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MONTE CARLO RENORMALIZATION: TEST ON THE TRIANGULAR ISING MODEL

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Abstract

We test the performance of the Monte Carlo renormalization method using the Ising model on the triangular lattice. We apply block-spin transformations which allow for adjustable parameters so that the transformation can be optimized. This optimization takes into account the relation between corrections to scaling and the location of the fixed point. To this purpose we determine corrections to scaling of the triangular Ising model with nearest- and next-nearest-neighbor interactions, by means of transfer matrix calculations and finite-size scaling. We find that the leading correction to scaling just vanishes for the nearest-neighbor model. However, the fixed point of the commonly used majority-rule block-spin transformation lies far away from the nearest-neighbour critical point. This raises the question whether the majority rule is suitable as a renormalization transformation, because corrections to scaling are supposed to be absent at the fixed point. We define a modified block-spin transformation which shifts the fixed point back to the vicinity of the nearest-neighbour critical Hamiltonian. This modified transformation leads to results for the Ising critical exponents that converge faster, and are more accurate than those obtained with the majority rule.

I. INTRODUCTION

Monte Carlo renormalization, or short MCRG, has contributed much to our understanding of phase transitions [1–4], especially of the three-dimensional Ising model [5–8]. However, the convergence of the method using the so-called majority rule as a block-spin transformation, depends significantly on the number and character of the couplings included in the renormalization analysis. Moreover, the convergence of the method using the majority rule has been observed to be rather irregular, especially in three dimensions. Swendsen [9] suggested a modified blocking rule, which approximately transforms the nearest-neighbour Ising model at its critical point onto itself (which thus also assumes the role of the fixed point), to improve the convergence. Corrections to scaling are usually associated with irrelevant scaling fields, but the reverse is not necessarily true. Here we recall the so-called redundant operators [10] which do not influence the critical singularity. Shankar and Gupta [11] identified redundant operators and their eigenvalues from the majority rule MCRG data of 2D Ising model on square lattice. A redundant operator associated with the majority blocking rule RG transformation of 3D Ising model has also been found by Baillie etc. [8]. These findings suggest that it could be possible to move the fixed point along the direction of redundant field by modifying the blocking rule, such that faster convergence can be realized. However, Fisher and Randeria [12] argued that a general point on a critical manifold cannot be transformed into a fixed point nor *vice versa*, because a thermodynamic function of a specific model must have unique corrections to scaling, irrespective of which RG transformation is used to analyze the system.

Blöte etc. [13] reviewed the question concerning the analyticity of the block transformation, which may be singular at the infinite system critical point, or even ill defined [14]. They related this question to whether the corrections to scaling vanish at the fixed point of the transformation. Investigating a 3D Ising model with suitably chosen second- and third-neighbour interactions, they ensured that the leading correction to scaling is strongly suppressed at the critical point. They found fast convergence and a high accuracy by using an optimized block rule, chosen such that the fixed point is close to the original Hamiltonian.

To investigate the problem further, and to avoid the complexity introduced by 3-dimensional model, it is better to make use of a 2-dimensional model. An Ising model on the 2-dimensional triangular lattice serves as a suitable model to be investigated. Since every blocking step reduces the number of spins by a factor of only 3, a considerable number of RG steps can be performed.

First, we shall demonstrate that the leading correction to scaling vanishes at the critical nearest neighbour Hamiltonian of this model. We do this by means of transfer matrix calculations and finite size scaling. On the other hand, we find that the fixed point of general used majority rule RG transformation has a fixed point that is well separated from the nearest neighbour critical point. This makes the modified block spin transformation which maps the nearest-neighbour model at its critical point onto itself more suitable than the general used majority rule, for the reason that it is consistent with the fact that the correction to scaling vanishes, while the majority rule is not. We will compare the results of the two different transformation rules. This can be done easily because the critical point and exponents of this model are exactly known. In addition, dealing with a two dimensional model with scaling factor $\sqrt{3}$, more blocking steps can be taken in our research, such that we can have a better chance to eliminate the finite size effect, and better determine the renormalization transformation effect.

This paper is organized as follows: we briefly review the MCRG method and technique details in section II. In section III, we find out the correction to scaling of the critical nearest neighbour Ising model on the 2-dimensional triangular lattice by means of the transfer matrix method and finite size scaling. Then, we calculate the fixed point of majority rule RG transformation, and suggest a modified block rule which maps the fixed point back to its critical point. In section IV, we compare the convergence and accuracy of the majority rule and the modified rule transformations. Finally, we conclude in section V.

II. MODEL AND METHOD

The MCRG method has amply been reviewed [3, 4], we only briefly outline the method here. The reduced Hamiltonian of the Ising model can be written using lattice sums:

$$H(S) = -\sum_{\alpha=0}^{\infty} K_\alpha S_\alpha \tag{1}$$

where S is a spin configuration, the K_α are couplings, and the S_α are the conjugate lattice sums over spin products, e.g., K_1 is the magnetic field and $S_1 = \sum_i s_i$ is the sum over all spins; K_2 is the nearest-neighbor coupling and $S_2 = \sum_{<nn>} s_i s_j$ the sum over all nearest-neighbor pairs (s_i, s_j) . A special "coupling" is the background energy density K_0 and S_0 is the number of spins.

Application of a block-spin transformation to Monte Carlo generated configurations S leads to configurations S' described by a Hamiltonian $H' = H(K'_0, K'_1, K'_2, ...; S')$. The renormalized couplings K'_α are assumed to be analytic functions of the original ones, even at the infinite system critical point.

For the case of Ising model on a triangular lattice (shown in Fig. 1), three spins in one elementary lattice face is renormalized to a new single spin. The transformation is defined by the probability $P(s'; s_1, s_2, s_3)$ of a block spin s', where $s_i, i = 1, 2, 3$ are the spins in a triangular lattice face. This transformation can be the majority rule, or the others we called modified rule, depending on the probability P.

A $L * L$ lattice system under periodic boundary condition $s_{L+1,j} = s_{1,j}; s_{i,L+1} = s_{i,1}$ in both the i, j directions will be renormalized to a $L * L/3$ lattice with a shifted periodic boundary in j direction ($s'_{L+1,j} = s'_{1,j}; s'_{i,L/3+1} = s'_{i+\delta,1}$, where $\delta = L/3$), and a rotation of angle $\pi/6$ of lattice also happens. The renormalization procedure can be iterated. A $L * L/3$ lattice system with shifted periodic boundary in j direction will be renormalized to a periodic $L/3 * L/3$ system. This procedure is shown in Fig. 1.

We denote the renormalization level by superscripts. Thus, after i renormalization transformations the Hamiltonian is $H^{(i)} = H(K_0^{(i)}, K_1^{(i)}, K_2^{(i)}, \ldots; S^{(i)})$.

The linearized Renormalization Group transformation matrix

$$\frac{\partial K_\alpha^{(i)}}{\partial K_\beta^{(i-1)}} = T_{\alpha\beta}^{(i)} \tag{2}$$

can be related to lattice sum correlations via

$$\sum_\gamma B_{\alpha\gamma}^{(i)} T_{\gamma\beta}^{(i)} = C_{\alpha\beta}^{(i)} \tag{3}$$

where the correlations

$$B_{\alpha\beta}^{(i)} = \langle\langle S_\alpha^{(i)} S_\beta^{(i)} \rangle\rangle = \langle S_\alpha^{(i)} S_\beta^{(i)} \rangle - \langle S_\alpha^{(i)} \rangle \langle S_\beta^{(i)} \rangle = \frac{\partial \langle S_\alpha^{(i)} \rangle}{\partial K_\beta^{(i)}} \tag{4}$$

and

$$C_{\alpha\beta}^{(i)} = \langle\langle S_\alpha^{(i)} S_\beta^{(i-1)} \rangle\rangle \ = \ \langle S_\alpha^{(i)} S_\beta^{(i-1)} \rangle - \langle S_\alpha^{(i)} \rangle \langle S_\beta^{(i-1)} \rangle = \frac{\partial \langle S_\alpha^{(i)} \rangle}{\partial K_\beta^{(i-1)}} \tag{5}$$

can be calculated using the Monte Carlo method.

Since the even and odd lattice sums are not correlated, the analysis can be performed separately in the even and odd coupling subspaces.

The fixed point of the transformation can also be found, in the case that the distance $\delta\vec{K}$ of the original Hamiltonian to the fixed point is small. Consider the triangular Ising model with lattice size $\sqrt{3}^{2p}$, the lattice sums calculated after n renormalization steps on the remaining $\sqrt{3}^{2(p-n)}$ lattice are denoted $S_\alpha^{(p,n)}$. One can linearize as follows:

$$\langle S_\alpha^{(p+m,n+m)} \rangle - \langle S_\alpha^{(p,n)} \rangle = \sum_\beta [\langle\langle S_\alpha^{(p+m,n+m)} S_\beta^{(p+m,0)} \rangle\rangle - \langle\langle S_\alpha^{(p,n)} S_\beta^{(p,0)} \rangle\rangle] \delta K_\beta \tag{6}$$

thus find the distance $\delta\vec{K}$ by solving this equation.

The present MCRG calculations involve the following steps:

1. The generation of a spin configuration by means of a number of Metropolis sweeps and a number of Wolff cluster [15] steps. In our simulation, typically 2 Metropolis sweeps and 10 Wolff cluster steps are taken to generate a new spin configuration.

2. The calculation of the lattice sums S_α.

3. Execution of the block spin transformation which reduces the lattice size by a factor 3.

4. The same as step 2, using the reduced spin lattice.

5. Repetition of steps 3 and 4. This sequence stops at system size 3^2.

6. Calculation and accumulation of the cross products. $S_\alpha^{(i)} S_\beta^{(j)}$

7. Repetition of steps 1-6(called a 'cycle') for a large number of samples.

The transformation matrix T is approximated by solving Eq.3 after truncation to a finite number of couplings. We have included up to 10 even couplings and 5 odd couplings in our simulations, as shown in Fig. 2. This will be proved enough to reach a good convergence of eigenvalues later. Under iteration of the block-spin transformation, the $K_\alpha^{(i)}(\alpha > 0)$ are assumed to approach the fixed point of the transformation, where the eigenvalues of T determine the critical exponents.

III. CORRECTION TO SCALING AND THE FIXED POINT

Models belonging to the same universality class have the same leading singular behavior of a thermodynamic quantity as their critical points are approached. However, corrections to the leading singular behaviors of these models may be different.

According to the standard Renormalization Group theory, the singular part of free energy density has the following scaling behaviour:

$$f_s(t,u) = |t|^{d/y_t} f_s(\pm 1, |t|^{-y_i/y_t}u) = |t|^{d/y_t}[f_s(\pm 1, 0) + a|t|^{-y_i/y_t}u + \cdots] \tag{7}$$

where t is the relevant scaling field, and u is the irrelevant scaling field. y_t, y_i are the relevant and leading irrelevant renormalization exponents respectively. The so called correction to scaling is related to the irrelevant scaling field. A thermodynamic quantity, e.g., specific heat has the following scaling form

$$C_s(t,u) = |t|^{d/y_t-2}[b_0 + b_1 u|t|^{-y_i/y_t} + \cdots] \tag{8}$$

where the amplitude of the correction to scaling is proportional to the irrelevant scaling field u, which is zero only at the fixed point of the RG transformation and a general point on a critical manifold cannot be transformed into a fixed point nor *vice versa* [12].

The irrelevant scaling field can be estimated by using the transfer matrix calculation and finite size scaling method. Consider a triangular lattice Ising model wrapped on a infinitely long cylinder with circumference L. The scaling equation for the magnetic scaled gap is:

$$X_h(L,t,u) = X_h + a_1 t L^{y_t} + a_2 u L^{y_i} + \ldots \tag{9}$$

where $X_h(L,t,u) = \frac{L}{2\pi\xi_h(L,t,u)}$ is the magnetic scaled gap, which can be calculated by using the transfer matrix method [16]. X_h is the magnetic scaling dimension, which is equal to 1/8 for the Ising model. t is the relevant scaling field with $y_t = 1$ the thermal renormalization exponent. u is the irrelevant scaling field and y_i is the leading irrelevant renormalization exponent, which is known as -2.

Consider an Ising model with nearest neighbour coupling K_{nn} and next nearest neighbour coupling K_{nnn}. For a given K_{nn}, solving the scaling equation

$$X_h(L,t,u) = X_h(L-1,t,u) \tag{10}$$

for sequential system sizes, we can find $t \propto uL^{-3}$, i.e. $K_{nnn}(L) = K_{nnn,c} + bL^{-3}$. The amplitude b which is proportional to the irrelevant scaling field can be found as well as critical next nearest neighbour coupling $K_{nnn,c}$ by extrapolating finite size solutions $K_{nnn}(L)$.

Fig.3 shows that correction to scaling amplitude b vs. the critical ratio K_{nnn}/K_{nn} for the Ising model on the square lattice and the triangular lattice, respectively. We can see that correction to scaling vanishes for critical nearest neighbour Ising model on the triangular lattice, while it is quite strong for the critical nearest neighbour Ising model on square lattice. In the latter case, correction to scaling vanishes at $K_{nnn}/K_{nn} \approx 0.3$.

A natural choice of RG transformation fitting triangular nearest neighbour Ising model is the one which has its fixed point very close to the nearest neighbour critical point.

On the other hand, the MCRG method relies on the assumption of analyticity of the transformation. A general belief is that local transformations can be arbitrarily defined supposing that this condition is satisfied. The general chosen block spin transformation is the so-called majority rule transformation, i.e.:

$$P(s';s_1,s_2,s_3) = \begin{cases} 1 \ \text{ if } s' = \text{sgn}(s_1+s_2+s_3) \\ 0 \ \text{ if } s' \neq \text{sgn}(s_1+s_2+s_3) \end{cases}$$

A question which arose naturally is: where is the fixed point of this majority rule RG transformation for the Ising model on the triangular lattice ? According to Eq.(6), by comparing two systems with compatible sizes, we can locate the fixed point. In fact, we determine the irrelevant scaling fields of the original Hamiltonian. Therefore, the best results are expected for n as small as possible. Thus we used $n=0, m=1$, and solved δK_α from two sequential sizes in the two couplings and three couplings space respectively. To best estimate the fixed point in a two- or three- dimensional coupling space, we did the simulation iteratively, i.e., we simulated the 'fixed point' Hamiltonian, which is estimated by solving Eq.(6), to find out the next estimation of 'fixed point' Hamiltonian. Good convergences happen in both the two- and three-dimensional coupling spaces. The final results are shown in Tables I and II. Extrapolating these finite size data to infinity system, we have the location of fixed point listed in the last row of the two tables.

Our final estimations show that the fixed point of majority rule RG transformation for Ising model on the triangular lattice is far away from the nearest neighbour critical model: $K_2 = \frac{ln3}{4}$ and all other $K_\alpha = 0$ for $\alpha \neq 2$. However, this is inconsistent with the fact that the Ising model on the triangular lattice has no correction to scaling effects.

To restore the consistence, we try to introduce a modified block spin transformation with its fixed point close to the critical point of nearest-neighbour model, and this can be done following

Blöte etc. [13]:

$$P(s') = \frac{\exp(\omega s' s_b)}{2\cosh(\omega s_b)} \tag{11}$$

where s_b is the sum of spins in the triangular face:$s_b = s_1 + s_2 + s_3$.

Adjusting ω, we can make sure that the solution δK_α of Eq.6 is approximately zero. Our simulations show that the value of ω is not sensitive to the finite system sizes involved. The best estimated ω is about 1.258. With this value the block-spin transformation does not move the Hamiltonian away from the nearest neighbour critical point. Thus, this is consistent with the fact that the correction to scaling of nearest neighbour Ising model on the triangular lattice does not exist.

IV. CRITICAL EXPONENTS

Now we have two block spin transformations with different fixed points. One is consistent with the vanishing of correction to scaling, one is not. To compare the two different block rules, we have done extensive MCRG simulations. 10^8 configurations were generated to do RG blocking and correlation procedures. Statistical errors of the lattice sums and correlations of the lattice sums are estimated by dividing the long run to several shorter runs, and calculating the standard deviation of the results from the averages.

From the standard theory, the renormalized Hamiltonian approaches the fixed point along the irrelevant direction in the way $a_1 b^{y_i} + a_2 b^{y_j} + \cdots$, where $b = \sqrt{3}$ is the scaling factor of the triangular lattice and $y_i, y_j, \cdots$ are the leading and sub-leading irrelevant RG exponents, etc. The leading eigenvalue of the even subspace of $T^{(i)}_{\alpha\beta}$, which is named $\lambda^{(i)}_e$, and the one of the odd subspace, $\lambda^{(i)}_o$, will finally approach the fixed point values $\sqrt{3}$ and $\sqrt{3}^{15/8}$, which correspond to thermal renormalization exponent $y_t = 1$ and magnetic renormalization exponent $y_h = 1.875$ respectively, after large enough renormalization transformation steps i.

In practice, we have to truncate the calculation of transformation matrix to a finite number of couplings and simulate on a finite size system. We have included up to 10 even couplings and 5 odd couplings in our simulations for both the majority blocking rule and modified blocking rule. We will show that this is enough for the present calculation later in this paper. The largest system size we have reached has 3^{10} sites, i.e. a 243×243 system. The smallest system size we used has 3^3 sites, i.e. a 9×3 system.

Solving Eq.3, we can get the finite size and finite number of couplings dependent linearized RG transformation matrix $T^{(i)}_{\alpha\beta}$ in the i-th renormalization level. By diagonalizing this transformation matrix, the relevant eigenvalues can be found, which depend on the renormalization level i, on the original system sizes 3^p we start block spin transformation (the system size shrinks to 3^{p-i} in the ith renormalization level), and on the dimensionality of the coupling space n_c. We thus denote them as $\lambda^{(i,n_c)}_e(p)$ and $\lambda^{(i,n_c)}_o(p)$.

The renormalization effects on these eigenvalues can be described by standard Renormalization theory as we mentioned above, but there is no systematic theory to describe the finite size effect of the RG transformation itself. This effect is believed due to the truncation of the space of operators, and has been found be smooth and to decay fast [17], so that we attempt to descirbe the finite-size effect in terms of an expansion in the inverse number of sites.

Besides these two effects, one can also include the joint effects of them.

Including all these effects, we can extrapolate the eigenvalues of fixed point RG transformation matrix from the finite size data according to the following fit formula:

$$\lambda_s^{(i,n_c)}(p) = \lambda_s + b_1\sqrt{3}^{iy_i} + b_2\sqrt{3}^{iy_j} + c_1 3^{(p-i)y_1} + c_2 3^{(p-i)y_2} + c_3 3^{(p-i)y_3} + c_4 3^{(p-i)y_4} + mt\sqrt{3}^{iy_i} 3^{(p-i)y_1} \tag{12}$$

where subscript s of λ can be e for the eigenvalue of even subspace transformation matrix or o for the odd subspace. y_1, y_2, y_3, y_4 are the finite-size effect exponents, which are set as $-1, -2, -3, -4$ respectively during our fitting. y_i and y_j are the leading and subleading irrelevant renormalization exponents. The last term of this fit formula describes the joint effect of renormalization and finite size.

A. Exponents calculated with the majority rule and with the modified block-spin transformation

Let's first show the eigenvalues of the RG transformation matrix in the 5-dimensional even and odd coupling space and fit the data with the formula (12) to find out the fixed point values, then compare the estimated results of the two different transformations.

First, the results of majority rule MCRG.

Table III lists the eigenvalues $\lambda_e^{(i,5)}(p)$ of the majority rule RG transformation matrix in the 5-dimensional even coupling space. We make use of nonlinear least square fit via function (12) to the data. Preliminary fits show that amplitudes of the terms L^{-3} and L^{-4} are extremely small, and this is true in all other fits we made in this research. Thus we can neglect these terms in our fits. We also set y_i to be free in the fitting to investigate the convergence of the data along the irrelevant field direction. The fit yields the extrapolated eigenvalue $\lambda_e = 1.7330(4)$, while y_i is fit as $-1.69(8)$, which is not the expected value -2. This fit has $\chi^2 = 100$, and the statistical error of the raw data are about 0.0003, the degree of freedom in the fit is 29. The fit result λ_e is slightly outside the right region of the exact value $\sqrt{3}$, considering the errorbar of the fit.

Table IV lists the eigenvalues $\lambda_o^{(i,5)}(p)$ of the majority rule RG transformation matrix in the 5-dimensional odd coupling space. Similar fit of the data yields estimated eigenvalue $\lambda_o = 2.80067(7)$, with $\chi^2 = 36$(the statistic errors of the raw data are about 0.00015, and the degree of freedome in the fit is 29). The leading irrelevant exponent y_i is fit as $-1.99(2)$. The extrapolated odd eigenvalue is also slightly outside the right region of the exact value $3^{15/16}$.

Then the results of the modified rule MCRG.

Table V lists eigenvalues $\lambda_e^{(i,5)}(p)$ of the modified rule RG transformation with $\omega = 1.258$ in the 5-dimensional even coupling space. Nonlinear least square fit yields the eigenvalue λ_e as 1.7319(4), with $\chi^2 = 14$. The degree of freedom in this fit is 29, while statistic errors of the raw data are about 0.0003. The leading irrelevant exponent y_i is fit as $-1.2(2)$. This result of λ_e is much closer to the exact value than the result of the majority rule. Besides, the exact value sits inside of the error bar of the fit result.

Table VI lists eigenvalues $\lambda_o^{(i,5)}(p)$ of the modified rule RG transformation with $\omega = 1.258$ in the 5-dimensional odd coupling space. A similar fit generates the fixed point eigenvalue $\lambda_o = 2.80078(14)$, with $\chi^2 = 6$(the statistic errors of the raw data are about 0.00015, the degree of freedom in this fit is 29), and $y_i = -1.6(2)$. Again, the extrapolated value is much closer to the exact value than the one of majority rule and the exact value sits inside the error bar of the fit result.

The values of y_i of the fits for λ_e and λ_o of the modified block spin transformation are suggestive. They are very close to Potts subleading thermal exponent $-4/3$ [18] in the 2-dimensional Ising model [19], which is very difficult to observe. Fixed $y_i = -4/3$, fitting the data $\lambda_e^{(i,5)}(p)$ and $\lambda_o^{(i,5)}(p)$ via function (12) yields $\lambda_e = 1.7318(3)$ and $\lambda_o = 2.80108(20)$, respectively.

To compare the results of the two different block spin transformations, we include the main results in Table VII. Apparently, modified blocking rule MCRG simulation generates much better estimation of leading eigenvalues of RG transformation matrix, both in the even and odd subspace.

B. Convergence with the dimensionality of the coupling subspace

In order to investigate the convergence with the dimensionality of the coupling subspace, the number of couplings n_c used in the MCRG analyses on 3D Ising model has increased from 7 in Ref. [5] to 99 in Ref. [8]. A criterion to distinguish "important" and "less" important couplings was introduced in Ref. [7]. Fast apparent convergence with increasing n_c has been found [7]. To test the convergence with the dimensionality of the coupling space in the present 2-dimensional Ising model, we have included up to 10 even couplings in our analysis, see Fig. 2. We do the similar data fitting on finite size eigenvalues of both majority and modified rule transformation matrix in coupling space dimensionality from 1 to 10 in the even subspace.

The fit results are shown in Fig. 4 for the majority block spin transformation and in Fig. 5 for the modified block spin with $\omega = 1.258$, respectively.

It is clear that eigenvalues converge to their best estimations when the number of couplings included reaches 5 for both of the two block spin transformations, although accuracies are different. This is also true in the odd subspace. Thus our analysis given in the preceding section, based on data in the 5-dimensional coupling space, is enough.

V. CONCLUSIONS

Our MCRG calculations show that the use of a modified block spin transformation which maps the fixed point Hamiltonian back to its nearest neighbour critical point leads to better estimation of RG exponents than using the commonly used majority rule transformation. This is achievable because the leading correction to scaling of nearest neighbour Ising model on the triangular lattice vanishes. The majority rule leads to a fixed point with significant antiferromagnetic interactions as found in our analysis. Although we have not explicitly calculated the corrections to scaling in this fixed point, the transfer-matrix data presented for the model with nearest- and next-nearest neighbors indicate that the corrections do not vanish at the fixed point. This result is inconsistent with the prevailing interpretation of the fixed point in the renormalization group theory. However, we mention an alternative point of view [20] which uses a mechanism that generates corrections even at the fixed point, and that generates nonanalytic contributions in the renormalized Hamiltonian. In this work, we have attempted to avoid such problems by restoring consistency with the usual picture of vanishing correction amplitudes at the fixed point. Indeed we observe better convergence and improved accuracies.

In addition, we find that the convergence to the fixed point is controlled by an irrelevant exponent close to $y_i' = -4/3$, which is different from the known irrelevant exponent -2. It is tempting to relate this value $-4/3$ to the subleading Potts temperature exponent [18]. The associated amplitudes seem to vanish for the q-state Potts model at $q = 2$, i.e. the Ising model. Corrections to scaling with this exponent have been reported only in quantities involving explicit differentiations with respect to the number q [21], and in partial differential approximants [19]. Corrections with exponent $-4/3$ are supposed to describe geometric properties of Ising configurations and not to enter into the thermal properties of the model. Perhaps such corrections could arise from non-thermodynamic geometric aspects introduced by the block spin transformation.

The effect of the nonvanishing corrections-to-scaling at the fixed point of the majority rule RG transformation seems less serious in the two-dimensional Ising model than that in three dimensions. In our present work, we have strongly reduced the amplitudes of the leading corrections describing the approach to the fixed point. In contrast, in earlier work in three dimensions, it seems that also strong nonlinear effects, perhaps reflecting nonanalyticity, are suppressed. The difference between the two- and three-dimensional Ising models may be related to the observation of Aharony and Fisher [22] that irrelevant fields are difficult to observe in Ising models because the nonlinearity of the temperature field, for which one can read $t = (T - T_c)/T_c$ in first order, generates corrections with the same powers.

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TABLE I: Fixed point estimated in the two-dimensional coupling space: finite size data and extrapolated values.

L	S	K_1^*	K_2^*
9×9	9×3	0.3611(2)	−0.0543(2)
27×9	9×9	0.3518(2)	−0.0488(2)
27×27	27×9	0.3485(2)	−0.0466(2)
81×27	27×27	0.3468(2)	−0.0455(3)
81×81	81×27	0.3464(2)	−0.0452(2)
∞		0.346(2)	−0.045(1)

TABLE II: Fixed point estimated in three-dimensional coupling space: finite size data and extrapolated values.

L	S	K_1^*	K_2^*	K_3^*
9×9	9×3	0.4267(2)	−0.0463(2)	−0.0389(2)
27×9	9×9	0.3881(2)	−0.0325(2)	−0.0326(2)
27×27	27×9	0.3792(2)	−0.0291(2)	−0.0312(2)
81×27	27×27	0.3731(2)	−0.0269(2)	−0.0301(2)
81×81	81×27	0.3714(2)	−0.0263(2)	−0.0298(2)
∞		0.370(1)	−0.026(2)	−0.030(1)

TABLE III: Eigenvalues of majority rule transformation matrix in the $n_c = 5$ even coupling space. The statistic error of the data are approximately 0.0003.

i\p	10	9	8	7	6	5	4	3
1	1.653704	1.653505	1.653410	1.653081	1.652479	1.650893	1.645550	1.695418
2	1.711253	1.711037	1.710827	1.710254	1.708959	1.703025	1.759714	
3	1.724861	1.724722	1.723996	1.722573	1.716360	1.777823		
4	1.728788	1.728342	1.726711	1.720288	1.784270			
5	1.729631	1.728416	1.722031	1.786435				
6	1.728706	1.722093	1.787168					
7	1.722542	1.787104						
8	1.787676							

TABLE IV: Eigenvalues of majority rule transformation matrix in the $n_c = 5$ odd coupling space. The statistic error of the data are approximately 0.00015.

i\p	10	9	8	7	6	5	4	3
1	2.730199	2.730195	2.730211	2.730201	2.730195	2.730163	2.730076	2.726260
2	2.773272	2.773272	2.773260	2.773260	2.773204	2.773176	2.768895	
3	2.791232	2.791231	2.791204	2.791194	2.791069	2.786765		
4	2.797330	2.797324	2.797274	2.797198	2.792882			
5	2.799578	2.799540	2.799433	2.795107				
6	2.800388	2.800293	2.795968					
7	2.800598	2.796270						
8	2.796371							

TABLE V: Eigenvalues of the modified rule RG transformation matrix ($\omega = 1.258$) in the $n_c = 5$ even coupling space. The estimated statistic error of the data are approximately 0.0003.

i\p	10	9	8	7	6	5	4	3
1	1.708660	1.708615	1.708684	1.708735	1.708734	1.708597	1.709245	1.694470
2	1.725449	1.725540	1.725570	1.725460	1.725277	1.725550	1.710963	
3	1.729391	1.729291	1.729405	1.729316	1.729749	1.715674		
4	1.731048	1.730904	1.730914	1.731122	1.717573			
5	1.731546	1.731354	1.732069	1.718616				
6	1.731564	1.732176	1.718922					
7	1.732312	1.719175						
8	1.719252							

TABLE VI: Eigenvalues of the modified rule RG transformation matrix ($\omega = 1.258$) in the $n_c = 5$ odd coupling space. The estimated statistic error of the data are approximately 0.00015.

i\p	10	9	8	7	6	5	4	3
1	2.791732	2.791727	2.791731	2.791723	2.791697	2.791697	2.791701	2.790458
2	2.795678	2.795680	2.795682	2.795680	2.795620	2.795615	2.794304	
3	2.798618	2.798604	2.798606	2.798573	2.798551	2.797178		
4	2.799832	2.799824	2.799811	2.799739	2.798382			
5	2.800394	2.800351	2.800366	2.798950				
6	2.800635	2.800596	2.799194					
7	2.800731	2.799349						
8	2.799424							

TABLE VII: Comparing of results of majority rule transformation and of the modified transformation.

	exact value	majority rule	modified rule (fit with y_i free)	modified rule(fit with $y_i = -4/3$)
λ_e	$\sqrt{3}$	1.7330(4)	1.7319(4)	1.7318(3)
λ_o	$\sqrt{3}^{15/8}$	2.80067(7)	2.80078(14)	2.80108(20)

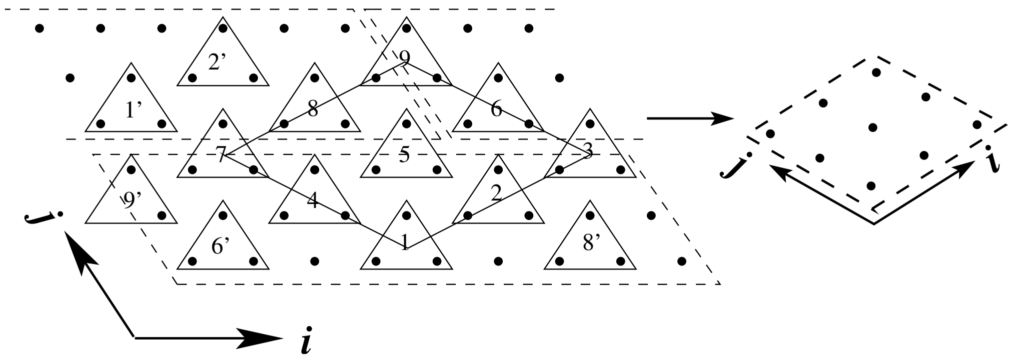


FIG. 1: Triangular lattice and blocking procedure. The dashed boxes before blocking represent shifted periodic boundary condition of the system.

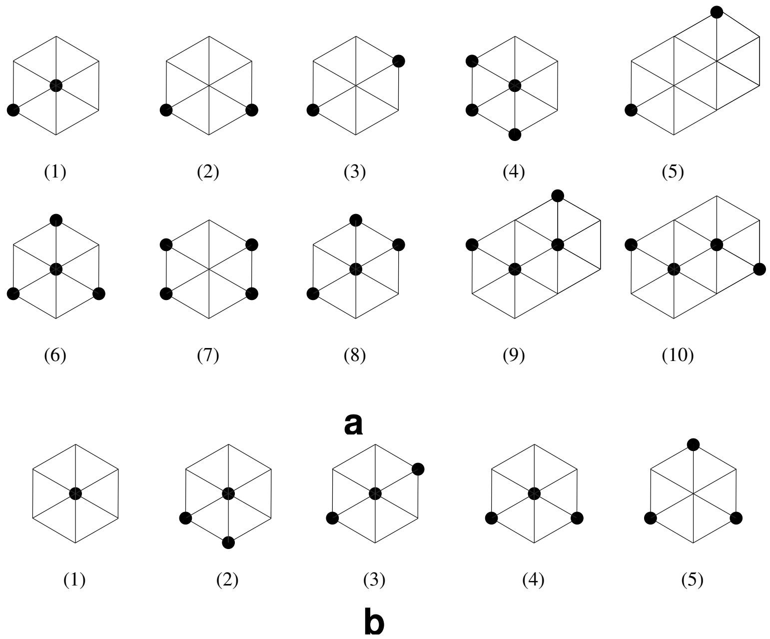
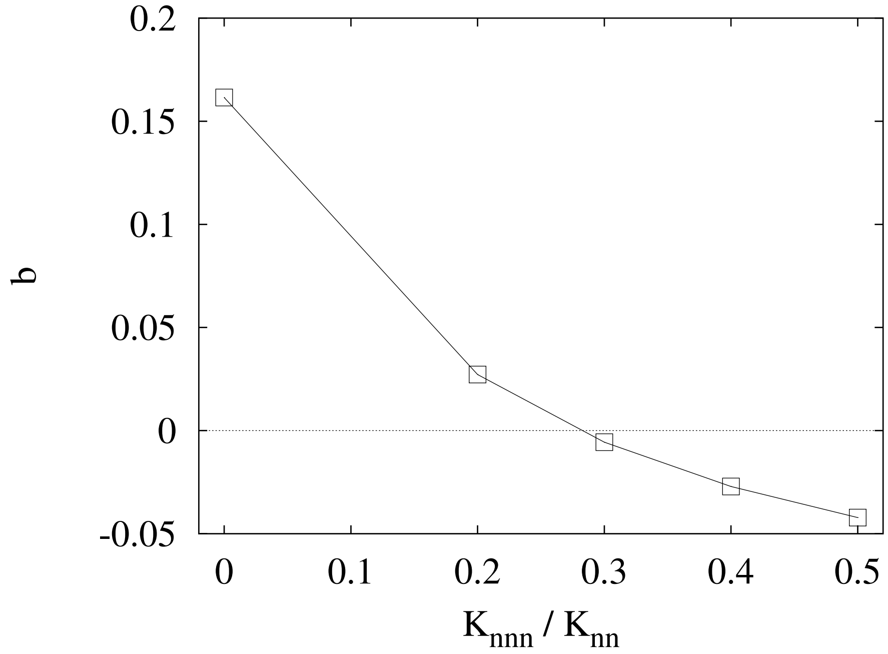
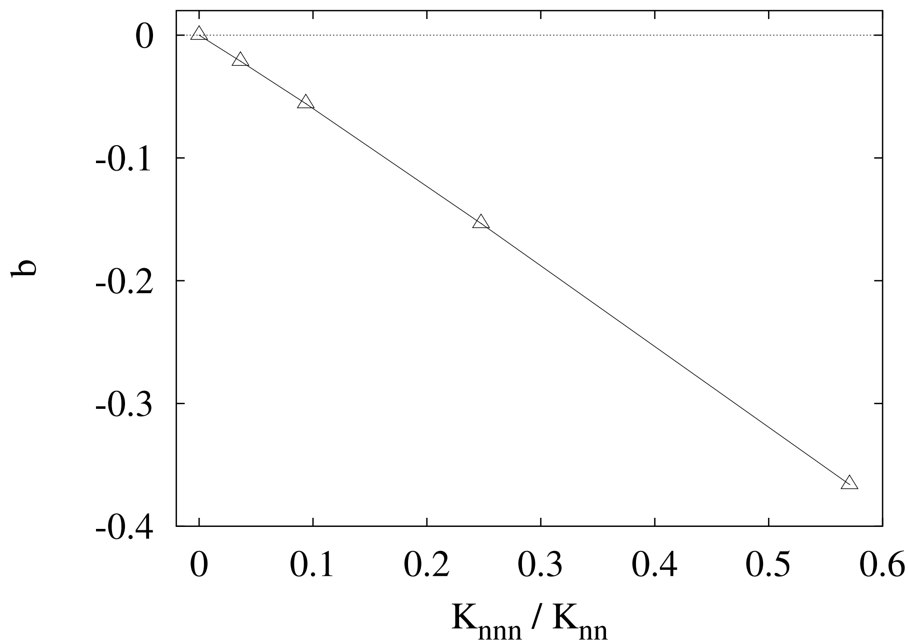


FIG. 2: The even (a) and odd (b)couplings used in present work. The circles represent the spins participating in the interactions.



(a)



(b)

FIG. 3: The amplitude of the finite size correction of K_{nnn}, which is proportional to the correction to scaling of the system, vs. the ratio K_{nnn}/K_{nn} of a critical system.(a) Ising model on square lattice.(b) Ising model on the triangular lattice.

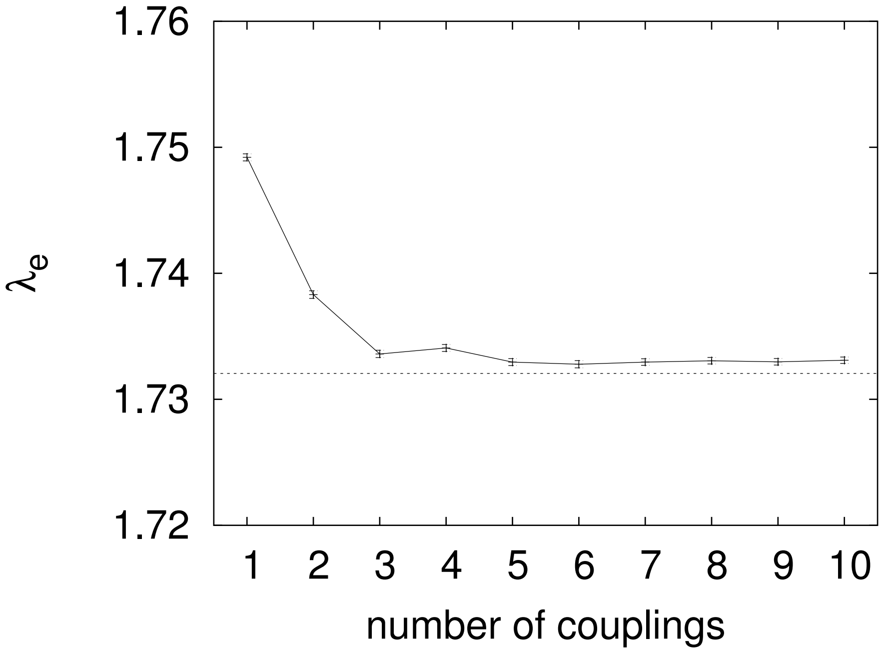


FIG. 4: The extrapolated even fixed point eigenvalues of majority rule RG transformation as a function of the dimensionality of even coupling space included in the analysis. The dashed line indicates the exact value $\sqrt{3}$.

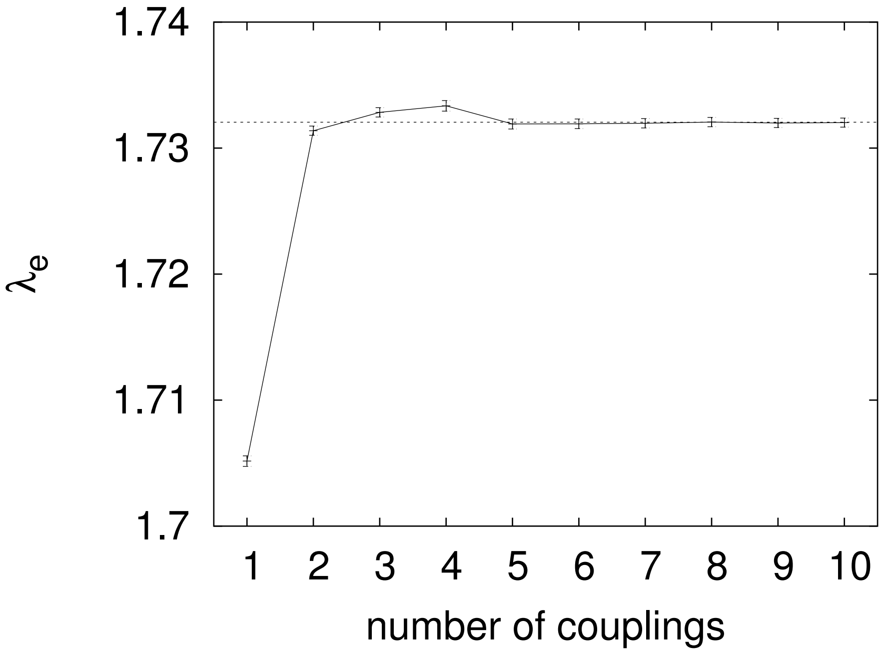


FIG. 5: The extrapolated even fixed point eigenvalues of modified rule RG transformation ($\omega = 1.258$) as a function of the dimensionality of even coupling space included in the analysis. Two almost coincided lines are the fit results with y_i free and with $y_i = -4/3$ respectively. The dashed line indicates the exact value $\sqrt{3}$.