{\partial^\alpha}{\partial x^\alpha}\cdot\frac{\partial^\alpha}{\partial x^\alpha} - \frac{\partial^\alpha}{\partial y^\alpha}\cdot\frac{\partial^\alpha}{\partial y^\alpha} - \frac{\partial^\alpha}{\partial z^\alpha}\cdot\frac{\partial^\alpha}{\partial z^\alpha}\right) \tag{118}$$

From the definition of the fractional derivative we know that

$$\frac{\partial^\alpha}{\partial x^\alpha}\cdot\frac{\partial^\alpha}{\partial x^\alpha} \neq \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}$$

Thus, the action of $\Box^\alpha$ on a scalar function is given by

$$\begin{aligned} \Box^\rho\varphi = div^\rho grad^\rho\varphi = \\ = \left(\frac{1}{\rho}\right)^{2\rho}\varphi^{(1-\rho)^2}\left\{\left(\frac{\partial}{\partial t}\varphi\right)^{\rho-\rho^2}\left[\frac{\partial}{\partial t}\left(\varphi^{1-\rho}\left(\frac{\partial\varphi}{\partial t}\right)^\rho\right)\right]^\rho - \right. \\ - \left(\frac{\partial}{\partial x}\varphi\right)^{\rho-\rho^2}\left[\frac{\partial}{\partial x}\left(\varphi^{1-\rho}\left(\frac{\partial\varphi}{\partial x}\right)^\rho\right)\right]^\rho - \left(\frac{\partial}{\partial y}\varphi\right)^{\rho-\rho^2}\left[\frac{\partial}{\partial y}\left(\varphi^{1-\rho}\left(\frac{\partial\varphi}{\partial y}\right)^\rho\right)\right]^\rho - \end{aligned}$$

$$-\left(\frac{\partial}{\partial z}\varphi\right)^{\rho-\rho^2}\left[\frac{\partial}{\partial z}\left(\varphi^{1-\rho}\left(\frac{\partial\varphi}{\partial z}\right)^{\rho}\right)\right]^{\rho}\Bigg\} \tag{119}$$

For example, in the limit $\rho = 1 - \varepsilon$, $\varepsilon - \to 0$ the fractional Klein - Gordon equation has the form

$$(\Box^{\rho} + m^2)\varphi = (\Box + m^2)\varphi + \frac{\varepsilon}{\varphi}\partial^{\nu}\varphi\partial_{\nu}\varphi + \varepsilon\partial_{\nu}^2\varphi\ln(\varphi\partial^{\nu}\varphi) = 0 \tag{120}$$

This is an equation with self - interaction for the scalar particle. Eq. (120) may play an important role in dynamics of quarks inside their confinement region.

In conclusion, it is worth noting that noncommutative and fractional derivative methods will find many applications in nature phenomena [simulation of chemical reactions (Zabadal et al., 2001), fractional (quantum) Hall effect (Susskind, 2001) etc.], especially in physical and chemical processes under extremal conditions like phase transitions, superdense, supercold and hot matter conditions, nonlinear and turbulent processes, even in predictions of weather conditions and earthquakes and so on.

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