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A LOCAL HOMOLOGY THEORY FOR LINEARLY COMPACT MODULES

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Abstract

We introduce a local homology theory for linearly modules which is in some sense dual to the local cohomology theory of A. Grothendieck. Some basic properties of local homology modules are shown such as: the vanishing and non-vanishing, the noetherianness of local homology modules. By using duality, we extend some well-known results in theory of local cohomology of A. Grothendieck.

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1. Introduction

Let R be a noetherian commutative ring and I an ideal of R. We knew that the local cohomology theory of A. Grothendieck has proved to be an important tool in algebraic geometry and commutative algebra. Therefore, many authors try to study a theory which is its dual. However, this work is not easy, because it relates to inverse limits which are in general not right exact on the category of modules. At first, E. Matlis [17], [18] studied left derived functors of the $I-$adic completion functor in which the ideal I was generated by an $R-$regular sequence. Next, J. P. C. Greenlees and J. P. May [9] using the homotopy colimit, or telescope, of the cochain of Koszul complexes to define *local homology groups* of a module M by $H^I_\bullet(M) = H_\bullet(\mathrm{Hom}(\mathrm{Tel}K^\bullet(\underline{x}^t); M))$, where $\underline{x}$ is a finitely generated system of I. Later, L. Alonso Tarrío, A. Jeremias López and J. Lipman [1] have presented a sheafified derived-category generalization of Greenlees-May results for a quasi-compact separated scheme. Recently, we defined in [6] the *local homology modules* $H^I_i(M)$ of an $R-$module M with respect to the ideal I by $H^I_i(M) = \varprojlim_t \mathrm{Tor}^R_i(R/I^t; M)$. This definition is in some sense dual to A. Grothendieck's definition of local cohomology modules. Also in [6], we showed some fundamental properties of local homology modules when M was artinian. We know that artinian modules are linearly compact with discrete topology (see [16] (4.2)]). There is a natural question: how to define a local homology theory for linearly compact modules? The concept of linearly compact spaces was first introduced by Lefschetz [13] for vector spaces of infinite dimension and it was then generalized for modules by D. Zelinsky [29]. It was also studied by authors such as: H. Leptin, I. G. Macdonald, C. U. Jensen, H. Zöschinger ... . The class of linearly compact modules is very large, it contains many important classes of modules in commutative algebra. Even its subclass of semi-discrete linearly compact modules contains also artinian modules, moreover it contains also noetherian modules over a complete ring. Note that the inverse limits are exact on the category of linearly compact modules.

The purpose of this paper is to show basic properties about local homology of linearly compact modules. By using duality, we extend some well-known results in theory of local cohomology of A. Grothendieck. Throughout, R will be a noetherian commutative ring and has a topological structure.

In sections 2 and 3 we recall and prove some properties of linearly compact modules and semi-discrete linearly compact modules that we shall use. In case $(R, \mathfrak{m})$ is a local ring and the topology on R is the $\mathfrak{m}-$adic topology, each linearly compact $R-$module M has a natural structure as a module over $\widehat{R}$. Moreover if M is a semi-discrete linearly compact $R-$module, then M is also a semi-discrete linearly compact $\widehat{R}-$module (Theorem 3.5).

In section 4 we show some basic properties of local homology modules. We will see that in the case of linearly compact modules, our definition is coincident with the definition of J. P. C. Greenlees and J. P. May [9, 2.4] (Proposition 4.5). Theorem 4.9 shows a character of the $I-$separated linearly compact modules.

In section 5 we give the vanishing and non-vanishing of local homology modules. Let M be a linearly compact $R-$module with Krull dimension $\operatorname{Ndim} M = d$, then $H_i^I(M) = 0$ for all $i > d$ (Theorem 5.5). If M is a semi-discrete linearly compact $R-$module over a local ring $(R, \mathfrak{m})$ with $\operatorname{Ndim} L(M) = d \geq 0$ ($L(M)$ is the sum of all artinian $R-$submodules of M), then $H_d^{\mathfrak{m}}(M) \neq 0$ and $H_i^{\mathfrak{m}}(M) = 0$ for all $i > d$ (Theorem 5.6). Let $\operatorname{Width}_I(M)$ denote the length of the longest $M-$coregular sequences in I. Theorem 5.8 shows that if M is a semi-discrete linearly compact $R-$module such that $0 :_M I \neq 0$, then $\operatorname{Width}_I(M) = \inf\{i / H_i^I(M) \neq 0\}$.

In section 6 we show the noetherianness of local homology modules $H_i^{\mathfrak{m}}(M)$ of a semi-discrete linearly compact module M over a local ring $(R, \mathfrak{m})$ (Theorem 6.1). Moreover, if M is a semi-discrete linearly compact $R-$module with $\operatorname{Ndim} M = d$, then $H_d^I(M)$ is a noetherian $\Lambda_I(R)-$module (Theorem 6.4). We also give a criterion for a module to be noetherian (Lemma 6.3).

The last section is devoted to study duality. In this section $(R, \mathfrak{m})$ is a local ring and the topology on R is the $\mathfrak{m}-$adic topology. Let $E(R/\mathfrak{m})$ be the injective envelope of $R/\mathfrak{m}$, the Matlis dual of an $R-$module M is $D(M) = \operatorname{Hom}(M; E(R/\mathfrak{m}))$. Let M be a Hausdorff linearly topologized $R-$module, in [16] I. G. Macdonald defined $M^* = Hom(M; E(R/\mathfrak{m}))$ the set of continuous homomorphisms of $R-$modules. Note that M is semi-discrete (i.e., all submodules are closed) if and only if $D(M) = M^*$ (see [16, 5.8]). Theorem 7.8 gives us duality formulaes between local cohomology modules and local homology modules. From the properties of local homology modules, by using duality, we get back some well-known properties of local cohomology modules. Proposition 7.12 says that if M is a semi-discrete linearly compact module with $\operatorname{Ndim} L(M^*) = d \geq 0$, then $H_{\mathfrak{m}}^d(M) \neq 0$ and $H_{\mathfrak{m}}^i(M) = 0$ for all $i > d$. From Proposition 7.12, we get back A. Grothedieck's Non-vanishing theorem (Corollary 7.13). Proposition 7.14 gives us the artinianness of local cohomology modules $H_{\mathfrak{m}}^i(M)$ of a semi-discrete linearly compact module M. This is also a slight extension of a well-known result in local cohomology theory, since a noetherian module over a complete local ring is semi-discrete linearly compact (see [16, 7.3]).

In this paper, the terminology "isomorphism" means "algebraic isomorphism" and the terminology "topological isomorphism" means "algebraic isomorphism with the homomorphisms (and its inverse) are continuous".

2. Linearly compact modules

In this section we recall the concept of *linearly compact* modules by terminology of I. G. Macdonald (see [16]) and some their basic properties.

Let M be a topological $R-$module. A *nucleus* of M is a neighbourhood of the zero element of M, and a *nuclear base* of M is a base for the nuclei of M. If N is a submodule of M which contains a nucleus then N is open (and therefore closed) in M and M/N is discrete. M is Hausdorff if and only if the intersection of all the nuclei of M is 0. M is said to be *linearly topologized* if M has a nuclear base $\mathcal{M}$ consisting of submodules.

Definition 2.1. A Hausdorff linearly topologized $R-$module M is said to be *linearly compact* if M has the following property: if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection.

It should be noted that an artinian $R-$module is linearly compact with the discrete topology (see [16, 3.10]).

Remark 2.2. Let M be an $R-$module. If $\mathcal{M}$ is a family of submodules of M satisfies the conditions:

(i) For all $N_1,\ N_2 \in \mathcal{M}$ there is an $N_3 \in \mathcal{M}$ such that $N_3 \subseteq N_1 \cap N_2$,

(ii) For an element $x \in M$ and $N \in \mathcal{M}$ there is a nucleus U of R such that $Ux \subseteq N$,

then $\mathcal{M}$ is a base of a linear topology on M (see [16, 2.1]).

The following properties of linearly compact modules are often used in this paper.

Lemma 2.3. (see [16, §3]) *(i) Let M be a Hausdorff linearly topologized $R-$module, N a closed submodule of M. Then M is linearly compact if and only if N and M/N are linearly compact.*

(ii) Let $f : M \longrightarrow N$ be a continuous homomorphism of Hausdorff linearly topologized $R-$modules. If M is linearly compact, then $f(M)$ is linearly compact and f is a closed map.

(iii) If $\{M_i\}_{i\in I}$ is a family of linearly compact $R-$modules, then $\prod\limits_{i\in I} M_i$ is linearly compact.

(iv) The inverse limit of a system of linearly compact $R-$modules and continuous homomorphisms is linearly compact.

If $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, then $\varprojlim_t{}^1 M_t = 0$ by [11, 7.1]. Therefore we have the following lemma.

Lemma 2.4. *Let*

$$0 \longrightarrow \{M_t\} \longrightarrow \{N_t\} \longrightarrow \{P_t\} \longrightarrow 0$$

be a short exact sequence of inverse systems of $R-$modules. If $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, then the sequence of inverse limits

$$0 \longrightarrow \varprojlim_t M_t \longrightarrow \varprojlim_t N_t \longrightarrow \varprojlim_t P_t \longrightarrow 0$$

is exact.

Let M be a linearly compact $R-$module and F a free $R-$module with the base $\{e_i\}_{i\in J}$. Then $\mathrm{Hom}_R(F;M) \cong M^J$, where $M^J = \prod_{i\in J} M_i$ and $M_i = M$ for all $i \in J$. Thus $\mathrm{Hom}_R(F;M)$ is a linearly compact $R-$module by (2.3) (iii). Moreover, if $h : F \longrightarrow F'$ is a homomorphism of free $R-$modules, then the induced homomorphism $h^* : \mathrm{Hom}_R(F';M) \longrightarrow \mathrm{Hom}_R(F;M)$ is continuous by [11, 7.1]. We now take a free resolution $\mathbf{F}_\bullet$ of N :

$$\mathbf{F}_\bullet : \quad \ldots \longrightarrow F_i \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0.$$

Because $\mathrm{Ext}^i_R(N;M)$ is a quotient of submodules of $\mathrm{Hom}(F_i;M)$, we get a topology on $\mathrm{Ext}^i_R(N;M)$.

Lemma 2.5. *Let M be a linearly compact $R-$module and N an $R-$module, then for all $i \geq 0$, $\mathrm{Ext}^i_R(N;M)$ is a linearly compact $R-$module with the topology induced by a free resolution of N and this topology is independent to free resolutions of N. Moreover, if $f : N \longrightarrow N'$ is a homomorphism of $R-$modules, then the induced homomorphism $\mathrm{Ext}^i_R(N';M) \longrightarrow \mathrm{Ext}^i_R(N;M)$ is continuous.*

Proof. Let $\mathbf{F}_\bullet$ be a free resolution of N. It follows that $\mathrm{Hom}_R(\mathbf{F}_\bullet;M)$ forms a complex of linearly compact modules with continuous homomorphisms. For all $i \geq 0$, we have $\mathrm{Ext}^i_R(N;M) = H^i(\mathrm{Hom}_R(\mathbf{F}_\bullet;M))$ is a quotient of closed submodules of $\mathrm{Hom}(F_i;M)$, hence linearly compact by

(2.3) (i) (ii). Now, if $\mathbf{F}'_\bullet$ is a free resolutions of N, then the homomorphism $f : N \longrightarrow N'$ induces a homomorphism of complexes $\varphi_\bullet : \mathbf{F}_\bullet \longrightarrow \mathbf{F}'_\bullet$ lifting f. Therefore the induced homomorphism

$$\mathrm{Hom}(\mathbf{F}'_i; M) \longrightarrow \mathrm{Hom}_R(\mathbf{F}_i; M)$$

is continuous by [11, 7.1]. So the induced homomorphism

$$\mathrm{Ext}^i_R(N; M) \longrightarrow \mathrm{Ext}^i_R(N'; M)$$

is also continuous.

Especially, if $\mathbf{F}_\bullet$ and $\mathbf{G}_\bullet$ are both free resolutions of N, then the identical homomorphism $id : N \longrightarrow N$ induces an isomorphism

$$\varphi_i : H^i(\mathrm{Hom}_R(\mathbf{F}_\bullet; M)) \longrightarrow H^i(\mathrm{Hom}_R(\mathbf{G}_\bullet; M)).$$

Moreover by the above argument, the homomorphism φ_i and its inverse are continuous. Thus φ_i is a topological isomorphism. That means the topology defined on $\mathrm{Ext}^i_R(N; M)$ is independent to free resolutions of N. $\square$

If N is a finitely generated $R-$module, there is a free resolution $\mathbf{F}_\bullet$ of N with the finitely generated free modules. If M is a linearly compact $R-$module, we have an induced topology on $\mathrm{Tor}^R_i(N; M)$. By an argument analogous to that used for the proof of Lemma 2.5, we get the following lemma.

Lemma 2.6. *Let N be a finitely generated $R-$module and M a linearly compact $R-$module, then for all $i \geq 0$, $\mathrm{Tor}^R_i(N; M)$ is a linearly compact $R-$module with the topology induced by a free resolution of N (consisting of finitely generated free modules) and this topology is independent to free resolutions of N. Moreover, if $f : N \longrightarrow N'$ is a homomorphism of finitely generated $R-$modules, then the induced homomorphism $\psi_{i,M} : \mathrm{Tor}^R_i(N; M) \longrightarrow \mathrm{Tor}^R_i(N'; M)$ is continuous.*

Next we show that the functor $\mathrm{Tor}^R_i(N; -)$ commutes with inverse limits of inverse systems of linearly compact $R-$modules with continuous homomorphisms in case N is a finitely generated $R-$module.

Lemma 2.7. *If N is a finitely generated $R-$module and $\{M_t\}$ an inverse system of linearly compact $R-$modules with continuous homomorphisms, then for all $i \geq 0$, $\{\mathrm{Tor}_i^R(N; M_t)\}$ forms an inverse system of linearly compact modules with continuous homomorphisms. Moreover, we have*

$$\mathrm{Tor}_i^R(N; \varprojlim_t M_t) \cong \varprojlim_t \mathrm{Tor}_i^R(N; M_t).$$

Proof. Let $\mathbf{F}_\bullet$ be a free resolution of N with finitely generated $R-$modules. Since $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, $\{F_i \otimes_R M_t\}$ forms an inverse system of linearly compact modules with continuous homomorphisms for all $i \geq 0$ by (2.3) (iii). Then $\{\mathrm{Tor}_i^R(N; M_t)\}$ forms an inverse system of linearly compact modules with continuous homomorphisms. Moreover

$$\mathbf{F}_\bullet \otimes_R \varprojlim_t M_t \cong \varprojlim_t (\mathbf{F}_\bullet \otimes_R M_t),$$

since the inverse limit commutes with the direct product and

$$H_i(\varprojlim_t (\mathbf{F}_\bullet \otimes_R M_t)) \cong \varprojlim_t H_i(\mathbf{F}_\bullet \otimes_R M_t)$$

by (2.4) and [20, 6.1, Th. 1]. This finishes the proof. □

3. Semi-discrete linearly compact modules

A Hausdorff linearly topologized $R-$module M is called *semi-discrete* if every submodule of M is closed. Thus a discrete $R-$module is semi-discrete. The class of semi-discrete linearly compact modules contains all artinian modules. Moreover, it also contains all finitely generated modules in case R is a complete local ring (see [14, 7.3]).

Denote by $L(M)$ the sum of all artinian submodules of M, we have the following lemma.

Lemma 3.1. (see [31]) *If M is a semi-discrete linearly compact $R-$module, then $L(M)$ is an artinian module.*

Lemma 3.2. *Let M be a semi-discrete linearly compact $R-$module. If M has an irredundant sum: $M = \sum\limits_{i\in V} M_i$, in which $M_i \nsubseteq \sum\limits_{i\neq j\in V} M_j$ for all $i \in V$, then the set $\{M_i\}_{i\in V}$ is finite.*

Proof. Set $N_i = \sum\limits_{i\neq j\in V} M_j$, for all $i \in V$. Then every finite subcollection $\{N_1, \dots, N_n\}$ satisfies $N_i + \bigcap\limits_{j\neq i} N_j = M$ $(i = 1, \dots, n)$, in other word by terminology of Zelinsky [29, §1], $\{N_i\}$ is independent in M. Now by an argument analogous to [29, Prop. 5] we get the set V is finite. □

A prime ideal $\mathfrak{p}$ is called *co-associated* to a non-zero $R-$module M if there is an artinian homomorphic image L of M with $\mathfrak{p} = \operatorname{Ann}_R L$. The set of co-associated primes to M is denoted by $\operatorname{Coass}_R(M)$. M is called $\mathfrak{p}-$*coprimary* if $\operatorname{Coass}_R(M) = \{\mathfrak{p}\}$. A module is called *hollow* if it cannot be written as a sum of two proper submodules. A hollow module M is $\mathfrak{p}-$coprimary, where $\mathfrak{p} = \{x \in R/xM \neq M\}$ (see [5, 2]). Let N be a submodule of M, a supplement of N in M is a submodule K of M minimal such that $N + K = M$. If M is a semi-discrete linearly compact $R-$module, then every submodule of M has a supplement in M (see [30]).

Lemma 3.3. *If M is a semi-discrete linearly compact $R-$module, then M can be written as a finite sum of hollow modules.*

Proof. Assume the contrary, that M cannot be written as a finite sum of hollow modules. Then M can be written as a sum of proper submodules: $M = M_1 + M_1'$, in which M_1' is a supplement of M_1 in M and M_1' cannot be written as a finite sum of hollow modules. Next, M_1' can be written as a sum of proper submodules: $M_1' = M_2 + M_2'$, in which M_2' is a supplement of M_2 in M_1' and M_2' cannot be written as a finite sum of hollow modules. Thus $M = M_1 + M_2 + M_2'$ and it is easy to check that this is an irredundant sum. Continueing in this fashion we obtain the fact that M can be written as an infinite irredundant sum of submodules which contradicts (3.2). The proof is complete. □

Corollary 3.4. *If M is a semi-discrete linearly compact $R-$module, then the set $\operatorname{Coass}(M)$ is finite.*

Proof. By (3.3), M can be written as a finite sum of hollow modules

$$M = \sum_{i=1}^{n} M_i.$$

Set $\mathfrak{p}_i = \{x \in R \mid xM_i \neq M_i\}$. Then [5, Prop. 2] gives us

$$\mathrm{Coass}_R(M) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

This finishes the proof. $\square$

Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{m}$ its maximal ideal and the topology on R is the $\mathfrak{m}-$adic topology. Denote by $\widehat{R}$ the $\mathfrak{m}-$adic completion of R and the topology on $\widehat{R}$ is the $\hat{\mathfrak{m}}-$adic topology. We know that an artinian module over a local ring has a natural structure as a module over $\widehat{R}$ (see [24, 1.11]). The following theorem gives us a more general result for linearly compact $R-$modules and semi-discrete linearly compact $R-$modules.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a local ring with the $\mathfrak{m}-$adic topology and M a linearly compact $R-$module. We have the following statements:*
(i) M has a natural structure as a linearly compact module over $\widehat{R}$. Moreover, if N is a closed $R-$module of M, then N is an linearly compact $\widehat{R}-$submodule.
(ii) If M is a semi-discrete linearly compact $R-$module, then a subset N of M is an $R-$submodule if and only if it is an $\widehat{R}-$submodule, and then M is also a semi-discrete linearly compact $\widehat{R}-$module.

Proof. (i) Assume that $\{U_i\}_{i \in J}$ is a nuclear base of M consisting of submodules. Then $M = \varprojlim_{i \in J} M/U_i$, in which M/U_i is an artinian $R-$module for all $i \in J$ by [16, 5.5]. It should be noted that an artinian module over a local ring $(R, \mathfrak{m})$ has a natural structure as an artinian module over $\widehat{R}$. So that, a homomorphism of artinian $R-$modules over a local ring $(R, \mathfrak{m})$ can be considered as a homomorphism of artinian $\widehat{R}-$modules. Thus $\{M/U_i\}$ can be regard as an inverse system of artinian $\widehat{R}-$modules. Therefore M has a natural structure as a linearly compact module over $\widehat{R}$.

Now, if N is a closed $R-$module of M, then $N = \varprojlim_{i \in J} N/(N \cap U_i)$. We have $N/(N \cap U_i) \cong (N + U_i)/U_i \subseteq M/U_i$, thus $N/(N \cap U_i)$ can be considered as an artinian $R-$submodule of M/U_i, so it is an artinian $\widehat{R}-$submodule. Moreover, the homomorphisms of the inverse system $\{M/(N \cap U_i)\}$ induce from the inverse system $\{M/U_i\}$. Therefore N is an linearly compact $\widehat{R}-$submodule of M.
(ii) It is clear that if N is an $\widehat{R}-$submodule, then N is an $R-$submodule. The converse follows from (3.5) (i) and the fact that all submodules of a semi-discrete linearly compact module are closed.

Let $\mathcal{M}$ be a nuclear base of the linear topology $R-$module M. Then $\mathcal{M}$ consists $R-$submodules which are also $\widehat{R}-$submodules. It is clear that $\mathcal{M}$ satisfies the conditions (2.2) (i) (ii) for a nuclear base of a linear topology $\widehat{R}-$module. Therefore, M is also a semi-discrete linearly compact $\widehat{R}-$module because M is a semi-discrete linearly compact $R-$module. $\square$

Remark 3.6. The inverse of (3.5) (i) is not true. For example, Let $(R, \mathfrak{m})$ be a local ring and $(R, \mathfrak{m})$ is not complete in $\mathfrak{m}-$adic topology. Then $\widehat{R}$ is a semi-discrete linearly compact $\widehat{R}-$module and R can be considered as an $R-$submodule of $R-$module $\widehat{R}$. However, R is not an $\widehat{R}-$submodule.

4. Local homology modules of linearly compact modules

Let I be an ideal of R, the *i-th local homology* module $H_i^I(M)$ of an $R-$module M with respect to I is defined by (see [6, 3.1])

$$H_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(R/I^t; M).$$

It is clear that $H_0^I(M) \cong \Lambda_I(M)$, in which $\Lambda_I(M) = \varprojlim_t M/I^tM$ the $I-$adic completion of M.

Assume that I is generated by elements $x_1, x_2, \dots, x_r$. Let $H_i(\underline{x}(t); M)$ be the $i-$th Koszul homology module of M with respect to $\underline{x}(t) = (x_1^t, \dots, x_r^t)$. We proved the following theorem in [6, §3].

Theorem 4.1. (see [6, §3]) *Let M be an $R-$module. Then for all $i \geq 0$,*
(i) $H_i^I(M) \cong \varprojlim_t H_i(\underline{x}(t); M)$,
(ii) $H_i^I(M)$ is $I-$separated, i.e., $\bigcap_{t>0} I^t H_i^I(M) = 0$.

Remarks 4.2. (i) It follows from the definition of local homology modules that $H_i^I(M)$ has a natural structure as a module over the $I-$adic completion $\Lambda_I(R)$ of R. Indeed, let

$$0 \longrightarrow K \longrightarrow P_{i-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence with P_j projective $(j = 1, 2, \dots, i-1)$. We have following exact sequences for all $t > 0$

$$0 \longrightarrow \operatorname{Tor}_i^R(R/I^t; M) \longrightarrow K/I^tK \longrightarrow P_{i-1}/I^tP_{i-1}.$$

Therefore, the sequence

$$0 \longrightarrow H_i^I(M) \longrightarrow \Lambda_I(K) \longrightarrow \Lambda_I(P_{i-1})$$

is exact, since the inverse limit is left exact. We know that $\Lambda_I(K)$ has the structure of $\Lambda_I(R)-$module, hence $H_i^I(M)$ has a structure of $\Lambda_I(R)-$module as a submodule of $\Lambda_I(K)$.
(ii) If M is a noetherian $R-$module, then $H_i^I(M) = 0$ for all $i > 0$ (see [6, 3.2 (ii)]).

Let M be a linearly compact $R-$module. Then $\operatorname{Tor}_i^R(R/I^t; M)$ is also a linearly compact $R-$module by the topology defined as in (2.6), so we have a induced topology on the local homology module $H_i^I(M)$. Moreover, we will see that $H_i^I(M)$ is also a linearly compact $R-$module.

Proposition 4.3. *If M is a linearly compact $R-$module, then for all $i \geq 0$, $H_i^I(M)$ is a linearly compact $R-$module.*

Proof. It follows from (2.6) that $\{\operatorname{Tor}_i^R(R/I^t; M)\}_t$ forms an inverse system of linearly compact modules with continuous homomorphisms. Hence $H_i^I(M)$ is also a linearly compact $R-$module by (2.3) (iv). $\square$

The following proposition shows that local homology modules commute with inverse limits of inverse systems of linearly compact $R-$modules with continuous homomorphisms.

Proposition 4.4. *If $\{M_s\}$ is a system of linearly compact $R-$modules with the continuous homomorphisms, then*

$$H_i^I(\varprojlim_s M_s) \cong \varprojlim_s H_i^I(M_s).$$

Proof. By (2.7) we have

$$\begin{aligned} H_i^I(\varprojlim_s M_s) &= \varprojlim_t \operatorname{Tor}_i^R(R/I^t; \varprojlim_s M_s) \\ &\cong \varprojlim_t \varprojlim_s \operatorname{Tor}_i^R(R/I^t; M_s) \\ &\cong \varprojlim_s \varprojlim_t \operatorname{Tor}_i^R(R/I^t; M_s) = \varprojlim_s H_i^I(M_s). \end{aligned}$$

The proof is complete. $\square$

Let $L_i^I(M)$ be the $i-$th derived module of the $I-$adic completion of M. We will show that in case M is linearly compact, the local homology module $H_i^I(M)$ is isomorphic to the module $L_i^I(M)$, hence our definition of local homology modules is coincident with the definition of J. P. C. Greenlees and J. P. May (see [9, 2.4]).

Proposition 4.5. *Let M be a linearly compact $R-$module. Then*

$$H_i^I(M) \cong L_i^I(M)$$

for all $i \geq 0$.

Proof. For all $i \geq 0$ we have a short exact sequence by [9, 1.1],

$$0 \longrightarrow \varprojlim_t{}^1 \operatorname{Tor}_{i+1}^R(R/I^t; M) \longrightarrow L_i^I(M) \longrightarrow H_i^I(M) \longrightarrow 0.$$

Moreover, it follows from (2.6) that $\{\operatorname{Tor}_{i+1}^R(R/I^t; M)\}$ forms an inverse system of linearly compact modules with continuous homomorphisms. Hence

$$\varprojlim_t{}^1 \operatorname{Tor}_{i+1}^R(R/I^t; M) = 0$$

by [11, 7.1]. This finishes the proof. $\square$

The following corollary is an immediate consequence of (4.5).

Corollary 4.6. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$$

be a short exact sequence of linearly compact modules. Then we have a long exact sequence of local homology modules

$$\cdots \longrightarrow H_i^I(M') \longrightarrow H_i^I(M) \longrightarrow H_i^I(M") \longrightarrow$$

$$\cdots \longrightarrow H_0^I(M') \longrightarrow H_0^I(M) \longrightarrow H_0^I(M") \longrightarrow 0.$$

The following lemma shows that local homology modules $H_i^I(M)$ are Λ_I-acyclic for all $i > 0$.

Lemma 4.7. *Let M be a linearly compact $R-$module. Then for all $j \geq 0$,*

$$H_i^I(H_j^I(M)) \cong \begin{cases} H_j^I(M), & i = 0, \\ 0, & i > 0. \end{cases}$$

Proof. It follows from (2.6) that $\left\{ \operatorname{Tor}_j^R(R/I^t; M)\right\}_t$ forms an inverse system of linearly compact $R-$modules with the continuous homomorphisms. Then we have by (4.4) and (4.1) (i),

$$\begin{aligned} H_i^I(H_j^I(M)) &= H_i^I(\varprojlim_t \operatorname{Tor}_j^R(R/I^t; M)) \\ &\cong \varprojlim_t H_i^I(\operatorname{Tor}_j^R(R/I^t; M)) \\ &\cong \varprojlim_t \varprojlim_s H_i(\underline{x}(s); \operatorname{Tor}_j^R(R/I^t; M)), \end{aligned}$$

in which $\underline{x} = (x_1, \ldots, x_r)$ be a system of generators of I and $\underline{x}(s) = (x_1^s, \ldots, x_r^s)$. Since $\underline{x}(s)\operatorname{Tor}_j^R(R/I^t; M) = 0$ for all $s \geq t$, we get

$$\varprojlim_s H_i(\underline{x}(s); \operatorname{Tor}_j^R(R/I^t; M)) \cong \begin{cases} \operatorname{Tor}_j^R(R/I^t; M), & i = 0, \\ 0, & i > 0. \end{cases}$$

This finishes the proof. □

Lemma 4.8. *Let M be a linearly compact $R-$module. Then*

$$H_i^I(\bigcap_{t>0} I^t M) \cong \begin{cases} 0, & i = 0, \\ H_i^I(M), & i > 0. \end{cases}$$

Proof. The short exact sequence of linearly compact $R-$modules for all $t > 0$,

$$0 \longrightarrow I^t M \longrightarrow M \longrightarrow M/I^t M \longrightarrow 0$$

induces by (2.4) a short exact sequence of linearly compact $R-$modules

$$0 \longrightarrow \bigcap_{t>0} I^t M \longrightarrow M \longrightarrow \Lambda_{\mathrm{I}}(M) \longrightarrow 0,$$

since $\varprojlim_{\mathrm{t}} I^t M = \bigcap_{t>0} I^t M$ by [23, exercise 2.50]. Then we get a long exact sequence of local homology modules

$$\cdots \longrightarrow H_{i+1}^I(\Lambda_{\mathrm{I}}(M)) \longrightarrow H_i^I(\bigcap_{t>0} I^t M) \longrightarrow H_i^I(M) \longrightarrow H_i^I(\Lambda_{\mathrm{I}}(M)) \longrightarrow$$

$$\cdots \longrightarrow H_1^I(\Lambda_{\mathrm{I}}(M)) \longrightarrow H_0^I(\bigcap_{t>0} I^t M) \longrightarrow H_0^I(M) \longrightarrow H_0^I(\Lambda_{\mathrm{I}}(M)) \longrightarrow 0.$$

The result now follows from (4.7). □

The following theorem gives us a character of $I-$separated modules.

Theorem 4.9. *Let M be a linearly compact $R-$module. The following statements are equivalent:*

(i) M is $I-$separated, i.e., $\bigcap_{t>0} I^t M = 0$.

(ii) $\Lambda_{\mathrm{I}}(M) \cong M$.

(iii) $H_0^I(M) \cong M$, $H_i^I(M) = 0$ for all $i > 0$.

Proof. $(ii) \Leftrightarrow (i)$ is clear from the short exact sequence

$$0 \longrightarrow \bigcap_{t>0} I^t M \longrightarrow M \longrightarrow \Lambda_{\mathrm{I}}(M) \longrightarrow 0.$$

$(i) \Rightarrow (iii)$. From (i) we have $H_0^I(M) = \Lambda_{\mathrm{I}}(M) \cong M$. Combining (4.8) with (i) gives $H_i^I(M) \cong H_i^I(\bigcap_{t>0} I^t M) = 0$ for all $i > 0$.

$(iii) \Rightarrow (ii)$ is trivial. □

Corollary 4.10. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated $R-$module. Then M is a linearly compact $R-$module if and only if M is complete in $\mathfrak{m}-$adic topology (i.e., $\Lambda_{\mathfrak{m}}(M) \cong M$).*

Proof. Since M is a finitely generated $R-$module, M is $\mathfrak{m}-$separated. Thus If M is a linearly compact $R-$module, then $\Lambda_{\mathfrak{m}}(M) \cong M$ by (4.9). Conversely, if M is complete in $\mathfrak{m}-$adic topology, we have $M \cong \varprojlim_t M/\mathfrak{m}^t M$. Therefore M is a linearly compact $R-$module by (2.3) (iv), because $M/\mathfrak{m}^t M$ are artinian $R-$modules. □

5. Vanishing and non-vanishing of local homology modules

We first recall the concept of *Krull dimension* of an $R-$module M, denoted by $\operatorname{Ndim} M$, due to R. N. Roberts [22]: Let M be an $R-$module. When $M = 0$ we put $\operatorname{Ndim} M = -1$. Then by induction, for any ordinal α, we put $\operatorname{Ndim} M = \alpha$ when (i) $\operatorname{Ndim} M < \alpha$ is false, and (ii) for every ascending chain $M_0 \subseteq M_1 \subseteq \dots$ of submodules of M, there exists a positive integer m_0 such that $\operatorname{Ndim}(M_{m+1}/M_m) < \alpha$ for all $m \geq m_0$. Thus M is non-zero and noetherian if and only if $\operatorname{Ndim} M = 0$. If $0 \longrightarrow M" \longrightarrow M \longrightarrow M' \longrightarrow 0$ is a short exact sequence of $R-$modules, then $\operatorname{Ndim} M = \max\{\operatorname{Ndim} M", \operatorname{Ndim} M'\}$. In case M is an artinian module, $\operatorname{Ndim} M < \infty$ (see [22]). Moreover if M is a semi-discrete linearly compact module, then $\operatorname{Ndim} M < \infty$ (see [7]).

Lemma 5.1. *Let M be an $R-$module with $\operatorname{Ndim} M = d > 0$ and $x \in R$ such that $xM = M$. Then*

$$\operatorname{Ndim} 0 :_M x \leq d - 1.$$

Proof. The argument is analogous to the proof of [22, 4]. Consider the ascending chain

$$0 \subseteq 0 :_M x \subseteq 0 :_M x^2 \subseteq \dots .$$

Since $\operatorname{Ndim} M = d$, there exists a positive integer n such that

$$\operatorname{Ndim}(0 :_M x^{n+1}/0 :_M x^n) \leq d - 1.$$

But $xM = M$, so the homomorphism

$$0 :_M x^{n+1}/0 :_M x^n \overset{x^n}{\longrightarrow} 0 :_M x$$

is an isomorphism. Therefore $\operatorname{Ndim} 0 :_M x \leq d - 1$. □

In order to state the vanishing theorem and non-vanishing theorem, we need the following lemmas. Recall that a module M is called *simple* if M have only trivial submodules. Denote by $\operatorname{Soc}(M)$ the sum of all simple submodules of M.

Lemma 5.2. *Let M be a semi-discrete linearly compact $R-$module. Then $H_0^I(M) = 0$ if and only if $xM = M$ for some $x \in I$.*

Proof. By [6, 2.5], $H_0^I(M) = 0$ if and only if $IM = M$. Therefore, the result follows from (3.3) and [5, 2.9]. □

Lemma 5.3. *If M is a semi-discrete linearly compact $R-$module such that* $\operatorname{Soc}(M) = 0$, *then*

$$H_i^I(M) = 0$$

for all $i > 0$.

Proof. Combining (4.8) with (5.2), we may assume that there is an $x \in I$ such that $xM = M$. It follows from [31, 1.6(b)] that $0 :_M x = 0$. Thus we have an isomorphism $M \overset{x}{\cong} M$. It induces the following isomorphism

$$H_i^I(M) \overset{x}{\cong} H_i^I(M)$$

for all $i > 0$. Therefore we have by (4.1) (ii),

$$H_i^I(M) = xH_i^I(M) = \bigcap_{t>0} x^t H_i^I(M) = 0$$

for all $i > 0$. □

Lemma 5.4. *Let M be a semi-discrete linearly compact $R-$module. Then*

$$H_i^I(M) \cong H_i^I(L(M))$$

for all $i > 0$, and the following sequence is exact

$$0 \longrightarrow H_0^I(L(M)) \longrightarrow H_0^I(M) \longrightarrow H_0^I(M/L(M)) \longrightarrow 0.$$

Proof. The short exact sequence of linearly compact $R-$modules

$$0 \longrightarrow L(M) \longrightarrow M \longrightarrow M/L(M) \longrightarrow 0$$

gives rise to a long exact sequence of local homology modules

$$\ldots \longrightarrow H_{i+1}^I(M/L(M)) \longrightarrow H_i^I(L(M)) \longrightarrow H_i^I(M) \longrightarrow H_i^I(M/L(M)) \longrightarrow \ldots.$$

Since $\operatorname{Soc}(M/L(M)) = 0$, we have $H_i^I(M/L(M)) = 0$ for all $i > 0$ by (5.3). This finishes the proof. □

In [6, 4.8] we proved that if M is an artinian $R-$module with $\operatorname{Ndim} M = d$, then $H_i^I(M) = 0$ for all $i > d$. In the case of linearly compact $R-$modules we also have a similar result but more general.

Theorem 5.5. *Let M be a linearly compact $R-$module with* $\operatorname{Ndim} M = d$. *Then*

$$H_i^I(M) = 0$$

for all $i > d$.

Proof. We know that a linearly compact $R-$module is an inverse limit of discrete linearly compact $R-$modules, i. e., $M = \varprojlim_{U \in \mathcal{M}} M/U$ (see [16, 3.11]). Then

$$H_i^I(M) \cong \varprojlim_{U \in \mathcal{M}} H_i^I(M/U)$$

by (4.4). M/U is a discrete linearly compact $R-$module with $\operatorname{Ndim} M/U \leq \operatorname{Ndim} M$. Thus we only need to prove in the case M is a discrete linearly compact $R-$module. Therefore we have the isomorphisms for all $i > 0$ by (5.4),

$$H_i^I(M) \cong H_i^I(L(M)).$$

Since $L(M)$ is an artinian module and $\operatorname{Ndim} L(M) \leq \operatorname{Ndim} M = d$, the result follows from [6, 4.8]. □

In [6, 4.10] we proved that if M is a non-zero artinian $R-$module over a local ring $(R, \mathfrak{m})$ with $\operatorname{Ndim} M = d$, then $H_d^{\mathfrak{m}}(M) \neq 0$. In the following theorem we get a more general result for semi-discrete linearly compact $R-$modules.

Theorem 5.6. *Let $(R, \mathfrak{m})$ be a local ring and M a semi-discrete linearly compact $R-$module with* $\operatorname{Ndim} L(M) = d \geq 0$. *Then $H_d^{\mathfrak{m}}(M) \neq 0$ and $H_i^{\mathfrak{m}}(M) = 0$ for all $i > d$.*

Proof. In virtue of (5.4), we have the isomorphisms $H_i^{\mathfrak{m}}(M) \cong H_i^{\mathfrak{m}}(L(M))$ for all $i > 0$ and the exact $0 \longrightarrow H_0^{\mathfrak{m}}(L(M)) \longrightarrow H_0^{\mathfrak{m}}(M)$. Because $L(M)$ is an artinian module, we get $H_d^{\mathfrak{m}}(L(M)) \neq 0$ and $H_i^{\mathfrak{m}}(L(M)) = 0$ for all $i > d$ by [6, 4.8, 4.10]. This finishes the proof. □

Remark 5.7. Let $(R, \mathfrak{m})$ and M as in (5.6). Then $\operatorname{Soc}(M/L(M)) = 0$, so $\dim R/\mathfrak{p} \leq 1$ for all $\mathfrak{p} \in \operatorname{Coass}(M)$ by [31,1.6 (a)]. Therefore $\operatorname{Ndim}(M/L(M)) \leq 1$ by [7, 2.9]. Thus if $\operatorname{Ndim} M > 1$ or $\operatorname{Ndim} M = 0$, then $\operatorname{Ndim} L(M) = \operatorname{Ndim} M$, hence

$$\operatorname{Ndim} M = \max\big\{i \mid H_i^{\mathfrak{m}}(M) \neq 0\big\}.$$

However, in case $\operatorname{Ndim} M = 1$ the equality may not true.

A sequence of elements $x_1, \dots, x_r$ in R is said to be an $M-$*coregular* sequence (see [21, 3.1]) if $0 :_M (x_1, \dots, x_r) \neq 0$ and $0 :_M (x_1, \dots, x_{i-1}) \xrightarrow{x_i} 0 :_M (x_1, \dots, x_{i-1})$ is surjective for $i = 1, \dots, r$. We denote by $\mathrm{Width}_I(M)$ the length of the longest $M-$coregular sequences in I. In case M is a semi-discrete linearly compact $R-$module, we know that $\mathrm{Ndim}\, M < \infty$. Therefore $\mathrm{Width}_I(M) < \infty$ by (5.1). We have the following theorem.

Theorem 5.8. *Let M be a semi-discrete linearly compact $R-$module and I an ideal of R such that $0 :_M I \neq 0$. Then all maximal $M-$coregular sequences in I have the same length. Moreover*

$$\mathrm{Width}_I(M) = \inf\{i / H_i^I(M) \neq 0\}.$$

proof. It is sufficient to prove that if $(x_1, x_2, \dots, x_n) \subseteq I$ is a maximal $M-$coregular sequence, then (i) $H_i^I(M) = 0$ for all $i < n$, and (ii) $H_n^I(M) \neq 0$. We prove both (i) and (ii) by the induction on n. When $n = 0$, there does not exists any element x in I such that $xM = M$. Therefore $H_0^I(M) \neq 0$ by (5.2).

Let $n > 0$. The short exact sequence

$$0 \longrightarrow 0 :_M x_1 \longrightarrow M \xrightarrow{x_1} M \longrightarrow 0$$

gives rise to a long exact sequence

$$\dots \longrightarrow H_i^I(0 :_M x_1) \longrightarrow H_i^I(M) \xrightarrow{x_1} H_i^I(M) \longrightarrow H_{i-1}^I(0 :_M x_1) \longrightarrow \dots .$$

By the inductive hypothesis, we have $H_i^I(0 :_M x_1) = 0$ for all $i < n-1$ and
$H_{n-1}^I(0 :_M x_1) \neq 0$. Hence $H_i^I(M) = x_1 H_i^I(M)$ for all $i < n$. From (4.1), (ii) we have $H_i^I(M) = \bigcap_{t>0} x_1^t H_i^I(M) = 0$ for all $i < n$. We now have the exact sequence

$$\dots \longrightarrow H_n^I(M) \xrightarrow{x_1} H_n^I(M) \longrightarrow H_{n-1}^I(0 :_M x_1) \longrightarrow 0.$$

Since $H_{n-1}^I(0 :_M x_1) \neq 0$, we get $H_n^I(M) \neq 0$. This finishes the proof. $\square$

6. Noetherian local homology modules

We know that $H_i^I(M)$ has the natural structure as a module over $\Lambda_I(R)$. Especially, $H_i^{\mathfrak{m}}(M)$ is an $\widehat{R}-$module. In [6, 4.6] we proved that if M is an artinian $R-$module over a local ring $(R,\mathfrak{m})$, then $H_i^{\mathfrak{m}}(M)$ is a noetherian $\widehat{R}-$module for all $i \geq 0$. This is also true in the case of semi-discrete linearly compact $R-$modules.

Theorem 6.1. *Let $(R,\mathfrak{m})$ be a local noetherian ring and M a semi-discrete linearly compact $R-$module. Then $H_i^{\mathfrak{m}}(M)$ is a noetherian $\widehat{R}-$module for all $i \geq 0$.*

Proof. When $i = 0$, we have

$$H_0^{\mathfrak{m}}(M) \cong \Lambda_{\mathfrak{m}}(M).$$

Since M is a semi-discrete linearly compact $R-$module, $M/\mathfrak{m}M$ is also a linearly compact semi-discretee $k-$module ($k = R/\mathfrak{m}$ the residue field). Then by [16, 5.2] $M/\mathfrak{m}M$ is a finite dimensional vector $k-$space. Therefore $\Lambda_{\mathfrak{m}}(M)$ is a Noetherian $\widehat{R}-$module because of [8, 7.2.9].

If $i > 0$, we have the isomorphism $H_i^I(M) \cong H_i^I(L(M))$ by (5.4). Since $L(M)$ is artinian, the result follows from [6, 4.6]. □

Two lemmas are used to prove Theorem 6.4.

Lemma 6.2. (see [12, Theorem 1, (i)]) *Let A be a commutative ring and $A[T_1,\dots,T_s]$ a polynomial ring of variables $T_1,\dots,T_s$. Let $K = \bigoplus_{t=-\infty}^{+\infty} K_t$ be a graded $A[T_1,\dots,T_s]-$module. Then K is a noetherian $A[T_1,\dots,T_s]-$module if and only if there exist integers k,l such that*

(i) $K_t = 0$ *for* $t < k$;

(ii) $K_{t+1} = \sum_{i=1}^{s} T_i K_t$ *for* $t \geq l$;

(iii) K_t *is a noetherian $A-$module for* $k \leq t \leq l$.

The following lemma is a criterion for a module to be noetherian.

Lemma 6.3. *Let J be a finitely generated ideal of a commutative ring A such that A is complete in the $J-$adic topology and M an $A-$module. If M/JM is a noetherian $A-$module and M is $J-$separated (i.e., $\bigcap_{t>0} J^tM = 0$), then M is a noetherian $A-$module.*

Proof. Set

$$K = \bigoplus_{t\geq 0} J^tM/J^{t+1}M$$

the associated graded module over the graded ring

$$Gr_J(A) = \bigoplus_{t\geq 0} J^t/J^{t+1}.$$

Let $x_1, x_2, \dots, x_s$ be a system of generators of J and $(A/J)[T_1, \dots, T_s]$ the polynomial ring of variables $T_1, T_2, \dots, T_s$. The natural epimorphism

$$g : (A/J)[T_1, \dots, T_s] \longrightarrow Gr_J(A)$$

leads K to be an $(A/J)[T_1, \dots, T_s]-$module. We write $K_t = J^tM/J^{t+1}M$ for all $t \geq 0$, then $K_0 = M/JM$ is a noetherian $A/J-$module by the hypothesis. On the other hand, it is easy to check that

$$K_{t+1} = \sum_{i=1}^{s} T_i K_t$$

for all $t \geq 0$. Thus K satisfies the conditions of (6.2). Therefore K is a noetherian $(A/J)[T_1, \dots, T_s]-$module and so K is a noetherian $Gr_J(A)-$module. Moreover M is $J-$separated by the hypothesis. It follows from [2, 10.25] that, M is a noetherian $A-$module. This finishes the proof. □

Theorem 6.4. *Let M be a semi-discrete linearly compact $R-$module with* $\operatorname{Ndim} M = d$, *then $H_d^I(M)$ is a noetherian $\Lambda_I(R)-$module.*

Proof. When $d = 0$, we have M is a noetherian $R-$module. So M is $I-$separated. Thus $H_0^I(M) \cong M$ by (4.9). Therefore $H_0^I(M)$ is a noetherian $\Lambda_I(R)-$module.

Let $d > 0$. Combining (4.8) with (5.2), we may assume that there is an element $x \in I$ such that $xM = M$. Now the short exact sequence

$$0 \longrightarrow 0 :_M x \longrightarrow M \xrightarrow{x} M \longrightarrow 0$$

induces an exact sequence

$$H_d^I(M) \xrightarrow{x} H_d^I(M) \xrightarrow{\delta} H_{d-1}^I(0 :_M x).$$

If $0 :_M x = 0$, then $H_i^I(M) = xH_i^I(M) = \bigcap_{t>0} x^t H_i^I(M) = 0$ for all $i \geq 0$ by (4.1) (ii). Suppose now that $0 :_M x \neq 0$. Since $\operatorname{Ndim}(0 :_M x) \leq d-1$ by (5.1), $H_{d-1}^I(0 :_M x)$ is a noetherian $\Lambda_I(R)-$module because of the inductive hypothesis. Set $G = H_d^I(M)$, then $G/xG =$

$G/\operatorname{Ker}\delta \cong \operatorname{Im}\delta \subseteq H^I_{d-1}(0:_M x)$. Hence G/xG is a noetherian $\Lambda_I(R)-$module. Thus G/JG is a noetherian $\Lambda_I(R)-$module ($J = I\Lambda_I(R)$). Moreover, $\bigcap_{t>0} J^tG = \bigcap_{t>0} I^tG = 0$ by (4.1) (ii) and $\Lambda_I(R)$ is complete in $J-$adic topology. Therefore, G is a noetherian $\Lambda_I(R)-$module by (6.3). This finishes the proof. □

7. Duality

Henceford $(R,\mathfrak{m})$ will be a local ring, $\mathfrak{m}$ its maximal ideal. Suppose now that the topology on R is the $\mathfrak{m}-$adic topology.

a) Duality

Let M be an $R-$module and $E(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. The module $D(M) = \operatorname{Hom}(M; E(R/\mathfrak{m}))$ is called *Matlis dual* of M.

If M is a Hausdorff linearly topology $R-$module, then in [16] I. G. Macdonald defined $M^* = Hom(M; E(R/\mathfrak{m}))$ the set of continuous homomorphisms of $R-$modules. It's clear that $M^* \subseteq D(M)$, the equality holds when M is semi-discrete (Lemma 7.1). In case $(R,\mathfrak{m})$ is local complete, the topology on M^* is defined as in [16, 8.1]. Moreover, if M is semi-discrete, then the topology of M^* coincides with that induced on it as a submodule of $E(R/\mathfrak{m})^M$, where $E(R/\mathfrak{m})^M = \prod_{x\in M}(E(R/\mathfrak{m}))_x$, $(E(R/\mathfrak{m}))_x = E(R/\mathfrak{m})$ for all $x \in M$ (see [16, 8.6]).

A homomorphism of Hausdorff linearly topologized modules $f: M \longrightarrow N$ is called open if the image of an open set of M is an open set of $f(M)$. If M is a linearly compact $R-$module, then the homomorphism f is always open (see [16, §4, §5]).

Lemma 7.1. (see [16, 5.8]) *A Hausdorff linearly topologized $R-$module M is semi-discrete if and only if $D(M) = M^*$.*

Lemma 7.2. (see [16, 5.7]) *Let M be a Hausdorff linearly topologized $R-$module and $u: M \longrightarrow A^*$ a homomorphism. Then the following statements are equivalent:*

a) u is continuous, *b)* $\operatorname{Ker} u$ *is open,* *c)* $\operatorname{Ker} u$ *is closed.*

A Hausdorff linearly topologized$R-$module is $\mathfrak{m}-$*primary* if each element of M is annihilated by a power of $\mathfrak{m}$. A Hausdorff linearly topologized $R-$module M is *linearly discrete* if every $\mathfrak{m}-$primary quotient of M is discrete. If M is linearly discrete, then M is semi-discrete (see [16, 6.2]). The direct limit of a direct system of linearly discrete $R-$modules is linearly discrete (see [16, 6.7]). If $f : M \longrightarrow N$ is an epimorphism of Hausdorff linearly topologized $R-$modules in which M is linearly discrete, then f is continuous (see [16, 6.8]).

Lemma 7.3. (see [16, 9.12, 9.13]) *Let $(R, \mathfrak{m})$ be a complete local ring.*
(i) If M is linearly compact, then M^* is linearly discrete (hence semi-discrete). If M is semi-discrete, then M^* is linearly compact.
(ii) If M is linearly compact or linearly discrete, then we have a topological isomorphism $M^{**} \cong M$.

If M is a linearly topologized $R-$module, then the module $\mathrm{Ext}^i_R(R/I^t; M)$ is also a linearly topologized $R-$module by the topology defined as in (2.5). Therefore we have a induced topology on the local cohomology module $H^i_I(M) = \varinjlim_t \mathrm{Ext}^i_R(R/I^t; M)$. The following lemma shows that local homology modules $H^i_I(M)$ are linearly discrete when M is linearly discrete.

Lemma 7.4. *If M is a linearly discrete $R-$module, then local homology modules $H^i_I(M)$ are also linearly discrete $R-$modules for all $i \geq 0$.*

Proof. It should be noted that the direct of a direct system of linearly discrete $R-$modules and continuous homomorphisms is linearly discrete by [16, 6.7] and

$$H^i_I(M) = \varinjlim_t \mathrm{Ext}^i_R(R/I^t; M).$$

In virtue of (2.5) , we only need to prove that the modules $\mathrm{Ext}^i_R(R/I^t; M)$ are linearly discrete for all i, t. Indeed, Let $\mathbf{F}_{\bullet,t}$ be a free resolution of the module R/I^t in which the free modules are finitely generated. We have a complex $\mathrm{Hom}(\mathbf{F}_{\bullet,t}; M)$ of linearly discrete modules, since direct sums of linearly discrete modules are also linearly discrete by [16, 6.6]. Therefore the modules $\mathrm{Ext}^i_R(R/I^t; M) = H^i(\mathrm{Hom}(\mathbf{F}_{\bullet,t}; M))$ are linearly discrete, because quotient modules of linearly discrete modules are linearly discrete. The proof is complete. □

Next, we show that the functor $(-)^*$ is exact on the category of linearly compact $R-$modules and continuous homomorphisms.

Lemma 7.5. *Let*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

be a short exact sequence of linearly compact $R-$modules, in which the homomorphisms f, g are continuous. Then the induced sequence

$$0 \longrightarrow P^* \xrightarrow{g^*} N^* \xrightarrow{f^*} M^* \longrightarrow 0$$

is exact also.

Proof. By [16, 5.5], f is an open (continuous) mapping. Then there is a topological isomorphism $M \cong f(M)$. Replacing M by $f(M)$, we may assume that M is a close submodule of N. It is easy to see that g^* is injective. For every continuous homomorphism $h : M \longrightarrow E(R/\mathfrak{m})$, there is a continuous homomorphism $\varphi : N \longrightarrow E(R/\mathfrak{m})$ which extends h by [16, 5.9]. Thus f^* is surjective. We now proceed to show that $\operatorname{Ker} f^* = \operatorname{Im} g^*$. It is clear that $\operatorname{Im} g^* \subseteq \operatorname{Ker} f^*$. So the lemma is completely proved if we show that $\operatorname{Ker} f^* \subseteq \operatorname{Im} g^*$. Let $\psi \in \operatorname{Ker} f^*$, we have $\psi(\operatorname{Ker} g) = \psi(f(M)) = 0$. Then ψ induces a homomorphism $\phi : P \longrightarrow E(R/\mathfrak{m})$ such that $\phi g = \psi$. It follows $\operatorname{Ker} \phi = g(\operatorname{Ker} \psi)$. Since ψ is continuous, $\operatorname{Ker} \psi$ is open by (7.2). Moreover, g is open, so $\operatorname{Ker} \phi$ is also open. Therefore ϕ is continuous by (7.2). Thus $\psi \in \operatorname{Im} g^*$. This finishes the proof. □

Submodules and quotient modules of a semi-discrete module are also semi-discrete. In case of linearly compact $R-$modules we have the following converse.

Lemma 7.6. *Let*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

be a short exact sequence of linearly compact $R-$modules, in which the homomorphisms f, g are continuous. Then N is semi-discrete if and only if M and P are semi-discrete.

Proof. The "only if" part is clear. We now prove the "if" part. It follows from (7.1) and the hypothesis that $P^* = D(P)$ and $M^* = D(M)$. We now have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P^* & \xrightarrow{g^*} & N^* & \xrightarrow{f^*} & M^* & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle j} & & \| & & \\
0 & \longrightarrow & D(P) & \xrightarrow{D(g)} & D(N) & \xrightarrow{D(f)} & D(M) & \longrightarrow & 0,
\end{array}$$

in which j is an inclusion and rows are exact by (7.5) and [23, 3.16]. It follows that $N^* = D(N)$. Thus N is semi-discrete by (2.1) (i). The proof is complete. □

Lemma 7.7. *Let N be a finitely generated $R-$module and M a linearly compact $R-$module. Then*

$$(\mathrm{Tor}_i^R(N;M))^* \cong \mathrm{Ext}_R^i(N;M^*),$$

$$\mathrm{Tor}_i^R(N;M^*) \cong (\mathrm{Ext}_R^i(N;M))^*$$

for all $i \geq 0$.

Proof. Let

$$\mathbf{F}_\bullet : \cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

be a free resolution of N, in which the free $R-$modules F_i are finitely generated. We have the complex $\mathbf{F}_\bullet \bigotimes_R M$ of linearly compact $R-$modules, in which the differentials are continuous. Since the functor $(-)^*$ is exact on the category of linearly compact $R-$modules and the continuous homomorphisms by (7.5), we have

$$(H_i(\mathbf{F}_\bullet \otimes_R M))^* \cong H^i((\mathbf{F}_\bullet \otimes_R M)^*)$$

by [20, 6.1]. But it follows from [16, 2.5] that

$$(\mathbf{F}_\bullet \otimes_R M)^* \cong \mathrm{Hom}_R(\mathbf{F}_\bullet; M^*).$$

Therefore

$$\begin{aligned}(\mathrm{Tor}_i^R(N;M))^* &\cong (H_i(\mathbf{F}_\bullet \otimes_R M))^* \\ &\cong H^i(\mathrm{Hom}_R(\mathbf{F}_\bullet; M^*)) \\ &\cong \mathrm{Ext}_R^i(N;M^*).\end{aligned}$$

The proof of the second isomorphism is similar. □

We now can prove the following duality theorem.

Theorem 7.8. *(i)* (see [6, 3.3 (ii)]) *Let M be an $R-$module. Then for all $i \geq 0$,*

$$H_i^I(D(M)) \cong D(H_I^i(M)).$$

(ii) If M is a linearly compact $R-$module, then for all $i \geq 0$,

$$H_i^I(M^*) \cong (H_I^i(M))^*.$$

Moreover, if $(R, \mathfrak{m})$ *is a complete local ring, then*

$$H_I^i(M^*) \cong (H_i^I(M))^*.$$

(iii) If $(R, \mathfrak{m})$ *is a complete local ring and* M *a semi-discrete linearly compact* $R-$*module, then we have topological isomorphisms of* $R-$*modules for all* $i \geq 0$,

$$H_I^i(M^*) \cong (H_i^I(M))^*,$$

$$H_i^I(M^*) \cong (H_I^i(M))^*.$$

Proof. (i) was proved in [6, 3.3 (ii)].

(ii). To prove the first isomorphism we note that for a direct system $\{N_t\}$ of Hausdorff linearly topologized $R-$modules with the continuous homomorphisms, $\varprojlim_{\mathrm{t}} N_t^* \cong (\varinjlim_{\mathrm{t}} N_t)^*$ by [16, 2.6]. Moreover, it follows from (2.5) that $\{\operatorname{Ext}_R^i(R/I^t; M)\}$ is an inverse system of linearly compact $R-$modules with continuous homomorphisms. Therefore we get by (7.7),

$$\begin{aligned} H_i^I(M^*) &= \varprojlim_t \operatorname{Tor}_i^R(R/I^t; M^*) \\ &\cong \varprojlim_t (\operatorname{Ext}_R^i(R/I^t; M))^* \\ &\cong (\varinjlim_t \operatorname{Ext}_R^i(R/I^t; M))^* \\ &= H_I^i(M)^*. \end{aligned}$$

The second isomorphism is proved similarly. Note that for an inverse system $\{M_t\}$ of linearly compact $R-$modules with continuous homomorphisms we have $(\varprojlim_{\mathrm{t}} M_t)^* \cong \varinjlim_{\mathrm{t}} M_t^*$ by [16, 9.14], and $\{\operatorname{Tor}_i^R(R/I^t; M)\}$ is an inverse system of linearly compact $R-$modules with continuous homomorphisms by (2.6). Therefore in virtue of (7.7) we get

$$\begin{aligned} H_I^i(M^*) &= \varinjlim_{\mathrm{t}} \operatorname{Ext}_R^i(R/I^t; M^*) \\ &\cong \varinjlim_{\mathrm{t}} (\operatorname{Tor}_i^R(R/I^t; M))^* \\ &\cong (\varprojlim_{\mathrm{t}} \operatorname{Tor}_i^R(R/I^t; M))^* \\ &= (H_i^I(M))^*. \end{aligned}$$

(iii). We shall prove the first isomorphism. We've just proved it's an algebraic isomorphism. Therefore we only need to show that both $H^i_I(M^*)$ and $(H^I_i(M))^*$ are linearly discrete by [16, 6.8]. Indeed, combining (4.3) with (7.3) (i) we have $H^I_i(M)^*$ is linearly discrete. On the other hand M^* is also linearly discrete. Then the local homology modules $H^i_I(M^*)$ are linearly discrete, because of (7.4). Thus the first isomorphism is proved completely. The second isomorphism follows from the first isomorphism and (7.3) (ii). □

Corollary 7.9. *Let $(R, \mathfrak{m})$ be a complete local ring.*
(i) If M is linearly compact $R-$module, then for all $i \geq 0$,

$$H^i_I(M) \cong (H^I_i(M^*))^*,$$

$$H^I_i(M) \cong (H^i_I(M^*))^*.$$

(i) If M is a semi-discrete linearly compact $R-$module, then we have topological isomorphisms of $R-$modules for all $i \geq 0$,

$$H^i_I(M) \cong (H^I_i(M^*))^*,$$

$$H^I_i(M) \cong (H^i_I(M^*))^*.$$

Proof. (i) follows from (7.7) (ii), (7.3) (ii) and (7.4).
(ii) follows from (7.8) (iii) and (7.3) (ii). □

Corollary 7.10. *If M is a semi-discrete linearly compact $R-$module, then for all $i \geq 0$,*

$$H^i_I(M)^* = D(H^i_I(M)).$$

Proof. Since M is a semi-discrete linearly compact $R-$module, $M^* = D(M)$ by (7.1). We have the commutative diagram

$$\begin{array}{ccc} (H^i_I(M))^* & \stackrel{j}{\hookrightarrow} & D(H^i_I(M)) \\ f \uparrow\cong & & g \uparrow\cong \\ H^I_i(M^*) & = & H^I_i(D(M)), \end{array}$$

in which j is the inclusion and f, g are the isomorphism in (7.8). It follows

$$(H^i_I(M))^* = D(H^i_I(M)). \quad \square$$

b) Applications for local cohomology

We now can extend some well-known results in local cohomology theory. To prove proposition 7.12, the following lemma is necessary.

Lemma 7.11. (see [16, 5.6]) *Let M be a Hausdorff linearly topologized $R-$module. Then* $M^* = 0 \Leftrightarrow M = 0$.

Proposition 7.12. *Let M be a semi-discrete linearly compact $R-$module with* $\operatorname{Ndim} L(M^*) = d \geq 0$. *Then $H^d_{\mathfrak{m}}(M) \neq 0$ and $H^i_{\mathfrak{m}}(M) = 0$ for all $i > d$.*

Proof. We first note that M is a semi-discrete linearly compact $\widehat{R}-$module by (3.5) (ii). Moreover, the natural homomorphism $f : R \longrightarrow \widehat{R}$ gives by [3, 4.2.1] an isomorphism $H^d_{\mathfrak{m}}(M) \cong H^d_{\hat{\mathfrak{m}}}(M)$. Therefore, we may assume without any loss of generality that $(R, \mathfrak{m})$ is a complete local ring. Since M is a semi-discrete linearly compact $R-$module, M^* is also a semi-discrete linearly compact $R-$module by (7.3) (i). Then $(H^d_{\mathfrak{m}}(M))^* \neq 0$ and $(H^i_{\mathfrak{m}}(M))^* = 0$ for all $i > d$ because of (5.6) and (7.8) (ii), so $H^d_{\mathfrak{m}}(M) \neq 0$ and $H^i_{\mathfrak{m}}(M) = 0$ for all $i > d$ by (7.11). □

¿From (7.12) we get back the Grothendieck's non-vanishing Theorem in the local cohomology theory.

Corollary 7.13. (see [3, 6.1.4]) *Let M be a non-zero noetherian $R-$module with* $\dim M = d$. *Then $H^d_{\mathfrak{m}}(M) \neq 0$.*

Proof. Let $\widehat{M}$ be the $\mathfrak{m}-$adic completion of M, then $\widehat{M}$ is a finitely generated $\widehat{R}-$module with $\dim_{\hat{R}} M = d$. It follows from [4, 3.5.4 (d)] that $H^d_{\mathfrak{m}}(M) \cong H^d_{\hat{\mathfrak{m}}}(\widehat{M})$. Thus we may assume that R is complete. Since M is a noetherian $R-$module, $M^* = D(M)$ is an artinian $R-$module. Then $L(M^*) = M^* = D(M)$. Therefore

$$\operatorname{Ndim} L(M^*) = \operatorname{Ndim} D(M) = \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Coass}(D(M))\}$$

by [7, §2]. On the other hand, [28, 2.3] gives $\operatorname{Coass}(D(M)) = \operatorname{Ass} M$. Thus

$$\begin{aligned} \operatorname{Ndim} L(M^*) &= \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass} M\} \\ &= \dim M = d. \end{aligned}$$

Now the result follows from (7.12), because a finitely generated module over a complete local ring is semi-discrete linearly compact by [16, 7.3]. □

Proposition 7.14. *Let M be a semi-discrete linearly compact $R-$module. Then the local cohomology modules $H^i_{\mathfrak{m}}(M)$ are artinian $R-$modules for all $i \geq 0$.*

Proof. We first consider in case R is a complete ring. It follows from (7.1) and (7.3) (i) that $D(M)$ is a semi-discrete linearly compact $R-$module. Then $H_i^{\mathfrak{m}}(D(M))$ is a noetherian $R-$module for all $i \geq 0$ by (6.1). In virtue of (7.8) (i), $H_i^{\mathfrak{m}}(D(M)) \cong D(H^i_{\mathfrak{m}}(M))$. Therefore $H^i_{\mathfrak{m}}(M)$ is an artinian $R-$module by [26, 3.4.12].

Now let $(R, \mathfrak{m})$ be an arbitrarily local ring. By an argument analogous to the proof of (7.12), we have an isomorphism $H^d_{\mathfrak{m}}(M) \cong H^i_{\widehat{\mathfrak{m}}}(M)$ and M is a semi-discrete linearly compact $\widehat{R}-$module. By our claim $H^i_{\widehat{\mathfrak{m}}}(M)$ an artinian $\widehat{R}-$module, so $H^i_{\mathfrak{m}}(M)$ is an artinian $\widehat{R}-$module. But we know that $H^i_{\mathfrak{m}}(M)$ is $\mathfrak{m}-$primary (i.e., $H^i_{\mathfrak{m}}(M) = \bigcup_{t>0}(0 :_{H^i_{\mathfrak{m}}(M)} \mathfrak{m}^t)$). Then a subset of $H^i_{\mathfrak{m}}(M)$ is an $R-$submodule if and only if it is an $\widehat{R}-$module. Therefore $H^i_{\mathfrak{m}}(M)$ is an artinian $R-$module. □

The following consequence of Proposition 7.14 is a well-known result in the local cohomology theory.

Corollary 7.15. *Let M be a finitely generated $R-$module. Then the local cohomology modules $H^i_{\mathfrak{m}}(M)$ are artinian $R-$modules for all $i \geq 0$.*

Proof. By an argument analogous to that used for the beginning of the proof of Corollary 7.13, we may assume that R is complete. Therefore the proof follows immediately from (7.14), since a finitely generated module over a complete local ring is semi-discrete linearly compact by [16, 7.3]. □

Remark 7.16. There are many semi-discrete linearly compact $R-$modules which are neither artinian nor noetherian. For example, in case $(R, \mathfrak{m})$ is complete, let X_1 be an artinian $R-$module but not a finitely generated $R-$module and X_2 a finitely generated $R-$module but not an artinian $R-$module. Then both X_1 and X_2 are semi-discrete linearly compact, so the module $X = X_1 \oplus X_2$ is also semi-discrete linearly compact by (7.6), but X is neither artinian nor finitely generated. Further, let Q be a quotient module of X, then Q is also a semi-discrete linearly compact $R-$module.

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