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IC/2004/117

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

UNIQUENESS OF GIBBS MEASURE FOR POTTS MODEL WITH COUNTABLE SET OF SPIN VALUES

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Abstract

We consider a nearest-neighbor Potts model, with countable spin values $0, 1, \ldots$, and non zero external field, on a Cayley tree of order k (with $k+1$ neighbors). We study translation-invariant 'splitting' Gibbs measures. We reduce the problem to the description of the solutions of some infinite system of equations. For any $k \geq 1$ and any fixed probability measure ν with $\nu(i) > 0$ on the set of all non negative integer numbers $\Phi = \{0, 1, ...\}$ we show that the set of translation-invariant splitting Gibbs measures contains at most one point, independently on parameters of the Potts model with countable set of spin values on Cayley tree. Also we give a full description of the class of measures ν on Φ such that with respect to each element of this class our infinite system of equations has unique solution $\{a^i, i = 1, 2, ...\}$, where $a \in (0, 1)$.

MIRAMARE – TRIESTE
November 2004

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1 Introduction

The Potts model (with $q \geq 2$ spin values) was introduced as a generalization of the Ising model. The idea came from the representation of the Ising model as interacting spins which can be either parallel or antiparallel. An obvious generalization was to extend the number of directions of the spins. Such a model was proposed by C.Domb as a PhD thesis for his student R.Potts in 1952. At present the Potts model encompasses a number of problems in statistical physics and lattice theory (see, e.g. [10]). It has been a subject of increasing intense research interest in recent years.

One of the central problems in the theory of Gibbs measures is to describe infinite-volume (or limiting) Gibbs measures corresponding to a given Hamiltonian. The existence of such measures for a wide class of Hamiltonians was established in the ground-breaking work of Dobrushin (see, e.g. [5]). However, a complete analysis of the set of limiting Gibbs measures for a specific Hamiltonian is often a difficult problem.

The Potts model can be studied on both Z^d and Cayley tree Γ^k. In [2] for the Potts model with $q \geq 2$ spin values on the Cayley tree it was proven that there are q translation-invariant and uncountable many non translation-invariant extreme Gibbs measures. In [3] the Potts model with countable set Φ of spin values on Z^d was considered and it was proven that with respect to Poisson distribution on Φ the set of limiting Gibbs measure is not empty.

In this paper we consider Potts model with a nearest neighbor interaction and countable set of spin values on a Cayley tree.

A Cayley tree $\Gamma^k = (V, L)$ of order $k \geq 1$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k+1$ edges incident to each vertex. Here V is the set of vertices and L that of edges (arcs).

We consider a model where the spin takes values in the set of all non negative integer numbers $\Phi := \{0, 1, \dots\}$, and is assigned to the vertices of the tree. A configuration σ on V is then defined as a function $x \in V \mapsto \sigma(x) \in \Phi$; the set of all configurations is Φ^V. The (formal) Hamiltonian of the Potts model is :

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)} - \alpha \sum_{x\in V} \delta_{0\sigma(x)}, \tag{1}$$

where $J, \alpha \in R$ are constants. As usual, $\langle x, y\rangle$ stands for nearest neighbor vertices and δ is the Kroneker's symbol.

We consider a standard sigma-algebra $\mathcal{B}$ of subsets of Φ^V generated by the cylinder subsets. A probability measure μ on $(\Phi^V, \mathcal{B})$ is called a Gibbs measure (with Hamiltonian H) if it satisfies the DLR equation, namely for any $n = 1, 2, \dots$ and $\sigma_n \in \Phi^{V_n}$:

$$\mu\left(\left\{\sigma \in \Phi^V : \sigma|_{V_n} = \sigma_n\right\}\right) = \int_{\Phi^V} \mu(\mathrm{d}\omega)\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n),$$

where $\nu^{V_n}_{\omega|_{W_{n+1}}}$ is the conditional probability

$$\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n) = \frac{1}{Z_n\left(\omega|_{W_{n+1}}\right)} \exp\left(-\beta H\left(\sigma_n \,||\, \omega|_{W_{n+1}}\right)\right),$$

and $\beta = \frac{1}{T}$, $T > 0$ is temperature. Here and below, W_l stands for a 'sphere' and V_l for a 'ball' on the tree, of radius $l = 1, 2, \ldots$, centered at a fixed vertex x^0 (an origin):

$$W_l = \{x \in V : d(x, x^0) = l\}, \ \ V_l = \{x \in V : d(x, x^0) \leq l\};$$

and

$$L_n = \{\langle x, y\rangle \in L : x, y \in V_n\};$$

distance $d(x, y)$, $x, y \in V$, is the length of (i.e. the number of edges in) the shortest path connecting x with y. Φ^{V_n} is the set of configurations in V_n (and Φ^{W_n} that in W_n; see below). Furthermore, $\sigma|_{V_n}$ and $\omega|_{W_{n+1}}$ denote the restrictions of configurations $\sigma, \omega \in \Phi^V$ to V_n and W_{n+1}, respectively. Next, $\sigma_n : \ x \in V_n \mapsto \sigma_n(x)$ is a configuration in V_n and $H\left(\sigma_n \,||\, \omega|_{W_{n+1}}\right)$ is defined as the sum $H(\sigma_n) + U\left(\sigma_n, \omega|_{W_{n+1}}\right)$ where

$$H(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \delta_{\sigma_n(x)\sigma_n(y)} - \alpha \sum_{x \in V_n} \delta_{0\sigma_n(x)},$$

$$U\left(\sigma_n, \omega|_{W_{n+1}}\right) = -J \sum_{\langle x,y\rangle :\ x \in V_n, y \in W_{n+1}} \delta_{\sigma_n(x)\omega(y)}.$$

Finally, $Z_n\left(\omega|_{W_{n+1}}\right)$ stands for the partition function in V_n, with the boundary condition $\omega|_{W_{n+1}}$:

$$Z_n\left(\omega|_{W_{n+1}}\right) = \sum_{\widetilde{\sigma}_n \in \Phi^{V_n}} \exp\left(-\beta H\left(\widetilde{\sigma}_n \,||\, \omega|_{W_{n+1}}\right)\right).$$

According to the nearest-neighbour character of the interaction, the Gibbs measures possess a natural Markov property: for given a configuration ω_n on W_n, random configurations in V_{n-1} (i.e., 'inside' W_n) and in $V \setminus V_{n+1}$ (i.e., 'outside' W_n) are conditionally independent.

We use a standard definition of a translation-invariant measure (see, e.g., [5]). The main object of study in this paper are translation-invariant Gibbs measures for Potts model with countable set of spin values on Cayley tree .

2 An infinite system of functional equations

Following [4], [6] we consider a special class of Gibbs measures. These measures are called in [4], [6] Markov chains, in [8], [9] - entrance laws and in [7] - splitting Gibbs measures. In this paper we also call them splitting Gibbs measures.

Write $x < y$ if the path from x^0 to y goes through x. Call vertex y a direct successor of x if $y > x$ and x, y are nearest neighbors. Denote by $S(x)$ the set of direct successors of x. Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k + 1$.

For $A \subset V$ denote by Φ^A the configuration space on A. Let $h:\ x \mapsto h_x = (h_{0,x}, h_{1,x}, ...) \in R^\infty$ be a real sequence-valued function of $x \in V \setminus \{x^0\}$. Fix a probability measure $\nu = \{\nu(i) > 0, i \in \Phi\}$. Given $n = 1, 2, \ldots$, consider the probability distribution μ_n on Φ^{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\right) \prod_{x \in V_n} \nu(\sigma(x)), \tag{2}$$

Here, as before, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function:

$$Z_n = \sum_{\widetilde{\sigma}_n \in \Phi^{V_n}} \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x \in W_n} h_{\widetilde{\sigma}(x),x}\right) \prod_{x \in V_n} \nu(\widetilde{\sigma}(x)). \tag{3}$$

Remark. Note that Z_n is finite, since ν is a probability measure and $\exp(-\beta H(\widetilde{\sigma}_n) + \sum_{x \in W_n} h_{\widetilde{\sigma}(x),x})$ is bounded on Φ^{V_n}.

We say that the probability distributions $\mu^{(n)}$ are compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega_n \in \Phi^{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu^{(n-1)}(\sigma_{n-1}). \tag{4}$$

Here $\sigma_{n-1} \vee \omega_n \in \Phi^{V_n}$ is the concatenation of σ_{n-1} and ω_n. In this case there exists a unique measure μ on Φ^V such that, for any n and $\sigma_n \in \Phi^{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$. Such a measure is called a splitting Gibbs measure corresponding to Hamiltonian (1) and function $x \mapsto h_x$, $x \neq x^0$.

The following statement describes conditions on h_x guaranteeing compatibility of distributions $\mu^{(n)}(\sigma_n)$.

Proposition 1. *Probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, in (2) are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$h_x^* = \sum_{y \in S(x)} F(h_y^*, \theta). \tag{5}$$

Here, and below $\theta = \exp(J\beta)$, $h_x^* = (h_{1,x} - h_{0,x} - \alpha\beta + \ln\frac{\nu(1)}{\nu(0)}, h_{2,x} - h_{0,x} - \alpha\beta + \ln\frac{\nu(2)}{\nu(0)}, ...)$ and the function $F(\,\cdot\,, \theta):\ R^\infty \to R^\infty$ is $F(h, \theta) = (F_1(h, \theta), F_2(h, \theta), ...)$, with

$$F_i(h, \theta) = -\alpha\beta + \ln\frac{\nu(i)}{\nu(0)} + \ln\frac{(\theta - 1)\exp(h_i) + \sum_{j=1}^\infty \exp(h_j) + 1}{\theta + \sum_{j=1}^\infty \exp(h_j)},$$

$h = (h_1, h_2, ...,),\quad i = 1, 2, \ldots$

Proof. *Necessity.* Suppose that (4) holds; we want to prove (5). Substituting (2) in (4), we obtain that for any configurations σ_{n-1}: $x \in V_{n-1} \mapsto \sigma_{n-1}(x) \in \Phi$:

$$\frac{Z_{n-1}}{Z_n} \sum_{\omega_n \in \Phi^{W_n}} \exp\left(\sum_{x \in W_{n-1}} \sum_{y \in S(x)} (J\beta\delta_{\sigma_{n-1}(x)\omega_n(y)} + \alpha\delta_{0\omega_n(y)} + h_{\omega_n(y),y})\right) \times$$

$$\prod_{y\in W_n}\nu(\omega_n(y))=\exp\left(\sum_{x\in W_{n-1}}h_{\sigma_{n-1}(x),x}\right), \quad (6)$$

where ω_n: $x\in W_n\mapsto\omega_n(x)$.

From (6) we get:

$$\frac{Z_{n-1}}{Z_n}\sum_{\omega_n\in\Phi^{W_n}}\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\exp\left(J\beta\delta_{\sigma_{n-1}(x)\omega_n(y)}+\alpha\beta\delta_{0\omega_n(y)}+\right.$$

$$\left.h_{\omega_n(y),y}+\ln\nu(\omega_n(y))\right)=\prod_{x\in W_{n-1}}\exp\left(h_{\sigma_{n-1}(x),x}\right).$$

Consequently, for any $i\in\Phi$,

$$\prod_{y\in S(x)}\frac{\sum_{j\in\Phi}\exp\left(J\beta\delta_{ij}+\alpha\beta\delta_{0j}+h_{j,y}+\ln\nu(j)\right)}{\sum_{j\in\Phi}\exp\left(J\beta\delta_{0j}+\alpha\beta\delta_{0j}+h_{j,y}+\ln\nu(j)\right)}=\exp\left(h_{i,x}-h_{0,x}\right),$$

so that

$$\prod_{y\in S(x)}\frac{1+\sum_{j=1}^{\infty}\exp\left(h^*_{j,y}\right)+(\theta-1)\exp\left(h^*_{i,y}\right)}{\theta+\sum_{j=1}^{\infty}\exp\left(h^*_{j,y}\right)}=$$

$$\exp\left(h^*_{i,x}+\alpha\beta-\ln\frac{\nu(i)}{\nu(0)}\right),$$

where $h^*_{i,x}=h_{i,x}-h_{0,x}+\ln\frac{\nu(j)}{\nu(0)}-\alpha\beta$, which implies (5).

Sufficiency. Suppose that (5) holds. It is equivalent to the representations

$$\prod_{y\in S(x)}\sum_{j\in\Phi}\exp\left(J\beta\delta_{ij}+\alpha\beta\delta_{0j}+h_{j,y}+\ln\nu(j)\right)=a(x)\exp\left(h_{i,x}\right), i=0,1,... \quad (7)$$

for some function $a(x)>0, x\in V$. We have

$$\text{LHS of (4)}=\frac{1}{Z_n}\exp(-\beta H(\sigma_{n-1}))\prod_{x\in V_{n-1}}\nu(\sigma(x))\times$$

$$\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{j\in\Phi}\exp\left(J\beta\delta_{\sigma_{n-1}(x)j}+\alpha\beta\delta_{0j}+h_{j,y}+\ln\nu(j)\right). \quad (8)$$

Substituting (7) into (8) and denoting $A_n(x)=\prod_{x\in W_{n-1}}a(x)$, we get

$$\text{RHS of (8)}=\frac{A_{n-1}}{Z_n}\exp(-\beta H(\sigma_{n-1}))\prod_{x\in V_{n-1}}\nu(\sigma(x))\prod_{x\in W_{n-1}}h_{\sigma_{n-1}(x),x}. \quad (9)$$

Since $\mu^{(n)}$, $n\geq 1$ is a probability, we should have

$$\sum_{\sigma_{n-1}}\sum_{\sigma^{(n)}}\mu^{(n)}(\sigma_{n-1},\sigma^{(n)}))=1$$

Hence from (9) we get $Z_{n-1}A_{n-1}=Z_n$, and (4) holds.

From Proposition 1 it follows that for any $h=\{h_x,\ \ x\in V\}$ satisfying (5) there exists a unique Gibbs measure μ and vice versa. However, the analysis of solutions to (5) is not easy.

It is natural to begin with translation-invariant solutions where $h_x=h\in R^{\infty}$ is constant.

3 Translation-invariant solutions of (5)

Assume $h_x = h = (h_1, h_2, ...)$ for any $x \in V$. In this case we obtain from (5):

$$h_i = -\alpha\beta + \ln\frac{\nu(i)}{\nu(0)} + k\ln\frac{(\theta-1)\exp(h_i) + \sum_{j=1}^{\infty}\exp(h_j) + 1}{\theta + \sum_{j=1}^{\infty}\exp(h_j)}, i = 1, 2, ... \tag{10}$$

Set $u_i = \exp(h_i), i = 1, 2, ...$ From (10) we have

$$u_i = \frac{\nu(i)}{\nu(0)}\exp(-\alpha\beta)\left(\frac{(\theta-1)u_i + \sum_{j=1}^{\infty}u_j + 1}{\theta + \sum_{j=1}^{\infty}u_j}\right)^k, i = 1, 2, ... \tag{11}$$

In this section we give full analysis of the system of equations (11).

3.1 The set of solutions $\{u_i\}$ with $\sum_{j=1}^{\infty} u_j = \infty$

In this subsection we shall describe solutions of (11) with property $\sum_{j=1}^{\infty} u_j = \infty$. In this case from (11) we get

$$u_i = \frac{\nu(i)}{\nu(0)}\exp(-\alpha\beta), i = 1, 2, ... \tag{12}$$

Since $\sum_{j=1}^{\infty}\nu(j) = 1$ by (12) we get

$$\sum_{j=1}^{\infty} u_j = \frac{1-\nu(0)}{\nu(0)\exp(\alpha\beta)} < +\infty.$$

Thus there is no solution of (11) with $\sum_{j=1}^{\infty} u_j = \infty$.

3.2 The set of solutions with $\sum_{j=1}^{\infty} u_j < +\infty$

Now we want to describe solutions of (11) with $\sum_{j=1}^{\infty} u_j = A < +\infty$, where A is some fixed positive number. In this case from (11) we obtain

$$\eta_i u_i = \left(\frac{(\theta-1)u_i + A + 1}{\theta + A}\right)^k, \tag{13}$$

where $\eta_i = \frac{\nu(0)}{\nu(i)}\exp(\alpha\beta)$. Denote $B_i = \eta_i(\theta + A)^k$. Note that

$$u_i \to 0 \text{ and } B_i \to \infty \text{ as } i \to \infty. \tag{14}$$

From (13) we obtain

$$B_i u_i = ((\theta-1)u_i + A + 1)^k, i = 1, 2, ... \tag{15}$$

3.2.1 Case $\theta > 1$. As function $u \to ((\theta-1)u + A + 1)^k$ is concave increasing, we conclude that (15) has a unique positive solution, say u_i^* if u_i^* satisfies the following equations

$$\begin{cases} B_i u_i = ((\theta-1)u_i + A + 1)^k, \\ B_i = k(\theta-1)((\theta-1)u_i^* + A + 1)^{k-1}. \end{cases} \tag{16}$$

In other words, if (16) is satisfied for $u_i^* > 0$ then u_i^* is a unique positive solution to (15).

Assume $k \geq 2$. From (16) we have $u_i^* = u^*$ where $u^* = \frac{A+1}{(k-1)(\theta-1)}$ and $B_i = B^*$ where

$$B^* = (\theta - 1)\Big(\frac{A+1}{k-1}\Big)^{k-1} k^k.$$

We conclude that (15) has two solutions $0 < u_{i,1}^* < u_{i,2}^*$ for $B_i > B^*$.

Note that B^* and u^* do not depend on i and for $B_i > B^*$ we have

$$0 < u_{i,1}^* < u^* < u_{i,2}^*, \quad \text{for any} \quad i = 1, 2, ... \tag{17}$$

By (17) we have $\sum_{i=1}^{\infty} u_{i,2}^* = \infty$ thus $u_{i,2}^*$ does not satisfy the convergence condition.

Note that $u_{i,1}^*$ depends on $k, J, \alpha, \beta, \nu(i), \nu(0)$. Assume (condition on $\nu(i), i = 0, 1, ...$)

$$\sum_{i=1}^{\infty} u_{i,1}^* = A. \tag{18}$$

Thus we have proved

Theorem 2. *If for any* $i = 1, 2, ..., B_i > B^*$ *and* ν *satisfies the condition (18) then the system of equations (11) has unique solution* $\{u_{i,1}^*\}$ *with* $\sum_{i=1}^{\infty} u_{i,1}^* = A$.

In order to demonstrate the conditions of Theorem 2 we will consider cases $k = 1$ and $k = 2$ separately.

Assume $k = 1$. In this case from (15) we get

$$u_i = \frac{(A+1)\nu(i)}{\nu(0)e^{\alpha\beta}(\theta + A) - (\theta - 1)\nu(i)},$$

which is positive if $\frac{\nu(0)}{\nu(i)} e^{\alpha\beta}(\theta + A) > \theta - 1$. This condition corresponds to $B_i > B^*$ for $k = 1$. In this case the condition (18) can be written as

$$\sum_{i=1}^{\infty} \frac{(A+1)\nu(i)}{\nu(0)e^{\alpha\beta}(\theta + A) - (\theta - 1)\nu(i)} = A. \tag{19}$$

Assume $k = 2$. In this case condition $B_i > B^*$ has the form

$$B_i = \frac{\nu(0)}{\nu(i)} e^{\alpha\beta}(\theta + A)^2 > 4(\theta - 1)(A + 1). \tag{20}$$

The solutions $u_{i,m}^*$, $m = 1, 2$ are

$$u_{i,1}^* = \frac{B_i - 2(\theta-1)(A+1) - \sqrt{B_i[B_i - 4(\theta-1)(A+1)]}}{2(\theta-1)^2},$$

$$u_{i,2}^* = \frac{B_i - 2(\theta-1)(A+1) + \sqrt{B_i[B_i - 4(\theta-1)(A+1)]}}{2(\theta-1)^2}.$$

By (14) we have $u_{i,2}^* \to \infty$ if $i \to \infty$. The condition (18) (for $k = 2$) on ν can be rewritten as

$$\sum_{i=1}^{\infty} \Big(\nu(0)(\theta + A)^2 e^{\alpha\beta} - 2(\theta - 1)(A + 1)\nu(i) -$$

$$(\theta+A)\sqrt{\nu(0)e^{\alpha\beta}[\nu(0)e^{\alpha\beta}(\theta+A)-4(\theta-1)(A+1)\nu(i)]}\Big)(2(\theta-1)^2\nu(i))^{-1}=A. \qquad (21)$$

3.2.2. Case $\theta \leq 1$. If $\theta = 1$ then from (11) we obtain $u_i = \frac{\nu(i)}{\nu(0)e^{\alpha\beta}}$. This is unique solution of (11). Note that this case corresponds to zero interaction case and is not interesting.

If $\theta < 1$ then function $\varphi(u) = ((\theta-1)u + A + 1)^k$ is convex and decreasing for odd k and equation (15) has unique solution. For even k the function φ is decreasing for $u < \frac{A+1}{1-\theta}$ and increasing for $u > \frac{A+1}{1-\theta}$. Thus for $\theta < 1$ equation (11) has unique solution u_i' which may satisfy condition $\sum_{i=1}^{\infty} u_i' = A$. Note that for $\theta < 1$ we do not need a condition like $B_i > B^*$. Here we just need to condition (18).

Remark. Summarizing we note that for a given measure $\nu = \{\nu(i), i \in \Phi\}$ the equation (11) has at most one solution for any $\alpha, J \in R$ and $k \geq 1$. In the next section we shall describe exact values of a unique exponential solution of (11) and corresponding measure ν.

4 Exponential solutions of (11)

In this section we shall describe solutions of (11) such that $u_i = a^i$ for some $a \in (0,1)$. In this case $\sum_{i=1}^{\infty} u_i = A = \frac{a}{1-a}$.

4.1. Case $\theta > 1$.

From (11) we have

$$\nu(i) \equiv \nu(i,a) = \nu(0)a^i e^{\alpha\beta}\left(\frac{a+(1-a)\theta}{(\theta-1)(1-a)a^i+1}\right)^k. \qquad (22)$$

Now we shall choose a such that $\sum_{i=1}^{\infty} \nu(i) < +\infty$. Consider

$$\frac{\nu(i+1)}{\nu(i)} = a \cdot \left(\frac{(\theta-1)(1-a)a^i+1}{(\theta-1)(1-a)a^{i+1}+1}\right)^k. \qquad (23)$$

Using d'Alembert's convergence condition we should get

$$\frac{\nu(i+1)}{\nu(i)} \leq q < 1. \qquad (24)$$

If $k = 1$ from (23) we have $a \in (0,1)$. If $k \geq 2$ using AM-GM inequality from (23) we have

$$\text{RHS of (23)} \leq \left(\frac{a[(\theta-1)(1-a)a^{i+1}+1]+k[(\theta-1)(1-a)a^i+1]}{(k+1)[(\theta-1)(1-a)a^{i+1}+1]}\right)^{k+1} \leq$$

$$\left(\frac{a+k}{k+1}\cdot T_i\right)^{k+1},$$

where

$$T_i = \frac{(\theta-1)(1-a)a^i+1}{(\theta-1)(1-a)a^{i+1}+1}, \quad i = 1,2,...$$

It is easy to see that $T_{i+1} < T_i, \quad i = 1,2,...$ Hence in order to obtain (24) it is enough to solve $\frac{a+k}{k+1}\cdot T_1 < 1$ which is equivalent to

$$(a-1)(a^2 - a + \frac{1}{k(\theta-1)}) < 0. \qquad (25)$$

From (25) we get $a \in (0,1)$ if $1 < \theta \leq 1 + \frac{4}{k}$ and $a \in (0, a_-^*) \cup (a_+^*, 1)$ if $\theta > 1 + \frac{4}{k}$, where $a_\pm^* = \frac{1 \pm \sqrt{1 - 4((\theta-1)k)^{-1}}}{2}$.

4.2. Case $\theta \leq 1$. Denote $b_i = (\theta - 1)(1-a)a^i + 1$. It is easy to see that $b_{i+1} \geq b_i > 0$ for any $i = 1, 2, ...,$ $\theta \leq 1$ and $a \in (0,1)$. Using $b_{i+1} > b_i$ from (23) we get (24) for any $a \in (0,1)$.

Thus we have

Theorem 3. (i) *If* $k = 1$ *or* $k \geq 2$ *and* $1 < \theta \leq 1 + \frac{4}{k}$ *(resp.* $\theta > 1 + \frac{4}{k}$*) then for any* $a \in (0,1)$ *(resp.* $a \in (0, a_-^*) \cup (a_+^*, 1)$*) and* $\nu(i) = \nu(i, a)$ *(see (22)) there exists unique translation-invariant Gibbs measure* μ_a *which corresponds to solution* $\{u_i = a^i\}$ *of (11).*

(ii) *If* $\theta \leq 1$ *then for any* $a \in (0,1)$ *and* $\nu(i) = \nu(i, a)$ *there exists unique translation-invariant Gibbs measure* μ_a *.*

Denote by $G(H)$ the set of all splitting translation-invariant Gibbs measures for Hamiltonian (1).

Summarizing, we obtain the following

Theorem 4. *For any parameters* $\alpha, J \in R$, $k \geq 1$, $\beta > 0$ *and any fixed probability measure* ν *on* Φ *the set* $G(H)$ *contains at most one point.*

Remark. Note that (see [2]) for the Potts model with $q \geq 2$ spin values on Cayley tree exist q distinct translation-invariant Gibbs measures. Theorem 4 shows that the result is not true if $q \to \infty$.

Acknowledgments. The final part of this work was done at the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy and the authors thank ICTP for providing financial support and all facilities. They would also like to thank IMU/CDE-program for travel support. The first author was particularly supported by the IIUM short term research Grant and the second author by Italian scheme of Borsa di Studio NATO-CNR. The second author thanks CNR for providing financial support and the Physics Department of “La Sapienza” University in Rome for all facilities.

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