United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON THE STABILITY OF MOTION OF A GYROSTAT ABOUT A FIXED POINT UNDER THE ACTION OF NON SYMMETRIC FIELDS: II

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Abstract

The Problem of motion under more general (not necessarily axisymmetric) fields was touched in a few occasions, mainly in the search of integrable cases. In [9] we have studied the problem of motion of a heavy magnetized gyrostat carrying electric charges and acted upon by uniform electric and magnetic fields in addition to gravity. The equilibrium positions of the gyrostat have been found. The stability analysis was performed for some positions of equilibrium when the body is dynamically symmetric and the gyrostatic moment is directed along the axis of symmetry. In this work we study the stability for all the equilibrium positions under no restriction on the moments of inertia and the gyrostatic moment.

MIRAMARE – TRIESTE

December 2004

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1 Introduction

Despite its practical importance, the problem of motion under the action of nonsymmetric forces has escaped attention for a long time. Despite the richness in its structure, integrable cases of this problem are still rare. The first one was found in [5] (see also [3]). A few more cases were introduced in [2, 4, 15, 16, 18, 19, 21, 23]. The cases presented in [19, 20] is the only ones that involve, in addition to potential forces, not only a gyrostatic moment fixed in the body, but also gyroscopic moments depending on the orientation of the body. A physical interpretation of such moments is possible as a result of the Lorentz effect on a permanent distribution of charges carried by the moving body [14, 19]. An alternative explanation of those moments pointed out in [14] assumes the presence of (non-isotropic) dielectric parts of the body under the joint action of electric and magnetic fields.

In [21] the general problem of motion of a rigid body of complete dynamical symmetry about a fixed point under the action of a system of potential and gyroscopic forces admitting no axial symmetry was studied. Generally speaking, this problem can be modelled by the motion of an electric field, magnetized gyrostat under the action of a skew combination of Newtonian, Coulomb, magnetic and Lorentz forces. The main objective was to establish a certain equivalence between versions of this problem and the well-studied case of axisymmetric forces. This equivalence reveals two new integrable cases of the problem and a certain connection between other cases known before. In addition, it furnishes a simple way for certain analytical and qualitative studies of the motion and usually enables complete solution of the new problem just by transforming a known solution. In both new cases, as in the case of [19], the Lorentz forces play an essential role [21].

In [17], the problem of motion of a dynamically symmetric gyrostat acted upon by non symmetric potential forces admitting a cyclic integral was brought into equivalence with another one concerning the motion of a similar gyrostat under the action of axisymmetric potential forces. This method furnishes a simple way for obtaining certain analytical and qualitative results about the motion in one problem using known results about the other. In [8] a reduction of the order of the equation of motion in the problem with non symmetric potential to two degrees of freedom is obtained using this analogy. The reduced equations of motion are written in terms of the isometric variables. They are used to study the stability of the stationary motion and equilibrium positions of the body. The almost stationary solutions and the periodic solutions in the neighborhood of the stationary points or equilibrium positions are obtained. Also in [8] we use Euler-Poisson' s equations of motion of a rigid body - gyrostat about a fixed point in a field of force with a non- axisymmetric potential to study the motion about one of the principal axes of the body which takes a permanent position in space.

In [23] a generalization of an integrable case of the dynamics of a rigid body-gyrostat acted upon by a skew combination of forces introduced in [19]. The body has Kovalveskaya configu-

ration and a singular term was added to the potential. An interesting geometric transformation that reduces certain problems admitting space axial symmetry to problems which have neither space nor body axis of symmetry was explored in [24] and thus a new integrable problems of the latter was constructed. The solution of one of the two problems always be obtained from that of the other although the two types of the problems are completely different from the physical point of view. In [25] the problem of motion of a rigid body- gyrostat acted upon by two homogenous gravity and magnetic fields was considered. The gyrostatic moment is directed along a principal axis of inertia while the centre of mass and the magnetic moment lie in the orthogonal principal plane. The conditions imposed by Dragovic [7] on the motion in the integrable case out by him and considered as a generalization of the famous Gorychev-Chaplygin case [12] lead only to a pendulum-like motion about a fixed axis , in which the presence of the magnetic field is not significant. Also the motion is possible even when Gorychev-Chaplygin conditions on the moments of inertia are completely removed.

The problem of motion of a heavy magnetized gyrostat carrying electric charges and acted upon by uniform electric and magnetic fields in addition of gravity was considered in [9]. In this problem the gyrostatic moment is a constant vector $\mathbf{k} = (k_1, k_2, k_2)$. The equilibrium positions of the gyrostat have been found in the case in which the three fields are perpendicular to each other. In [9] the stability analysis was performed for some positions of equilibrium when the body is dynamically symmetric and the gyrostatic moment is directed along the axis of symmetry.

In this paper we study the stability for all the equilibrium positions under no restriction on the moments of inertia and the gyrostatic moment.

2 Equations of motion

Let OXYZ and Oxyz be two Cartesian coordinate systems fixed in the space and in the body respectively. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors in the direction of the xyz axes, $\omega = (p, q, r)$ be the angular velocity of the body and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be the unit vectors in the direction of the XYZ - axes all refered to the body system oxyz which we take as the system of principal axes of inertia. The inertia Matrix of the body has the form $I = (A, B, C)$. Let $\mathbf{k} = (k_1, k_2, k_3)$ be the gyrostatic moment of intrinsic cyclic motions in the body (due to rotors or holes completely filled with an ideal incompressible fluid). The relative position of the two systems will be specified by Eulerian angles: ψ -the angle of precession around the Z -axis, θ - the angle of nutation between z and Z, and φ the angle of rotation of the body around the z-axis. Thus:

$$
\begin{aligned}
\alpha &= (\cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi, \ -\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi, \ \sin\theta\sin\psi), \\
\beta &= (\sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi, \ -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi, \ \sin\theta\cos\psi), \\
\gamma &= (\sin\theta\sin\varphi, \ \sin\theta\cos\varphi, \ \cos\theta), \\
\omega &= (\dot\theta\cos\varphi + \dot\psi\sin\theta\sin\varphi, \ -\dot\theta\sin\varphi + \dot\psi\sin\theta\cos\varphi, \ \dot\psi\cos\theta + \dot\varphi).
\end{aligned}
\tag{1}
$$

When the body moves under the action of conservative forces with potential $V(\alpha, \beta, \gamma)$ the

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equations of motion can be written in the Euler Poisson form [22]:

$$\dot{\omega}I + \omega \times (\omega I + \mu) = \alpha \times \frac{\partial V}{\partial \alpha} + \beta \times \frac{\partial V}{\partial \beta} + \gamma \times \frac{\partial V}{\partial \gamma},$$
$$\dot{\alpha} + \omega, \times \alpha = 0, \quad \dot{\beta} + \omega \times \beta = 0, \quad \dot{\gamma} + \omega \times \gamma = 0 \tag{2}$$

where V is depending on the Eulerian angles through the nine components of the vectors α, β, γ through the relation:

$$\mu = \ell + (\alpha \times \frac{\partial}{\partial \alpha} + \beta \times \frac{\partial}{\partial \beta} + \gamma \times \frac{\partial}{\partial \gamma})\ell, \quad V = \mathbf{a}.\alpha + \mathbf{b}.\beta + \mathbf{c}.\gamma \tag{3}$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$ are constant vectors in the body where $\ell = \ell(\alpha, \beta, \gamma)$ is the gyrostatic moment. The above system admits in addition to Jacobi's integral

$$\tfrac{1}{2}\omega I.\omega + V = h \tag{4}$$

a set of geometrical integrals

$$\alpha^2 = 1, \quad \beta^2 = 1, \quad \gamma^2 = 1, \quad \alpha.\beta = 0, \quad \beta.\gamma = 0, \quad \alpha.\gamma = 0. \tag{5}$$

The problem under consideration admits an equivalent formulation as a Hamiltonian system with three degrees of freedom . For the complete integration of the system (2) in the sense of Liouvile [1], we need two first integrals independent of (4), (5).

The Problem of motion of a heavy magnetized gyrostat carrying electric charges and acted upon by uniform electric and magnetic fields in addition of gravity was considered in [21]. In this problem the gyrostatic moment $\mu = \ell = \mathbf{k}$ is a constant vector where $\mathbf{k} = (k_1, k_2, k_3)$. The equations of motion (2) reduced to [9]:

$$\dot{\omega}I + \omega \times (\omega I + \mathbf{k}) = \alpha \times \frac{\partial V}{\partial \alpha} + \beta \times \frac{\partial V}{\partial \beta} + \gamma \times \frac{\partial V}{\partial \gamma},$$
$$\dot{\alpha} + \omega \times \alpha = 0, \quad \dot{\beta} + \omega \times \beta = 0, \quad \dot{\gamma} + \omega \times \gamma = 0. \tag{6}$$

3 Equilibrium positions

Equations (6) admit several equilibrium solutions. To determine these solutions we start by assuming the following equilibrium state:

$$\omega = 0, \, \alpha_i = const. = \alpha_i^0, \, \beta_i = const. = \beta_i^0 \text{ and } \gamma_i = const. = \gamma_i^0.$$

The equilibrium positions are the solutions of the following equations:

$$a_3\alpha_2 - a_2\alpha_3 + b_3\beta_2 - b_2\beta_3 + c_3\gamma_2 - c_2\gamma_3 = 0,$$
$$a_1\alpha_3 - a_3\alpha_1 + b_1\beta_3 - b_3\beta_1 + c_1\gamma_3 - c_3\gamma_1 = 0,$$
$$a_2\alpha_1 - a_1\alpha_2 + b_2\beta_1 - b_1\beta_2 + c_2\gamma_1 - c_1\gamma_2 = 0,$$
$$\dot{\alpha}_i = 0, \dot{\beta}_i = 0, \dot{\gamma}_i = 0, i = 1, 2, 3. \tag{7}$$

We consider the case in which the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ characterize the centre of mass, the magnetic moment and electric dipole moment take the form [9]:

$$\mathbf{a} = (a, 0, 0), \quad \mathbf{b} = (0, b, 0), \quad \mathbf{c} = (0, 0, c). \tag{8}$$

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This means that those vectors are assumed orthogonal to each other and coinciding with the principal axes of the body. Without loss of generality we assume that:

$$a \geq b \geq 0 \tag{9}$$

as this choice is equivalent to choosing the direction of the axes moving with the body.

Solving the above system of equations (7) for the case (8) we can get the equilibrium positions in the form:

(I) If a, b and c are different. In this case we have four equilibrium positions given by:

$$\textbf{(i)} \quad \alpha = (1, \ 0, \ 0), \quad \beta = (0, \ 1, \ 0), \quad \gamma = (0, \ 0, 1), \tag{10}$$

$$\textbf{(ii)} \quad \alpha = (-1, \ 0, \ 0), \quad \beta = (0, \ -1, \ 0), \quad \gamma = (0, \ 0, 1), \tag{11}$$

$$\textbf{(iii)} \quad \alpha = (1, \ 0, \ 0), \quad \beta = (0, \ -1, \ 0), \quad \gamma = (0, \ 0, -1), \tag{12}$$

$$\textbf{(iv)} \quad \alpha = (-1, \ 0, \ 0), \quad \beta = (0, \ 1, \ 0), \quad \gamma = (0, \ 0, -1), \tag{13}$$

In each of these configurations each of the body axes is directed along or opposite to the corresponding space axis.

(II) When two of the parameters a, b and c are equal, two of the equilibrium positions remain unchanged while the other two positions degenerate into a continuum of natural equilibrium positions in the following manner:

(1) when $a = b$, the two positions **(i)** and **(ii)** remain. Instead of the two positions **(iii)** and **(vi)** we have the one parameter family of equilibrium positions:

$$\alpha = (\cos\Theta, \ \sin\Theta, \ 0), \ \beta = (\sin\Theta, - \ \cos\Theta, \ 0), \ \gamma = (0, \ 0, \ -1), \ 0 \leq \Theta \leq 2\pi \tag{14}$$

(2) when $c = b$, instead of the two positions **(ii)** and **(vi)** we have one parameter family of equilibrium positions given by:

$$\alpha = (-1, \ 0, \ 0), \ \beta = (0, \ \cos\Phi, \ \sin\Phi), \ \gamma = (0, \ \sin\Phi, \ -\cos\Phi), \ 0 \leq \Phi \leq 2\pi \tag{15}$$

where the two positions **(i)** and **(iii)** still exist.

(3) when $c = -b$, the two positions **(ii)** and **(vi)** do not degenerate. The two positions **(i)** and **(iii)** degenerate to the one parameter family of equilibrium positions:

$$\alpha = (1, \ 0, \ 0), \ \beta = (0, \ \cos\Phi, \ \sin\Phi), \ \gamma = (0, \ -\sin\Phi, \ \cos\Phi), \tag{16}$$

(4) when $c = a$, we have the two positions **(ii)** and **(iv)**. These positions exist beside the family of positions of equilibrium:

$$\alpha = (\cos\Psi, \ 0, \ \sin\Psi), \ \beta = (0, \ -1, \ 0), \ \gamma = (\sin\Psi, \ 0, \ -\cos\Psi), \ 0 \leq \Psi \leq 2\pi \tag{17}$$

instead of the two positions **(ii)** and **(iii)**.

(5) when $c = -a$, the two positions **(ii)** and **(iii)** remain. The two positions **(i)** and **(iv)** degenerate to one parameter family of equilibrium positions:

$$\alpha = (\cos \Psi, \ 0, \ \sin \Psi), \quad \beta = (0, \ 1, \ 0), \quad \gamma = (-\sin \Psi, \ 0, \ \cos \Psi), \tag{18}$$

(III) When $a = b = -|c|$ it can be shown that one isolated equilibrium position remains while the other three degenerate into a two - parameter family of equilibrium positions.

4 Type of Equilibrium positions given in (I) (The Extrema Points of V)

We can determine that the equilibrium positions mentioned in **(I)** are either extremum (maximum or minimum) or saddle points of V. In the plane of parameters $\frac{b}{a}, \frac{c}{a}$ we can find that, Fig. 1:

(1) In the regions $\frac{c}{a} > 1$, $1 > \frac{c}{a} > \frac{b}{a}$ ($\chi_1, \ \chi_2$, respectively) the position **(i)** is maximum point of V, the position **(iv)** is minimum while the two positions **(ii)** and **(iii)** are saddle.

(2) In the regions $1 > \frac{b}{a} > \frac{c}{a} > 0$ and $0 > \frac{c}{a} > -\frac{b}{a}$, $(\chi_3, \ \chi_4$ respectively) the position **(i)** is maximum point of V, the position **(ii)** is minimum while the two positions **(iii)** and **(iv)** are saddle.

(3) In the region $-\frac{b}{a} > \frac{c}{a} > -1$ and $-1 > \frac{c}{a}$, $(\chi_5, \ \chi_6$ respectively) the position **(iii)** is maximum point of V, the position **(ii)** is minimum while the two positions **(i)** and **(iv)** are saddle.

It is well known that an isolated minimum of the potential of a natural mechanical system is a stable equilibrium position, while equilibrium at maximum or at saddle points is unstable [11]. Gyroscopic forces can lead, if added to the system, to stabilization of equilibrium in both last cases of instability. The effect of gyroscopic forces on the stability of the equilibrium was studied by Lord Kelvin [10] where he proved that:

(i) An equilibrium, which is stable under purely potential forces, remains stable with the addition of gyroscopic forces.

(ii) If the instability of an isolated position of equilibrium under the action of exclusively potential forces has an odd degree, gyroscopic stabilization of the equilibrium is not possible.

(iii) For certain conditions of the equilibrium, which are unstable under the action of purely potential forces, it is possible to improve or stabilize the system by the addition of suitable gyroscopic forces if the degree of instability is even.

By gyroscopic stability we mean stability in the linear approximation due to the presence of gyroscopic forces. This however does not make any conclusion about stability of the nonlinear equations of motion. One way of resolving this difficulty is to construct suitable Lyapunov's function.

The classification of the equilibrium positions according to the type of the extremum of the potential leads to:

(4.1) - The positions **(ii)** in the regions $\chi_3, ..., \chi_6$ and position **(iv)** in the regions χ_1, χ_2 correspond to potential minima and thus they are Lyapunov stable, in these regions.

(4.2) The equilibrium positions **(i)** in the regions $\chi_1, ..., \chi_4$ and the position **(iii)** in regions χ_5, χ_6 are at potential maximum and have two degree of instability. It may be stabilized due to the presence of gyroscopic forces [6].

(4.3)- Each of the equilibrium positions at saddle points have two degree of instability and hence are unstable and can be stabilized by the addition of gyroscopic forces. Those positions are:

1- position **(i)** in regions χ_5, χ_6,   2- position **(ii)** in the regions χ_1, χ_2,

3- position **(iii)** in regions $\chi_1, ..., \chi_4$,   4- position **(iv)** in regions $\chi_3, ..., \chi_6$.

Whether each position is stable or unstable can be decided by two methods. The linear approximation method will detect unstable positions and the use of Lyapunov function may find conditions for stability.

Stability analysis is performed for the equilibrium positions mentioned in the cases **(4.2)** and **(4.3)**.

5   Equilibrium positions stability in the linear approximation

To study the stability of the small motions about the equilibrium positions we shall linearize the equations of motion. The objective is to give a first picture on how stability depends on the physical parameters of the system (a, b, c). To this end the system is slightly displaced from its equilibrium position by small amounts $\overline{\alpha_i}$, $\overline{\beta_i}$, $\overline{\gamma_i}$ the equations for small motions about the equilibrium point are then linearized by drooping terms that consist of products of the small quantities. Stability is studied by using methods of linear theory. Let:

$$\alpha_i = \alpha_i^0 + \overline{\alpha_i}, \quad \beta_i = \beta_i^0 + \overline{\beta_i} \quad , \gamma_i = \gamma_i^0 + \overline{\gamma_i} \tag{19}$$

where α_i^0, β_i^0 and γ_i^0 are the values of α_i, β_i and γ_i at the equilibrium positions and $\overline{\alpha_i}$, $\overline{\beta_i}$, $\overline{\gamma_i}$ are the perturbations. Substituting equation (19) into the equations of motion (7), we then obtain the equations of perturbed motions in the form:

$$\dot{\overline{p}} = \frac{C}{A}[-k_3\ \overline{q} + k_2\ \overline{r} - b\ \overline{\beta_3} + c\ \ \overline{\gamma_2}]$$

$$\dot{\overline{q}} = \frac{C}{B}[k_3\ \overline{p} - k_1\ \overline{r} + a\ \overline{\alpha_3} - c\ \ \overline{\gamma_1}] \tag{20}$$

$$\dot{\overline{r}} = -k_2\ \overline{p} + k_1\ \overline{q} - a\ \overline{\alpha_2} + b\ \ \overline{\beta_1}$$

$$\dot{\alpha_1} = \alpha_2^0\,\bar{r} - \alpha_3^0\,\bar{q}, \qquad \dot{\alpha_2} = \alpha_3^0\,\bar{p} - \alpha_1^0\,\bar{r}, \qquad \dot{\alpha_3} = \alpha_1^0\,\bar{q} - \alpha_2^0\,\bar{p},$$

$$\dot{\beta_1} = \beta_2^0\,\bar{r} - \beta_3^0\,\bar{q}, \qquad \dot{\beta_2} = \beta_3^0\,\bar{p} - \beta_1^0\,\bar{r}, \qquad \dot{\beta_3} = \beta_1^0\,\bar{q} - \beta_2^0\,\bar{p}, \tag{21}$$

$$\dot{\gamma_1} = \gamma_2^0\,\bar{r} - \gamma_3^0\,\bar{q}, \qquad \dot{\gamma_2} = \gamma_3^0\,\bar{p} - \gamma_1^0\,\bar{r}, \qquad \dot{\gamma_3} = \gamma_1^0\,\bar{q} - \gamma_2^0\,\bar{p}$$

where $a = \frac{a}{C}$, $b = \frac{b}{C}$, $c = \frac{c}{C}$, $k_i = \frac{k_i}{C}$, $i = 1,2,3$.

The characteristic equation takes the form:

$$\lambda^6 P(\lambda) = 0 \tag{22}$$

where

$$P(\lambda) = \lambda^6 + \upsilon_4\,\lambda^4 + \upsilon_3\,\lambda^3 + \upsilon_2\,\lambda^2 + \upsilon_1\,\lambda + \upsilon_0, \tag{23}$$

and

$$\upsilon_4 = \frac{C}{A}\left[\frac{A}{B}\,k_1^2 + k_2^2 + \frac{C}{B}\,k_3^2\right] - \left[\,a\,(1 + \frac{C}{B})\alpha_1^0 + b\,(1 + \frac{C}{A})\,\beta_2^0 + c\,(\frac{C}{A} + \frac{C}{B})\,\gamma_3^0\right],$$

$$\upsilon_3 = \frac{C}{B}\,(c\,\gamma_2^0 - b\,\beta_3^0)\,k_1 + \frac{C}{A}\,(a\,\alpha_3^0 - c\,\gamma_1^0)\,k_2 + \frac{C^2}{AB}\,(-a\,\alpha_2^0 + b\,\beta_1^0)\,k_3,$$

$$\upsilon_2 = \frac{C^2}{AB}\,\{\,a[k_1\,(k_2\,\alpha_2^0 + k_3\,\alpha_3^0) - \alpha_1^0\,(k_2^2 + k_3^2) + b\,\gamma_3^0 + c\,\alpha_1^0\gamma_3^0] + c^2\,\gamma_3^{0\,2}$$

$$+ b[k_2\,(k_1\,\beta_1^0 + k_3\,\beta_3^0) - \beta_2^0\,(k_1^2 + k_3^2)] + c[b\,\gamma_3^0\,\beta_2^0 + k_3\,(k_1\,\gamma_1^0 + k_2\,\gamma_2^0) -$$

$$\gamma_3^0\,(k_1^2 + k_2^2)]\} + \frac{C}{A}\,\{\frac{A}{B}\,a^2\,\alpha_1^{0\,2} + b^2\,\beta_2^{0\,2} + a\,[c\,(\beta_2^0 + \frac{A}{B}\,\gamma_3^0\,\alpha_1^0)$$

$$+ b\,(1 + \frac{A}{B})\,\alpha_1^0\beta_2^0] + b\,c\,[\frac{A}{B}\alpha_1^0 + \beta_2^0\,\gamma_3^0]\}, \tag{24}$$

$$\upsilon_1 = \frac{C^2}{A}\{k_1\,[\,b^2\,\beta_2^0\,\beta_3^0 - c^2\,\gamma_2^0\,\gamma_3^0 - c\,a\alpha_2^0\,\gamma_1^0 + b\,(\,a\,\alpha_3^0\,\beta_1^0 + c\,(\,\beta_3^0\,\gamma_3^0 - \beta_2^0\,\gamma_2^0\,)\,)]$$

$$+ k_2\,[\,c^2\,\gamma_1^0\,\gamma_3^0 - a^2\,\alpha_1^0\,\alpha_3^0 + a\,c(\,\alpha_1^0\,\gamma_1^0 - \alpha_3^0\,\gamma_3^0) + b\,(\,c\,\gamma_2^0\,\beta_1^0 - a\,\alpha_2^0\,\beta_3^0\,)\,]$$

$$+ k_3\,[\,a^2\,\alpha_1^0\,\alpha_2^0 - b^2\,\beta_1^0\,\beta_2^0 + a\,c\,\alpha_3^0\,\gamma_2^0 + b\,(\,a\,(\alpha_2^0\,\beta_2^0 - \alpha_1^0\,\beta_1^0) - c\,\beta_3^0\,\gamma_1^0\,)\,]\,\},$$

$$\upsilon_0 = \frac{C^2}{AB}\{c\,[-\,b^2\,\beta_2^0\,\alpha_1^0 + a^2\,\alpha_1^0\,\beta_2^0 + a\,b\,(\,\alpha_2^0\,\beta_3^0\,\gamma_1^0 + \alpha_3^0\,\beta_1^0\,\gamma_2^0 - 2\,\alpha_1^0\,\beta_2^0\,\gamma_3^0\,)]$$

$$- c^2\,\gamma_3^0\,[\,a\,\beta_2^0 + b\,\alpha_1^0\,] + a\,b\,(a\,\alpha_1^0 + b\,\beta_2^0\,)\gamma_3^0\}$$

The factor λ^6 on the left-hand side of equation (22) is a consequence of the existence of six integrals (5) of the equations of motions in the redundant coordinates used. The equilibrium position will be stable (in linear approximation) if all the solutions of the corresponding characteristic equation are pure imaginary. The position will be unstable if $P(\lambda)$ has at least one real root with positive real part.

For the equilibrium positions given in **(I)** we can notice that:

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$$\alpha_2^0 = \alpha_3^0 = 0, \qquad \beta_1^0 = \beta_3^0 = 0, \qquad \gamma_1^0 = \gamma_2^0 = 0.$$

This leads to

$$v_1 \;=\; 0, \qquad v_3 \;=\; 0, \tag{25}$$

and $P(\lambda)$ reduces to:

$$P(\lambda) = \; \lambda^6 + v_4\,\lambda^4 + v_2\,\lambda^2 + v_0 \tag{26}$$

where:

$$v_4 = \tfrac{C}{A}[\tfrac{A}{B}\,k_1^2 + k_2^2 + \tfrac{C}{B}\,k_3^2] - [a\,\alpha_1^0 + b\,\beta_2^0 + \tfrac{C}{A}(b\,\beta_2^0 + c\,\gamma_3^0) + \tfrac{C}{B}(a\,\alpha_1^0 + c\,\gamma_3^0)],$$

$$v_2 = \tfrac{C^2}{AB}\,\{(a\,\alpha_1^0 + c\,\gamma_3^0)(b\,\beta_2^0 + c\,\gamma_3^0) - [(b\,\beta_2^0 + c\,\gamma_3^0)\,k_1^2 + (a\,\alpha_1^0 + c\,\gamma_3^0)\,k_2^2$$

$$+(a\,\alpha_1^0 + b\,\beta_2^0)\,k_3^2]\} + \tfrac{C}{A}\,(a\,\alpha_1^0 + b\,\beta_2^0)[b\,\beta_2^0 + c\,\gamma_3^0 + \tfrac{A}{B}\,(a\,\alpha_1^0 + c\,\gamma_3^0)], \tag{27}$$

$$v_0 = -\tfrac{C^2}{AB}(a\,\alpha_1^0 + b\,\beta_2^0)(a\,\alpha_1^0 + c\,\gamma_3^0)(b\,\beta_2^0 + c\,\gamma_3^0)$$

To study the stability of these equilibrium positions we discuss the existence of the pure imaginary roots of the general equation (26).

Under the condition:

$$v_0 < 0 \tag{28}$$

the characteristic equation has at least one positive real root. Since v_0 depends only on the parameters a, b, c, thus for certain values of these parameters the equilibrium position will be unstable for all values of the moment of inertia A, B, C and gyrostatic moment k_i, $i = 1, 2, 3$. When:

$$v_0 > 0 \tag{29}$$

we find that: (1) the solutions of equation (26) are pure imaginary under the conditions:

$$v_2 > 0, \; v_4 > 0, \tag{30}$$

and

$$\Delta \le 0, \tag{31}$$

$$\Delta = (\tfrac{M}{2})^2 + (\tfrac{N}{3})^3, \qquad M = \tfrac{2}{27}\,v_4^2 - \tfrac{1}{3}\,v_2^2\,v_4^2 + v_0, \qquad N = \tfrac{1}{3}\,v_4^2 - v_2$$

In this case the equilibrium positions are stable.

(2) There exists at least one positive real root if the condition (30) is satisfied and (31) is not. The positions of equilibrium are unstable.

(3) There exist both pure imaginary roots and complex conjugate roots if both the conditions (30) and (31) are not satisfied. The equilibrium positions will be stable if the real part of the complex conjugate roots are negative.

In the space of the parameters of equation (26) the regions of stability I and instability II are

shown in Fig. 2.

Now we study the stability of equilibrium positions. For those positions in the first case **(I)** we find that:

(1)- The stability of the position (i):

For this position given by the equation given by (10), v_4, v_2, v_0 in the characteristic equation are given by:

$$v_{4_1} = \frac{C}{A}[\frac{A}{B}\,k_1{}^2 + k_2{}^2 + \frac{C}{B}\,k_3{}^2] - a\,[\,(1+\frac{b}{a}) + \frac{C}{A}\,(\frac{b}{a}+\frac{c}{a}) \; + \; \frac{C}{B}\,(1+\frac{c}{a})\,],$$

$$v_{2_1} = \frac{C^2}{AB}\,\{a^2\,(\frac{b}{a}+\frac{c}{a})(1+\frac{b}{a}) + a\,[k_2{}^2(1+\frac{c}{a}) + k_3^2(1+\frac{b}{a}) + \; k_1^2(\frac{b}{a}+\frac{c}{a})]\}$$

$$+ \frac{a^2 C}{A}(1+\frac{b}{a})[\frac{A}{B}\,(1+\frac{c}{a}) + \frac{b}{a}+\frac{c}{a}], \tag{32}$$

$$v_{0_1} = -\frac{a^3 C}{AB}(1+\frac{b}{a})(1+\frac{c}{a})(\frac{b}{a}+\frac{c}{a})$$

Using equation (32) and applying the condition (28) and (29) respectively we find that this position is unstable for

$$c < -a, \qquad c > -b \tag{33}$$

while for

$$-a < c < -b \tag{34}$$

this position is stable under the conditions:

$$\frac{C}{A}[\frac{A}{B}\,k_1{}^2 + k_2{}^2 + \frac{C}{B}\,k_3{}^2 - a\,(\frac{b}{a}+\frac{c}{a})\,] > a\,[\,1+\frac{b}{a} + \; \frac{C}{B}\,(1+\frac{c}{a})\,],$$

$$\frac{C}{A}\,\{-\frac{B}{C}\,a\,(\frac{b}{a}+\frac{c}{a})(1+\frac{b}{a}) + k_2{}^2(1+\frac{c}{a}) + k_3^2(1+\frac{b}{a})\} > \tag{35}$$

$$a\,(1+\frac{b}{a})\,(1+\frac{c}{a}) - \frac{C}{B}\,k_1^2(\frac{b}{a}+\frac{c}{a}),$$

$$27\,(\frac{2}{27}\,v_{4_1}^2 - \frac{1}{3}\,v_{2_1}^2\,v_{4_1}^2 + v_{0_1})^2 + 4\,(\frac{1}{3}\,v_{4_1}^2 - v_{2_1})^3 \leq 0 \tag{36}$$

where $v_{4_1}, v_{2_1}, v_{0_1}$ are given by (32). Otherwise the position is unstable.

According to the condition (9) it suffices to consider the strip $1 \geq \frac{b}{a} \geq 0$ in the plane of the parameters $\frac{b}{a}, \frac{c}{a}$.

Thus this position is unstable in the two blank regions (Fig. 3) where $-1 > \frac{c}{a}, \frac{c}{a} > -\frac{b}{a}$. In the dotted region $\frac{c}{a} > -\frac{b}{a}$ the gyroscopic stabilization occurs when the gyrostatic moment exceeds a certain limit depending on the parameters.

When $k_1 = k_2 = 0$ the above conditions (35) and (36) are not satisfied (where the equation (26) has a positive real root) and the position is unstable, as we get in [9].

(2)- The stability of the position (ii):

The corresponding characteristic equation for this position v_4, v_2, v_0 in the form:

$$v_{4_2} = \frac{C}{A}[\frac{A}{B}\,k_1{}^2 + k_2{}^2 + \frac{C}{B}\,k_3{}^2] + a\,[\,(1 + \frac{b}{a}) + \frac{C}{A}\,(\frac{b}{a} - \frac{c}{a}) + \frac{C}{B}\,(1 - \frac{c}{a})\,],$$

$$v_{2_2} = \frac{C^2}{AB}\,\{a^2\,(\frac{b}{a} - \frac{c}{a})(1 - \frac{c}{a}) + a\,[\,k_2{}^2(1 - \frac{c}{a}) + k_3^2(1 + \frac{b}{a}) + \ k_1^2(\frac{b}{a} - \frac{c}{a})]\}$$

$$+ \frac{a^2\,C}{A}(1 + \frac{b}{a})[\frac{A}{B}\,(1 - \frac{c}{a}) + \frac{b}{a} - \frac{c}{a}], \tag{37}$$

$$v_{0_2} = \frac{a^3\,C^2}{AB}(1 + \frac{b}{a})(1 - \frac{c}{a})(\frac{b}{a} - \frac{c}{a})$$

Equation (37) and conditions (28) and (29) give us: this position is unstable for

$$a > c > b \tag{38}$$

While for

$$c > a > b \tag{39}$$

this position is stable if the conditions:

$$\frac{C}{A}[\frac{A}{B}\,k_1{}^2 + k_2{}^2 + \frac{C}{B}\,k_3{}^2] + a\,(1 + \frac{b}{a}) > a\,[\,\frac{C}{A}\,(\frac{c}{a} - \frac{b}{a}) + \ \frac{C}{B}\,(\frac{c}{a} - 1)\,],$$

$$\frac{C^2}{AB}\,\{a^2\,(\frac{b}{a} - \frac{c}{a})(1 - \frac{c}{a}) + a\,k_3^2(1 + \frac{b}{a})\} > \frac{a^2\,C}{A}(1 + \frac{b}{a})[\frac{A}{B}\,(\frac{c}{a} - 1)$$

$$+ \frac{c}{a} - \frac{b}{a}\,] + a\,[\,k_2{}^2(\frac{c}{a} - 1) + \ k_1^2(\frac{c}{a} - \frac{b}{a})] \tag{40}$$

$$27\,(\frac{2}{27}\,v_{4_2}^2 - \frac{1}{3}\,v_{2_2}^2\,v_{4_2}^2 + v_{0_2})^2 + 4\,(\frac{1}{3}\,v_{4_2}^2 - v_{2_2})^3 \leq 0 \tag{41}$$

are satisfied. For

$$a > b > c \tag{42}$$

we can find that:

$$v_{4_2} > 0,\, v_{2_2} > 0,\, v_{0_2} > 0 \tag{43}$$

and also the condition (41) will be satisfied since the position corresponds to minimum of the potential. The regions of stability and instability, in the plane of parameters $\frac{b}{a}, \frac{c}{a}$ are illustrated in Fig. 4. The hatched region is the region of stability in the Lyapunove sense.

(3)- The stability of the position (iii):

For this position the corresponding characteristic equation has:

$$v_{4_3} = \frac{C}{A}[\frac{A}{B}\,k_1{}^2 + k_2{}^2 + \frac{C}{B}\,k_3{}^2] - a\,[\,(1 - \frac{b}{a}) - \frac{C}{A}\,(\frac{b}{a} + \frac{c}{a}) + \ \frac{C}{B}\,(1 - \frac{c}{a})\,],$$

$$v_{2_3} = \frac{C^2}{AB}\,\{a^2\,(\frac{b}{a} + \frac{c}{a})(\frac{c}{a} - 1) - a\,[\,k_2{}^2(1 - \frac{c}{a}) + k_3^2(1 - \frac{b}{a}) - \ k_1^2(\frac{b}{a} + \frac{c}{a})]\}$$

$$+ \frac{a^2\,C}{A}(1 - \frac{b}{a})[\frac{A}{B}\,(1 - \frac{c}{a}) - (\frac{b}{a} + \frac{c}{a}\,)], \tag{44}$$

$$v_{0_3} = \frac{a^3\,C^2}{AB}(1 - \frac{b}{a})(1 - \frac{c}{a})(\frac{b}{a} + \frac{c}{a})$$

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From equations (44) and (28) we find that this position is unstable for

$$c > a \quad or \quad c < -b \tag{45}$$

For

$$a > c > -b \tag{46}$$

this position is stable under the conditions:

$$\frac{C}{A}[\frac{A}{B} k_1{}^2 + k_2{}^2 + \frac{C}{B} k_3{}^2] > a[\,(1 - \frac{b}{a}) - \frac{C}{A}\,(\frac{b}{a} + \frac{c}{a}) \; + \; \frac{C}{B}\,(1 - \frac{c}{a})\,],$$

$$\frac{C^2}{AB}\{a^2\,(\frac{b}{a} + \frac{c}{a})(\frac{c}{a} - 1) - a\,[k_2{}^2(1 - \frac{c}{a}) + k_3^2(1 - \frac{b}{a}) - k_1^2(\frac{b}{a} + \frac{c}{a})]\} > \tag{47}$$

$$-\frac{a^2\,C}{A}(1 - \frac{b}{a})[\frac{A}{B}\,(1 - \frac{c}{a}) - (\frac{b}{a} + \frac{c}{a})],$$

$$27\,(\frac{2}{27}\,v_{4_3}^2 - \frac{1}{3}\,v_{2_3}^2\,v_{4_3}^2 + v_{0_3})^3 + 4\,(\frac{1}{3}\,v_{4_3}^2 - v_{2_3})^3 \leq 0. \tag{48}$$

Otherwise the position is unstable. In Fig. 5 the regions of stability and instability are illustrated.

When $k_1 = k_2 = 0$ the above conditions (47) and (48) are not satisfied (where the equation (26) has a positive real root) and the position is unstable, as we get in [9].

(4)- The stability of the position (iv):

The corresponding characteristic equation has:

$$v_{4_4} = \frac{C}{A}[\frac{A}{B} k_1{}^2 + k_2{}^2 + \frac{C}{B} k_3{}^2] + a[\,(1 - \frac{b}{a}) + \frac{C}{A}\,(\frac{c}{a} - \frac{b}{a}) \; + \; \frac{C}{B}\,(1 + \frac{c}{a})\,],$$

$$v_{2_4} = \frac{C^2}{AB}\{a^2\,(\frac{c}{a} - \frac{b}{a})(\frac{c}{a} + 1) + a\,[k_2{}^2(1 + \frac{c}{a}) + k_3^2(1 - \frac{b}{a}) - k_1^2(\frac{c}{a} - \frac{b}{a})]\}$$

$$+ \frac{a^2\,C}{A}(1 - \frac{b}{a})[\frac{A}{B}\,(1 + \frac{c}{a}) - (\frac{c}{a} - \frac{b}{a})], \tag{49}$$

$$v_{0_4} = \frac{a^3\,C^2}{AB}(1 - \frac{b}{a})(1 + \frac{c}{a})(\frac{c}{a} - \frac{b}{a})$$

From equations (49), (28) and (29) we find that this position is unstable for

$$b > c > -a \tag{50}$$

It is stable for

$$-a > c \quad or \quad c > b \tag{51}$$

under the conditions:

$$\frac{C}{A}[\frac{A}{B} k_1{}^2 + k_2{}^2 + \frac{C}{B} k_3{}^2] > -a[\,(1 - \frac{b}{a}) + \frac{C}{A}\,(\frac{c}{a} - \frac{b}{a}) \; + \; \frac{C}{B}\,(1 + \frac{c}{a})\,],$$

$$\frac{C^2}{AB}\{a^2\,(\frac{c}{a} - \frac{b}{a})(\frac{c}{a} + 1) + a\,[k_2{}^2(1 + \frac{c}{a}) + k_3^2(1 - \frac{b}{a}) - k_1^2(\frac{c}{a} - \frac{b}{a})]\} > \tag{52}$$

$$-\frac{a^2\,C}{A}(1 - \frac{b}{a})[\frac{A}{B}\,(1 + \frac{c}{a}) - (\frac{c}{a} - \frac{b}{a})],$$

$$27 \left(\tfrac{2}{27} v_{4_4}^2 - \tfrac{1}{3} v_{2_4}^2 v_{4_4}^2 + v_{0_4}\right)^2 + 4 \left(\tfrac{1}{3} v_{4_4}^2 - v_{2_4}\right)^3 \leq 0 \tag{53}$$

and unstable otherwise. The conditions (53) will also be satisfied since the position corresponds to minimum of the potential. The regions of stability and instability are illustrated, Fig. 6. The hatched region are the region of stability in the Lyapunove sense.

When $k_1 = k_2 = 0$ we get the same results obtained in [9].

For all the families of equilibrium positions given in **(II)** we find that the characteristic equation (26) reduces to:

$$\lambda^8 \, Q(\lambda) = 0 \tag{54}$$

where

$$Q(\lambda) = \lambda^4 + \rho_4 \, \lambda^2 + \rho_2. \tag{55}$$

Thus the characteristic equation has a pair of zero eigenvalues. In the space of coefficients of the characteristic equation the other eigenvalues are pure imaginary if the conditions:

$$\rho_4 > 0, \qquad \rho_2 > 0, \qquad \rho_4{}^2 - 4 \, \rho_2 \geq 0 \tag{56}$$

are satisfied.
If the above conditions (56) are not satisfied, there exists at least one eigenvalue with positive real part. In the plane of parameters of equation (55) the region I in Fig. 7 is the region in which the characteristic equation has no pure imaginary roots. The critical case where the characteristic equation has a zeros root and pure imaginary roots appears in the region II.

Thus in the first region, I, the family of equilibrium positions is unstable while in the second one, II, we cannot decide if this position is stable or unstable. The stability of the families of equilibrium positions **(II)** in general requires further analysis of nonlinear terms in (6).

Now we determine the region in which these families are unstable and consequently the other one in which critical cases appear.

(1) The stability of the family of equilibrium positions (14)
The values of ρ_2, ρ_4 in the characteristic equation (55) corresponding to this family are given by:

$$\rho_{4_1} = \frac{C^2}{AB} k_3{}^2 + \frac{C}{A} \left\{\frac{A}{B} k_1{}^2 + k_2{}^2 + a\left[\frac{A}{B}\left(\frac{c}{a} - \cos\Theta\right) + \left(\frac{c}{a} + \cos\Theta\right)\right]\right\},$$

$$\rho_{2_1} = \frac{a\,C^2}{AB} \left\{a\left(\frac{c}{a} - \cos\Theta\right)\left(\frac{c}{a} + \cos\Theta\right) + k_1{}^2\left(\frac{c}{a} + \cos\Theta\right) + k_2{}^2\left(\frac{c}{a} - \cos\Theta\right)\right\} \tag{57}$$

When the conditions

$$\rho_{4_1} > 0, \qquad \rho_{2_1} > 0, \qquad \rho_{4_1}^2 - 4 \, \rho_{2_1} \geq 0 \tag{58}$$

are satisfied we have the critical case. When the above condition, (58), is not satisfied this family of positions will be unstable.

(2) The stability of the family of equilibrium positions given by (15)

The corresponding characteristic equation has:

$$\rho_{4_2} = \frac{C^2}{AB}\,k_3{}^2 + \frac{C}{A}\,[\,k_2{}^2 + \frac{A}{B}\,[k_1{}^2 + a\,(\,1 - \frac{c}{a}\,\cos\Phi)]] + a\,(\,1 + \frac{c}{a}\,\cos\Phi),$$

$$\rho_{2_2} = \frac{a\,C^2}{AB}\,\{\,a\,\frac{A}{C}\,(\,1 - \frac{c}{a}\,\cos\Phi)(\,1 + \frac{c}{a}\,\cos\Phi) + k_2{}^2(\,1 - \frac{c}{a}\,\cos\Phi) + k_3{}^2(\,1 + \frac{c}{a}\,\cos\Phi)\}$$

(59)

The critical case occurs under the conditions:

$$\rho_{4_2} > 0, \qquad \rho_{2_2} > 0, \qquad \rho_{4_2}^2 - 4\,\rho_{2_2} \geq 0. \tag{60}$$

The family of positions (15) will be unstable when the conditions (60) are not satisfied.

(3) The stability of the family of the equilibrium positions given by (16)

The corresponding characteristic equation has:

$$\rho_{4_3} = \frac{C^2}{AB}\,k_3{}^2 + \frac{C}{A}\,[\,k_2{}^2 + \frac{A}{B}\,[k_1{}^2 - a\,(\,1 + \frac{c}{a}\,\cos\Phi)]] - a\,(\,1 - \frac{c}{a}\,\cos\Phi),$$

$$\rho_{2_3} = \frac{a\,C^2}{AB}\,\{\,a\,\frac{A}{C}\,(\,1 - \frac{c}{a}\,\cos\Phi)(\,1 + \frac{c}{a}\,\cos\Phi) - k_2{}^2(\,1 + \frac{c}{a}\,\cos\Phi) + k_3{}^2(\frac{c}{a}\,\cos\Phi - 1\,)\}$$

(61)

The conditions of existence of the critical case are given by:

$$\rho_{4_3} > 0, \qquad \rho_{2_3} > 0, \qquad \rho_{4_3}^2 - 4\,\rho_{2_3} \geq 0. \tag{62}$$

When the above conditions are not satisfied this family of positions will be unstable.

(4) The stability of family of the equilibrium positions (17)

The corresponding characteristic equation has:

$$\rho_{4_4} = \frac{a\,C^2}{AB}\,k_3{}^2 + \frac{C}{A}\,[\frac{A}{B}\,k_1{}^2 + k_2{}^2 + a\,(\frac{b}{a} + \cos\Psi)] + a\,(\frac{b}{a} - \cos\Psi),$$

$$\rho_{2_4} = \frac{a\,C}{A}\,\{\,a\,(\frac{b}{a} - \cos\Psi)(\frac{b}{a} + \cos\Psi) + \frac{C}{B}\,[k_1{}^2(\frac{b}{a} + \cos\Psi) + k_3{}^2(\frac{b}{a} - \cos\Psi)]\}$$

(63)

The conditions:

$$\rho_{4_4} > 0, \qquad \rho_{2_4} > 0, \qquad \rho_{4_4}^2 - 4\,\rho_{2_4} \geq 0 \tag{64}$$

are satisfied for the critical case and when the conditions (64) are not valid this family of positions is unstable.

(5) The stability of the family of the equilibrium positions given by (18)

The corresponding characteristic equation has:

$$\rho_{4_5} = \frac{a\,C^2}{AB}\,k_3{}^2 + \frac{C}{A}\,[\frac{A}{B}\,k_1{}^2 + k_3{}^2 - a\,(\frac{b}{a} - \cos\Psi)] - a\,(\frac{b}{a} + \cos\Psi),$$

$$\rho_{2_5} = \frac{a\,C}{A}\,\{\,a\,(\frac{b}{a} - \cos\Psi)(\frac{b}{a} + \cos\Psi) - \frac{C}{B}\,[k_1{}^2(\frac{b}{a} - \cos\Psi) + k_3{}^2(\frac{b}{a} + \cos\Psi)]\}$$

(65)

This family of equilibrium positions is unstable when the conditions:

$$\rho_{4_5} > 0, \qquad \rho_{2_5} > 0, \qquad \rho_{4_5}^2 - 4\,\rho_{2_5} \geq 0 \qquad (66)$$

are not satisfied. The critical case appear under the above conditions.

6   Sufficient conditions for stability

Via a theorem of Lyapunov [6] it follows that the equilibrium positions in the linear approximation will remain unstable when the nonlinear system is considered. Thus each of the following positions are unstable:

1-the position **(i)** in all the regions $\chi_1, \ldots, \chi_4, \chi_6$

2-the position **(ii)** in the regions χ_2 ,

3-the position**(iii)** in the regions χ_1, χ_5, χ_6,

4-the position **(iv)** in the regions χ_3, χ_4, χ_5

5- all families of equilibrium positions, (14) - (18) when the conditions (58), (60), (62), (64) and (66) are not satisfied, respectively.

The stability of positions which are stable in linear approximation in general requires further analysis of nonlinear terms in (6). These Positions are :

1- position **(i)** in the region χ_5

2- position **(ii)** in the region χ_1,

3- position **(iii)** in the regions χ_2, χ_3, χ_4

4- position **(iv)** in the region χ_6.

The two cases of unconditional stability are those mentioned in Section **(4.1)** which correspond to minimum of the potential. Hence according to a theorem of Lagrange [6] the conditions of appearance the minimum positions are also sufficient for the stability of position.

Figures 3-6 provided here summarize the results in the plane of parameters $\frac{b}{a}$, $\frac{c}{a}$. The hatched regions correspond to positions that are stable in the sense of Lyapunov and the blank regions to unstable positions. In dotted regions gyroscopic stabilization occurs when the gyrostatic moment exceeds a certain limit depending on the parameters but the stability in the sense of Lyapunov requires further consideration of the nonlinear terms in the equations of motion and applying the method of KAM theory [1, 13] .

Acknowledgments

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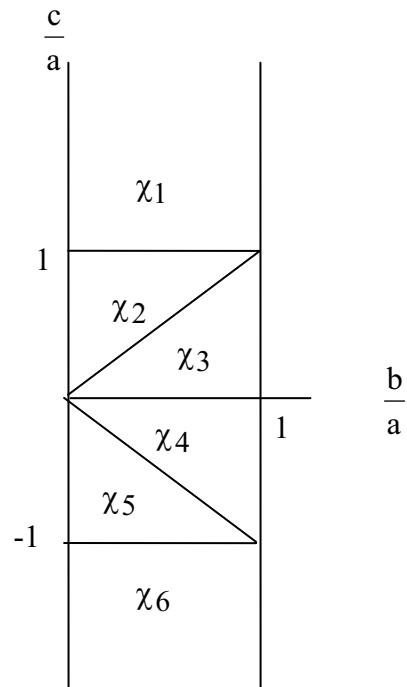
Fig. (1. 1)
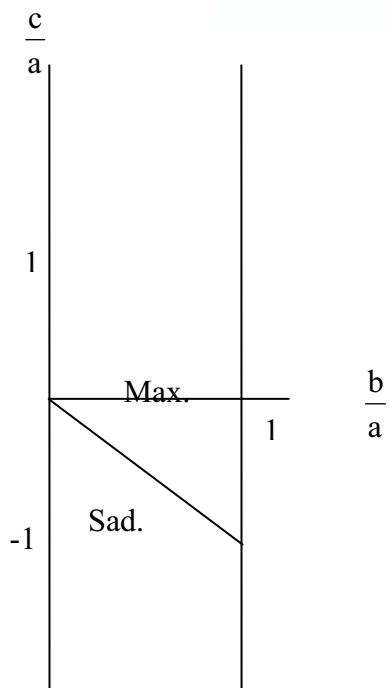
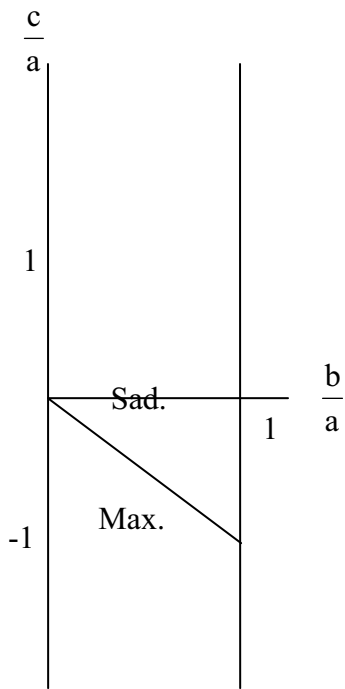Fig. (1.2)
Position (i)
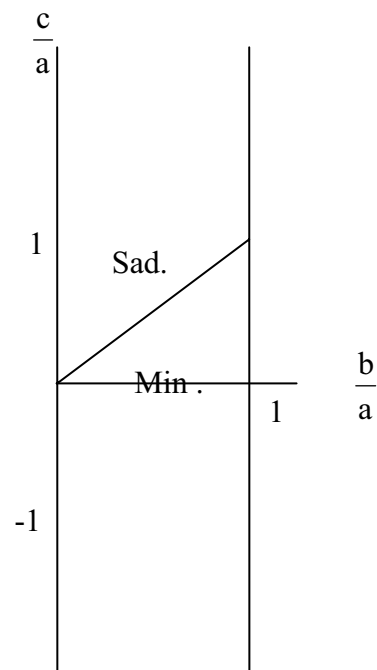
Fig. (1.3)
Position (ii)
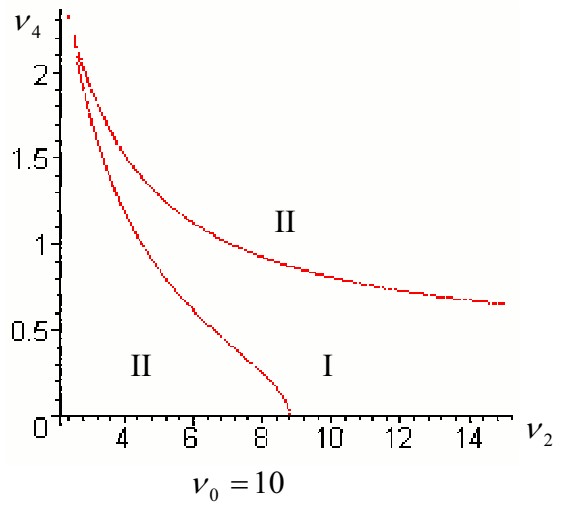
Fig. (1.4)
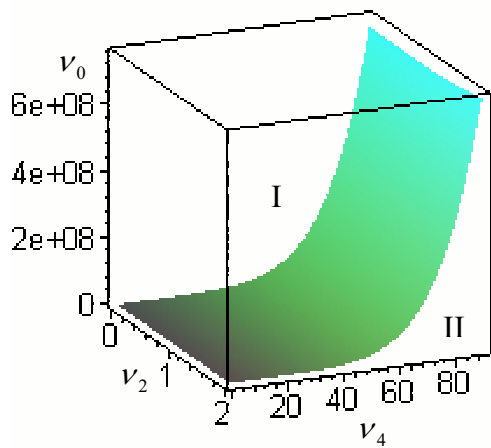Position (iii)
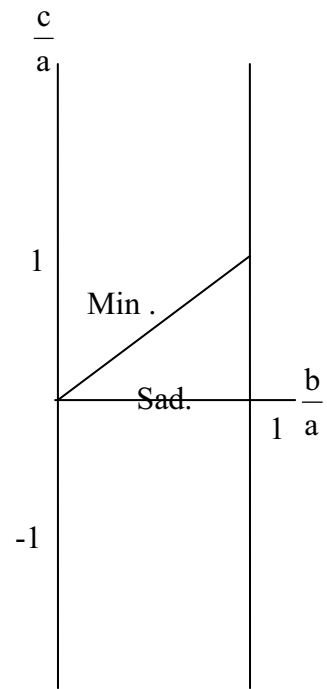
Fig. (1.5)
Position (iv)

Fig. 1

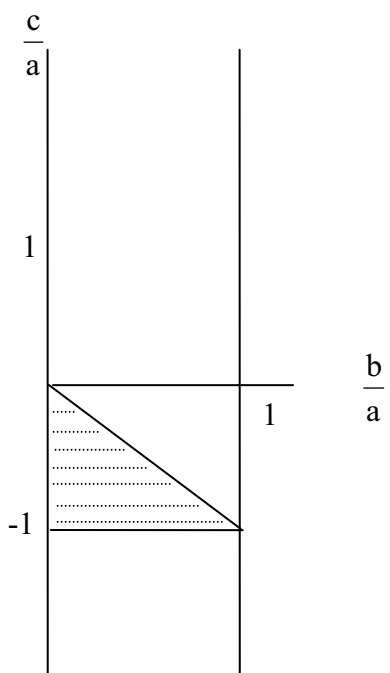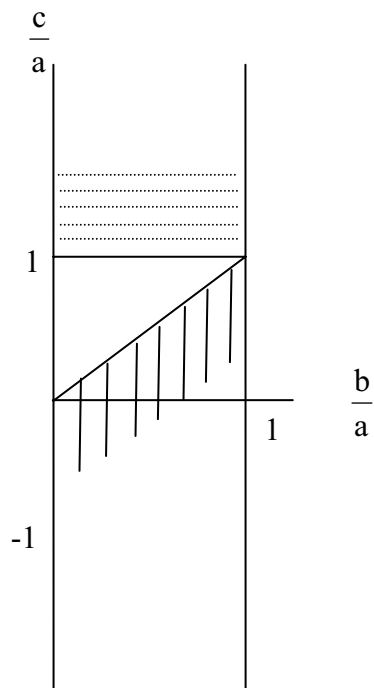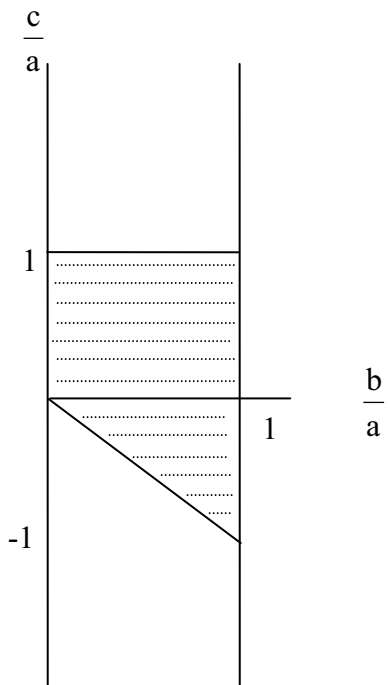$\nu_0 = 10$

Fig. 2.
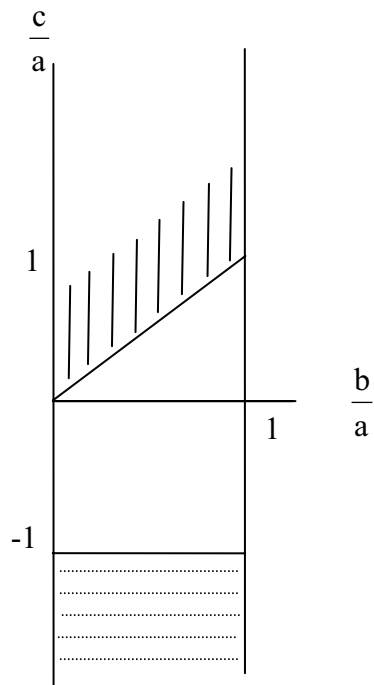
Fig. 3
Position (i)
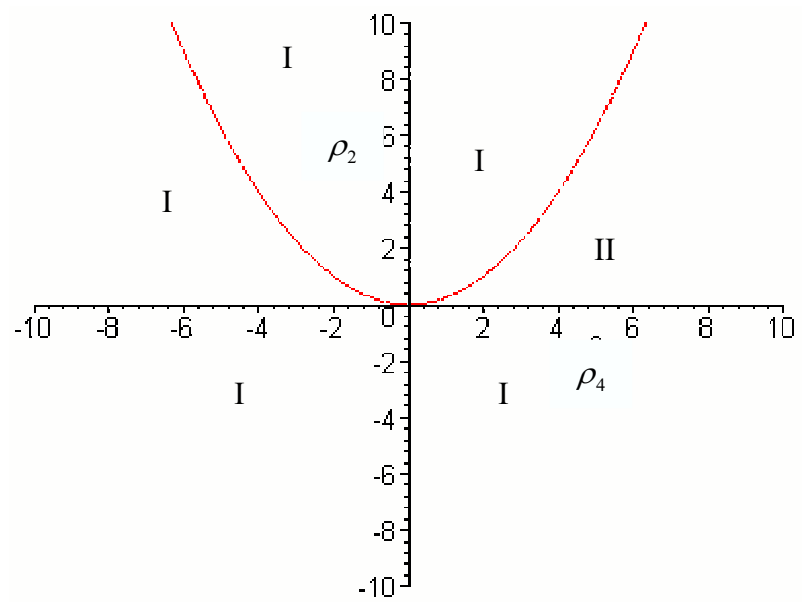
Fig. 4
Position (ii)

Fig. 5
Position (iii)

Fig. 6
Position (iv)

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Fig . 7