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# ON CORESTRICTION PRINCIPLE IN NON-ABELIAN GALOIS COHOMOLOGY OVER LOCAL AND GLOBAL FIELDS. II : CHARACTERISTIC $p > 0$

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## Abstract

We show the validity of the Corestriction Principle for non-abelian cohomology of connected reductive groups over local and global fields of characteristic $p > 0$, by extending some results by Kneser and Douai.

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**Introduction.** Let $k$ be a field, $G$ a linear algebraic $k$-group and $K/k$ a finite field extension. We denote by $\mathrm{H}^1(K,G) = \mathrm{H}^1(Gal(K_s/K, G(K_s))$ the usual 1-Galois cohomology set, where $K_s$ denotes the separable closure of $K$ in a fixed algebraic closure $\bar{K}$. Assume that we have a map which is functorial in $K$ :

$$\alpha_K : \mathrm{H}^p(K,G) \to \mathrm{H}^q(K,T),$$

i.e., a map of functors $\alpha = (\alpha_K) : (K \mapsto \mathrm{H}^p(K,G)) \to (K \mapsto \mathrm{H}^q(K,T))$ where $K$ runs over all finite field extensions of $k$, $T$ is a commutative linear algebraic $k$-group. If $K/k$ is a finite separable extension of $k$, then it is well-known that there exists corestriction homomorphism

$$Cores_{K/k,T} : \mathrm{H}^q(K,T) \to \mathrm{H}^q(k,T).$$

It is natural to ask whether or not the following inclusion holds

$$Cores_{K/k,T}(\mathrm{Im}\ (\alpha_K)) \subset \mathrm{Im}\ (\alpha_k).$$

If it is the case for all $K$, then we say that the Corestriction Principle holds for the image of the map $\alpha_k : \mathrm{H}^p(k,G) \to \mathrm{H}^q(k,T)$. We may also consider similar notion for kernel of $\alpha_k$, when $G$ is commutative and $T$ may be not. It is most natural to consider the class of maps $(\alpha_K)$ which arise as connecting maps in exact sequences of Galois cohomology induced from an exact sequence of $k$-groups. We refer to [Gil], [Me], [T1] - [T3] and references therein for the discussion of some results related to this notion. In [T2] we showed that over local and global fields $k$ of characteristic 0 the Corestriction Principle holds for the image (and kernel) of connecting maps, where $G$ is any connected linear algebraic $k$-group and $T$ is a linear commutative algebraic $k$-group.

In this note we prove the Corestriction Principle for image (and kernel) in the case $k$ is a local (resp. global) field of characteristic $p > 0$ (which are called also local or global function fields) for connected reductive $k$-groups. We mention that there are two difficulties arising in this case.

Firstly, we note that if $k$ is perfect, then it is well-known (see, e.g., [Se], Ch. III) that 1-Galois cohomology for unipotent groups is trivial and the proof of the main results of [T1] - [T3] is reduced to the case of reductive groups. However, in the case $k$ is non-perfect, this is no longer the case, and moreover, the unipotent radical of a $k$-group does not need to be defined over $k$. Thus in this case we have to restrict ourselves to the case of connected reductive groups $G$ and tori $T$.

Secondly, one of main tools used in the case char. $k = 0$ is the abelian Galois cohomology theory of Borovoi and of Kottwitz ([Bo1-2], [Ko]), which has no analogue (for the time being) in the case char. $k > 0$. So instead of using abelian Galois cohomology, we need to make some further reductions.

# 1 Statement of Theorem

We keep the following convention. All algebraic groups considered are linear algebraic groups, i.e., absolutely reduced affine group schemes, except possibly for certain group schemes of multiplicative type whenever they encounter (which will be clearly indicated). For them, (only in the characteristic $p > 0$ case) we need to use the flat (or the same, Amitsur) cohomology (denoted by $\mathrm{H}^i_{fl}(.,.)$) instead of Galois cohomology. Recall that for linear algebraic groups $G$ over $k$, it is well-known (see e. g., [Mi], Chap. III) that the flat and Galois cohomology of $G$ are canonically isomorphic.

Recall that for a given exact sequence of algebraic $k$-groups $A, B$ and quotient $k$-variety $C$

$$1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1,$$

we have a long exact sequence of pointed sets for any field extension $K/k$

$$(1) \qquad 1 \to A(K) \xrightarrow{f_K} B(K) \xrightarrow{g_K} C(K) \xrightarrow{\delta_K} \mathrm{H}^1_{fl}(K,A) \xrightarrow{f'_K} \mathrm{H}^1_{fl}(K,B).$$

Here $A$ may not be a normal $k$-subgroup of $B$, so $C$ may not be a $k$-group. If $A$ is a normal $k$-subgroup, we may consider a longer sequence involving $\mathrm{H}^1_{fl}(K,C)$

$$(2) \qquad 1 \to A(K) \xrightarrow{f_K} B(K) \xrightarrow{g_K} C(K) \xrightarrow{\delta_K}$$
$$\xrightarrow{\delta_K} \mathrm{H}^1_{fl}(K,A) \xrightarrow{f'_K} \mathrm{H}^1_{fl}(K,B) \xrightarrow{g'_K} \mathrm{H}^1_{fl}(K,C).$$

If, moreover, $A$ is a central $k$-subgroup of $B$, we may also consider a longer exact sequence involving also $\mathrm{H}^2_{fl}(k,A)$ :

$$(3) \qquad 1 \to A(K) \xrightarrow{f_K} B(K) \xrightarrow{g_K} C(K) \xrightarrow{\delta_K}$$
$$\xrightarrow{\delta_K} \mathrm{H}^1_{fl}(K,A) \xrightarrow{f'_K} \mathrm{H}^1_{fl}(K,B) \xrightarrow{g'_K} \mathrm{H}^1_{fl}(K,C) \xrightarrow{\Delta_K} \mathrm{H}^2_{fl}(K,A).$$

All the maps $f_K, g_K, \delta_K, f'_K, \ldots$ arising this way are called simply *connecting maps*. We have the following

**Theorem A.** *Let $k$ be a local or global field of characteristic $p > 0$.*
a) *Let $\alpha_k : \mathrm{H}^p(k,G) \to \mathrm{H}^q(k,T)$ be a connecting map induced from an exact sequence involving $k$-groups as in* (1), (2) *or* (3). *Assume that $G$ is connected, reductive and $T$ is a torus. Then the Corestriction Principle for the image of $\alpha_k$ holds.*

b) *Let $\alpha_k : \mathrm{H}^p(k,T) \to \mathrm{H}^q(k,G)$ be a connecting map induced from an exact sequence involving $k$-groups as in* (1), (2) *or* (3). *Assume that $G$ is connected, reductive and $T$ is a torus. Then the Corestriction Principle for the kernel of $\alpha_k$ holds.*

Theorem A makes use of, among other things, the following theorem (see Theorem 2.4 below) which is an analogue of a theorem of Kneser in the number field case, and it seems to be of independent interest.

**Theorem B.** *Let $G$ be a semisimple group over a local or global function field $k$, $\pi : \tilde{G} \to G$ the universal covering of $G$, $F = \mathrm{Ker}\,(\pi)$. Then the coboundary map*

$$\Delta_k : \mathrm{H}^1_{fl}(k, G) \to \mathrm{H}^2_{fl}(k, F)$$

*is bijective.*

Notice that in [Do], Douai has announced that the map $\Delta_k$ above in the case of global function field is always *surjective*. The proof seems quite different from ours, which makes use band (gerbes) theory of Giraud.

# 2 Preliminaries

**2.0.** We recall (cf. [Bo1,Bo2], [Ko]) that for a connected reductive group $G$ defined over a field $k$, a *z-extension* of $G$ is a connected reductive $k$-group $H$ such that the semisimple part of $H$ (the derived subgroup of $H$) is simply connected and $H$ is an extension (in the sense of algebraic groups) of $G$ by means of an induced $k$-torus $Z$, i.e., we have an exact sequence of $k$-groups

$$1 \to Z \to H \to G \to 1.$$

This notion was introduced (and the existence of such extensions for any given $G$ was proved) by Langlands in the case of characteristic 0, but one checks that the same also holds in the case of positive characteristic. If $K$ is a field extension of $k$, $x \in \mathrm{H}^1(K, G)$, then a $z$-extension $H$ of $G$ over $k$ is called $x$-lifting, if $x \in \mathrm{Im}\,(\mathrm{H}^1(K, H) \to \mathrm{H}^1(K, G))$.

We need the following lemma in the sequel, which extends some results regarding $z$-extensions in the case of char. 0 to that of char. $p > 0$ (cf. [Bo1]).

**2.1. Lemma.** *Let $k$ be any field.*

a) *Let $G$ be a connected reductive $k$-group. Then there exist $z$-extensions of $G$ over $k$.*

b) *Given an exact sequence $1 \to G_0 \to G_1 \to G_2 \to 1$ of connected reductive $k$-groups there exists a $z$-extension of this sequence, i.e., an exact sequence $1 \to H_0 \to H_1 \to H_2 \to 1$ of connected reductive $k$-groups and a commutative diagram*

$$\begin{array}{ccccccccc}
1 & \to & H_0 & \to & H_1 & \to & H_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & G_0 & \to & G_1 & \to & G_2 & \to & 1
\end{array}$$

*of connected reductive $k$-groups such that each group $H_i$ is a $z$-extension of $G_i$, $i = 0, 1, 2$.*

c) *Let $K$ be a finite separable extension of $k$, $G$ a connected reductive $k$-group. Then for any element $x \in \mathrm{H}^1(K, G)$ there exists a $x$-lifting $z$-extension $H$ of $G$.*

d) *Let $K$ be as above and let $\pi : G_1 \to G_2$ be a $k$-homomorphism of connected reductive $k$-groups. Then there exists a $z$-extension $\pi' : H_1 \to H_2$ of $\pi : G_1 \to G_2$, such that $H_1$ is $x$-lifting $z$-extension of $G_1$ for any given $x \in \mathrm{H}^1(K, G_1)$.*

*Proof.* a) Let $G = SG'$, where $S$ is a central $k$-subtorus of $G$, and $G' = [G, G]$ is connected, semisimple, and the product is almost simple. Denote by $\tilde{G}$ the simply connected $k$-covering of $G'$, $\mu$ the (schematic) kernel of the central $k$-isogeny $\tilde{G} \times S \to G'S$. Then it is known ([Ha1], Satz 1.2.1) that $\mu$ can be embedded into a maximal torus $M$ lying in a Borel $k$-subgroup $B$ of a quasi-split simply connected semisimple $k$-group $G_q$ of the same Dynkin type as $\tilde{G}$. One may check also that such a $k$-torus is an induced $k$-torus (see [BrT1], Proposition 4.4.16). Then $\mu$ is embedded diagonally into the direct product $M \times (\tilde{G} \times S)$, and we may identify $\mu$ with a central finite $k$-subgroup scheme of multiplicative type in $M \times (\tilde{G} \times S)$. We denote by $H$ the quotient $M \times (\tilde{G} \times S)/\mu$. One checks as in [Ha1], Satz 1.2.1 that we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \mu & \to & \tilde{G} \times S & \stackrel{\alpha}{\to} & G & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
1 & \to & M & \to & H & \stackrel{\alpha}{\to} & G & \to & 1 \\
 & & \gamma\downarrow & & \downarrow\gamma & & & & \\
 & & T & = & T & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

(which is nothing else than the Ono's cross diagram ([O], [Sa])), where $H$ is a connected reductive $k$-group which is an extension of $G$ by an induced $k$-torus $M$.

b) Once the existence of $z$-extensions is established, many other results regarding $z$-extensions may be extended to the case of characteristic $p > 0$ too. The proofs of many of them are the same as in the case of char. 0. For the convenience of readers we recall briefly the argument (cf. [Bo1] for more details).

First choose $z$-extensions $H \to G_1$ and $H_2 \to G_2$. Then we set

$$H_1 := H \times_{G_2} H_2, H_0 := \mathrm{Ker}\,(H_1 \to H_2).$$

Then one checks that $H_i \to G_i$, $i = 0, 1$, are $z$-extensions and we have the commutative diagram as desired.

The proof of c) and d) follow from a) and b) above by repeating the same proof in the case of characteristic 0 (cf. the proof of Lemmas 2.3 and 2.4 of [T2]). ∎

The following is an analogue of an important result of Kneser ([Kn1], Sec. 15) for the case of local function fields.

**2.2. Lemma.** *Let $G$ be a semisimple group defined over a local function field $k$. Then there exist maximal $k$-tori of $G$ which are anisotropic over $k$ and have trivial 2-dimensional Galois cohomology.*

The proof is essentially the same as in the case of characteristic 0, so we omit it. (For the second assertion one needs Tate - Nakayama duality for tori over local function fields, which has been proved in [Sh], Chap. VI, Sec. 5). Also, we need Lemma 2.2 basically only in the case of groups of type $\mathrm{A} \times \mathrm{A} \times \cdots \times \mathrm{A}$, which can be reduced to the case of a single type A only and can be proved directly just as in [Kn2], pp. 64 - 65, or [PR], Chap. VI.)

**2.3. Lemma.** *Let $G$ be a semisimple group* (*resp. and of Dynkin type* $\mathrm{A} \times \mathrm{A} \times \cdots \times \mathrm{A}$) *defined over a local* (*resp. global*) *function field $k$, $F$ a central $k$-subgroup of $G$. Then the coboundary map*

$$\mathrm{H}^1_{fl}(k, G/F) \to \mathrm{H}^2_{fl}(k, F)$$

*is surjective.*

*Proof.* The case of global field is proved in [Ha2], Sec. 3, Lemma 2. So we assume that $k$ is a local function field. By Lemma 2.2, there exists a maximal $k$-torus $T \subset G$ which has trivial Galois (hence flat) cohomology in dimension 2.

From the exact sequence

$$\mathrm{H}^1_{fl}(k, T) \to \mathrm{H}^1_{fl}(k, T/F) \xrightarrow{\Delta} \mathrm{H}^2_{fl}(k, F) \to \mathrm{H}^2_{fl}(k, T) = 0,$$

we see that $\Delta$ is surjective, hence the map

$$\mathrm{H}^1_{fl}(k, G/F) \to \mathrm{H}^2_{fl}(k, F)$$

is also surjective as desired. ∎

The following is an analogue of another important results of Kneser for the case of local and global function fields (see [Kn2], Theorem 2, p. 60, and Theorem 2, p. 77). Its validity itself is already of independent interest and some ideas of its proof have been already indicated in [BH], p. 523 and p. 528.

**2.4. Theorem.** *Let $G$ be a semisimple group over a local or global function field $k$, $\pi : \tilde{G} \to G$ the universal covering of $G$, $F = \mathrm{Ker}\ (\pi)$. Then the coboundary map*

$$\Delta_k : \mathrm{H}^1_{fl}(k, G) \to \mathrm{H}^2_{fl}(k, F)$$

*is bijective.*

*Proof.* Since $\mathrm{H}^1_{fl}(k, \tilde{G}) \simeq \mathrm{H}^1(k, \tilde{G}) = 0$ if $k$ is a local (resp. global) function field by Bruhat - Tits [BrT2] (resp. by Harder [Ha2]), the usual twisting argument shows that it suffices to prove the surjectivity of the coboundary map $\Delta_k$.

We show first that if $G$ is quasi-split $k$-group then the assertion of Theorem 2.4 holds. Given $F$ as above, we claim that there exists a semisimple $k$-subgroup $H$ of $\tilde{G}$ of Dynkin type $\mathrm{A} \times \mathrm{A} \times \cdots \times \mathrm{A}$ of $\tilde{G}$ such that $F \subset H$. Since $\tilde{G}$ is simply connected, to prove our claim we may assume that $\tilde{G}$ is absolutely almost simple over $k$. Set $\tilde{F} = Cent(\tilde{G})$ the (schematic) center of $\tilde{G}$. It suffices then to find such a subgroup $H$ of given type in $\tilde{G}$ such that $\tilde{F} \subset H$. We consider the following cases by distinguishing the Dynkin type (Tits index) (see [Ti]) of $\tilde{G}$. We may assume that $\tilde{G}$ is not of type $\mathrm{A}_n$. For a maximal $k$-torus $T$ containing a maximal $k$-split torus $S$ of $\tilde{G}$, let $\Phi = \Phi(T, \tilde{G})$ be the root system of $\tilde{G}$ with respect to $T$, $\{\alpha_1, ..., \alpha_n\}$ be a system of simple roots of $\Phi$, such that the corresponding Tits index is given below, where $\alpha_i$ corresponds to the vertex $i$, and $\tilde{\alpha}$ corresponds to the maximal root of the corresponding root system.

We extend an argument of Serre to the case of a non-algebraically closed fields (see [CT], Proof of Prop. 8.2, which treats the case of groups over algebraically closed fields), by claiming that there exists a semisimple $k$-subgroup $H$ of $\tilde{G}$ of type $\mathrm{A} \times \cdots \times \mathrm{A}$ and $\mathrm{rank}(H) = \mathrm{rank}(\tilde{G})$. Then $H$ contains a maximal $k$-torus of $\tilde{G}$, hence also the center of $\tilde{G}$. (Below we indicate also another choice of the group $H$ containing the center of $\tilde{G}$, which is not necessary of maximal rank.)

2.4.1. Type $\mathrm{B}_n, n \geq 2$. The group $\tilde{G}$ is split over $k$ and its Tits index is as follows

$${}^1\circ \Longrightarrow \circ^2 \Longleftarrow \circ^{\tilde{\alpha}}$$

for $n = 2$, and

$$
\begin{array}{l}
{}^{1}\circ - - {}^{2}\circ - - \cdots - - \circ - - {}^{n-1}\circ \Longrightarrow \circ^{n} \\
\qquad\;\; | \\
\qquad\;\; \tilde{\alpha}
\end{array}
$$

for $n \geq 3$. Here the maximal root $\tilde{\alpha}$ is given by (see [Bou], Table II)

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n.$$

We use induction on $n$. If $n = 2$ (resp. $n = 3$) let $H$ be the regular simply connected semisimple $k$-subgroup of $\tilde{G}$ with the root system $\{\tilde{\alpha}, \alpha_1\}$. (resp. $\{\tilde{\alpha}, \alpha_1, \alpha_3\}$). Then $H$ is of type $\mathrm{A}_1 \times \mathrm{A}_1$ (resp. $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$) and of maximal rank. Assume that $n \geq 4$. Then we consider the regular $k$-subgroups $H_1, H_2, H_3$ of $\tilde{G}$ with the root system $\{\tilde{\alpha}\}$, $\{\alpha_1\}$, $\{\alpha_3, ..., \alpha_n\}$, respectively, and take $H' = H_1 \times H_2 \times H_3$. Then it is clear that $H'$ is defined over $k$, semisimple $k$-split of type $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{B}_{n-2}$ and of maximal rank. By induction, $H_3$ contains a regular $k$-subgroup $H'_3$ which is semisimple $k$-split of type $\mathrm{A} \times \cdots \times \mathrm{A}$ and of maximal rank in $H_3$. Then one may take $H = H_1 \times H_2 \times H'_3$, which is of desired type.

(One may take also $H$ to be the regular semisimple $k$-subgroup of $\tilde{G}$ with the root system $\{\alpha_n\}$. Then $H$ is of type $\mathrm{A}_1$ and contains the center of $\tilde{G}$.)

2.4.2. Type $\mathrm{C}_n, n \geq 3$. $\tilde{G}$ is $k$-split and the Tits index of $\tilde{G}$ is as follows

$$ {}^{\tilde{\alpha}}\circ \Longrightarrow \circ^{1} - - \circ^{2} - - \cdots - - {}^{n-1}\circ \Longleftarrow \circ^{n} $$

Here the maximal root $\tilde{\alpha}$ is given by (see [Bou], Table III)

$$\tilde{\alpha} = 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n.$$

We use induction on $n$. If $n = 3$, we consider the regular semisimple $k$-subgroups $H_1, H_2$ with the root system $\{\tilde{\alpha}\}$, $\{\alpha_2, \alpha_3\}$, respectively. Since $\mathrm{B}_2 \simeq \mathrm{C}_2$, $H_2$ contains a regular $k$-subgroup $H'_2$ of type $\mathrm{A}_1 \times \mathrm{A}_1$, so we may take $H = H_1 \times H'_2$. If $n > 3$, let $H_1, H_2$ be the regular semisimple $k$-subgroups of $\tilde{G}$ with the root system $\{\tilde{\alpha}\}$, $\{\alpha_2, ..., \alpha_n\}$, respectively. Then $H_2$ is of type $\mathrm{C}_{n-1}$, which, by induction hypothesis, contains a regular semisimple $k$-subgroup $H'_2$ of type $\mathrm{A} \times \cdots \times \mathrm{A}$ of maximal rank in $H_2$. Then $H = H_1 \times H'_2$ is of type $\mathrm{A} \times \cdots \times \mathrm{A}$, defined over $k$ and of maximal rank in $\tilde{G}$.

(One may also take $H_i$ to be the regular semisimple $k$-subgroup of $\tilde{G}$ with root system $\{\alpha_{2i-1}\}$, $1 \leq i \leq [n+1]/2$, and $H = \prod_{1 \leq i \leq [n+1]/2} H_i$. Then one checks that $H$ contains the center of $\tilde{G}$ and is of type $\mathrm{A} \times \cdots \times \mathrm{A}$.)

2.4.3. Type $\mathrm{D}_n, n \geq 4$. If $\tilde{G}$ is $k$-split then it has the following Tits index

$$\begin{array}{lr}
 & \circ^{n} \\
\circ^{1} -- \circ^{2} -- \circ -- \cdots -- \circ^{n-3} -- \circ^{n-2} \langle & \\
\quad\quad | & \circ^{n-1} \\
\quad\quad \tilde{\alpha} &
\end{array}$$

If $\tilde{G}$ is $k$-quasi-split, then it has the following Tits index

$$\begin{array}{lr}
 & \circ^{n} \\
\circ^{1} -- \circ^{2} -- \circ -- \cdots -- \circ^{n-3} -- \circ^{n-2} \big( \updownarrow & \\
\quad\quad | & \circ^{n-1} \\
\quad\quad \tilde{\alpha} &
\end{array}$$

Here the maximal root $\tilde{\alpha}$ is given by (cf. [Bou], Table IV)

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.$$

In both cases we denote by $H_1, H_2, H_3$ the regular simply connected semisimple $k$-subgroups of $\tilde{G}$ with root system $\{\tilde{\alpha}\}$, $\{\alpha_1\}$, $\{\alpha_3, ..., \alpha_n\}$, respectively and use induction on $n$. If $n = 4$ (resp. $n = 5$), then $H_3$ is of type $\mathrm{A}_1 \times \mathrm{A}_1$ (resp. is of type $\mathrm{A}_3$), hence $H = H_1 \times H_2 \times H_3$ is of type $\mathrm{A} \times \cdots \times \mathrm{A}$ and is regular, semisimple of maximal rank. If $n > 5$, $H_3$ is of type $\mathrm{D}_{n-2}$ and by induction, it contains a regular semisimple $k$-subgroup $H'_3$ of type $\mathrm{A} \times \cdots \times \mathrm{A}$. Then we may take $H = H_1 \times H_2 \times H'_3$. It rests to show that all the groups $H_i$ are defined over $k$.

It is clear for $H_2$ and $H_3$, and also for $H_1$ in the case $\tilde{G}$ is $k$-split. Assume that $\tilde{G}$ is $k$-quasi-split. There exists a separable quadratic extension $K$ of $k$, which splits $\tilde{G}$. Let $\Gamma = Gal(K/k) = \{1, \sigma\}$ be the corresponding Galois group. Then $T$ is defined over $k$ and is $K$-split, and the action of $Gal(k_s/k)$ on the character group $X^*(T)$ of $T$ factors through $\Gamma$. The same is true for the action of $Gal(k_s/k)$ on the cocharacter group $X_*(T)$. For $\alpha \in X^*(T), h \in X_*(T)$ (which are always defined over $K$), the $\Gamma$-action (hence the action of $\sigma$) is defined as follows

$$\alpha^{\sigma}(t) = \sigma(\alpha(\sigma^{-1}(t))), t \in T(K),$$

$$h^{\sigma}(z) = \sigma(h(\sigma^{-1}(z))), z \in K^*.$$

In [T] there was given the action of $\sigma$ in the case $k = \mathbf{R}$, $K = \mathbf{C}$, but one can check that the results proved there (and their proofs) also hold in our situation. In particular, we have (see [T], p. 1105)

$$\alpha_i^{\sigma} = \begin{cases} \alpha_i & \text{if } i < n-1, \\ \alpha_n & \text{if } i = n-1, \\ \alpha_{n-1} & \text{if } i = n. \end{cases}$$

Thus $\tilde{\alpha}^{\sigma} = \tilde{\alpha}$, hence $H_1$ (the root subgroup corresponding to $\tilde{\alpha}$) is defined (in fact split) over $k$.

(We may also take $H$ to be the regular semisimple $k$-subgroup of $\tilde{G}$ with the root system $\{\tilde{\alpha}, \alpha_2, ..., \alpha_n\}$. Then $H$ is of maximal rank in $\tilde{G}$. If $n$ is even, let $H$ be the regular semisimple $k$-subgroup of $\tilde{G}$ with the root system $\{\alpha_1, \alpha_3, ..., \alpha_{n-3}, \alpha_{n-1}, \alpha_n\}$. Then $H$ is of type $\mathrm{A}_1 \times \cdots \times \mathrm{A}_1$ and is $k$-split (resp. $k$-quasi-split) if $\tilde{G}$ is so.) If $n$ is odd, let $H$ be the regular semisimple $k$-subgroup of $\tilde{G}$ with the root system $\{\alpha_1, \alpha_3, ..., \alpha_{n-4}, \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$. Then $H$ is of type $\mathrm{A}_1 \times \cdots \times \mathrm{A}_1 \times \mathrm{A}_3$ and is $k$-split (resp. $k$-quasi-split) if $\tilde{G}$ is so. In all cases above, $H$ contains the center of $\tilde{G}$ as desired.)

2.4.4. Type $\mathrm{E}_6$. If $\tilde{G}$ is $k$-split, its Tits index is as follows

$$\begin{array}{ccccccccc} & & & & \tilde{\alpha} & & & & \\ & & & & | & & & & \\ & & & & {}^2\circ & & & & \\ & & & & | & & & & \\ \circ^1 & -- & \circ^3 & -- & \circ^4 & -- & \circ^5 & -- & \circ^6 \end{array}$$

and if $\tilde{G}$ is $k$-quasi-split, its Tits index is as follows

$$\tilde{\alpha}\circ - -{}^2\circ - -{}^4\circ \left(\begin{array}{ccc} \circ^1 & - - & \circ^3 \\ \updownarrow & & \updownarrow \\ \circ_5 & - - & \circ_6 \end{array}\right.$$

Here the maximal root $\tilde{\alpha}$ is given by (cf. [Bou], Table V)

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6.$$

We consider the regular semisimple subgroups $H_1, H_2$ of $\tilde{G}$ with the root system $\{\tilde{\alpha}\}, \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, respectively. Then it is clear that $H = H_1 \times H_2$ is semisimple of type $\mathrm{A}_1 \times \mathrm{A}_5$. To check that $H$ is defined over $k$ we may proceed as in 2.4.3. Namely it suffices to treat the case $\tilde{G}$ is quasi-split (non-split). Let $K$ be a separable quadratic extension of $k$ which splits $\tilde{G}$. Let $\Gamma = Gal(K/k) = \{1, \sigma\}$ be the corresponding Galois group. Then as in [T], p. 1107, we have the following action of $\sigma$ on simple roots $\{\alpha_1, ..., \alpha_6\}$.

$$\alpha_i^\sigma = \begin{cases} \alpha_i & \text{if } i = 2, 4, \\ \alpha_5 & \text{if } i = 3, \\ \alpha_6 & \text{if } i = 1, \\ \alpha_1 & \text{if } i = 6, \\ \alpha_3 & \text{if } i = 5. \end{cases}$$

Hence one sees that the root subgroup $H_1$ corresponding to $\tilde{\alpha}$ is defined (and split) over $k$. Thus $H$ is defined over $k$.

(For these cases we can take the regular semisimple $k$-subgroup $H$ of $\tilde{G}$ with the root system $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$. Then one checks that $H$ is of type $\mathrm{A}_2 \times \mathrm{A}_2$ and contains the center of $\tilde{G}$.

2.4.5. Type $\mathrm{E}_7$. The group $G$ is $k$-split and its Tits index is given by

$$\begin{array}{ccccccccccccc} & & & & & & {}^2\circ & & & & & & \\ & & & & & & | & & & & & & \\ \tilde{\alpha}\circ - - \circ^1 & -- & \circ^3 & -- & & & \circ^4 & -- & \circ^5 & -- & \circ^6 & -- & \circ^7 \end{array}$$

Here the maximal root $\tilde{\alpha}$ is given by (cf. [Bou], Table VI)

$$\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7.$$

We consider the regular semisimple subgroups $H_1, H_2$ of $\tilde{G}$ with the root system $\{\tilde{\alpha}\}$, $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, respectively. Then it is clear that $H_2$ is a semisimple $k$-split subgroup of $\tilde{G}$ of type $\mathrm{D}_6$, which contains a regular simply connected semisimple $k$-subgroup $H'_2$ of type $\mathrm{A} \times \cdots \times \mathrm{A}$ of maximal rank in $H_2$ (by 2.4.3 above), hence $H = H_1 \times H'_2$ is a semisimple $k$-subgroup of type $\mathrm{A} \times \cdots \times \mathrm{A}$ of maximal rank in $\tilde{G}$.

(We may also take the regular semisimple regular $k$-subgroup $H$ of $\tilde{G}$ with root system $\{\alpha_2, \alpha_5, \alpha_7\}$ (or root system $\{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$). Then one checks that $H$ is of type $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$ (or $\mathrm{A}_5$) and contains the center of $\tilde{G}$ and the claim follows.)

From Lemma 2.3 above, for $H' = H/F$, we see that the coboundary map $\Delta_{H'} : \mathrm{H}^1_{fl}(k, H') \to H^2_{fl}(k, F)$ is surjective, hence the same is true for $\Delta_G : \mathrm{H}^1_{fl}(k, G) \to \mathrm{H}^2_{fl}(k, F)$, since $\Delta_{H'}$ factors through $\Delta_G$.

In the general case, $\tilde{G}$ is obtained from a quasi-split $k$-group $\tilde{G}_q$ by an inner twisting. Such a twisting does not change the center of $\tilde{G}$, so $G$ is an inner twisting of $G_q = \tilde{G}_q/F$. The following lemma will finish the proof of Theorem 2.4.

**2.5. Lemma.** *If the coboundary map* $\Delta_q : \mathrm{H}^1_{fl}(k, G_q) \to \mathrm{H}^2_{fl}(k, F)$ *is surjective then so is the map* $\Delta : \mathrm{H}^1_{fl}(k, G) \to \mathrm{H}^2_{fl}(k, F)$.

*Proof .* We have the following commutative diagram, where all vertical maps are bijections (the right one is just a "translation map") (see [Se], Chap. I, Prop. 44, in the case of Galois cohomology, and [Gi], Chap. IV, Prop. 4.3.4, in the case of "general" (including flat) cohomology)

$$\begin{array}{ccc} \mathrm{H}^1_{fl}(k, G) & \xrightarrow{\Delta} & \mathrm{H}^2_{fl}(k, F) \\ \tau_q \downarrow & & \downarrow \tau \\ \mathrm{H}^1_{fl}(k, G_q) & \xrightarrow{\Delta_q} & \mathrm{H}^2_{fl}(k, F). \end{array}$$

Since $\Delta_q$ is surjective, it follows readily that $\Delta$ is also. ∎

In the case of flat cohomology, we need the following application of the trace theory (for abelian sheaves in flat topology) due to Deligne [De1,De2] applied to the case of commutative group schemes over fields, which was pointed out by Gille in [Gil], Sec. 0.4.

**2.6. Lemma.** (Cf. [Gil, Sec. 0.4]) *Let $G$ be a commutative group scheme over a field $k$, $L$ a finite extension of $k$. Then for $i \geq 0$, there exists corestriction homomorphism*

$$Cores_{L/k,G} : \mathrm{H}^i_{fl}(L, G) \to \mathrm{H}^i_{fl}(k, G)$$

*which are functorial in $G$.* ∎

# 3 Proof of Theorem A, part a)

We consider the following cases.

I) *Case* $p = q = 0$ (cf. also [De2] and [MS]). We are given an exact sequence of $k$-groups

$$1 \to K \to G \to T \to 1,$$

where $G$ is connected reductive, $K$ is a closed subgroup scheme of $G$ and $T$ is a torus, and $k$ is a local or global field of characteristic $p > 0$. Since $T \simeq G/K$ is commutative, $K$ contains the derived subgroup $G' = [G, G]$ of $G$. We have $G = G'S$, where $S$ is a central $k$-torus of $G$, and $K = G'S'$, where $S'$ is a $k$-subgroup scheme of $S$.
$a$) Assume that $G'$ is simply connected. Assume also $K = G'$. For any finite extension $L/k$ we have the following exact sequence of pointed sets in Galois cohomology

$$G(L) \to T(L) \to \mathrm{H}^1(L, K) = 0,$$

where the triviality of $\mathrm{H}^1(L, K)$ follows from a theorem of Bruhat - Tits [BrT2] (in the local field case), and a theorem of Harder [Ha2] (in the global field case). Therefore the Corestriction Principle holds in this case.

Assume that $K \neq G'$ . Then we have the following commutative diagram with exact rows in flat cohomology

$$\begin{array}{ccccccc} G'(L) & \to & G(L) & \stackrel{\pi_L}{\to} & (G/G')(L) & \to & 0 \\ \downarrow & & \downarrow = & & \downarrow \beta_L & & \\ K(L) & \to & G(L) & \stackrel{\alpha_L}{\to} & T(L) & \to & \mathrm{H}^1_{fl}(L, K) \end{array} .$$

Since $\alpha_L = \beta_L \pi_L$ and $\pi_L$ is surjective, so everything is reduced to $\beta_L$ and the assertion trivially holds.

$b)$ $G'$ is not simply connected. Consider a $z$-extension of $G$, which exists by Lemma 2.1 :

$$1 \to Z \to H \to G \to 1.$$

Denote by $T_1 = H/[H,H]$, which is a $k$-torus and consider the following commutative diagrams related with the above exact sequence, for any extension field $L/k$

$$\begin{array}{ccccccc} Z(L) & \to & H(L) & \stackrel{\pi_L}{\to} & G(L) & \to & 1 \\ & & \downarrow \beta_L & & \downarrow \alpha_L & & \\ & & T_1(L) & \stackrel{\gamma_L}{\to} & T(L) & & \end{array}$$

Here $\beta_L$ and $\pi_L$ are surjective for any $L$, thus the image of $\alpha_L$ is just the image of $\gamma_L$. Since the Corestriction Principle holds automatically for $\gamma_k$, it also holds for $\alpha_k$. Therefore the case $p = q = 0$ is proved.

II) *Case* $p = 0, q = 1$. In this case we are given the following exact sequence

$$1 \to T \to G_1 \to G \to 1,$$

where $T$ is a torus and $G$ is connected, reductive, hence so is $G_1$. By Lemma 1.1 there is a $z$-extension $1 \to H_0 \to H_1 \stackrel{\pi}{\to} H_2 \to 1$ of the above sequence. Since for any field extension $L/k$ we have surjective homomorphism $\beta_L : H_2(L) \to G(L)$, so from the following commutative diagram with exact rows

$$\begin{array}{ccc} H_2(L) & \stackrel{\delta_L}{\to} & \mathrm{H}^1(L, H_0) \\ \downarrow \beta_L & & \downarrow \alpha_L \\ G(L) & \stackrel{\delta'_L}{\to} & \mathrm{H}^1(L, T) \end{array}$$

it follows that it suffices to prove the assertion for the connecting map $\delta_k : H_2(k) \to \mathrm{H}^1(k, H_0)$.

Notice that since $T$ is a torus, $H_0$ is also a torus. Also the semisimple part of $H_1$ and $H_2$ are simply connected $k$-groups. Therefore the restriction of $\pi$ to the semsimple part $\tilde{G}$ of $H_1$ is an isomorphism, so we may assume that $\tilde{G} = [H_2, H_2]$ is the semsimple part of $H_2$, and we have the following decompositions into almost direct product

$$H_1 = \tilde{G}S_1, H_2 = \tilde{G}S_2,$$

where $S_i$ is a central $k$-subtorus of $H_i$, $i = 1, 2$. From this we derive the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \tilde{G} & \simeq & \tilde{G} & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \to & H_0 & \to & \tilde{G}S_1 & \to & \tilde{G}S_2 & \to & 1 \\
 & & \downarrow\simeq & & \downarrow & & \downarrow & & \\
1 & \to & \bar{H}_0 & \to & \bar{S}_1 & \to & \bar{S}_2 & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

where $\bar{S}_i$ is the corresponding quotient of $S_i$, $i = 1, 2$. From this diagram by making use the vanishing of $\mathrm{H}^1$ for simply connected groups over local or global fields of positive characteristic as above, we derive the following commutative diagram for any finite field extension $L/k$

$$
\begin{array}{ccc}
(\tilde{G}S_2)(L) & \stackrel{\delta_L}{\to} & \mathrm{H}^1(L, H_0) \\
\downarrow & & \downarrow\simeq \\
\bar{S}_2(L) & \stackrel{\delta'_L}{\to} & \mathrm{H}^1(L, \bar{H}_0) \\
\downarrow & & \\
1 & &
\end{array}
$$

It follows that the image of $\delta_L$ is just the image of the composite map $\bar{S}_2(L) \stackrel{\delta'_L}{\to} \mathrm{H}^1(L, \bar{H}_0) \simeq \mathrm{H}^1(L, H_0)$, and the assertion follows.

III) *Case* $p = q = 1$. We are given an exact sequence of $k$-groups

$$1 \to G_1 \to G \to T \to 1.$$

Since $T$ is a torus, it follows that it suffices to prove the assertion for the case $G_1 = G'$. Let $L$ be a finite field extension of $k$. Let $G = G'S$, where $S$ is a central $k$-subtorus of $G$, $G' = [G, G]$. Denote by $Z(G)$ (resp. $F'$) the schematic center of $G$ (resp. $G'$), $\bar{G} = G'/F'$ the corresponding adjoint group, then we have

$$G/F' \simeq \bar{G} \times S',$$

where $S'$ is a torus quotient of $S$, namely $S' = S/F$ with $F$ the (schematic) intersection $F := F' \cap S$. Consider the exact sequences

$$1 \to F' \to G' \to \bar{G} \to 1,$$

$$1 \to F' \to Z(G) \to S' \to 1,$$

and

$$1 \to F' \to G'S \to \bar{G} \times S' \to 1,$$

and the long exact sequence of cohomology deduced from it, we have the following exact sequences of cohomology :

$$\mathrm{H}^1_{fl}(k, F') \to \mathrm{H}^1_{fl}(k, G') \to \mathrm{H}^1_{fl}(k, \bar{G}) \stackrel{\Delta_{1,k}}{\to} \mathrm{H}^2_{fl}(k, F'),$$

$$\mathrm{H}^1_{fl}(k, F') \to \mathrm{H}^1_{fl}(k, Z(G)) \to \mathrm{H}^1_{fl}(k, S') \stackrel{\Delta_{2,k}}{\to} \mathrm{H}^2_{fl}(k, F'),$$

and also similar sequences when $k$ is replaced by $L$. We have also the following commutative diagram with the exact first row

$$\begin{array}{ccccc}
\mathrm{H}^1_{fl}(L, G) & \stackrel{\beta_L}{\to} & \mathrm{H}^1_{fl}(L, \bar{G}) \times \mathrm{H}^1_{fl}(L, S') & \stackrel{\Delta_L}{\to} & \mathrm{H}^2_{fl}(L, F') \\
\downarrow = & & \downarrow & & \\
\mathrm{H}^1_{fl}(L, G) & \stackrel{\alpha_L}{\to} & \mathrm{H}^1_{fl}(L, S') & &
\end{array}$$

Since $\bar{G} \times S'$ is a direct product and $S$ is in the center of $G'S$, one checks by computing the corresponding 2-cocycles that the map $\Delta_L$ is given by

$$\Delta_L(g, s) = \Delta_{1,L}(g) + \Delta_{2,L}(s),$$

where the "+" is taken in $\mathrm{H}^2_{fl}(L, F')$. Let $x' \in \mathrm{H}^1_{fl}(L, G)$, $\beta_L(x') = (g', s')$, $g' \in \mathrm{H}^1_{fl}(L, \bar{G}), s' \in \mathrm{H}^1_{fl}(L, S'), s = Cores_{L/k,S'}(s') \in \mathrm{H}^1_{fl}(k, S')$. Then we have $\Delta_L(g', s') = 0$, so $\Delta_{2,L}(s') = -\Delta_{1,L}(g')$. By Theorem 2.4, for $\tilde{F}$ the schematic center of $\tilde{G}$, the coboundary map (in the long exact sequence induced from the exact sequence $1 \to \tilde{F} \to \tilde{G} \to \bar{G} \to 1$)

$$\Delta^*_L : \mathrm{H}^1_{fl}(L, \bar{G}) \to \mathrm{H}^2_{fl}(L, \tilde{F})$$

is surjective for any finite extension $L$ of $k$. We have the following exact sequence for finite group schemes

$$1 \to F_0 \to \tilde{F} \to F' \to 1,$$

where $F_0 = \mathrm{Ker}\ (\tilde{G} \to G')$, which induces a homomorphism $p_L : \mathrm{H}^2_{fl}(L, \tilde{F}) \to \mathrm{H}^2_{fl}(L, F')$. Since $\Delta_{1,L} = p_L \circ \Delta^*_L$, and by Theorem 2.4, $\Delta^*_L$ is always surjective, so by Lemma 2.6, the Corestriction Principle holds for the image of $\Delta_{1,k}$ and via corestriction map we have

$$\begin{aligned}
Cores_{L/k,F'}(\Delta_{2,L}(s')) &= \Delta_{2,k}(s) \\
&= Cores_{L/k,F'}(-\Delta_{1,L}(g')) \\
&= -Cores_{L/k,F'}(\Delta_{1,L}(g')) \\
&= -\Delta_{1,k}(\bar{g})
\end{aligned}$$

for some $\bar{g} \in \mathrm{H}^1_{fl}(k, \bar{G})$. Therefore $\Delta_k(\bar{g}, s) = 0$, i.e., $(\bar{g}, s) = \beta_k(g), g \in \mathrm{H}^1_{fl}(k, G)$, or equivalently $s = \alpha_k(g) \in \mathrm{Im}\ (\alpha_k)$. ■

# 4 Proof of Theorem A, part b)

We consider separately the possible values of $p, q$, so we have the following cases.

a) *Case* $p = q = 0$. We are given the following exact sequence of $k$-groups

$$1 \to T \to G \to G_1 \to 1$$

and the assertion is trivial in this case.

b) *Case* $p = 0, q = 1$. We are given the exact sequence

$$1 \to G \to G_1 \to T \to 1.$$

Since $T, G$ are connected, reductive $k$-group, the same is true for $G_1$. Since we are interested in the kernel of the coboundary map

$$\delta_k : T(k) \to \mathrm{H}^1(k, G),$$

which is nothing else than the image of $G_1(k) \to T(k)$, so we are back to the first case a) of Theorem.

c) *Case* $p = q = 1$. We consider the following exact sequence of $k$-groups

$$1 \to T \to G \to G_1 \to 1.$$

Since $T$ is a $k$-subtorus of $G$, it is contained in a maximal $k$-torus of $G$, so we may assume that $T$ is already a maximal one. Then by making use of Lemma 2.1, the proof in the case *char*.0 (see [T2], p. 296) carries over to the case of characteristic $p > 0$. ■

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