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ON THE ROBUSTNESS OF ASYMPTOTIC STABILITY FOR A CLASS OF SINGULARLY PERTURBED SYSTEMS WITH MULTIPLE DELAYS

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Abstract

This paper is concerned with the stability robustness for a class of singularly perturbed systems of linear functional differential equations. First, the stability radius for the reduced systems is proposed. Then, asymptotic behavior of the structured complex stability radius for the singularly perturbed systems is established as the small parameter tends to zero.

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1 Introduction

In this paper we consider the singularly perturbed system (SPS) of functional differential equations (FDE-s)

$$\begin{aligned} \dot{x}(t) &= L_{11}x_t + L_{12}y_t \\ \varepsilon\dot{y}(t) &= L_{21}x_t + L_{22}y_t \end{aligned} \tag{1.1}$$

where $x \in \mathbb{C}^{n_1},\ y \in \mathbb{C}^{n_2},\ \varepsilon > 0$ is a small parameter;

$$\begin{aligned} L_{j1}x_t &= \sum_{i=0}^{l} A_{j1}^i x(t-\tau_i) + \int_{-\tau_l}^{0} D_{j1}(\theta)x(t+\theta)d\theta \\ L_{j2}y_t &= \sum_{k=0}^{m} A_{j2}^k y(t-\varepsilon\mu_k) + \int_{-\mu_m}^{0} D_{j2}(\theta)y(t+\varepsilon\theta)d\theta \end{aligned} \tag{1.2}$$

$j = 1, 2,\ A_{jk}^i$ are constant matrices of appropriate dimensions, $D_{jk}(.)$ are integrable matrix-valued functions, and $0 \le \tau_0 \le \tau_1 \le ... \le \tau_p,\ 0 \le \mu_0 \le \mu_1 \le ... \le \mu_m$.

A lot of problems arising in various fields of science and engineering can be modelled by SPS-s of differential equations with or without delay, e.g., see [8] and the references cited therein. The system (1.1) was analyzed by Dragan and Ionita in [2]. By extending classical results of Klimusev and Krasovskii, e.g., see [14], the authors gave a parameter-independent sufficient condition ensuring the exponential-asymptotic stability of the zero solution of (1.1) for all sufficiently small ε. For characterizing the robustness of asymptotic stability for linear systems, an appropriate measure is the so-called stability radii introduced by Hinrichsen and Pritchard [10, 11, 12]. A formula of the complex structured stability radius for linear systems was easily obtained in [11]. The result was extended to linear functional systems in [17]. The real stability radius for linear systems, which is a more difficult issue, was investigated in a remarkable paper of Qiu et al. [15]. Recently, this result was extended to linear time-delay systems [13]. See also a fairly complete reference list on the topic in [1]. In this paper, we focus on the complex stability radius, only.

Let us assume system (1.1),(1.2) is asymptotically stable for all sufficiently small ε. Following the notions introduced in [17, 13], we consider the system (1.1) with the coefficients subjected to structured perturbations as follows

$$\begin{aligned} \tilde{L}_{j1}x_t &= \sum_{i=0}^{l} (A_{j1}^i + B_j\Delta_1^i C_1^i)x(t-\tau_i) + \\ &\qquad \int_{-\tau_l}^{0} (D_{j1}(\theta) + B_j\delta_1(\theta)C_1^{l+1})x(t+\theta)d\theta \\ \tilde{L}_{j2}y_t &= \sum_{k=0}^{m} (A_{j2}^k + B_j\Delta_2^k C_2^k)y(t-\varepsilon\mu_k) + \\ &\qquad \int_{-\mu_m}^{0} (D_{j2}(\theta) + B_j\delta_2(\theta)C_2^{m+1})y(t+\varepsilon\theta)d\theta \end{aligned} \tag{1.3}$$

where

$$\{\Delta_1^i\}_{i=0}^l \in \mathbb{C}^{p_1\times q_{1i}}, \{\Delta_2^k\}_{k=0}^m \in \mathbb{C}^{p_2\times q_{2k}}, \delta_1(\theta) \in \mathbb{C}^{p_1\times q_{1(l+1)}}, \delta_2(\theta) \in \mathbb{C}^{p_2\times q_{2(m+1)}}$$

are uncertain perturbations, $\delta_1(.), \delta_2(.)$ are integrable matrix-valued functions on the indicated intervals; $B_j \in \mathbb{C}^{n_j \times p_j}, j = 1, 2;\ C_1^i \in \mathbb{C}^{q_{1i} \times n_1}, i = 0, 1, ..., l + 1; C_2^k \in \mathbb{C}^{q_{2k} \times n_2}, k = 0, 1, ..., m + 1$ are sets of matrices determining perturbation structure. For brevity, let us denote

$$\begin{aligned}
\mathbf{A} &= \left\{ \{A_{j1}^i\}_{i=0}^l, \{A_{j2}^k\}_{k=0}^m, \{D_{ij}(.)\}_{i,j=1}^2 \right\}, \\
\mathbf{B} &= \{B_1, B_2\}, \\
\mathbf{C} &= \left\{ \{C_1^i\}_{i=0}^l, \{C_2^k\}_{k=0}^m \right\}, \\
\mathbf{\Delta} &= \left\{ \{\Delta_1^i\}_{i=0}^l, \{\Delta_2^k\}_{k=0}^m, \delta_1(.), \delta_2(.) \right\}.
\end{aligned}$$

Our aim is to determine the complex structured stability radius for (1.1),(1.2), which is defined by

$$r_\varepsilon(\mathbf{A}, \mathbf{B}, \mathbf{C}) := \inf\{\|\mathbf{\Delta}\|, \text{the perturbed system (1.1),(1.3) is not asym. stable}\}, \tag{1.4}$$

where

$$\|\mathbf{\Delta}\| := \sum_{i=0}^{l} \|\Delta_1^i\| + \sum_{k=0}^{m} \|\Delta_2^k\| + \int_{-\tau_l}^{0} \|\delta_1(\theta)\| d\theta + \int_{-\mu_m}^{0} \|\delta_2(\theta)\| d\theta,$$

and $\|.\|$ is a matrix norm induced by vector norms. By multiplying both sides of the second equation in (1.1) with ε^{-1}, one obtains a regular explicit system of FDE-s. By applying an extended variant of the result in [17], a formula of the stability radius for the system of differential equation with multiple time-delays can easily be obtained. However, this formula may hardly be realized in practical computation because of the appearance of ε^{-1}. Therefore, we are interested in the asymptotic behavior of the stability radius as the parameter tends to zero. Such a robust stability analysis was done for the classical SPS of ordinary differential equations by Dragan in [3]. Recently, by using an approach different from that in [3], Du and Linh have extended the result to a more general class of singularly perturbed differential equations [4] and to index-1 DAE-s containing a small parameter [5]. Here, by using the same approach as in our preceding papers, we will obtain a similar result for system (1.1),(1.2). That is, the stability radius of the SPS-s is shown to converge to the minimum of the stability radii of the "reduced slow" system and of the "boundary layer fast" system as the parameter tends to zero.

The paper is organized as follows. In the next section, we first recall the sufficient condition obtained in [2] for the exponential-asymptotic stability of system (1.1),(1.2). Then, a formula of the stability radius for implicit systems of differential equations with multiple time-delays is proposed. This is in fact an extended variant of the formula obtained in [17]. The main results come in Section 3. First, we analyze the robust stability of the reduced slow system, which is a semi-explicit index-1 system of functional differential-algebraic equations (FDAE-s). Secondly, the asymptotic behavior of the stability radius for the SPS is characterized as the parameter tends to zero. Finally, a conclusion will close the paper.

2 Preliminary

2.1 A sufficient stability condition

It is well-known that a linear system of functional differential equations is asymptotic stable if and only if all roots of the associated characteristic equation are located in the open half plane $\mathbb{C}^-$, see [9]. However, in the case of the SPS (1.1),(1.2) it is not easy to check this condition. As we mentioned above, we should multiply both sides of the second equation with ε^{-1} in order to get a regular explicit system. Hence, the characteristic equation should contain ε^{-1}, too, which make the computation of roots become difficult.

Taking $\varepsilon = 0$ in (1.1), we obtain

$$\begin{array}{rcl} \dot{x}(t) & = & L_{11}x_t + \bar{L}_{12}y(t) \\ 0 & = & L_{21}x_t + \bar{L}_{22}y(t) \end{array} \tag{2.1}$$

where

$$\bar{L}_{j2} = \sum_{k=0}^{m} A_{j2}^{k} + \int_{-\mu_m}^{0} D_{j2}(\theta)d\theta, \quad j = 1, 2. \tag{2.2}$$

That is, the second equation becomes an algebraic equation. Let us assume that $\bar{L}_{22}$ is invertible. The reduced slow system (2.1),(2.2) is called an index-1 FDAE of semi-explicit form, see [6]. By substituting $y(t) = \bar{L}_{22}^{-1}L_{21}x_t$ into the first equation, we obtain a linear functional differential equation

$$\dot{x}(t) = L_S x_t, \tag{2.3}$$

with

$$L_S x_t = \sum_{i=0}^{l} A_S^i x(t - \tau_i) + \int_{-\tau_p}^{0} D_S(\theta)x(t + \theta)d\theta, \tag{2.4}$$

where $A_S^i = A_{11}^i - \bar{L}_{12}\bar{L}_{22}^{-1}A_{21}^i,\ i = 0, 1, ..., l, \quad D_S(\theta) = D_{11}(\theta) - \bar{L}_{12}\bar{L}_{22}^{-1}D_{21}(\theta)$.

We also consider the fast boundary layer system

$$\dot{z}(\zeta) = L_F z_\zeta, \tag{2.5}$$

where

$$L_F z_\zeta = \sum_{k=0}^{m} A_{22}^{k} z(\zeta - \mu_k) + \int_{-\mu_m}^{0} D_{22}(\theta)y(\zeta + \theta)d\theta$$

and $\zeta = \varepsilon^{-1}t$ is the scaled time.
We assume the following

Assumption A1. All the roots of the equation

$$\det\,(\lambda I_{n_2} - \sum_{k=0}^{m} A_{22}^{i} e^{-\lambda\mu_k} - \int_{-\mu_m}^{0} D_{22}(\theta)e^{\lambda\theta}d\theta) = 0$$

are located in the open left half plane $\mathbb{C}^-$ and

Assumption A2. All the roots of the equation

$$\det\,(\lambda I_{n_1} - \sum_{i=0}^{p} A_S^i e^{-\lambda\tau_i} - \int_{-\tau_p}^{0} D_S(\theta)e^{\lambda\theta}d\theta) = 0$$

are located in the open left half plane $\mathbb{C}^-$.

Note that these equations are the characteristic equations associated with the systems (2.5) and (2.3), respectively. Furthermore, they are independent of the small parameter ε.

Theorem 1 *(Dragan and Ionita [2]) Let Assumptions A1-2 hold. There exists $\varepsilon_0 > 0$ such that for arbitrary $\varepsilon \in (0, \varepsilon_0)$, the zero solution of the system (1.1),(1.2) is exponential-asymptotically stable.*

We remark that Assumption A1 implies the nonsingularity of $\bar{L}_{22}$. Furthermore, it is possible to replace the open interval $(0, \varepsilon_0)$ by the closed one $[0, \varepsilon_0]$ (the case of $\varepsilon = 0$ is discussed in details in the next section).

2.2 The complex stability radius for implicit FDE-s

Consider an implicit system of FDE-s

$$E\dot{x}(t) = \sum_{i=0}^{l} A_i x(t-\tau_i) + \sum_{k=0}^{m} \int_{-\mu_k}^{0} D_k(\theta)x(t+\theta)d\theta, \tag{2.6}$$

where $x \in \mathbb{C}^n$, E, $A_i \in \mathbb{C}^{n\times n}$, $i = 0, 1., ...l$, are constant matrices, the leading term E is supposed to be nonsingular; $D_k(.) : [-\mu_k, 0] \to \mathbb{C}^{n\times n}$, $k = 0, 1, ..., m$, are integrable matrix-valued functions, and $0 \le \tau_0 \le \tau_1 \le ... \le \tau_l$, $0 \le \mu_0 \le \mu_1 \le ... \le \mu_m$. If one multiplies both sides of (2.6) with E^{-1}, one obtains an explicit systems of FDE-s which is well-known in the literature, e.g., see [9]. However, the computation of E^{-1} may be expensive and complicated in practice, especially when E contains one or some small parameters and may be nearly singular.

Suppose that system (2.6) is asymptotically stable. It is easy to see the asymptotic stability is equivalent to the condition that all the roots of the generalized characteristic equation

$$\det\,(\lambda E - \sum_{i=0}^{l} A_i e^{-\lambda\tau_i} + \sum_{k=0}^{m} \int_{-\mu_k}^{0} D_k(\theta)e^{\lambda\theta}d\theta) = 0$$

are located in the open left half plane $\mathbb{C}^-$. We also consider the perturbed system

$$E\dot{x}(t) = \sum_{i=0}^{l}(A_i + B\Delta_i C_i)x(t-\tau_i) + \sum_{k=0}^{m} \int_{-\mu_k}^{0} (D_k(\theta) + B\delta_k(\theta)C_{l+1+k})x(t+\theta)d\theta, \tag{2.7}$$

where $\Delta_i \in \mathbb{C}^{p\times q_i}$, $i = 0, 1, ..., l$, are uncertain perturbations, $\delta_k(.) : [-\mu_k, 0] \to \mathbb{C}^{p\times q_{l+1+k}}$, $i = 0, 1, ..., m$, are integrable perturbation functions, and $B \in \mathbb{C}^{n\times p}$ and $C_i \in \mathbb{C}^{q_i\times n}$, $i = 0, 1, ...l + m + 2)$, are matrices determining the perturbation structure. Denote

$$\boldsymbol{\Delta} = \left\{ \{\Delta_i\}_{i=0}^{l}, \{\delta_k(.)\}_{k=0}^{m} \right\}$$

and define

$$\|\mathbf{\Delta}\| = \sum_{i=0}^{l} \|\Delta_i\| + \sum_{k=0}^{m} \int_{-\mu_k}^{0} \|\delta(\theta)\| d\theta,$$

where $\|.\|$ is a matrix norm induced by vector norms. Following the notion of the complex structured stability radius in [11, 17], we define

$$r_{\mathbb{C}} := \inf\{\|\mathbf{\Delta}\|, \text{the perturbed system (2.7) is not asym. stable}\}.$$

We will make the use of the following auxiliary functions

$$\begin{array}{rcl} H(s) & = & sE - \sum_{i=0}^{l} A_i e^{-s\tau_i} + \sum_{k=0}^{m} \int_{-\mu_k}^{0} D_k(\theta) e^{s\theta} d\theta, \\ G_i(s) & = & C_i [H(s)]^{-1} B, \quad i = 0, 1, ..., l+m+2. \end{array}$$

with $s \in \mathbb{C},\ \Re(s) \geq 0$.

Theorem 2 *Suppose that system (2.6) is asymptotically stable and subjected to structured perturbations of the form (2.7). Then*

$$r_{\mathbb{C}} = (\max_{i=0,1,...,l+m+2} \sup_{s \in i\mathbb{R}} \|G_i(s)\|)^{-1}.$$

This result is an extension of that given in [17], where $E = I_n$, $l = m = 0$, $\tau_0 = 0$, $\mu_0 = 1$ were set. The proof can be carried out straightforward on a similar way.

3 Main Results

3.1 The complex stability radius for index-1 FDAE-s

Now let us consider the reduced slow system (2.1) again. This system of FDAE-s has index-1 if and only if $\bar{L}_{22}$ defined in (2.2) is nonsingular [6]. In this case, as we can see in the previous section, (2.1) can be reduced to a regular linear FDE by eliminating $y(t)$. Hence, we have

Proposition 1 *Suppose that $\bar{L}_{22}$ is nonsingular. There exists the unique solution of the initial value problem for the FDAE (2.1), $t \geq 0$, with initial condition*

$$x(t) = \varphi(t), \quad t \in [-\tau_l, 0], \tag{3.1}$$

where $\varphi(.) \in C([-\tau_l, 0], \mathbb{C}^{n_1})$ is arbitrarily given.

Note that the initial condition should be assigned to the differential component $x(.)$, only. The algebraic component $y(.)$ can be determined uniquely and explicitly by $x(.)$.

Definition 1 *Suppose that $\bar{L}_{22}$ is nonsingular. The zero solution of the initial value problem (2.1),(3.1) is said to be (exponential-)asymptotically stable if for any $\varphi(.) \in C([-\tau_l, 0], \mathbb{C}^{n_1})$, there exist positive constants c and α such that*

$$\|(x(t), y(t))\| \leq c|\varphi| e^{-\alpha t}$$

holds $\forall t \geq 0$ with $|\varphi| = \sup_{-\tau_l \leq t \leq 0} \|\varphi(t)\|$.

We also say that the linear time-invariant system (2.1) is asymptotically stable. It is easy to check the following

Proposition 2 *Suppose that $\bar{L}_{22}$ is nonsingular. The system (2.1) is asymptotically stable if and only if all the roots of the characteristic equation*

$$\det\left(\lambda\begin{pmatrix} I_{n_1} & 0 \\ 0 & 0\end{pmatrix} - \begin{pmatrix} \sum_{i=0}^{l} A_{11}^{i} e^{-\tau_i\lambda} + \int_{-\tau_l}^{0} D_{11}(\theta)e^{\theta\lambda}d\theta & \bar{L}_{12} \\ \sum_{i=0}^{l} A_{21}^{i} e^{-\tau_i\lambda} + \int_{-\tau_l}^{0} D_{21}(\theta)e^{\theta\lambda}d\theta & \bar{L}_{22}\end{pmatrix}\right) = 0 \tag{3.2}$$

are located in $\mathbb{C}^-$.

Clearly, equation (3.2) is equivalent to that in Assumption A2. Consider system (2.1) subjected to structured perturbations described as follows

$$\begin{aligned} \tilde{L}_{j1}x_t &= \sum_{i=0}^{l}(A_{j1}^{i} + B_j\Delta_1^i C_1^i)x(t-\tau_i) + \\ &\qquad \int_{-\tau_l}^{0}(D_{j1}(\theta) + B_j\delta_1(\theta)C_1^{l+1})x(t+\theta)d\theta, \\ \tilde{\tilde{L}}_{j2}y(t) &= \left(\sum_{k=0}^{m}(A_{j2}^{k} + B_j\Delta_2^k C_2^k) + \int_{-\mu_m}^{0}(D_{j2}(\theta) + B_j\delta_2(\theta)C_2^{m+1})d\theta\right)y(t), \end{aligned} \tag{3.3}$$

where $j = 1, 2$. The definition of the stability radius is slightly different from that for implicit regular systems. Namely, we define

$$\begin{aligned} r_0(\mathbf{A},\mathbf{B},\mathbf{C}) := \inf\{&\|\mathbf{\Delta}\|, \text{the perturbed system (2.1),(3.3) is not asym. stable} \\ &\text{or } \tilde{\tilde{L}}_{22} \text{ is singular}\}. \end{aligned} \tag{3.4}$$

First, we look for the index-1 preserving radius defined by

$$r_{ind} := \inf\{\sum_{k=0}^{m}\|\Delta_2^k\| + \int_{-\mu_m}^{0}\|\delta_2(\theta)\|d\theta,\ \tilde{\tilde{L}}_{22} \text{ is singular}\}.$$

The singularity of $\tilde{\tilde{L}}_{22}$ means exactly that at least one eigenvalue of this matrix moves to zero under the effect of perturbation. It is obvious that

$$r_0(\mathbf{A},\mathbf{B},\mathbf{C}) \le r_{ind}.$$

Using the same techniques used in [11, 12, 17], it is easy to prove

Proposition 3 *Suppose that $\bar{L}_{22}$ is nonsingular. Then*

$$r_{ind} = \{\max_{k=0,1,\dots,m+1}\|C_2^k\bar{L}_{22}^{-1}B_2\|\}^{-1}.$$

Furthermore, there exists a minimal norm perturbation under which $\tilde{\tilde{L}}_{22}$ is singular.

Note that in this "robust stability" problem, the stable and unstable regions are $\mathbb{C}_g = \mathbb{C}\setminus\{0\}$ and $\mathbb{C}_b = \{0\}$, respectively.

We introduce the following auxiliary functions

$$H_S(s) = s\begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \sum_{i=0}^{l} A_{11}^i e^{-\tau_i s} + \int_{-\tau_l}^{0} D_{11}(\theta)e^{\theta s}d\theta & \bar{L}_{12} \\ \sum_{i=0}^{l} A_{21}^i e^{-\tau_i s} + \int_{-\tau_l}^{0} D_{21}(\theta)e^{\theta s}d\theta & \bar{L}_{22} \end{pmatrix}$$

and

$$G_{S1}^i(s) = \begin{pmatrix} C_1^i & 0 \end{pmatrix} H_S(s)^{-1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad i = 0, 1, ..., l+1;$$

$$G_{S2}^k(s) = \begin{pmatrix} 0 & C_2^k \end{pmatrix} H_S(s)^{-1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad k = 0, 1, ..., m+1;$$

with $s \in \mathbb{C},\ \Re s \geq 0$.

For computing the inverse matrix, we use a well-known factorization of block matrices, e.g., see [7]. By some matrix calculations, these functions can be reformulated as follows

$$\begin{aligned} G_{S1}^i(s) &= C_1^i \left(sI - \bar{L}_{11}(s) + \bar{L}_{12}\bar{L}_{22}^{-1}\bar{L}_{21}(s)\right)^{-1}(B_1 - \bar{L}_{12}\bar{L}_{22}^{-1}B_2), \\ G_{S2}^k(s) &= -C_2^k\bar{L}_{22}^{-1}B_2 - C_2^k\bar{L}_{22}^{-1}\bar{L}_{21}(s)\left(sI - \bar{L}_{11}(s) + \bar{L}_{12}\bar{L}_{22}^{-1}\bar{L}_{21}(s)\right)^{-1} \\ &\qquad \times(B_1 - \bar{L}_{12}\bar{L}_{22}^{-1}B_2), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \bar{L}_{11}(s) &= \sum_{i=0}^{l} A_{11}^i e^{-\tau_i s} + \int_{-\tau_l}^{0} D_{11}(\theta)e^{\theta s}d\theta, \\ \bar{L}_{21}(s) &= \sum_{i=0}^{l} A_{21}^i e^{-\tau_i s} + \int_{-\tau_l}^{0} D_{21}(\theta)e^{\theta s}d\theta. \end{aligned}$$

Lemma 1 *Assume that $\bar{L}_{22}$ is nonsingular and the reduced slow system (2.1) is asymptotically stable. Then*

$$r_{ind} \geq \left(\max\{\max_{0\leq i\leq l+1}\sup_{s\in i\mathbb{R}}\|G_{S1}^i(s)\|, \max_{0\leq k\leq m+1}\sup_{s\in i\mathbb{R}}\|G_{S2}^k(s)\|\}\right)^{-1}.$$

Proof Taking into consideration that $\bar{L}_{11}(.)$, $\bar{L}_{12}(.)$ are bounded in $i\mathbb{R}$ and

$$\lim_{|s|\to+\infty}\|\left(sI_{n_1} - \bar{L}_{11}(s) + \bar{L}_{12}\bar{L}_{22}^{-1}\bar{L}_{21}(s)\right)^{-1}\| = 0,$$

the inequality is easily obtained from (3.5). ■

For brevity, let us denote the right-hand side of the inequality in Lemma 1 by r_{stab}.

Theorem 3 *Assume that $\bar{L}_{22}$ is nonsingular and the reduced slow system (2.1) is asymptotically stable. Then*

$$r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) = r_{stab}.$$

Proof There are two cases: either $r_{stab} < r_{ind}$ or $r_{stab} = r_{ind}$.

Case A. If $r_{stab} < r_{ind}$:

1. Suppose there exists no destabilizing perturbation set $\mathbf{\Delta}$ such that $\|\mathbf{\Delta}\| < r_{ind}$. By definition, it would imply immediately

$$r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) = r_{ind}.$$

We will see later that this subcase is impossible.

2. Otherwise, there exists a perturbation set $\mathbf{\Delta}$ such that $\|\mathbf{\Delta}\| < r_{ind}$ and the perturbed system is not asymptotically stable. Since the perturbed system remains index-1, it follows that the associated characteristic equation has a root outside $\mathbb{C}^-$. Hence, there exist s_0, $\Re s_0 \geq 0$ and a nonzero vector $x_0 \in \mathbb{C}^n$ such that

$$\begin{aligned} H_S(s_0)x_0 \;\; = \;\; & \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \Big\{ \textstyle\sum_{i=0}^{l} \Delta_1^i e^{-\tau_i s_0} \begin{pmatrix} C_1^i & 0 \end{pmatrix} + \int_{-\tau_l}^{0} \delta_1(\theta) e^{\theta s_0} d\theta \begin{pmatrix} C_1^{l+1} & 0 \end{pmatrix} \\ & + \textstyle\sum_{k=0}^{m} \Delta_2^k \begin{pmatrix} 0 & C_2^k \end{pmatrix} + \int_{-\mu_m}^{0} \delta_2(\theta) d\theta \begin{pmatrix} 0 & C_2^{m+1} \end{pmatrix} \Big\} x_0. \end{aligned} \tag{3.6}$$

Multiplying both sides of (3.6) with $H_S(s_0)^{-1}$ from the left, we have

$$\begin{aligned} x_0 \;\; = \;\; & H_S(s_0)^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \Big\{ \textstyle\sum_{i=0}^{l} \Delta_1^i e^{-\tau_i s_0} \begin{pmatrix} C_1^i & 0 \end{pmatrix} + \\ & + \textstyle\int_{-\tau_l}^{0} \delta_1(\theta) e^{\theta s_0} d\theta \begin{pmatrix} C_1^{l+1} & 0 \end{pmatrix} + \sum_{k=0}^{m} \Delta_2^k \begin{pmatrix} 0 & C_2^k \end{pmatrix} + \\ & + \textstyle\int_{-\mu_m}^{0} \delta_2(\theta) d\theta \begin{pmatrix} 0 & C_2^{m+1} \end{pmatrix} \Big\} x_0. \end{aligned} \tag{3.7}$$

For simplicity, denote

$$\begin{array}{ll} \mathcal{C}^i = \begin{pmatrix} C_1^i & 0 \end{pmatrix}, \; i = 0, 1, ..., l+1; & \mathcal{C}^{l+2+k} = \begin{pmatrix} 0 & C_2^k \end{pmatrix}, \; k = 0, 1, ...m, \\ \mathcal{G}_S^i(s) = G_{S1}^i(s), \; i = 0, 1, ..., l+1; & \mathcal{G}_S^{l+2+k}(s) = G_{S2}^k(s), \; k = 0, 1, ...m, \end{array}$$

and let N be the index such that

$$\|\mathcal{C}^N x_0\| = \max_{0 \leq i \leq l+m+2} \{\|\mathcal{C}^i x_0\|\}.$$

It is clear that $\mathcal{C}^N x_0 \neq 0$. Multiplying both sides of equality (3.7) with $\mathcal{C}^n$ from the left and taking norm, we obtain

$$\|\mathcal{C}^N x_0\| \leq \|\mathcal{G}_S^N(s_0)\| \|\mathbf{\Delta}\| \|\mathcal{C}^N x_0\|.$$

To verify this inequality, we use the estimates

$$\|\Delta_1^i e^{-\tau_i s_0}\| \leq \|\Delta_1^i\| \text{ and } \|\int_{-\tau_l}^{0} \delta_1(\theta) e^{\theta s_0} d\theta\| \leq \|\delta_1(.)\|$$

and the definition of $\|\mathbf{\Delta}\|$. It follows that

$$\|\mathbf{\Delta}\| \geq \|\mathcal{G}_S^N(s_0)\|^{-1} \geq \left(\max_{1 \leq i \leq l+m+2} \sup_{\Re s \geq 0} \|\mathcal{G}_S^i(s)\| \right)^{-1}.$$

Since each function $\mathcal{G}_S^i(s), \; i = 0, 1, ..., l+m+2$, is analytic in $\mathbb{C} \setminus \mathbb{C}^-$, due to the the maximum principle, their least upper bound is attained on $i\mathbb{R}$ (at a finite point or at infinity). Hence,

$$r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) \geq \left(\max_{1 \leq i \leq l+m+2} \sup_{s \in i\mathbb{R}} \|\mathcal{G}_S^i(s)\| \right)^{-1} = r_{stab}. \tag{3.8}$$

3. Now we prove the inverse inequality of (3.8). To this end, we construct a destabilizing perturbation which has the norm arbitrarily close to r_{stab}. Suppose that $\epsilon > 0$ is an arbitrary,

but sufficiently small number such that $r_{stab} + \epsilon < r_{ind}$. Then, there exist an index M and $s_1 \in i\mathbb{R}$ such that

$$\|\mathcal{G}_S^M(s_1)\|^{-1} \leq \left(\max_{1 \leq i \leq l+m+2} \sup_{s \in i\mathbb{R}} \|\mathcal{G}_S^i(s)\| \right)^{-1} + \epsilon < r_{ind}.$$

Due to the definition, there exists a vector $u \in \mathbb{C}^p$, $\|u\| = 1$ such that $\|\mathcal{G}_S^M(s_1)u\| = \|\mathcal{G}_S^M(s_1)\|$. Invoking a corollary of the Hahn-Banach theorem, there exists a column vector $v^* \in \mathbb{C}^q$, $\|v^*\| = 1$ such that $\|v^*\mathcal{G}_S^M(s_1)u\| = \|\mathcal{G}_S^M(s_1)u\|$. Let us define

$$\Delta_b := \|\mathcal{G}_S^M(s_1)\|^{-1} uv^* \in \mathbb{C}^{p \times q}.$$

It is easy to see that $\|\Delta_b\| = \|\mathcal{G}_S^M(s_1)\|^{-1}$. We construct a destabilizing perturbation $\mathbf{\Delta}$ as follows:

- If $M \leq l$, set $\Delta_1^M := \Delta_b e^{\tau_i s_1}$, and all the other perturbations are zero;
- If $M = l+1$, set $\delta_1(\theta) := \tau_l^{-1}\Delta_b e^{-\theta s_1}$, and all the others are zero;
- If $l+2 \leq M \leq l+m+1$, set $\Delta_2^M := \Delta_b$, and all the others are zero;
- If $M = l+m+2$, set $\delta_2(\theta) := \mu_m^{-1}\Delta_b$, and all the others are zero.

It is clear that, in any case, $\|\mathbf{\Delta}\| = \|\Delta_b\|$ holds. After some elementary calculations, one can easily verify that

$$\Delta_b \mathcal{G}_S^M(s_1)u = u \Rightarrow \Delta_b \mathbf{B} H_S(s_1)^{-1}\mathcal{C}^M u = u \Rightarrow \mathcal{C}^M \Delta_b \mathbf{B} w = H_S(s_1)w,$$

where $w := H_S(s_1)^{-1}\mathcal{C}^M u \neq 0$. From the construction of $\mathbf{\Delta}$ above, the characteristic equation associated with the perturbed system has the root s_1 located in $i\mathbb{R}$. Note that the perturbed system remains index-1. By Proposition 2, the perturbed system is not asymptotically stable. Since ϵ is arbitrarily chosen, we obtain

$$r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) \leq \left(\max_{1 \leq i \leq l+m+2} \sup_{s \in i\mathbb{R}} \|\mathcal{G}_S^i(s)\| \right)^{-1} \tag{3.9}$$

Inequality (3.9) implies that the case $r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) = r_{ind} > r_{stab}$ (discussed at Point 1) cannot occur. Thus, $r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) = r_{stab}$.

As another consequence of the above argument, if

$$\left(\max_{1 \leq i \leq l+m+2} \sup_{s \in i\mathbb{R}} \|\mathcal{G}_S^i(s)\| \right)$$

is attained at a finite number $s_2 \in i\mathbb{R}$, then the minimal norm destabilizing perturbation exists. Furthermore, it can be constructed as in Point 3 by setting $\epsilon = 0$, $s_1 = s_2$.

Case B. If $r_{stab} = r_{ind}$:

Take an arbitrary perturbation set $\mathbf{\Delta}$ such that $\|\mathbf{\Delta}\| < r_{ind}$. It is clear that $\mathbf{\Delta}$ cannot be a destabilizing perturbation. Otherwise, by repeating the argument in Point 2, we would have $\|\mathbf{\Delta}\| \geq r_{stab} = r_{ind}$ which yields contradictions. It means that the perturbed system remains index-1 and asymptotically stable. By the definition of the stability radius for index-1 FDAE-s, we obtain $r_0(\mathbf{A},\mathbf{B},\mathbf{C}) = r_{ind} = r_{stab}$. ■

The formula of the complex stability radius in Theorem 3 generalizes the result for index-1 DAE-s (without time-delay and perturbation structure) proposed in [16], where, by a different approach, an estimate analogous to that in Lemma 1 was given, too. Note also that Proposition 2 is proven for FDAE-s of index-1, only, which makes the proof of Theorem 3 a little bit more complicated. For more details on delay DAE-s, e.g., see [6, 18] and the references therein.

3.2 Asymptotic behavior of the stability radius for the SPS

Now, we turn to the main point of the paper, the asymptotic behavior of the stability radius for the SPS (1.1),(1.3).

First, we introduce the following auxiliary functions:

$$\begin{aligned}\bar{L}_{12}(\varepsilon,s) &= \sum_{k=0}^{m} A_{12}^k e^{-\varepsilon\mu_k s} + \int_{-\mu_m}^{0} D_{12}(\theta)e^{\varepsilon\theta s}d\theta,\\ \bar{L}_{22}(\varepsilon,s) &= \sum_{k=0}^{m} A_{22}^k e^{-\varepsilon\mu_k s} + \int_{-\mu_m}^{0} D_{22}(\theta)e^{\varepsilon\theta s}d\theta,\end{aligned}$$

$$H_\varepsilon(s) = s\begin{pmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2}\end{pmatrix} - \begin{pmatrix} \bar{L}_{11}(s) & \bar{L}_{12}(\varepsilon,s) \\ \bar{L}_{21}(s) & \bar{L}_{22}(\varepsilon,s)\end{pmatrix}$$

with $s \in \mathbb{C},\ \Re s \geq 0,\ \varepsilon \in (0,\varepsilon_0]$, where ε_0 is provided by Theorem 1. The functions $\bar{L}_{11}(s),\ \bar{L}_{21}(s)$ were introduced previously in (3.5). Furthermore,

$$G_{\varepsilon 1}^i(s) = \begin{pmatrix} C_1^i & 0 \end{pmatrix} H_\varepsilon(s)^{-1}\begin{pmatrix} B_1 \\ B_2\end{pmatrix}, \quad i = 0,1,...,l+1;$$

$$G_{\varepsilon 2}^k(s) = \begin{pmatrix} 0 & C_2^k \end{pmatrix} H_\varepsilon(s)^{-1}\begin{pmatrix} B_1 \\ B_2\end{pmatrix}, \quad k = 0,1,...,m+1;$$

By some matrix calculations, the latter functions can be reformulated as follows

$$\begin{aligned}G_{\varepsilon 1}^i(s) &= C_1^i\left[sI_{n_1} - \bar{L}_{11}(s) - \bar{L}_{12}(\varepsilon,s)(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon,s))^{-1}\bar{L}_{21}(s)\right]^{-1}\\ &\qquad \times(B_1 + \bar{L}_{12}(\varepsilon,s)(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon,s))^{-1}B_2),\\ G_{\varepsilon 2}^k(s) &= C_2^k(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon,s))^{-1}B_2 + C_2^k(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon,s))^{-1}\bar{L}_{21}(s)\\ &\qquad \times\left[sI_{n_1} - \bar{L}_{11}(s) - \bar{L}_{12}(\varepsilon,s)(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon,s))^{-1}\bar{L}_{21}(s)\right]^{-1}\\ &\qquad \times(B_1 + \bar{L}_{12}(\varepsilon,s)(\varepsilon s I_{n_2} - \bar{L}_{22})^{-1}B_2).\end{aligned} \tag{3.10}$$

Let us fix a closed interval $[0,\varepsilon_0]$ provided by Theorem 1. Applying Theorem 2 to the SPS (1.1),(1.3), we easily obtain

Proposition 4 *Let Assumption A1-A2 hold. Then*

$$r_\varepsilon(\mathbf{A},\mathbf{B},\mathbf{C}) = \left(\max\{\max_{0\leq i\leq l+1}\sup_{s\in i\mathbb{R}}\|G_{\varepsilon 1}^i(s)\|, \max_{0\leq k\leq m+1}\sup_{s\in i\mathbb{R}}\|G_{\varepsilon 2}^k(s)\|\}\right)^{-1}.$$

for all $\varepsilon \in (0,\varepsilon_0]$.

The following auxiliary result can also be easily proven.

Lemma 2 *Let Assumption A1-A2 hold. Then the matrix functions*

$$\bar{L}_{j1}(.),\ \bar{L}_{j2}(\varepsilon,.),\ j=1,2,\ \textit{and}\ (.\varepsilon I_{n_2}-\bar{L}_{22}(\varepsilon,.))^{-1}$$

are bounded in $i\mathbb{R}$ *and their bounds are independent of* $\varepsilon \in (0, \varepsilon_0]$.

Proof The uniform boundedness of $\bar{L}_{j1}(s)$, $\bar{L}_{j2}(\varepsilon, s)$, $j = 1, 2$, is obvious. To verify the uniform boundedness of $(.\varepsilon I_{n_2} - \bar{L}_{22}(\varepsilon, .))^{-1}$, we observe that

$$\sup_{s\in i\mathbb{R}} \|(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1}\| = \sup_{s\in i\mathbb{R}} \|(s I_{n_2} - \hat{L}_{22}(s))^{-1}\|,$$

where

$$\hat{L}_{22}(s) = \sum_{k=0}^{m} A_{22}^k e^{-\mu_k s} + \int_{-\mu_m}^{0} D_{22}(\theta) e^{\theta s} d\theta.$$

Furthermore,

$$\lim_{|s|\to+\infty} \|(s I_{n_2} - \hat{L}_{22}(s))^{-1}\| = 0.$$

Hence, the function in question is bounded in $i\mathbb{R}$ and its bound does not depend on ε. ■

Considering the fast boundary layer system (2.5) introduced in Section 2 again. We associate to this system the following auxiliary functions

$$G_F^k(s) = C_2^k (s I_{n_2} - \hat{L}_{22}(s))^{-1} B_2, \quad k = 0, 1, ..., m+1;\ \Re s \geq 0. \tag{3.11}$$

Applying Theorem 2 again to the boundary layer fast system (2.5) subjected to the corresponding structured perturbation, we have

$$r(\mathbf{A_{22}}, B_2, \mathbf{C_2}) = \left(\max_{0\leq k\leq m+1} \sup_{s\in i\mathbb{R}} \|G_F^k(s)\| \right)^{-1}, \tag{3.12}$$

where $\mathbf{A_{22}} = \left\{ \{A_{22}^k\}_{k=0}^m, D_{22}(.) \right\}$, $\mathbf{C_2} = \{C_2^k\}_{k=0}^{m+1}$ and $r(\mathbf{A_{22}}, B_2, \mathbf{C_2})$ denotes the structured complex stability radius for (2.5).

Our main result is the following

Theorem 4 *Let Assumption A1-A2 hold. Then,*

$$\lim_{\varepsilon\to+0} r_\varepsilon(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \min\{r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}), r(\mathbf{A_{22}}, B_2, \mathbf{C_2})\}.$$

Proof The key point of the proof is the uniform convergence

$$\| \left[s I_{n_1} - \bar{L}_{11}(s) - \bar{L}_{12}(\varepsilon, s)(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1} \bar{L}_{21}(s) \right]^{-1} \| \rightrightarrows 0 \tag{3.13}$$

as $|s| \to +\infty$ with respect to $\varepsilon \in [0, \varepsilon_0]$. We recall that, throughout the proof, the variable s is considered restrictively in the line $i\mathbb{R}$, only. Due to Lemma 2, (3.13) is evident.

Based on the results in Theorem 3, Proposition 4, and (3.12), it is sufficient to prove first,

$$\lim_{\varepsilon\to+0} \sup_{s\in i\mathbb{R}} \|G_{\varepsilon 1}^i(s)\| = \sup_{s\in i\mathbb{R}} \|G_{S1}^i(s)\|, \quad i = 0, 1, ..., l+1, \tag{3.14}$$

and secondly,

$$\lim_{\varepsilon\to+0}\sup_{s\in i\mathbb{R}}\|G^k_{\varepsilon 2}(s)\| = \max\{\sup_{s\in i\mathbb{R}}\|G^k_{S2}(s)\|, \sup_{s\in i\mathbb{R}}\|G^k_F(s)\|\}, \quad k=0,1,...,m+1. \tag{3.15}$$

1. Fix an arbitrary index $i,\ 0\le i\le l+1$ and an arbitrarily small number $\rho>0$. From (3.13), it is easy to see that $\|G^i_{\varepsilon 1}(s)\|$ converges uniformly to zero as $|s|$ tends to infinity. Therefore, there exists a bound T_1, T_1 is independent of ε, such that

$$\|G^i_{\varepsilon 1}(s)\|\le\rho, \quad \forall\ |s|\ge T_1.$$

On the other hand, in the compact domain $\{(s,\varepsilon),\ |s|\le T_1,\ 0\le\varepsilon\le\varepsilon_0\}$, $\left\|G^i_{\varepsilon 1}(s)\right\|$ is continuous as a two-variable function, hence uniformly continuous, too. Therefore, there exists a sufficiently small $\varepsilon_1=\varepsilon_1(\rho)$ such that for $\varepsilon\le\varepsilon_1$, we have

$$\sup_{|s|\le T_1}\left\|G^i_{\varepsilon 1}(s)\right\|\le\sup_{|s|\le T_1}\left\|G^i_{S1}(s)\right\|+\rho\le\sup_{s\in i\mathbb{R}}\left\|G^i_{S1}(s)\right\|+\rho.$$

Thus, for $\varepsilon\le\varepsilon_1$, we obtain

$$\sup_{s\in i\mathbb{R}}\left\|G^i_{\varepsilon 1}(s)\right\|\le\sup_{s\in i\mathbb{R}}\left\|G^i_{S1}(s)\right\|+\rho.$$

Since $\sup_{s\in i\mathbb{R}}\left\|G^i_{S1}(s)\right\|$ is finite, there exists a number $s_1=s_1(\rho)\in i\mathbb{R}$ such that

$$\left\|G^i_{S1}(s_1)\right\|\ge\sup_{s\in i\mathbb{R}}\left\|G^i_{S1}(s)\right\|-\rho.$$

Furthermore, because of the continuity of $\left\|G^i_{\varepsilon 1}(s_1)\right\|$ as a function of ε, there exists a sufficiently small $\varepsilon_2=\varepsilon_2(\rho)$ such that for $\varepsilon\le\varepsilon_2$, we obtain

$$\sup_{s\in i\mathbb{R}}\left\|G^i_{\varepsilon 1}(s)\right\|\ge\left\|G^i_{\varepsilon 1}(s_1)\right\|\ge\left\|G^i_{S1}(s_1)\right\|-\rho\ge\sup_{s\in i\mathbb{R}}\left\|G^i_{S1}(s)\right\|-2\rho.$$

Therefore, for $\varepsilon\le\min\{\varepsilon_1,\varepsilon_2\}$, the estimate

$$\sup_{s\in i\mathbb{R}}\left\|G^i_{S1}(s)\right\|-2\rho\le\sup_{s\in i\mathbb{R}}\left\|G^i_{\varepsilon 1}(s)\right\|\le\sup_{s\in i\mathbb{R}}\left\|G^i_{S1}(s)\right\|+\rho$$

holds. This proves (3.14).

2. To prove (3.15), we proceed as in [4] and [5]. Analogously to above, fix an index $k,\ 0\le k\le m+1$ and an arbitrarily small $\varrho>0$. We show that the inequalities

$$\begin{aligned}\max\left\{\sup_{s\in i\mathbb{R}}\left\|G^k_{S2}(s)\right\|,\sup_{s\in i\mathbb{R}}\left\|G^k_F(s)\right\|\right\}-2\varrho\le\sup_{s\in i\mathbb{R}}\left\|G^k_{\varepsilon 2}(s)\right\|\\ \le\max\left\{\sup_{s\in i\mathbb{R}}\left\|G^k_{S2}(s)\right\|,\sup_{s\in i\mathbb{R}}\left\|G^k_F(s)\right\|\right\}+\varrho\end{aligned} \tag{3.16}$$

hold for all sufficiently small ε.

a, First, we prove the last inequality in (3.16).

By a similar argument as in proving (3.14), there exists a sufficiently large number $T_2=T_2(\varrho)$, T_2 is independent of ε, such that

$$\begin{aligned}&\big\|C^k_2(\varepsilon sI_{n_2}-\bar L_{22}(\varepsilon,s))^{-1}\bar L_{21}(s)\\ &\times\left[sI-\bar L_{11}(s)-\bar L_{12}(\varepsilon,s)(\varepsilon sI_{n_2}-\bar L_{22}(\varepsilon,s))^{-1}\bar L_{21}(s)\right]^{-1}\\ &\quad\times(B_1+\bar L_{12}(\varepsilon,s)(\varepsilon sI_{n_2}-\bar L_{22})^{-1}B_2)\big\|\le\varrho,\quad |s|\ge T_2.\end{aligned}$$

Therefore, for s with $|s| \geq T_2$, we have

$$\left\|G_{\varepsilon 2}^{k}(s)\right\| \leq \left\|C_2^k(sI_{n_2} - \hat{L}_{22}(s))^{-1}B_2\right\| + \varrho.$$

Hence, we obtain

$$\begin{aligned} \sup_{|s|\geq T_2} \left\|G_{\varepsilon 2}^{k}(s)\right\| \leq \sup_{|s|\geq T_2} \left\|C_2^k(\varepsilon s I_{n_2} - \hat{L}_{22}(\varepsilon s))^{-1}B_2\right\| + \varrho = \\ = \sup_{|s|\geq \varepsilon T_2} \left\|G_F^k(s)\right\| + \varrho \leq \sup_{s\in i\mathbb{R}} \left\|G_F^k(s)\right\| + \varrho. \end{aligned} \tag{3.17}$$

On the other hand, in the compact domain $\{(s,\varepsilon),\ |s| \leq T_2,\ 0 \leq \varepsilon \leq \varepsilon_0\}$, $\left\|G_{\varepsilon 2}^{k}(s)\right\|$ is continuous as a two-variable function, hence uniformly continuous, too. Therefore, there exists a sufficiently small $\varepsilon_3 = \varepsilon_3(\varrho)$ such that for $\varepsilon \leq \varepsilon_3$, we have

$$\sup_{|s|\leq T_2} \left\|G_{\varepsilon 2}^{k}(s)\right\| \leq \sup_{|s|\leq T_2} \left\|G_{S2}^{k}(s)\right\| + \varrho \leq \sup_{s\in i\mathbb{R}} \left\|G_{S2}^{k}(s)\right\| + \varrho.$$

Thus, for $\varepsilon \leq \varepsilon_3$, we obtain

$$\sup_{s\in i\mathbb{R}} \left\|G_{\varepsilon 2}^{k}(s)\right\| \leq \max\left\{\sup_{s\in i\mathbb{R}} \left\|G_{S2}^{k}(s)\right\|,\ \sup_{s\in i\mathbb{R}} \left\|G_F^k(s)\right\|\right\} + \varrho.$$

b, Now, we prove the first inequality in (3.16).

Analogously to (3.17), we have

$$\sup_{|s|\geq T_2} \left\|G_{\varepsilon 2}^{k}(s)\right\| \geq \sup_{|s|\geq \varepsilon T_2} \left\|G_F^k(s)\right\| - \varrho.$$

Since $\left\|G_F^k(s)\right\|$ is continuous, $s \in i\mathbb{R}$, there exists a sufficiently small $\varepsilon_4 = \varepsilon_4(\varrho)$ such that for $\varepsilon \leq \varepsilon_4$, the inequality

$$\sup_{|s|\geq \varepsilon T_2} \left\|G_F^k(s)\right\| \geq \sup_{s\in i\mathbb{R}} \left\|G_F^k(s)\right\| - \varrho$$

holds. Hence, we obtain

$$\sup_{|s|\geq T_2} \left\|G_{\varepsilon 2}^{k}(s)\right\| \geq \sup_{s\in i\mathbb{R}} \left\|G_F^k(s)\right\| - 2\varrho.$$

On the other hand, since $\sup_{s\in i\mathbb{R}} \left\|G_{S2}^{k}(s)\right\|$ is finite, there exists a number $s_2 = s_2(\varrho) \in i\mathbb{R}$ such that

$$\left\|G_{S2}^{k}(s_2)\right\| \geq \sup_{s\in i\mathbb{R}} \left\|G_{S2}^{k}(s)\right\| - \varrho.$$

Furthermore, because of the continuity of $\left\|G_{\varepsilon 2}^{k}(s_2)\right\|$ as a function of ε, there exists a sufficiently small $\varepsilon_5 = \varepsilon_5(\varrho)$ such that for $\varepsilon \leq \varepsilon_5$, we obtain

$$\sup_{s\in i\mathbb{R}} \left\|G_{\varepsilon 2}^{k}(s)\right\| \geq \left\|G_{\varepsilon 2}^{k}(s_2)\right\| \geq \left\|G_{S2}^{k}(s_2)\right\| - \varrho \geq \sup_{s\in i\mathbb{R}} \left\|G_{S2}^{k}(s)\right\| - 2\varrho.$$

Therefore, for $\varepsilon \leq \min\{\varepsilon_4, \varepsilon_5\}$, the inequality

$$\sup_{s\in i\mathbb{R}} \left\|G_{\varepsilon 2}^{k}(s)\right\| \geq \max\left\{\sup_{s\in i\mathbb{R}} \left\|G_{S2}^{k}(s)\right\|,\ \sup_{s\in i\mathbb{R}} \left\|G_F^k(s)\right\|\right\} - 2\varrho$$

holds.

Then, for $\varepsilon \leq \min\{\varepsilon_3, \varepsilon_4, \varepsilon_5\}$, the inequalities in (3.16) hold. The proof of (3.15) is complete.

Since (3.14),(3.15) hold for all $i = 0, 1, ..., l+1$ and $k = 0, 1, ..., m+1$, the proof of Theorem 4 is complete. ■

4 Conclusion

In this paper, a class of SPS-s of differential equations with multiple delays has been considered. Motivated by and considered as a continuation of the stability analysis given in [2], the stability robustness of the SPS-s has been launched. The notion of the structured stability radius is extended to the reduced systems which are index-1 FDAE-s. By using the implicit-system approach, asymptotic behavior of the stability radius for the SPS-s is characterized as the parameter tends to zero. It is known that the complex stability radius for explicit linear systems depends continuously on data [12]. Here, we have shown that this property does not hold for the SPS-s, i.e., the stability radius may be discontinuous in parameter. The SPS analyzed here includes that in [3] as a special case. An extension of the results to more general systems of FDAE-s containing a small parameter would be of interest.

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