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**THREE TERM RECURSION RELATION FOR SPHERICAL FUNCTIONS  
ASSOCIATED TO THE COMPLEX HYPERBOLIC PLANE**

I. Pacharoni<sup>1</sup> and J. Tirao<sup>2</sup>  
*CIEM-FaMAF, Universidad Nacional de Córdoba, Córdoba 5000, Argentina*  
*and*  
*The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.*

**Abstract**

The symmetric space duality between the complex hyperbolic plane  $H_2(\mathbb{C}) = \text{SU}(2, 1)/\text{U}(2)$  and the complex projective plane  $P_2(\mathbb{C}) = \text{SU}(3)/\text{U}(2)$  also becomes apparent in the theory of matrix valued spherical functions associated to both spaces. This is stressed in this paper by proving a three term recursion relation for a family of matrix valued functions built up from the spherical functions associated to  $H_2(\mathbb{C})$ . In a very precise sense this extends the three term recursion relation which holds for  $P_2(\mathbb{C})$ .

MIRAMARE – TRIESTE

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<sup>1</sup>Junior Associate of the Abdus Salam ICTP. [pacharon@mate.uncor.edu](mailto:pacharon@mate.uncor.edu)

<sup>2</sup>Senior Associate of the Abdus Salam ICTP. [tirao@mate.uncor.edu](mailto:tirao@mate.uncor.edu)

## 1. INTRODUCTION

The analogy between the spherical geometries and the hyperbolic geometries is a special case of a general duality for symmetric spaces. This analogy reappears in the function theory on these spaces, as the one between the trigonometric functions and the hyperbolic functions, or between the Jacobi polynomials  $P_n^{(\alpha,\beta)}$  and the Jacobi functions  $F_u^{(\alpha,\beta)}$ . These analogies become apparent once we recall the expressions of the trigonometric and hyperbolic functions in terms of the exponential function, and the ones of the Jacobi polynomials and the Jacobi functions in terms of Gauss' hypergeometric function. In particular the last analogy, for  $\alpha = n$  and  $\beta = 1$ , is an instance of a more general one found studying the irreducible spherical functions of any type, associated to the complex projective plane  $P_2(\mathbb{C})$  or to the complex hyperbolic plane  $H_2(\mathbb{C})$  (see [GPT1] and [PRT]). Both are dual Hermitian symmetric spaces, and as such, the compact one  $P_2(\mathbb{C})$  contains the noncompact dual  $H_2(\mathbb{C})$  as an open submanifold (see Proposition 7.14 in [He]).

The group  $U = \text{SU}(3)$  acts in a natural way in the complex projective plane  $P_2(\mathbb{C})$ . This action is transitive and  $K = \text{S}(\text{U}(2) \times \text{U}(1))$  is the isotropy subgroup of the point  $(0, 0, 1) \in P_2(\mathbb{C})$ . Therefore  $P_2(\mathbb{C}) = U/K$ . We identify the complex plane with the affine plane

$$\mathbb{C}^2 = \{ (x, y, 1) \in P_2(\mathbb{C}) : (x, y) \in \mathbb{C}^2 \},$$

and we take full advantage of the  $K$ -orbit structure of  $P_2(\mathbb{C})$ . The affine plane  $\mathbb{C}^2$  is  $K$ -stable and the corresponding line at infinity  $L$  is a  $K$ -orbit. Moreover the  $K$ -orbits in  $\mathbb{C}^2$  are the spheres

$$S_r = \{ (x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = r^2 \}.$$

Thus we can take the points  $(r, 0) \in S_r$  and  $(1, 0, 0) \in L$  as representatives of  $S_r$  and  $L$ , respectively. Since  $(M, 0, 1) = (1, 0, \frac{1}{M}) \rightarrow (1, 0, 0)$  when  $M \rightarrow \infty$ , the closed interval  $[0, \infty]$  parameterizes the set of  $K$ -orbits in  $P_2(\mathbb{C})$ .

The noncompact dual of the Riemannian symmetric pair  $(U, K)$  is the pair  $(G, K)$  where  $G = \text{SU}(2, 1)$ . Since  $G$  is also a group of linear transformations of  $\mathbb{C}^3$  it acts naturally in  $P_2(\mathbb{C})$ . The  $G$ -orbit of the point  $(0, 0, 1)$  is the set

$$B = \{ (x, y, 1) \in P_2(\mathbb{C}) : |x|^2 + |y|^2 < 1 \},$$

and the corresponding isotropy subgroup is  $K$ . Thus  $H_2(\mathbb{C}) = G/K$  can be identified with the open ball of radius one centered at the origin in  $\mathbb{C}^2$ .

Let  $G$  be a locally compact unimodular group and let  $K$  be a compact subgroup of  $G$ . Let  $\hat{K}$  denote the set of all equivalence classes of complex finite dimensional irreducible representations of  $K$ ; for each  $\delta \in \hat{K}$ , let  $\xi_\delta$  denote the character of  $\delta$ ,  $d(\delta)$  the degree of  $\delta$ , i.e. the dimension of any representation in the class  $\delta$ , and  $\chi_\delta = d(\delta)\xi_\delta$ . We shall choose once and for all the Haar measure  $dk$  on  $K$  normalized by  $\int_K dk = 1$ . We shall denote by  $V$  a finite dimensional vector

space over the field  $\mathbb{C}$  of complex numbers and by  $\text{End}(V)$  the space of all linear transformations of  $V$  into  $V$ .

A *spherical function*  $\Phi$  on  $G$  of type  $\delta \in \hat{K}$  is a continuous function on  $G$  with values in  $\text{End}(V)$  such that  $\Phi(e) = I$  ( $I$  = identity transformation) and

$$\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky) dk,$$

for all  $x, y \in G$ . Then one has the following result.

**Proposition 1.1.** ([T],[GV]) *If  $\Phi : G \longrightarrow \text{End}(V)$  is a spherical function of type  $\delta$  then:*

- i)  $\Phi(kgk') = \Phi(k)\Phi(g)\Phi(k')$ , for all  $k, k' \in K, g \in G$ .
- ii)  $\pi : k \mapsto \Phi(k)$  is a representation of  $K$  such that any irreducible subrepresentation belongs to  $\delta$ .

Concerning the definition let us point out that the spherical function  $\Phi$  determines its type univocally (Proposition 1.1) and let us say that the number of times that  $\delta$  occurs in the representation  $k \mapsto \Phi(k)$  is called the *height* of  $\Phi$ .

Spherical functions of type  $\delta$  arise in a natural way upon considering representations of  $G$ . If  $g \mapsto U(g)$  is a continuous representation of  $G$ , say on a complete, locally convex, Hausdorff topological vector space  $E$ , then

$$P(\delta) = \int_K \chi_\delta(k^{-1})U(k) dk$$

is a continuous projection of  $E$  onto  $P(\delta)E = E(\delta)$ ;  $E(\delta)$  consists of those vectors in  $E$ , the linear span of whose  $K$ -orbit is finite dimensional and splits into irreducible  $K$ -subrepresentations of type  $\delta$ . Whenever  $E(\delta)$  is finite dimensional, the function  $\Phi : G \longrightarrow \text{End}(E(\delta))$  defined by  $\Phi(g)a = P(\delta)U(g)a, g \in G, a \in E(\delta)$  is a spherical function of type  $\delta$ . If the representation  $g \mapsto U(g)$  is topologically irreducible (i.e.  $E$  admits no non-trivial closed  $G$ -invariant subspace) then the associated spherical function  $\Phi$  is also irreducible.

If a spherical function  $\Phi$  is associated to a Banach representation of  $G$  then it is quasi-bounded, in the sense that there exists a semi-norm  $\rho$  on  $G$  and  $M \in \mathbb{R}$  such that  $\|\Phi(g)\| \leq M\rho(g)$  for all  $g \in G$ . Conversely, if  $\Phi$  is an irreducible quasi-bounded spherical function on  $G$ , then it is associated to a topologically completely irreducible (TCI) Banach representation of  $G$  (Godement, see [T]). Thus if  $G$  is compact any irreducible spherical function on  $G$  is associated to a Banach representation of  $G$ , which is finite dimensional by Peter-Weyl theorem.

When  $G$  is a connected Lie group then it is not difficult to prove that any spherical function  $\Phi : G \longrightarrow \text{End}(V)$  is differentiable ( $C^\infty$ ), and moreover that it is analytic. From the differential point of view a spherical function of type  $\delta$  can be characterized in the following way. Let  $D(G)$  denote the algebra of all left invariant differential operators on  $G$  and let  $D(G)^K$  denote the subalgebra of all operators in  $D(G)$  which are invariant under all right translation by elements in  $K$ . Let  $(V, \pi)$  be a finite dimensional representation of  $K$  such that any irreducible subrepresentation belongs to the same class  $\delta \in \hat{K}$ . Then we have

**Proposition 1.2.** ([T],[GV]) *A function  $\Phi : G \longrightarrow \text{End}(V)$  is a spherical function of type  $\delta$  if and only if*

- i)  $\Phi$  is analytic.
- ii)  $\Phi(k_1 g k_2) = \pi(k_1) \Phi(g) \pi(k_2)$ , for all  $k_1, k_2 \in K$ ,  $g \in G$ , and  $\Phi(e) = I$ .
- iii)  $[D\Phi](g) = \Phi(g)[D\Phi](e)$ , for all  $D \in D(G)^K$ ,  $g \in G$ .

It is also convenient to point out the following facts.

**Proposition 1.3.** ([T],[GV]) *The following properties are equivalent:*

- i)  $D(G)^K$  is commutative.
- ii) Every irreducible spherical function of  $(G, K)$  is of height one.

**Proposition 1.4.** ([T], Remark 4.7) *Let  $\Phi, \Psi : G \longrightarrow \text{End}(V)$  be two spherical functions on a connected Lie group  $G$  such that  $\Phi(k) = \Psi(k)$  for all  $k \in K$ . Then  $\Phi = \Psi$  if and only if  $[D\Phi](e) = [D\Psi](e)$  for all  $D \in D(G)^K$ .*

If  $G$  is a noncompact connected semisimple Lie group with finite center, and  $K$  is a maximal compact subgroup of  $G$ , then, from the Subquotient Theorem of Harish-Chandra (see [W] Theorem 5.5.1.5) or from Casselman's Subrepresentation Theorem (see [Kn] p. 238), we know that any TCI Banach representation of  $G$  is infinitesimally equivalent to a subquotient, respectively to a subrepresentation, of a nonunitary principal series  $U^{\sigma, \nu}$ . Thus, if  $\Phi$  is a quasi-bounded irreducible spherical function on  $G$  of type  $\delta \in \hat{K}$ , there exists  $(\sigma, \nu)$  and a  $K$ -projection  $Q(\delta)$ , of  $U^{\sigma, \nu}$  onto the  $\delta$ -isotypic component of an irreducible subrepresentation of  $U^{\sigma, \nu}$ , such that  $\Phi$  is equivalent to  $Q(\delta)U^{\sigma, \nu}Q(\delta)$ . This follows from Proposition 1.4. In particular when  $D(G)^K$  is abelian  $\Phi$  is equivalent to  $\Phi_\delta^{\sigma, \nu} = P(\delta)U^{\sigma, \nu}P(\delta)$ , because in such a case the multiplicity of  $\delta$  in  $U^{\sigma, \nu}$  is one.

From now on in this section  $(G, K)$  will be either  $(\text{SU}(3), \text{S}(\text{U}(2) \times \text{U}(1)))$  or  $(\text{SU}(2, 1), \text{S}(\text{U}(2) \times \text{U}(1)))$ . In these cases both algebras  $D(G)^K$  are isomorphic, and in fact isomorphic to  $D(G)^G \otimes D(K)^K$ ,  $D(G)^G$  and  $D(K)^K$  being polynomial algebras (Harish-Chandra's theorem) in two algebraically independent generators (see [GPT1]).

The set  $\hat{K}$  can be identified with the set  $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ . If  $k = \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}$ , with  $A \in \text{U}(2)$  and  $a = (\det A)^{-1}$ , then

$$(1) \quad \pi(k) = \pi_{n, \ell}(A) = (\det A)^n A^\ell,$$

where  $A^\ell$  denotes the  $\ell$ -symmetric power of  $A$ , defines an irreducible representation of  $K$  in the class  $(n, \ell) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ .

The representation  $\pi_{n, \ell}$  of  $\text{U}(2)$  extends to a unique holomorphic representation of  $\text{GL}(2, \mathbb{C})$  into  $\text{End}(V_\pi)$ , which we still denote by  $\pi_{n, \ell}$ . For any  $g \in \text{M}(3, \mathbb{C})$ , we denote by  $A(g)$  the left upper  $2 \times 2$  block of  $g$ , i.e.

$$A(g) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Let

$$\mathcal{A} = \{g \in G : \det A(g) \neq 0\}.$$

The canonical projection  $p : G \longrightarrow P_2(\mathbb{C})$  maps the open dense subset  $\mathcal{A} \subset G$  onto the affine plane  $\mathbb{C}^2$  when  $G = \text{SU}(3)$  and onto the unit ball  $B$  when  $G = \text{SU}(2, 1)$ , since  $\mathcal{A} = G$  in this case.

For any  $\pi = \pi_{(n, \ell)}$  let  $\Phi_\pi : \mathcal{A} \longrightarrow \text{End}(V_\pi)$  be defined by

$$\Phi_\pi(g) = \Phi_{n, \ell}(g) = \pi_{n, \ell}(A(g)).$$

If  $G = \text{SU}(3)$  and  $n \geq 0$  then  $\Phi_\pi$  extends to a unique irreducible spherical function on  $G$  of type  $(n, \ell)$ , and when  $G = \text{SU}(2, 1)$  and  $\Phi_\pi$  is an irreducible spherical function on  $G$  of type  $(n, \ell)$ .

To determine all spherical functions  $\Phi : G \longrightarrow \text{End}(V_\pi)$  of type  $\pi = \pi_{n, \ell}$ , we use the function  $\Phi_\pi$  in the following way: in the open set  $\mathcal{A} \subset G$  we define a function  $H$  by

$$(2) \quad H(g) = \Phi(g) \Phi_\pi(g)^{-1},$$

where  $\Phi$  is suppose to be a spherical function of type  $\pi$ . Then  $H$  satisfies

- i)  $H(e) = I$ .
- ii)  $H(gk) = H(g)$ , for all  $g \in \mathcal{A}, k \in K$ .
- iii)  $H(kg) = \pi(k)H(g)\pi(k^{-1})$ , for all  $g \in \mathcal{A}, k \in K$ .

Then property ii) says that  $H$  may be considered as a function on  $\mathbb{C}^2$  or  $B$ , and moreover from iii) it follows that  $H$  is determined by the function  $r \mapsto H(r, 0)$  on the intervals  $[0, +\infty)$  or  $[0, 1)$  when  $G = \text{SU}(3)$  or  $G = \text{SU}(2, 1)$ , respectively.

As we said above the algebra  $D(G)^G$ , of all differential operators on  $G$  which are invariant under left and right multiplication by elements in  $G$ , is a polynomial algebra in two algebraically independent generators  $\Delta_2$  and  $\Delta_3$ . A particular choice of these operators is given in Proposition 2.1 of [GPT1].

Now if we make the change of variables  $t = 1/(1 + r^2)$  when  $G = \text{SU}(3)$  and  $t = 1/(1 - r^2)$  when  $G = \text{SU}(2, 1)$ , then the fact that  $\Phi$  is an eigenfunction of  $\Delta_2$  and  $\Delta_3$ , makes  $H(t) = H(r, 0)$  into an eigenfunction of the following differential operators

$$(3) \quad \begin{aligned} DH &= t(1-t)H'' + (A_0 - tA_1)H' + \frac{1}{1-t}(B_0 - tB_1)H. \\ EH &= t(1-t)MH'' + (C_0 - tC_1)H' + \frac{1}{1-t}(D_0 + tD_1)H. \end{aligned}$$

for  $t \in (0, 1)$  when  $G = \text{SU}(3)$  or  $t \in (1, \infty)$  when  $G = \text{SU}(2, 1)$ . Observe that  $t = 1$  corresponds in both cases to the point  $p(e)$ . The coefficient matrices  $A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1$  and  $M$  are explicitly given in Section 10 of [GPT1].

Then it follows that the irreducible spherical functions of  $G$  of type  $(n, \ell)$  corresponds precisely to the simultaneous matrix valued eigenfunctions of  $D$  and  $E$  which are analytic in the intervals  $[0, 1]$  or  $[1, \infty)$ , respectively, when  $G = \text{SU}(3)$  or  $G = \text{SU}(2, 1)$ , and such that  $H(1) = I$  ([GPT1]

and [PRT]). From here we obtain that all irreducible spherical functions on  $SU(2, 1)$  are quasi-bounded since all appear associated to a principal series representation.

In [GPT2] and [PT] we proved a three term recursion relation for a sequence of matrix valued functions  $\tilde{H}(t, w)$  on  $SU(3)$  built up from  $\ell + 1$  spherical functions of a given type  $(n, \ell)$ .

In this paper we shall establish a similar result for  $SU(2, 1)$  which emphasizes the analogy between the function theories of our pairs  $(SU(3), U(2))$  and  $(SU(2, 1), U(2))$ .

In particular for  $G = SU(2, 1)$  one constructs, out of several spherical functions of a given type  $\pi_{n, \ell}$ ,  $n \in \mathbb{Z}$  and  $\ell \in \mathbb{N}_0$ , a family of  $(\ell + 1) \times (\ell + 1)$ -matrix valued functions  $\tilde{H}(t; \nu)$   $t \geq 1$ , such that as functions of the spectral parameter  $\nu$  they satisfy a three term recursion relation of the form

$$(4) \quad t\tilde{H}(t; \nu) = A_\nu\tilde{H}(t; \nu - \rho) + B_\nu\tilde{H}(t; \nu) + C_\nu\tilde{H}(t; \nu + \rho),$$

where  $A_\nu, B_\nu, C_\nu$  are matrices independent of  $t$ . These matrices are closely related to the corresponding matrices  $A_w, B_w, C_w$  in [PT] (see Section 6). On the other hand the functions  $\tilde{H}(t; \nu)$  satisfy a differential equation of the form

$$(5) \quad D\tilde{H}(t; \nu)^t = \tilde{H}(t; \nu)^t\Lambda,$$

where  $D$  is a second order differential operator in the variable  $t$  whose coefficients depend on  $t$  (and not on  $\nu$ ). Here  $\Lambda$  is a diagonal matrix with entries that depend on  $\nu$  but not on  $t$  and the superscript denote transpose.

Inspired on [PT] we construct a family of  $(\ell + 1)^2 \times (\ell + 1)$ -matrix valued functions  $\tilde{\Phi}(g; \nu)$  on  $G$  built up of  $\ell + 1$  spherical functions of a given type  $(n, \ell)$ . The three term recursion relation that constitutes our main result, Theorem 5.1, is given by

$$(6) \quad \phi(g)\psi(g)\tilde{\Phi}(g; \nu) = \tilde{A}_\nu\tilde{\Phi}(g; \nu - \rho) + \tilde{B}_\nu\tilde{\Phi}(g; \nu) + \tilde{C}_\nu\tilde{\Phi}(g; \nu + \rho)$$

and it generalizes (4). Here  $\tilde{A}_\nu = A_\nu \otimes I$ ,  $\tilde{B}_\nu = B_\nu \otimes I$ ,  $\tilde{C}_\nu = C_\nu \otimes I$ , where  $I$  denotes the  $(\ell + 1) \times (\ell + 1)$  identity matrix.

It is important to stress that this relation is valid on  $G$ , and not just on a one dimensional submanifold of  $G$ . On the other hand from Proposition 1.2 iii) it follows that

$$[D\tilde{\Phi}(\cdot; \nu)](g) = \tilde{\Phi}(g; \nu)[D\tilde{\Phi}(\cdot; \nu)](e)$$

for all  $D \in D(G)^K$ . Thus the family  $\tilde{\Phi}(g; \nu)$  provides an extension to  $G$  of the matrix valued version of the Bochner's problem. Moreover the recursion relation (4) follows easily from (6) by restriction, see Proposition 6.2.

The first step in the proof of (6) consists in establishing a multiplication formula of three terms (Theorem 4.4) for spherical functions, which is obtained from an explicit decomposition  $Y^{\sigma, \nu} \otimes W = Y_1 \oplus Y_2 \oplus Y_3$  into irreducible submodules of the tensor product of the Harish-Chandra module  $Y^{\sigma, \nu}$  of the principal series representation  $U^{\sigma, \nu}$  and the standard module  $W = \mathbb{C}^3$ . Here we use Kostant's results on the tensor product of a finite dimensional and an infinite dimensional representation (see [Ko]). Then the closures  $\bar{Y}_i$  of  $Y_i$  in  $U^{\sigma, \nu} \otimes W$  are linearly independent

$SU(2,1)$ -modules but  $\bar{Y}_1 \oplus \bar{Y}_2 \oplus \bar{Y}_3$  is not closed. Because of this the proof of Theorem 4.4 is delicate. Concerning the equivalence of  $\bar{Y}_i$  with certain principal series representations see Remark 3.19.

In this multiplication formula there appear spherical functions of types  $(n, \ell)$  and  $(n-1, \ell)$ . To take care of this it is necessary to combine the multiplication formula with its dual, obtaining a multiplication formula involving seven spherical functions of type  $(n, \ell)$ . Then these spherical functions are appropriately package into three  $(\ell+1)^2 \times (\ell+1)$  matrix valued function  $\tilde{\Phi}(g; \nu)$ ,  $\tilde{\Phi}(g; \nu + \rho)$ ,  $\tilde{\Phi}(g; \nu - \rho)$ , which yields to (6).

In the last section, by restricting to the abelian Iwasawa subgroup  $A$ , we derive a three term recursion relation for a family of  $(\ell+1) \times (\ell+1)$  matrix valued functions  $\tilde{H}(t; \nu)$  closely related to the  $\tilde{\Phi}(g; \nu)$ . Then by changing  $\nu$  to a new parameter  $w$  we explain why this three term recursion relation implies the one proved in the last section of [PT] (see Remark 6.3). Finally for  $\ell = 0$  this recursion relation comes down to the one which holds for the Jacobi functions

$$F_u^{\alpha, \beta} = {}_2F_1 \left( \begin{matrix} -u, u+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; x \right).$$

## 2. PRELIMINARIES

The group  $G = SU(2,1)$  consists of all  $3 \times 3$  complex matrices of determinant one that preserve the Hermitian form  $q(z) = z_1 \bar{z}_1 + z_2 \bar{z}_2 - z_3 \bar{z}_3$ . If we put  $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$  we have

$$G = SU(2,1) = \{ A \in SL(3, \mathbb{C}) : A^* J A = J \}.$$

Equivalently we have that  $A \in G$  if and only if the columns  $A_j$  of  $A$  are orthogonal with respect to

$$\langle z, w \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2 - z_3 \bar{w}_3,$$

and satisfy  $q(A_1) = q(A_2) = -q(A_3) = 1$ .

The subgroup  $K = S(U(2) \times U(1))$  consists of all unitary matrices of the form

$$k = \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}$$

where  $A \in U(2)$  and  $a = (\det A)^{-1}$ . Clearly the map  $k \mapsto A$  defines a Lie isomorphism of  $K$  onto  $U(2)$ .

The Lie algebra of  $G$  is

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}(3, \mathbb{C}) : J X J = -\bar{X}^t, \operatorname{tr} X = 0 \right\}.$$

Its complexification is  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$ . The Lie algebra  $\mathfrak{k}$  of  $K$  can be identified with  $\mathfrak{u}(2)$  and its complexification  $\mathfrak{k}_{\mathbb{C}}$  with  $\mathfrak{gl}(2, \mathbb{C})$ .

The following matrices form a basis of  $\mathfrak{g}$ .

$$\begin{aligned} H_1 &= \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}, & H_2 &= \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{bmatrix}, & Y_1 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & Y_2 &= \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ Y_3 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, & Y_4 &= \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, & Y_5 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, & Y_6 &= \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}. \end{aligned}$$

Let  $\mathfrak{h}_{\mathbb{C}}$  be the Cartan subalgebra of  $\mathfrak{g}_{\mathbb{C}}$  of all diagonal matrices. The corresponding root space decomposition is given by

$$\begin{aligned} X_{\alpha} &= E_{12}, & X_{-\alpha} &= E_{21}, & H_{\alpha} &= E_{11} - E_{22}, \\ X_{\beta} &= E_{23}, & X_{-\beta} &= E_{32}, & H_{\beta} &= E_{22} - E_{33}, \\ X_{\gamma} &= E_{13}, & X_{-\gamma} &= E_{31}, & H_{\gamma} &= E_{11} - E_{33}. \end{aligned}$$

where

$$\begin{aligned} \alpha(x_1 E_{11} + x_2 E_{22} + x_3 E_{33}) &= x_1 - x_2, \\ \beta(x_1 E_{11} + x_2 E_{22} + x_3 E_{33}) &= x_2 - x_3, \\ \gamma(x_1 E_{11} + x_2 E_{22} + x_3 E_{33}) &= x_1 - x_3. \end{aligned}$$

We have

$$\begin{aligned} X_{\alpha} &= \frac{1}{2}(Y_1 - iY_2), & X_{\beta} &= \frac{1}{2}(Y_5 - iY_6), & X_{\gamma} &= \frac{1}{2}(Y_3 - iY_4), \\ X_{-\alpha} &= -\frac{1}{2}(Y_1 + iY_2), & X_{-\beta} &= \frac{1}{2}(Y_5 + iY_6), & X_{-\gamma} &= \frac{1}{2}(Y_3 + iY_4). \end{aligned}$$

Let  $Z = H_{\alpha} + 2H_{\beta}$ ,  $\tilde{H}_1 = 2H_{\alpha} + H_{\beta}$  and  $\tilde{H}_2 = H_{\beta} - H_{\alpha}$ .

In [GPT1] we have proved the following proposition:

**Proposition 2.1.**  $D(G)^G$  as a polynomial algebra is generated by

$$\Delta_2 = -H_{\alpha}^2 - \frac{1}{3}Z^2 - 2H_{\alpha} - 2Z - 4X_{-\alpha}X_{\alpha} - 4X_{-\beta}X_{\beta} - 4X_{-\gamma}X_{\gamma}$$

and

$$\begin{aligned} \Delta_3 &= \frac{2}{9}H_{\alpha}^3 - \frac{2}{9}H_{\beta}^3 + \frac{1}{3}H_{\alpha}^2H_{\beta} - \frac{1}{3}H_{\alpha}H_{\beta}^2 + 2H_{\alpha}^2 + H_{\alpha}H_{\beta} + 4H_{\alpha} + 2H_{\beta} \\ &+ X_{-\alpha}X_{\alpha}H_{\alpha} + 2X_{-\alpha}X_{\alpha}H_{\beta} + 6X_{-\alpha}X_{\alpha} + 3X_{-\beta}X_{\beta} + 3X_{-\gamma}X_{\gamma} \\ &- X_{-\beta}X_{\beta}\tilde{H}_1 - X_{-\gamma}X_{\gamma}\tilde{H}_2 + 3X_{-\beta}X_{\gamma}X_{-\alpha} + 3X_{-\gamma}X_{\beta}X_{\alpha}. \end{aligned}$$

Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be the Cartan decomposition of  $\mathfrak{g}$  associated to the Cartan involution  $\theta(X) = -\overline{X}^t$ . Then

$$\mathfrak{k} = \left\{ \begin{pmatrix} k & 0 \\ 0 & y \end{pmatrix} : k \in \mathfrak{u}(2), y = -\operatorname{tr}(k) \right\} \text{ and } \mathfrak{p} = \left\{ \begin{pmatrix} 0 & b \\ \overline{b} & 0 \end{pmatrix} : b \in \mathbb{C}^2 \right\}.$$

Let

$$H_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus  $\mathfrak{a} = \mathbb{R}H_0$  is a maximal abelian subspace of  $\mathfrak{p}$ ,  $\mathfrak{m} = \mathbb{R}iT_0$  is the centralizer of  $\mathfrak{a}$  in  $\mathfrak{k}$  and  $\tilde{\mathfrak{h}} = \mathfrak{m} \oplus \mathfrak{a}$  is a Cartan subalgebra of  $\mathfrak{g}$ . The root space decomposition of  $\mathfrak{g}_{\mathbb{C}}$  with respect to  $\tilde{\mathfrak{h}}_{\mathbb{C}}$  is given by

$$\begin{aligned} \tilde{\alpha}(H_0) &= 1, & \tilde{\beta}(H_0) &= 1, & \tilde{\gamma}(H_0) &= 2, \\ \tilde{\alpha}(T) &= 3, & \tilde{\beta}(T) &= -3, & \tilde{\gamma}(T) &= 0, \end{aligned}$$



and the root vectors are

$$\begin{aligned}
X_{\tilde{\alpha}} &= E_{12} + E_{32}, & X_{-\tilde{\alpha}} &= E_{21} + E_{23}, \\
X_{\tilde{\beta}} &= E_{21} - E_{23}, & X_{-\tilde{\beta}} &= E_{12} - E_{32}, \\
X_{\tilde{\gamma}} &= E_{13} - E_{31} - E_{11} + E_{33}, & X_{-\tilde{\gamma}} &= E_{31} - E_{13} - E_{11} + E_{33}.
\end{aligned}$$

In what follows the following multiplication table will be useful:

$$\begin{aligned}
[X_{\tilde{\alpha}}, T] &= -3X_{\tilde{\alpha}}, & [X_{\tilde{\alpha}}, H_0] &= -X_{\tilde{\alpha}}, & [X_{\tilde{\alpha}}, X_{-\tilde{\alpha}}] &= H_0 + T, \\
[X_{\tilde{\alpha}}, X_{\tilde{\beta}}] &= -X_{\tilde{\gamma}}, & [X_{\tilde{\alpha}}, X_{-\tilde{\beta}}] &= 0, & [X_{\tilde{\alpha}}, X_{\tilde{\gamma}}] &= 0, \\
[X_{\tilde{\alpha}}, X_{-\tilde{\gamma}}] &= 2X_{\tilde{\beta}}, & [X_{\tilde{\beta}}, T] &= 3X_{\tilde{\beta}}, & [X_{\tilde{\beta}}, H_0] &= -X_{\tilde{\beta}}, \\
[X_{\tilde{\beta}}, X_{-\tilde{\beta}}] &= H_0 - T, & [X_{\tilde{\beta}}, X_{\tilde{\gamma}}] &= 0, & [X_{\tilde{\beta}}, X_{-\tilde{\gamma}}] &= -2X_{-\tilde{\alpha}}, \\
[X_{\tilde{\beta}}, X_{-\tilde{\alpha}}] &= 0, & [X_{\tilde{\gamma}}, T] &= 0, & [X_{\tilde{\gamma}}, H_0] &= -2X_{\tilde{\gamma}}, \\
[X_{\tilde{\gamma}}, X_{-\tilde{\alpha}}] &= 2X_{\tilde{\beta}}, & [X_{\tilde{\gamma}}, X_{-\tilde{\beta}}] &= -2X_{\tilde{\alpha}}, & [X_{\tilde{\gamma}}, X_{-\tilde{\gamma}}] &= 4H_0, \\
[X_{-\tilde{\alpha}}, T] &= 3X_{-\tilde{\alpha}}, & [X_{-\tilde{\alpha}}, H_0] &= X_{-\tilde{\alpha}}, & [X_{-\tilde{\alpha}}, X_{-\tilde{\beta}}] &= -X_{-\tilde{\gamma}}, \\
[X_{-\tilde{\alpha}}, X_{-\tilde{\gamma}}] &= 0, & [X_{-\tilde{\beta}}, T] &= -3X_{-\tilde{\beta}}, & [X_{-\tilde{\beta}}, H_0] &= X_{-\tilde{\beta}}, \\
[X_{-\tilde{\beta}}, X_{-\tilde{\gamma}}] &= 0, & [X_{-\tilde{\gamma}}, T] &= 0, & [X_{-\tilde{\gamma}}, H_0] &= 2X_{-\tilde{\gamma}}.
\end{aligned}$$

In this new basis, the differential operators  $\Delta_2$  and  $\Delta_3$  given in Proposition 2.1 are given by

$$\begin{aligned}
\Delta_2 &= -H_0^2 - \frac{1}{3}T^2 + 4H_0 - 2X_{\tilde{\alpha}}X_{-\tilde{\alpha}} - 2X_{\tilde{\beta}}X_{-\tilde{\beta}} - X_{\tilde{\gamma}}X_{-\tilde{\gamma}}, \\
4\Delta_3 &= -\frac{1}{9}T^3 + T^2 + 4T + 3H_0^2 + TH_0^2 - 4H_0T - 12H_0 \\
(7) \quad & - X_{\tilde{\alpha}}TX_{-\tilde{\alpha}} - X_{\tilde{\beta}}TX_{-\tilde{\beta}} + X_{\tilde{\gamma}}TX_{-\tilde{\gamma}} + 3X_{\tilde{\alpha}}H_0X_{-\tilde{\alpha}} \\
& - 3X_{\tilde{\beta}}H_0X_{-\tilde{\beta}} - 3X_{\tilde{\beta}}X_{\tilde{\alpha}}X_{-\tilde{\gamma}} - 3X_{\tilde{\gamma}}X_{-\tilde{\alpha}}X_{-\tilde{\beta}} \\
& + 12X_{\tilde{\beta}}X_{-\tilde{\beta}} + 6X_{\tilde{\gamma}}X_{-\tilde{\gamma}}.
\end{aligned}$$

Let  $\lambda \in \mathfrak{a}^*$  be the restricted root defined by  $\lambda(H_0) = 1$  and let  $\mathfrak{n} = \mathfrak{g}_\lambda + \mathfrak{g}_{2\lambda}$  be the sum of the corresponding restricted root subspaces of  $\mathfrak{g}$ . Then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$  is an Iwasawa decomposition of  $\mathfrak{g}$ .

Let  $A$  and  $N$  be the analytic subgroups of  $G$  with Lie algebras  $\mathfrak{a}$  and  $\mathfrak{n}$ , respectively, and let  $M$  be the centralizer of  $A$  in  $K$ . Then  $MAN$  is a minimal parabolic subgroup of  $G$ , and

$$M = \left\{ m_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \right\}, \quad A = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \right\}.$$

For  $r \in \mathbb{Z}$  and  $v \in \mathbb{C}$  we define  $\sigma \in \hat{M}$  and  $\nu \in \mathfrak{a}_{\mathbb{C}}^*$  by

$$(8) \quad \sigma(m_\theta) = e^{ir\theta} \quad \text{and} \quad \nu(tH_0) = vt.$$

Then  $man \mapsto e^{\nu(\log a)}\sigma(m)$  is a one dimensional representation of  $MAN$ , and it is this representation that we induce to  $G$  to construct its generalized principal series representation. Thus we put  $U^{\sigma, \nu} = \text{Ind}_{MAN}^G$ .

There are different “pictures” or realizations of these representations which have different uses and advantages. We choose the compact picture. In this case a dense subspace of the representation space of  $U^{r,v}$  is

$$\{ F : K \longrightarrow \mathbb{C} \text{ continuous} : F(km) = \sigma(m)^{-1}F(k); k \in K, m \in M \},$$

with norm

$$\|F\|^2 = \int_K |F(k)|^2 dk.$$

If  $g$  decomposes under  $G = KAN$  as  $g = \kappa(g) \exp(H(g))n$ , then the action is

$$U^{r,v}(g)F(k) = e^{-(\nu+\rho)H(g^{-1}k)}F(\kappa(g^{-1}k)).$$

The actual Hilbert space and representation are then obtain by completion. We recall that  $\rho \in \mathfrak{a}^*$  is the half-sum of the positive restricted roots counted with multiplicities. In this case it is given by  $\rho(H_0) = 2$ .

In this case, the action is more complicated than in the induced picture, but the space is independent of  $\nu$ .

Let  $Y^{r,v}$  be the Harish-Chandra module of all  $K$ -finite vectors in  $U^{r,v}$ . Then  $U^{r,v}$  admits an infinitesimal character since any element in  $D(G)^G$  reduces to an scalar operator on  $Y^{r,v}$  (see Proposition 8.22 of [Kn]). Then using the induced picture and the expressions (7) it is easy to prove the following proposition.

**Proposition 2.2.** *The infinitesimal character  $\chi_{r,v}$  of the principal series  $U^{r,v}$  is given by*

$$\begin{aligned} \chi_{r,v}(\Delta_2) &= -v^2 + 4 - \frac{1}{3}r^2 \\ \chi_{r,v}(\Delta_3) &= \frac{1}{4} \left( -\frac{1}{9}r^3 + r^2 + rv^2 + 3v^2 - 12 \right). \end{aligned}$$

We observe that  $U^{r,v}$  and  $U^{r,-v}$  have the same infinitesimal character. This is an instance of the general invariance of the infinitesimal character of the principal series representations under the restricted Weyl group.

Now we want to describe the structure of  $Y^{r,v}$  as a  $K$ -module. Since  $K$  is a compact group, by the Peter-Weyl theorem we know that we have the following unitary direct sum:

$$L^2(K) = \bigoplus_{\pi \in \hat{K}} V_\pi \otimes V_\pi^*,$$

where the identification of  $V_\pi \otimes V_\pi^*$  as a subspace of  $L^2(K)$  is given by  $(v \otimes \lambda) \mapsto (v \otimes \lambda)(k) = (k \cdot \lambda)(v)$ . The restrictions to  $V_\pi \otimes V_\pi^*$  of the left and right regular representations of  $K$  are, respectively,

$$L|_{V_\pi \otimes V_\pi^*} = \pi \otimes 1 \quad \text{and} \quad R|_{V_\pi \otimes V_\pi^*} = 1 \otimes \pi^*.$$

If  $\pi = \pi_{n,\ell}$  there exists a basis  $\{v_j^{(n,\ell)}\}_{j=0}^\ell$  of  $V_\pi$ , unique up to a multiplicative constant, such that

$$(9) \quad \begin{aligned} \dot{\pi}(H_\alpha)v_j^{(n,\ell)} &= (\ell - 2j)v_j^{(n,\ell)}, & \dot{\pi}(Z)v_j^{(n,\ell)} &= (2n + \ell)v_j^{(n,\ell)}, \\ \dot{\pi}(X_\alpha)v_j^{(n,\ell)} &= jv_{j-1}^{(n,\ell)}, & \dot{\pi}(X_{-\alpha})v_j^{(n,\ell)} &= (\ell - j)v_{j+1}^{(n,\ell)}. \end{aligned}$$

**Lemma 2.3.** *Let us consider a  $U(2)$  invariant inner product on  $V_{(n,\ell)}$ . Then the basis  $\{v_j^{(n,\ell)}\}_{j=0}^\ell$  described above is an orthogonal basis such that*

$$\|v_j^{(n,\ell)}\|^2 = \binom{\ell}{j}^{-1} \|v_0^{(n,\ell)}\|^2.$$

*Proof.* Let  $*$  denote the adjoint operator in  $\text{End}(V_{(n,\ell)})$  corresponding to an invariant inner product on  $V_{(n,\ell)}$ . Since  $\dot{\pi}(Y)^* = -\dot{\pi}(Y)$  for all  $Y \in \mathfrak{g}$  we have

$$\begin{aligned} \dot{\pi}(H_\alpha)^* &= i\dot{\pi}(H_1)^* = -i\dot{\pi}(H_1) = \dot{\pi}(H_\alpha), \\ \dot{\pi}(X_{-\alpha})^* &= -\frac{1}{2}(\dot{\pi}(Y_1) + i\dot{\pi}(Y_2))^* = -\frac{1}{2}(-\dot{\pi}(Y_1) + i\dot{\pi}(Y_2)) = \dot{\pi}(X_\alpha). \end{aligned}$$

Since  $\dot{\pi}(H_\alpha)^* = \dot{\pi}(H_\alpha)$  and the  $v_i$ 's are eigenvectors corresponding to different eigenvalues of  $\dot{\pi}(H_\alpha)$  they are orthogonal to each other.

Now the proof will be completed by induction on  $0 \leq i \leq \ell$ . The statement is clearly true for  $i = 0$ . Let us assume that the assertion is true for some  $0 \leq i \leq \ell - 1$ . Then

$$\begin{aligned} (\ell - i)\langle v_{i+1}^{(n,\ell)}, v_{i+1}^{(n,\ell)} \rangle &= \langle \dot{\pi}(X_{-\alpha})v_i^{(n,\ell)}, v_{i+1}^{(n,\ell)} \rangle = \langle v_i^{(n,\ell)}, \dot{\pi}(X_\alpha)v_{i+1}^{(n,\ell)} \rangle \\ &= (i + 1)\langle v_i^{(n,\ell)}, v_i^{(n,\ell)} \rangle. \end{aligned}$$

Thus

$$\langle v_{i+1}^{(n,\ell)}, v_{i+1}^{(n,\ell)} \rangle = \frac{i+1}{\ell-i} \binom{\ell}{i}^{-1} \langle v_0^{(n,\ell)}, v_0^{(n,\ell)} \rangle = \binom{\ell}{i+1}^{-1} \langle v_0^{(n,\ell)}, v_0^{(n,\ell)} \rangle.$$

This completes the proof of the proposition.  $\square$

Let  $\{\lambda_j^{(n,\ell)}\}_{j=0}^\ell$  be the dual basis of  $\{v_j^{(n,\ell)}\}_{j=0}^\ell$ .

**Proposition 2.4.** *The Harish-Chandra module  $Y^{r,v}$  as a  $K$ -module decomposes in the following way*

$$Y^{r,v} = \bigoplus_{\ell=0}^{\infty} \bigoplus_{j=0}^{\ell} \left( V_{(-r+\ell-3j,\ell)} \otimes \mathbb{C}\lambda_j^{(-r+\ell-3j,\ell)} \right).$$

*Proof.* We first note that

$$Y^{r,v} = \left\{ F \in \bigoplus_{\pi \in \hat{K}} V_\pi \otimes V_\pi^* : R(m_\theta)F = e^{-ir\theta}F, m_\theta \in M \right\}.$$

From  $T = (3H_\alpha - Z)/2$  it follows that

$$\dot{\pi}(T)v_j = (\ell - n - 3j)v_j, \quad \dot{\pi}^*(T)\lambda_j = -(\ell - n - 3j)\lambda_j.$$

Let us consider the representation of  $M$  in  $\bigoplus_{\pi \in \hat{K}} V_\pi \otimes V_\pi^*$  defined by  $m_\theta \mapsto e^{ir\theta}R(m_\theta)$ . Then

$$Y^{r,v} = \left( \bigoplus_{\pi \in \hat{K}} V_\pi \otimes V_\pi^* \right)^M = \bigoplus_{\pi \in \hat{K}} V_\pi \otimes (V_\pi^*)^M :$$

If  $\pi = \pi_{n,\ell}$  we have

$$(V_\pi^*)^M = \begin{cases} \mathbb{C}\lambda_j & \text{if } r = \ell - n - 3j, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of the proposition.  $\square$

### 3. ON THE STRUCTURE OF $U^{r,v} \otimes W$

As we shall see in the next section, a multiplication formula for the spherical functions associated to our pair  $(G, K)$  arises from a direct sum decomposition of  $U^{r,v} \otimes W$  into  $G$ -modules which admit infinitesimal characters. It is well known that even if  $U^{r,v}$  is irreducible  $U^{r,v} \otimes W$  need not have such a direct sum decomposition. Nevertheless it always has a finite composition series but the authors were not able to derive a multiplication formula in this general case. Moreover we shall give evidences that such a formula may not exist.

We shall start considering the  $D(G)$ -module  $Y^{r,v} \otimes W$ , taking into account Kostant's contribution [Ko] on the tensor product of a finite and an infinite dimensional representation. To quote what it is needed we introduce the following notation.

Let  $\mathfrak{g}_{\mathbb{C}}$  be a finite dimensional complex semisimple Lie algebra,  $U(\mathfrak{g}_{\mathbb{C}})$  be its universal enveloping algebra and  $Z(\mathfrak{g}_{\mathbb{C}})$  be the center of  $U(\mathfrak{g}_{\mathbb{C}})$ . Let  $V$  be a Harish-Chandra module with infinitesimal character  $\chi_{\xi}$  and let  $V_{\lambda}$  be an irreducible finite dimensional representation with highest weight  $\lambda$ . Let  $\Delta_{\lambda} = \{\mu_1, \dots, \mu_k\}$  be the set of all the distinct weights of  $V_{\lambda}$ . Now consider the sequence of  $k$  characters  $\chi_{\xi+\mu_1}, \dots, \chi_{\xi+\mu_k}$  and put

$$Y_i = \{y \in V \otimes V_{\lambda} : uy = \chi_{\xi+\mu_i}(u)y \text{ for all } u \in Z(\mathfrak{g}_{\mathbb{C}})\}.$$

Then the following is the content of Corollary 5.5 of [Ko].

**Theorem 3.1.** *If the characters  $\chi_{\xi+\mu_i}$ ,  $i = 1, \dots, k$  are distinct and  $Y_i$  is not zero, then  $Y_i$  is the maximal submodule of  $V \otimes V_{\lambda}$  which admits the infinitesimal character  $\chi_{\xi+\mu_i}$ , and*

$$V \otimes V_{\lambda} = Y_1 \oplus \dots \oplus Y_k.$$

Now we consider the standard irreducible representation of  $\mathfrak{g}_{\mathbb{C}}$  on  $W = \mathbb{C}^3$  and let  $\{e_1, e_2, e_3\}$  denote the canonical basis of  $\mathbb{C}^3$ . Then  $e_1 + e_3, e_2, e_1 - e_3$  are weight vectors with respect to  $\tilde{\mathfrak{h}}_{\mathbb{C}}$  of weights  $\mu_1, \mu_2, \mu_3$ , respectively, given by

$$\begin{aligned} \mu_1(H_0) &= 1, & \mu_2(H_0) &= 0, & \mu_3(H_0) &= -1 \\ \mu_1(T) &= 1, & \mu_2(T) &= -2, & \mu_3(T) &= 1. \end{aligned}$$

We observe that  $\lambda = \mu_1$  is the highest weight of  $W$ . In terms of the dual basis of  $\{H_0, T\}$  we have

$$\mu_1 = (1, 1), \quad \mu_2 = (0, -2), \quad \mu_3 = (-1, 1).$$

**Lemma 3.2.** *The infinitesimal character  $\chi_{\xi} = \chi_{r,v}$  of the Harish-Chandra module  $Y^{r,v}$  is given by  $\xi = (-v - 2, r)$ .*

*Proof.* We first recall the definition of the character  $\chi_{\xi}$  of  $Z(\mathfrak{g}_{\mathbb{C}})$  for  $\xi \in \tilde{\mathfrak{h}}_{\mathbb{C}}^*$ . We know that given  $u \in Z(\mathfrak{g}_{\mathbb{C}})$  there exists a unique  $f_u \in U(\tilde{\mathfrak{h}}_{\mathbb{C}})$  such that  $u - f_u \in U(\mathfrak{g}_{\mathbb{C}})\mathfrak{g}_{\mathbb{C}}^+$ . Then  $\chi_{\xi}(u) = f_u(\xi)$ . By using the isomorphism  $D(G)^G \simeq Z(\mathfrak{g}_{\mathbb{C}})$ , obtained by restricting the canonical isomorphism between  $D(G)$  and  $U(\mathfrak{g}_{\mathbb{C}})$ , from Proposition 2.2 we can compute  $\chi_{\xi}(\Delta_2)$  and  $\chi_{\xi}(\Delta_3)$ . But first

we need to rewrite these operators in such a way that the positive root vectors appear on the right. To do this we use the transpose anti-automorphism of  $U(\mathfrak{g})$  defined by: if  $X \in \mathfrak{g}$  then  $X' = -X$ , thus

$$(X_1 \cdots X_r)' = (-1)^r X_r \cdots X_1, \quad \text{for any } X_i \in \mathfrak{g}.$$

Then  $\Delta'_2 = \Delta_2$ , since  $\Delta_2$  is the Casimir operator of  $\mathfrak{g}_{\mathbb{C}}$ , and furthermore we have that  $\Delta'_3 = -\Delta_3 - 6\Delta_2$ . Thus from (7), we get

$$f_{\Delta_2} = -(H_0^2 + \frac{1}{3}T^2 + 4H_0),$$

and

$$f_{\Delta_3} = -\frac{1}{9}T^3 + T^2 + 4T + 3H_0^2 + 4H_0T + TH_0^2 + 12H_0.$$

Now it is easy to check that

$$f_{\Delta_2}(-v-2, r) = \chi_{r,v}(\Delta_2),$$

$$f_{\Delta_3}(-v-2, r) = \chi_{r,v}(\Delta_3).$$

This completes the proof of the proposition. □

**Theorem 3.3.** *Let  $r \in \mathbb{Z}$  and  $v \in \mathbb{C}$ . If  $v(v+r)(v-r) \neq 0$  then*

$$Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3,$$

where  $Y_1, Y_2, Y_3$  are Harish-Chandra modules with infinitesimal characters  $\chi_{r+1, v-1}$ ,  $\chi_{r-2, v}$ ,  $\chi_{r+1, v+1}$ , respectively. Moreover for  $i = 1, 2, 3$

$$Y_i = \{y \in Y^{r,v} \otimes W : \Delta_2 y = c_i y\},$$

where  $c_1 = \chi_{r+1, v-1}(\Delta_2)$ ,  $c_2 = \chi_{r-2, v}(\Delta_2)$ ,  $c_3 = \chi_{r+1, v+1}(\Delta_2)$ .

*Proof.* By Theorem 3.1 we just need to understand when  $\chi_{\xi+\mu_i} = \chi_{\xi+\mu_j}$  for  $i \neq j$  and  $\xi = (-v-2, r)$  (Lemma 3.2).

We know that in general  $\chi_\lambda = \chi_{\lambda'}$  ( $\lambda, \lambda' \in \tilde{\mathfrak{h}}_{\mathbb{C}}^*$ ) if and only there exists an element  $\tilde{w}$  in the translated Weyl group  $\tilde{W}$  such that  $\tilde{w}(\lambda) = \lambda'$  (see Section 2.1 of [Ko]). We recall that for  $w$  in the Weyl group  $W$  we have

$$\tilde{w}(\lambda) = w(\lambda + \rho) - \rho.$$

If  $\lambda = (\lambda_1, \lambda_2)$ , in terms of the dual basis of  $\{H_0, T\}$ , then

$$\tilde{s}_{\tilde{\alpha}}(\lambda) = \frac{1}{2}(\lambda_1 - \lambda_2, -3\lambda_1 - \lambda_2),$$

$$\tilde{s}_{\tilde{\beta}}(\lambda) = \frac{1}{2}(\lambda_1 + \lambda_2 - 2, 3\lambda_1 - \lambda_2 + 6),$$

$$\tilde{s}_{\tilde{\gamma}}(\lambda) = (-\lambda_1 - 4, \lambda_2).$$

If  $\xi = (-v-2, r)$  we have  $\xi + \mu_1 = (-v-1, r+1)$ ,  $\xi + \mu_2 = (-v-2, r-2)$  and  $\xi + \mu_3 = (-v-3, r+1)$ .

Then it is easy to see that

$$\tilde{s}_{\tilde{\alpha}}(\xi + \mu_1) = \xi + \mu_2, \quad \text{if and only if } v = r,$$

$$\tilde{s}_{\tilde{\beta}}(\xi + \mu_2) = \xi + \mu_3, \quad \text{if and only if } v = -r,$$

$$\tilde{s}_{\tilde{\gamma}}(\xi + \mu_1) = \xi + \mu_3, \quad \text{if and only if } v = 0,$$

and that no further identifications under  $\tilde{W}$  occur in  $\{\xi + \mu_j : j = 1, 2, 3\}$ .

From Lemma 3.2 it follows that  $\chi_{\xi+\mu_1} = \chi_{r+1,v-1}$ ,  $\chi_{\xi+\mu_2} = \chi_{r-2,v}$ ,  $\chi_{\xi+\mu_3} = \chi_{r+1,v+1}$ , which completes the proof of the first assertion.

Now let

$$\tilde{Y}_i = \{y \in Y^{r,v} \otimes W : \Delta_2 y = c_i y\}.$$

Since  $\Delta_2 \in D(G)^G$  it follows that  $\tilde{Y}_i$  is a  $D(G)$ -submodule of  $Y^{r,v} \otimes W$  such that  $Y_i \subset \tilde{Y}_i$ . On the other hand it is easy to verify that under the hypothesis  $v(v+r)(v-r) \neq 0$  all  $c_1, c_2, c_3$  are different. Therefore  $\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3$  are direct summands of  $Y^{r,v} \otimes W$ . Thus

$$Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3 \subset \tilde{Y}_1 \oplus \tilde{Y}_2 \oplus \tilde{Y}_3,$$

which implies that  $Y_i = \tilde{Y}_i$  for  $i = 1, 2, 3$ , completing the proof of the theorem.  $\square$

**Theorem 3.4.** *Let  $r \in \mathbb{Z}$  and  $v \in \mathbb{C}$ . If  $v(v+r)(v-r) \neq 0$  then*

$$Y^{r,v} \otimes W^* = Z_1 \oplus Z_2 \oplus Z_3,$$

where  $Z_1, Z_2, Z_3$  are Harish-Chandra modules with infinitesimal characters  $\chi_{r-1,v-1}$ ,  $\chi_{r+2,v}$ ,  $\chi_{r-1,v+1}$ , respectively. Moreover for  $i = 1, 2, 3$

$$Z_i = \{y \in Y^{r,v} \otimes W : \Delta_2 y = d_i y\},$$

where  $d_1 = \chi_{r-1,v-1}(\Delta_2)$ ,  $d_2 = \chi_{r+2,v}(\Delta_2)$ ,  $d_3 = \chi_{r-1,v+1}(\Delta_2)$ .

*Proof.* The theorem follows from Theorem 3.3 by duality. First of all we recall that the dual  $G$ -module of  $U^{r,v}$  is isomorphic to  $U^{-r,-v}$ . In fact

$$\langle f, h \rangle = \int_K f(k)h(k) dk$$

defines a pairing between  $U^{r,v}$  and  $U^{-r,-v}$  which is  $G$ -invariant because

$$\begin{aligned} \langle U^{r,v}(g)f, U^{-r,-v}(g)h \rangle &= \int_K e^{-2\rho H(g^{-1}k)} f(\kappa(g^{-1}k))h(\kappa(g^{-1}k)) dk \\ &= \int_K f(k)h(k) dk = \langle f, h \rangle. \end{aligned}$$

The second equality is a known integral identity (see [W] Section 5.5.1). Moreover this pairing is nonsingular because an  $f$  is in the representation space of  $U^{r,v}$  if and only if  $\bar{f}$  is in the representation space of  $U^{-r,-v}$ .

On the other hand if a Harish-Chandra module  $Y$  has infinitesimal character  $\chi$  then its  $K$ -finite dual  $Y^*$  is a Harish-Chandra module  $Y$  with infinitesimal character  $\chi^*$  given by  $\chi^*(D) = \chi(D')$ ,  $D \in D(G)^G$ . Then using Proposition 2.2 it follows that  $\chi_{r,v}^* = \chi_{-r,v}$ .

Now from Theorem 3.3 we get

$$Y^{-r,-v} \otimes W^* = Y_1^* \oplus Y_2^* \oplus Y_3^*,$$

Then by taking  $Z_1 = Y_3^*$ ,  $Z_2 = Y_2^*$ ,  $Z_3 = Y_1^*$  and changing signs the theorem follows.  $\square$

**3.1. Explicit decomposition of the tensor product.** The aim of this subsection is to obtain an explicit description of the  $G$  modules  $Y_1, Y_2, Y_3$  appearing in Theorem 3.3.

From now on we shall choose a particular basis  $\{a_s^{j,\ell} : 0 \leq s \leq \ell\}$  of weight vectors of  $V_{(-r+\ell-3j,\ell)}$  such that (9) holds.

We realize the  $K$ -module  $V_{(-r+\ell-3j,\ell)}$  as the  $\ell$ -symmetric power of  $\mathbb{C}^2$  (see (1)) and take  $a_s^{j,\ell} = e_1^{\ell-s} e_2^s$ . The weight of  $a_s^{j,\ell}$  with respect to the diagonal Cartan subalgebra  $\mathfrak{h}_{\mathbb{C}}$  of  $\mathfrak{k}_{\mathbb{C}}$  is  $(-r + 2\ell - 3j - s)x_1 + (-r + \ell - 3j + s)x_2$ .

We shall identify  $a_s^{j,\ell} \in V_{(-r+\ell-3j,\ell)}$  with a function on  $K$ : If  $\{\lambda_s^{j,\ell}\}$  denotes the dual basis of  $\{a_s^{j,\ell}\}$  then we put

$$(10) \quad v_s^{j,\ell}(k) = \lambda_j^{j,\ell}(k^{-1}a_s^{j,\ell}).$$

In this way  $V_{(-r+\ell-3j,\ell)} \simeq V_{(-r+\ell-3j,\ell)} \otimes \mathbb{C}\lambda_j^{j,\ell} \subset Y^{r,v}$  and

$$(11) \quad Y^{r,v} = \bigoplus_{0 \leq j \leq \ell} \langle v_0^{j,\ell} \rangle_K,$$

in accordance with Proposition 2.4. Explicitly, if

$$k = \begin{pmatrix} \bar{a} & \bar{b} & 0 \\ -be^{-i\theta} & ae^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}, \quad k^{-1} = \begin{pmatrix} a & -\bar{b}e^{i\theta} & 0 \\ b & \bar{a}e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix},$$

then

$$(12) \quad v_s^{j,\ell}(k) = e^{i\theta(-r+\ell-3j)} \lambda_j^{j,\ell} \left( (ae_1 + be_2)^{\ell-s} (-\bar{b}e^{i\theta}e_1 + \bar{a}e^{i\theta}e_2)^s \right).$$

Then we know that  $\{v_s^{j,\ell}\}_s$  is an orthogonal basis of  $V_{(-r+\ell-3j,\ell)} \subset Y^{r,v}$  such that

$$(13) \quad \begin{aligned} H_{\alpha} v_s^{j,\ell} &= (\ell - 2s)v_s^{j,\ell}, & Z v_s^{j,\ell} &= (-2r + 3\ell - 6j)v_s^{j,\ell}, \\ X_{\alpha} v_s^{j,\ell} &= s v_{s-1}^{j,\ell}, & X_{-\alpha} v_s^{j,\ell} &= (\ell - s)v_{s+1}^{j,\ell}. \end{aligned}$$

The following lemma is a consequence of the so called Pieri's formula, see [Ze] §77.

**Lemma 3.5.** *The following decomposition of  $K$ -modules holds*

$$Y^{r,v} \otimes W \simeq \bigoplus_{\ell=0}^{\infty} \bigoplus_{j=0}^{\ell} V_{(-r+\ell-3j,\ell+1)} \oplus V_{(-r+\ell-3j+1,\ell-1)} \oplus V_{(-r+\ell-3j-1,\ell)}.$$

Moreover for  $0 \leq j \leq \ell$  let

$$\begin{aligned} u_0^{j,\ell} &= v_1^{j,\ell} \otimes e_1 - v_0^{j,\ell} \otimes e_2 \in V_{(-r+\ell+1-3j,\ell-1)} \\ z_0^{j,\ell} &= v_0^{j,\ell} \otimes e_3 \in V_{(-r+\ell-3j-1,\ell)} \\ w_0^{j,\ell} &= v_0^{j,\ell} \otimes e_1 \in V_{(-r+\ell-3j,\ell+1)}. \end{aligned}$$

Then the elements  $u_0^{j,\ell}, z_0^{j-1,\ell-1}, w_0^{j-1,\ell-2}$  are dominant vectors of weight

$$\mu^{j,\ell} = (-r + 2\ell - 3j)x_1 + (-r + \ell - 3j + 1)x_2.$$

Now we notice that the standard  $G$ -module  $W$  realizes in the principal series  $U^{1,-3}$  in the following way: let  $\lambda \neq 0$  be a dominant weight vector in  $W^*$  with respect to the Cartan subalgebra  $\tilde{\mathfrak{h}}_{\mathbb{C}}$  of  $\mathfrak{g}_{\mathbb{C}}$ , and for any  $w$  in  $W$  let  $h_w(k) = \lambda(k^{-1}w)$  for  $k \in K$ . Then

**Lemma 3.6.** *The map  $w \mapsto h_w$  defines an injective homomorphism of  $G$ -modules from  $W$  into  $U^{1,-3}$ .*

*Proof.* It is convenient to use the induce picture to realize  $U^{1,-3}$ . We extend our function  $h_w$  to  $G$  by  $h_w(g) = \lambda(g^{-1}w)$  for  $g \in G$ . Then we only need to prove that

$$h_w(gm_\theta an) = e^{-(\nu+\rho)\log a} \sigma(m_\theta)^{-1} h_w(g),$$

for  $m_\theta \in M, a \in A, n \in N$ , where  $\nu(H_0) = -3, \sigma(m_\theta) = e^{i\rho\theta}$ .

If  $X \in \mathfrak{g}$  then

$$\begin{aligned} [Xh_w](g) &= \left( \frac{d}{dt} \right)_{t=0} \lambda((g \exp tX)^{-1}w) \\ &= \left( \frac{d}{dt} \right)_{t=0} (\exp tX \lambda)(g^{-1}w) = [X\lambda](g^{-1}w). \end{aligned}$$

Since  $X\lambda = 0$  for any  $X \in \mathfrak{n}$  it follows that  $h_w$  is right invariant under  $N$ . In terms of the dual basis  $\{\lambda_i\}_{i=1}^3$  of the canonical basis  $\{e_i\}_{i=1}^3$  of  $W$ ,  $\lambda$  is a nonzero multiple of  $\lambda_1 - \lambda_3$ . Then  $H_0\lambda = \lambda$  and  $T\lambda = -\lambda$ . Thus  $H_0h_w = h_w$  and  $iTh_w = -ih_w$ . Therefore

$$h_w(g \exp tH_0) = e^t h_w(g) \quad \text{and} \quad h_w(g \exp(\theta iT)) = e^{-i\theta} h_w(g),$$

which are the required properties for  $A$  and  $M$ . The lemma is proved.  $\square$

If  $M$  is abelian we have a  $G$ -morphism from  $U^{\sigma,\nu} \otimes U^{\sigma',\nu'}$  into  $U^{\sigma+\sigma',\nu+\nu'+\rho}$  defined by the multiplication of functions. In particular we consider the  $G$ -morphism defined by

$$P : U^{r,v} \otimes W \longrightarrow U^{r+1,v-1}, \quad f \otimes w \longmapsto fh_w.$$

**Proposition 3.7.** *The  $G$ -morphism  $P : U^{r,v} \otimes W \longrightarrow U^{r+1,v-1}$  is surjective and the Harish-Chandra module of  $\ker(P)$  is given by*

$$\bigoplus_{\ell \geq 0} \bigoplus_{j=0}^{\ell} \langle u_0^{j,\ell} + z_0^{j-1,\ell-1}, (\ell-1)w_0^{j-1,\ell-2} + (\ell-j)z_0^{j-1,\ell-1} \rangle,$$

where  $\langle u, v \rangle$  denotes the  $K$ -module generated by  $\{u, v\}$ .

*Proof.* We recall that  $h_w(k) = \lambda(k^{-1}w)$  and we may assume that  $\lambda = \lambda_1 - \lambda_3 \in W^*$ . If

$$k^{-1} = \begin{pmatrix} a & -\bar{b}e^{i\theta} & 0 \\ b & \bar{a}e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}$$

then

$$h_{e_1}(k) = a, \quad h_{e_2}(k) = -\bar{b}e^{i\theta}, \quad h_{e_3}(k) = -e^{-i\theta}.$$

On the other hand, by (12) we have

$$(14) \quad \begin{aligned} v_0^{j,\ell}(k) &= \binom{\ell}{j} (e^{i\theta})^{-r+\ell-3j} a^{\ell-j} b^j \\ v_1^{j,\ell}(k) &= (e^{i\theta})^{-r+\ell-3j+1} a^{\ell-j-1} b^{j-1} \left( |a|^2 \binom{\ell-1}{j-1} - |b|^2 \binom{\ell-1}{j} \right). \end{aligned}$$



Therefore

$$\begin{aligned}
P(u_0^{j,\ell})(k) &= v_1^{j,\ell}(k)h_{e_1}(k) - v_0^{j,\ell}(k)h_{e_2}(k) \\
&= (e^{i\theta})^{-r+\ell-3j+1}a^{\ell-j}b^{j-1} \left( |a|^2 \binom{\ell-1}{j-1} - |b|^2 \binom{\ell-1}{j} + |b|^2 \binom{\ell}{j} \right) \\
&= (e^{i\theta})^{-r+\ell-3j+1}a^{\ell-j}b^{j-1} \left( |a|^2 \binom{\ell-1}{j-1} + |b|^2 \binom{\ell-1}{j-1} \right) \\
&= \binom{\ell-1}{j-1} (e^{i\theta})^{-r+\ell-3j+1}a^{\ell-j}b^{j-1}.
\end{aligned}$$

In particular we note that  $P(u_0^{j,\ell}) = v_0^{j-1,\ell-1} \in U^{r+1,v-1}$ , proving that  $P$  is surjective. In a similar way we compute

$$P(z_0^{j-1,\ell-1})(k) = v_0^{j-1,\ell-1}(k)h_{e_3}(k) = -\binom{\ell-1}{j-1} (e^{i\theta})^{-r+\ell-3j+1}a^{\ell-j}b^{j-1}$$

and

$$P(w_0^{j-1,\ell-2})(k) = v_0^{j-1,\ell-2}(k)h_{e_1}(k) = \binom{\ell-2}{j-1} (e^{i\theta})^{-r+\ell-3j+1}a^{\ell-j}b^{j-1}.$$

To prove the statement about the kernel of  $P$  it is enough to find the  $K$ -dominant vectors in  $\ker(P)$ . Now it is easy to see that  $u_0^{j,\ell} + z_0^{j-1,\ell-1}$  and  $(\ell-1)w_0^{j-1,\ell-2} + (\ell-j)z_0^{j-1,\ell-1}$  are in  $\ker(P)$ . Moreover if  $v \in \ker(P)$  is a  $K$ -dominant vector of weight  $\mu^{j,\ell}$  it follows that it must be a linear combination of  $(\ell-1)w_0^{j-1,\ell-2} + (\ell-j)z_0^{j-1,\ell-1}$  and  $u_0^{j,\ell} + z_0^{j-1,\ell-1}$ . The proposition is proved.  $\square$

**Lemma 3.8.** *The Iwasawa decomposition  $g = \kappa(g)a(g)n(g)$  of an element  $g = (g_{ij}) \in \text{SU}(2,1)$  is given by*

$$a(g) = \begin{pmatrix} \cosh s & 0 & \sinh s \\ 0 & 1 & 0 \\ \sinh s & 0 & \cosh s \end{pmatrix}, \quad \kappa(g) = \begin{pmatrix} \bar{a} & \bar{b} & 0 \\ -be^{-i\theta} & ae^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix},$$

with

$$\begin{aligned}
\bar{a} &= \frac{g_{11} + g_{13}}{|g_{31} + g_{33}|}, & b &= -\frac{(g_{21} + g_{23})(g_{31} + g_{33})}{|g_{31} + g_{33}|^2}, \\
e^s &= |g_{31} + g_{33}|, & e^{i\theta} &= \frac{g_{31} + g_{33}}{|g_{31} + g_{33}|}.
\end{aligned}$$

*Proof.* Evaluating both sides of  $g = \kappa(g)a(g)n(g)$  at  $e_1 + e_3$  we get

$$\begin{pmatrix} g_{11} + g_{13} \\ g_{21} + g_{23} \\ g_{31} + g_{33} \end{pmatrix} = \kappa(g)a(g) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (\cosh s + \sinh s) \begin{pmatrix} \bar{a} \\ -be^{-i\theta} \\ e^{i\theta} \end{pmatrix},$$

because  $e_1 + e_3$  is left fixed by  $N$ . Then, from the third row it follows that  $e^s = |g_{31} + g_{33}|$  and that  $e^{i\theta} = (g_{31} + g_{33})/|g_{31} + g_{33}|$ . Now from the first and the second rows we obtain

$$\bar{a} = \frac{g_{11} + g_{13}}{|g_{31} + g_{33}|}, \quad b = -\frac{(g_{21} + g_{23})(g_{31} + g_{33})}{|g_{31} + g_{33}|^2}.$$

$\square$

**Proposition 3.9.** *For any  $0 \leq s \leq \ell$  we have*

- i)  $X_\beta v_s^{j,\ell} = \frac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2(\ell+1)} v_{s+1}^{j,\ell+1} + \frac{(\ell-s)(r-v+2j)}{2(\ell+1)} v_s^{j-1,\ell-1}$ .
- ii)  $X_{-\beta} v_s^{j,\ell} = -\frac{(j+1)(r+v+2j+2)}{2(\ell+1)} v_s^{j+1,\ell+1} + \frac{s(r+v-2\ell+2j)}{2(\ell+1)} v_{s-1}^{j,\ell-1}$ .

*Proof.* We start by computing  $X_\beta v_0^{j,\ell}$ . We recall that  $X_\beta = \frac{1}{2}(Y_5 - iY_6)$  and we point out that from (12) we get

$$v_0^{j,\ell}(k) = \binom{\ell}{j} e^{i\theta(-r+\ell-3j)} a^{\ell-j} b^j,$$

where

$$k = \begin{pmatrix} \bar{a} & \bar{b} & 0 \\ -be^{-i\theta} & ae^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}.$$

Now

$$(Y_5 v_0^{j,\ell})(k) = \left( \frac{d}{dt} \right)_{t=0} \left( e^{-(\nu+\rho)H(\exp(-tY_5)k)} v_0^{j,\ell}(\kappa(\exp(-tY_5)k)) \right).$$

We have

$$\exp(-tY_5)k = \begin{pmatrix} \bar{a} & \bar{b} & 0 \\ -be^{-i\theta} \cosh t & ae^{-i\theta} \cosh t & -e^{i\theta} \sinh t \\ be^{-i\theta} \sinh t & -ae^{-i\theta} \sinh t & e^{i\theta} \cosh t \end{pmatrix}.$$

From Lemma 3.8 we obtain

$$H(\exp(-tY_5)k) = \log |\Delta(t)| H_0$$

and

$$\kappa(\exp(-tY_5)k) = \begin{pmatrix} \bar{\alpha} & \bar{\beta} & 0 \\ -\beta e^{-i\varphi} & \alpha e^{-i\varphi} & 0 \\ 0 & 0 & e^{i\varphi} \end{pmatrix},$$

where

$$\alpha = \frac{a}{\Delta(t)}, \quad \beta = \frac{\Delta(t)(be^{-i\theta} \cosh t + e^{i\theta} \sinh t)}{|\Delta(t)|^2},$$

$$e^{i\varphi} = \frac{\Delta(t)}{|\Delta(t)|}, \quad \Delta(t) = be^{-i\theta} \sinh t + e^{i\theta} \cosh t.$$

Then

$$Y_5 v_0^{j,\ell}(k) = \binom{\ell}{j} \left( \frac{d}{dt} \right)_{t=0} \left( \frac{\Delta(t)^{-r+\ell-2j}}{|\Delta(t)|^{-r+2\ell-2j+v+2}} a^{\ell-j} (be^{-i\theta} \cosh t + e^{i\theta} \sinh t)^j \right).$$

We observe that

$$\left( \frac{d}{dt} \right)_{t=0} \left( (be^{-i\theta} \cosh t + e^{i\theta} \sinh t)^j \right) = j b^{j-1} e^{i\theta(-j+2)},$$

$$\left( \frac{d}{dt} \right)_{t=0} \left( \frac{\Delta(t)^{-r+\ell-2j}}{|\Delta(t)|^{-r+2\ell-2j+v+2}} \right) = (-r + \ell - 2j) b e^{i\theta(-r+\ell-2j-2)}$$

$$- (-r + v + 2\ell - 2j + 2) \operatorname{Re}(\bar{b} e^{2i\theta}) e^{i\theta(-r+\ell-2j)}.$$

Therefore

$$Y_5 v_0^{j,\ell}(k) = \binom{\ell}{j} (-r + \ell - 2j) e^{i\theta(-r+\ell-3j-2)} a^{\ell-j} b^{j+1}$$

$$- \binom{\ell}{j} (-r + v + 2\ell - 2j + 2) e^{i\theta(-r+\ell-3j)} a^{\ell-j} b^j \operatorname{Re}(\bar{b} e^{2i\theta})$$

$$+ \binom{\ell}{j} j e^{i\theta(-r+\ell-3j+2)} a^{\ell-j} b^{j-1}.$$

Similarly for  $Y_6$  we obtain

$$Y_6 v_0^{j,\ell}(k) = -i \binom{\ell}{j} (-r + \ell - 2j) e^{i\theta(-r+\ell-3j-2)} a^{\ell-j} b^{j+1}$$

$$+ \binom{\ell}{j} (-r + v + 2\ell - 2j + 2) e^{i\theta(-r+\ell-3j)} a^{\ell-j} b^j \operatorname{Im}(\bar{b} e^{2i\theta})$$

$$+ i \binom{\ell}{j} j e^{i\theta(-r+\ell-3j+2)} a^{\ell-j} b^{j-1}.$$

Therefore

$$\begin{aligned} X_\beta v_0^{j,\ell} &= \frac{1}{2}(Y_5 - iY_6)v_0^{j,\ell} \\ &= \binom{\ell}{j} e^{i\theta(-r+\ell-3j+2)} a^{\ell-j} b^{j-1} \left( -\frac{(-r+v+2\ell-2j+2)}{2} |b|^2 + j \right). \end{aligned}$$

On the other hand, by (14), we have that

$$\begin{aligned} v_0^{j-1,\ell-1}(k) &= \binom{\ell-1}{j-1} e^{i\theta(-r+\ell-3j+2)} a^{\ell-j} b^{j-1}, \\ v_1^{j,\ell+1}(k) &= e^{i\theta(-r+\ell-3j+2)} a^{\ell-j} b^{j-1} \left( \binom{\ell}{j-1} |a|^2 - \binom{\ell}{j} |b|^2 \right) \\ &= e^{i\theta(-r+\ell-3j+2)} a^{\ell-j} b^{j-1} \left( -\binom{\ell+1}{j} |b|^2 + \binom{\ell}{j-1} \right). \end{aligned}$$

Then it is easy to verify that

$$X_\beta v_0^{j,\ell} = \frac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2(\ell+1)} v_1^{j,\ell+1} + \frac{\ell(r-v+2j)}{2(\ell+1)} v_0^{j-1,\ell-1}.$$

Now by using that  $X_{-\alpha}^s$  and  $X_\beta$  commute and the fact that

$$X_{-\alpha}^s v_0^{j,\ell} = \ell(\ell-1)\dots(\ell-s+1)v_s^{j,\ell}, \quad X_{-\alpha}^s v_1^{j,\ell} = (\ell-1)\dots(\ell-s)v_{s+1}^{j,\ell}$$

(see (13)) we prove that

$$X_\beta v_s^{j,\ell} = \frac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2(\ell+1)} v_{s+1}^{j,\ell+1} + \frac{(\ell-s)(r-v+2j)}{2(\ell+1)} v_s^{j-1,\ell-1}.$$

This completes the proof of i). In the same way ii) follows.  $\square$

**Corollary 3.10.** *The following relations hold*

$$\begin{aligned} X_{-\beta} w_0^{j,\ell} &= -\frac{(j+1)(r+v+2j+2)}{2(\ell+1)} u_0^{j+1,\ell+1} - z_0^{j,\ell} + \frac{(r+v+2j-2\ell)}{2(\ell+1)} w_0^{j,\ell-1}, \\ X_{-\beta} z_0^{j-1,\ell-1} &= -\frac{j(r+v+2j)}{2\ell} z_0^{j,\ell}, \\ X_{-\beta} w_0^{j-1,\ell-2} &= -\frac{j(r+v+2j)}{2(\ell-1)} w_0^{j,\ell-1}. \end{aligned}$$

*Proof.* These are straightforward consequences of the definitions given in Lemma 3.5 and the previous proposition.  $\square$

**Corollary 3.11.** *We have*

$$\begin{aligned} X_\beta(X_{-\alpha}^{\ell-1}(u_0^{j,\ell})) &= \frac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2\ell(\ell+1)} X_{-\alpha}^\ell(u_0^{j,\ell+1}) \\ &\quad + \frac{(-r+v-2j)}{2\ell(\ell+1)} X_{-\alpha}^\ell(w_0^{j-1,\ell-1}), \\ X_\beta(X_{-\alpha}^{\ell-1}(z_0^{j-1,\ell-1})) &= \frac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2\ell^2} X_{-\alpha}^\ell(z_0^{j-1,\ell}) + \frac{1}{\ell} X_{-\alpha}^\ell(w_0^{j-1,\ell-1}), \\ X_\beta(X_{-\alpha}^{\ell-1}(w_0^{j-1,\ell-2})) &= \frac{(\ell-j)(-r+v+2\ell-2j)}{2\ell(\ell-1)} X_{-\alpha}^\ell(w_0^{j-1,\ell-1}). \end{aligned}$$

*Proof.* First by induction on  $k$  we establish the following relations for  $k = 1, \dots, \ell-1$

$$\begin{aligned} X_{-\alpha}^k(u_0^{j,\ell}) &= (\ell-1)\dots(\ell-k) \left( v_{k+1}^{j,\ell} \otimes e_1 - v_k^{j,\ell} \otimes e_2 \right), \\ X_{-\alpha}^k(z_0^{j-1,\ell-1}) &= (\ell-1)\dots(\ell-k) \left( v_k^{j-1,\ell-1} \otimes e_3 \right), \\ X_{-\alpha}^k(w_0^{j-1,\ell-2}) &= (\ell-2)\dots(\ell-k) \left( (\ell-k-1)v_k^{j-1,\ell-2} \otimes e_1 \right. \\ &\quad \left. + kv_{k-1}^{j-1,\ell-2} \otimes e_2 \right). \end{aligned}$$

Then the corollary is a direct consequence of Proposition 3.9.  $\square$

We recall that (Theorem 3.3)  $Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3$  where

$$\begin{aligned} Y_1 &= \{f \in Y^{r,v} \otimes W : \Delta_2 f = \chi_{r+1,v-1}(\Delta_2)f\}, \\ Y_2 &= \{f \in Y^{r,v} \otimes W : \Delta_2 f = \chi_{r-2,v}(\Delta_2)f\}, \\ Y_3 &= \{f \in Y^{r,v} \otimes W : \Delta_2 f = \chi_{r+1,v+1}(\Delta_2)f\}. \end{aligned}$$

**Proposition 3.12.** *If  $r - v, -r - v \notin 2\mathbb{N}_0$  and  $r \neq 0$ , then the Harish-Chandra modules  $Y_1, Y_2, Y_3$  are cyclic  $D(G)$ -modules, in fact they are generated by the minimal  $K$ -type dominant vectors given below*

- i)  $Y_1 = \langle (r + v + 2)u_0^{1,1} + (r - v + 2)z_0^{0,0} \rangle$ ,
- ii)  $Y_2 = \langle u_0^{0,1} \rangle$ ,
- iii)  $Y_3 = \langle u_0^{1,1} + z_0^{0,0} \rangle$ .

*Proof.* The elements  $u_0^{0,1}$ ,  $u_0^{1,1}$  and  $z_0^{0,0}$  are  $K$ -dominant vectors of minimal type in  $U^{r,v} \otimes \mathbb{C}^3$ , see Lemma 3.5. In fact  $u_0^{1,1}$  and  $z_0^{0,0}$  are of type  $(-r - 1, 0)$  and the type of  $u_0^{0,1}$  is  $(-r + 2, 0)$ . Moreover, through a careful calculation, using Propositions 2.1, 3.9 and the normalization (13) one can verify that,

$$\begin{aligned} \Delta_2(u_0^{0,1}) &= (-v^2 + 4 - \frac{1}{3}(r - 2)^2)u_0^{0,1}, \\ \Delta_2(u_0^{1,1}) &= (-v^2 + 4 - \frac{1}{3}r^2 + \frac{4}{3}r + \frac{8}{3})u_0^{1,1} + 2(r - v + 2)z_0^{0,0}, \\ \Delta_2(z_0^{0,0}) &= (-v^2 + 4 - \frac{1}{3}r^2 - \frac{8}{3}r - \frac{16}{3})z_0^{0,0} - 2(r + v + 2)u_0^{1,1}. \end{aligned}$$

From the last two identities we obtain the following eigenvectors of  $\Delta_2$ :

$$\begin{aligned} \Delta_2(u_0^{1,1} + z_0^{0,0}) &= (-(v + 1)^2 + 4 - \frac{1}{3}(r + 1)^2)(u_0^{1,1} + z_0^{0,0}), \\ \Delta_2((r + v + 2)u_0^{1,1} + (r - v + 2)z_0^{0,0}) \\ &= (-(v - 1)^2 + 4 - \frac{1}{3}(r + 1)^2)((r + v + 2)u_0^{1,1} + (r - v + 2)z_0^{0,0}). \end{aligned}$$

The first eigenvector  $u_0^{0,1}$  is of weight  $\mu^{0,1}$ , the second and the third,  $u_0^{1,1} + z_0^{0,0}$  and  $(r + v + 2)u_0^{1,1} + (r - v + 2)z_0^{0,0}$ , are of weight  $\mu^{1,1}$ . It is worth to observe that the eigenvalues are respectively:  $\chi_{r-2,v}(\Delta_2)$ ,  $\chi_{r+1,v-1}(\Delta_2)$  and  $\chi_{r+1,v+1}(\Delta_2)$ , see Proposition 2.2.

Therefore

$$\langle (r + v + 2)u_0^{1,1} + (r - v + 2)z_0^{0,0} \rangle \subseteq Y_1, \quad \langle u_0^{0,1} \rangle \subseteq Y_2, \quad \langle u_0^{1,1} + z_0^{0,0} \rangle \subseteq Y_3.$$

The  $U(\mathfrak{g}_{\mathbb{C}})$ -module structure of  $Y^{r,v} \otimes W$  can be visualized in the following diagram of all the highest weights of its  $K$ -submodules.

$$\begin{array}{ccccccc}
& & \mu^{0,1} & \xrightarrow{X_\beta} & \mu^{0,2} & \xrightarrow{X_\beta} & \mu^{0,3} & \xrightarrow{X_\beta} & \dots \\
& & X_{-\beta} \downarrow & & X_{-\beta} \downarrow & & X_{-\beta} \downarrow & & \\
& \mu^{1,1} & \xrightarrow{X_\beta} & \mu^{1,2} & \xrightarrow{X_\beta} & \mu^{1,3} & \xrightarrow{X_\beta} & \mu^{1,4} & \xrightarrow{X_\beta} & \dots \\
& X_{-\beta} \downarrow & & X_{-\beta} \downarrow & & X_{-\beta} \downarrow & & X_{-\beta} \downarrow & & \\
& \mu^{2,2} & \xrightarrow{X_\beta} & \mu^{2,3} & \xrightarrow{X_\beta} & \mu^{2,4} & \xrightarrow{X_\beta} & \mu^{2,5} & \xrightarrow{X_\beta} & \dots \\
& X_{-\beta} \downarrow & & X_{-\beta} \downarrow & & X_{-\beta} \downarrow & & X_{-\beta} \downarrow & & \\
& \vdots & & \vdots & & \vdots & & \vdots & & 
\end{array}$$

Over each  $\mu^{j,\ell}$ , with  $0 \leq j \leq \ell$  we place the irreducible  $K$ -submodules of that highest weight contained, respectively, in  $Y_1$ ,  $Y_2$  and  $Y_3$ . In each place of the first row there is only one  $K$ -module of highest weight  $\mu^{0,\ell}$  contained in  $Y_2$ , and on the first column there are two irreducible  $K$ -modules of highest weight  $\mu^{\ell,\ell}$  one contained in  $Y_1$  and the other in  $Y_3$ .

Let  $f_0^{j,\ell} \in Y^{r,v} \otimes W$  be a  $K$ -dominant vector of weight  $\mu^{j,\ell}$  and let  $f_\ell^{j,\ell}$  be a corresponding lowest weight vector. Then  $X_{-\beta}(f_0^{j,\ell})$  is a dominant vector of weight  $\mu^{j+1,\ell+1}$ , and  $X_\beta(f_\ell^{j,\ell})$  is a lowest weight vector in a  $K$ -module of highest weight  $\mu^{j,\ell+1}$ . This follows from  $[X_{-\beta}, X_\alpha] = 0$  and  $[X_\beta, X_{-\alpha}] = 0$ .

In particular, by induction on  $\ell \geq 1$  and on  $k \geq 1$  it is not difficult to prove the following:

$$\begin{aligned}
X_\beta^{\ell-1}(u_0^{0,1}) &= \frac{1}{(\ell-1)!} \left(\frac{-r+v+4}{2}\right)_{\ell-1} X_{-\alpha}^{\ell-1}(u_0^{0,\ell}), \\
X_{-\beta}^k(u_0^{0,\ell}) &= (-1)^k \frac{k!}{(\ell)_{k+1}} \left(\frac{r+v+2}{2}\right)_{k-1} \left(\frac{1}{2}(r+v+2k)\ell u_0^{k,\ell+k} \right. \\
&\quad \left. + \ell(\ell+k)z_0^{k-1,\ell+k-1} - \frac{1}{2}(r+v-2\ell)(\ell+k-1)w_0^{k-1,\ell+k-2}\right).
\end{aligned}$$

From the first equality and the hypothesis we see that  $u_0^{0,\ell} \in Y_2$  for all  $\ell \geq 1$ . Now using the second one we get that the vector

$$\begin{aligned}
(15) \quad D_2^{j,\ell} &= (r+v+2j)(\ell-j)u_0^{j,\ell} + 2\ell(\ell-j)z_0^{j-1,\ell-1} \\
&\quad - (r+v-2\ell+2j)(\ell-1)w_0^{j-1,\ell-2}
\end{aligned}$$

is a dominant weight vector in  $Y_2$  of highest weight  $\mu^{j,\ell}$ . We observe that  $D_2^{j,\ell} \neq 0$  if and only if  $j \neq \ell$ .

In a similar way for  $\ell \geq 1$  and  $k \geq 0$  one establishes that

$$\begin{aligned}
X_{-\beta}^{\ell-1}(u_0^{1,1} + z_0^{0,0}) &= (-1)^{\ell-1} \left(\frac{r+v+4}{2}\right)_{\ell-1} (u_0^{\ell,\ell} + z_0^{\ell-1,\ell-1}), \\
X_\beta^k(X_{-\alpha}^{\ell-1}(u_0^{\ell,\ell} + z_0^{\ell-1,\ell-1})) &= \frac{k!}{(\ell)_k(\ell)_{k+1}} \left(\frac{-r+v+2}{2}\right)_k X_{-\alpha}^{\ell+k-1} \left(\ell u_0^{\ell,\ell+k} \right. \\
&\quad \left. + (\ell+k)z_0^{\ell-1,\ell+k-1} + (\ell+k-1)w_0^{\ell-1,\ell+k-2}\right).
\end{aligned}$$

From the first equality and the hypothesis we see that  $u_0^{k+1,k+1} + z_0^{k,k} \in Y_3$  for all  $k \geq 0$ . Now using the second one we get that the vector

$$(16) \quad D_3^{j,\ell} = ju_0^{j,\ell} + \ell z_0^{j-1,\ell-1} + (\ell-1)w_0^{j-1,\ell-2}$$

is a dominant weight vector in  $Y_3$  of highest weight  $\mu^{j,\ell}$ , moreover it is nonzero if  $j \neq 0$ .

Similarly for  $\ell \geq 1$  and  $k \geq 1$  one can prove that

$$\begin{aligned} X_{-\beta}^{\ell-1} \left( (r+v+2)u_0^{1,1} + (r-v+2)z_0^{0,0} \right) &= (-1)^{\ell-1} \left( \frac{r+v+2}{2} \right)_{\ell-1} \\ &\quad \times \left( (r+v+2\ell)u_0^{\ell,\ell} + (r-v+2\ell)z_0^{\ell-1,\ell-1} \right), \\ X_{\beta}^k \left( X_{-\alpha}^{\ell-1} \left( (r+v+2\ell)u_0^{\ell,\ell} + (r-v+2\ell)z_0^{\ell-1,\ell-1} \right) \right) \\ &= \frac{1}{2} \frac{k!}{(\ell)_k(\ell)_{k+1}} \left( \frac{-r+v+2}{2} \right)_{k-1} X_{-\alpha}^{\ell+k-1} \left( \ell(r+v+2\ell)(-r+v+2k)u_0^{\ell,\ell+k} \right. \\ &\quad \left. + (\ell+k)(r-v+2\ell)(-r+v+2k)z_0^{\ell-1,\ell+k-1} \right. \\ &\quad \left. - (\ell+k-1)(r-v+2\ell)(r+v-2k)w_0^{\ell-1,\ell+k-2} \right). \end{aligned}$$

From the first equality and the hypothesis we see that

$(r+v+2\ell)u_0^{\ell,\ell} + (r-v+2\ell)z_0^{\ell-1,\ell-1} \in Y_1$  for all  $k \geq 0$ . Now using the second one we get that the vector

$$(17) \quad \begin{aligned} D_1^{j,\ell} &= j(r+v+2j)(-r+v+2\ell-2j)u_0^{j,\ell} \\ &\quad + \ell(r-v+2j)(-r+v+2\ell-2j)z_0^{j-1,\ell-1} \\ &\quad - (\ell-1)(r-v+2j)(r+v-2\ell+2j)w_0^{j-1,\ell-2} \end{aligned}$$

is a dominant weight vector in  $Y_1$  of highest weight  $\mu^{j,\ell}$ . It is easy to prove that  $D_1^{j,\ell} \neq 0$  if  $j \neq 0$  and  $v(r+v)(r-v) \neq 0$ .

Finally, since  $Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3$  and as a  $K$ -module is the direct sum of all the  $K$ -modules generated by

$$\{D_1^{j,\ell}\}_{0 < j \leq \ell} \cup \{D_2^{j,\ell}\}_{0 \leq j < \ell} \cup \{D_3^{j,\ell}\}_{0 < j \leq \ell}$$

the proposition follows.  $\square$

In the following theorem we generalize a bit the previous proposition and we exhibit the  $K$ -module structure of  $Y_1$ ,  $Y_2$  and  $Y_3$ .

**Theorem 3.13.** *Let  $v(v+r)(v-r) \neq 0$  and let  $D_1^{j,\ell}$ ,  $D_2^{j,\ell}$  and  $D_3^{j,\ell}$  be defined respectively by (17), (15) and (16). Then they are  $K$ -dominant vectors of highest weight  $\mu^{j,\ell}$  and*

$$Y_1 = \bigoplus_{0 < j \leq \ell} \langle D_1^{j,\ell} \rangle_K, \quad Y_2 = \bigoplus_{0 \leq j < \ell} \langle D_2^{j,\ell} \rangle_K, \quad Y_3 = \bigoplus_{0 < j \leq \ell} \langle D_3^{j,\ell} \rangle_K.$$

Moreover

$$Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3.$$

We used above  $\langle D_i^{j,\ell} \rangle_K$  to denote the  $K$ -module generated by  $D_i^{j,\ell}$ .

*Proof.* That  $D_i^{j,\ell}$  is  $K$ -dominant and of highest weight  $\mu^{j,\ell}$  follows from (17), (15) and (16). Through a careful and long calculation, using Propositions 2.1, 3.9 and the normalization (13) one can verify that  $D_i^{j,\ell} \in Y_i$  for  $i = 1, 2, 3$ . Finally, since  $Y^{r,v} \otimes W = Y_1 \oplus Y_2 \oplus Y_3$  and as a  $K$ -module is the direct sum of all the  $K$ -modules generated by

$$\{D_1^{j,\ell}\}_{0 \leq j \leq \ell} \cup \{D_2^{j,\ell}\}_{0 < j \leq \ell} \cup \{D_3^{j,\ell}\}_{0 \leq j \leq \ell},$$

the theorem follows.  $\square$

We consider in  $U^{r,v} \otimes W$  the Hilbert structure given by tensoring the inner product of  $L^2(K)$  with the standard inner product of  $\mathbb{C}^3$ . Then the closure  $\overline{Y}_i$  of  $Y_i$  ( $i=1,2,3$ ) in  $U^{r,v} \otimes W$  is a  $G$  module, because  $G$  is connected. But it is worth to observe that  $\overline{Y}_1$ ,  $\overline{Y}_2$  and  $\overline{Y}_3$  are not orthogonal subspaces.

*Remark 3.14.* If  $v(r+v)(r-v) \neq 0$  then  $\overline{Y}_1$ ,  $\overline{Y}_2$ ,  $\overline{Y}_3$  are linearly independent but  $\overline{Y}_1 \oplus \overline{Y}_2 \oplus \overline{Y}_3$  is not closed in  $U^{r,v} \otimes W$ .

In fact, if  $\overline{Y}_1 \cap \overline{Y}_2$  were not zero then  $\overline{Y}_1$  and  $\overline{Y}_2$  would contain a common  $K$ -irreducible submodule which would imply that  $Y_1 \cap Y_2 \neq \{0\}$ , which is a contradiction. Similarly if  $(\overline{Y}_1 \oplus \overline{Y}_2) \cap \overline{Y}_3 \neq \{0\}$  then  $(Y_1 \oplus Y_2) \cap Y_3 \neq \{0\}$  which is also a contradiction. Thus  $\overline{Y}_1$ ,  $\overline{Y}_2$  and  $\overline{Y}_3$  are linearly independent.

For the second assertion in the remark we recall that one may have closed linearly independent subspaces  $M_1$  and  $M_2$  in a Hilbert space such that  $M_1 \oplus M_2$  is not closed. But the following is true (see [Tay], 4.8): If  $X$  is a Banach space and  $M_1$  and  $M_2$  are closed linearly independent subspaces of  $X$  then  $M_1 \oplus M_2$  is closed if and only if there exists  $d > 0$  such that  $\|x_1 - x_2\| \geq d$  whenever  $x_1 \in M_1, x_2 \in M_2$  and  $\|x_1\| = \|x_2\| = 1$ .

In our case if we let

$$y_\ell = \|D_2^{j,\ell}\|^{-1} D_2^{j,\ell} \quad \text{and} \quad z_\ell = \|D_3^{j,\ell}\|^{-1} D_3^{j,\ell},$$

one can verify, by using Lemma 3.22, that

$$\|y_\ell - z_\ell\|^2 = 2 - 2 \left( \frac{8j(\ell-j)}{|r+v|^2 + 8j(\ell-j)} \right)^{1/2}.$$

If we fix  $0 < j < \ell$  then  $\lim_{\ell \rightarrow \infty} \|y_\ell - z_\ell\| = 0$ . Therefore  $\overline{Y}_2 \oplus \overline{Y}_3$  and  $\overline{Y}_2 \oplus \overline{Y_1 \oplus \overline{Y}_3}$  are not closed in  $U^{r,v} \otimes W$ . Therefore

$$U^{r,v} \otimes W \supsetneq \overline{Y}_2 \oplus \overline{Y_1 \oplus \overline{Y}_3} \supset \overline{Y}_1 \oplus \overline{Y}_2 \oplus \overline{Y}_3,$$

which implies that  $\overline{Y}_1 \oplus \overline{Y}_2 \oplus \overline{Y}_3$  is not closed, since  $Y_1 \oplus Y_2 \oplus Y_3$  is dense in  $U^{r,v} \otimes W$ .

The following lemmas are consequences of Proposition 3.9.

**Lemma 3.15.** *We have*

$$\begin{aligned} X_\beta(D_1^{j,\ell}) &= \frac{(\ell-j+1)(-r+v+2\ell-2j)}{2\ell(\ell+1)} X_{-\alpha}(D_1^{j,\ell+1}) + \frac{(r-v+2j)}{2} D_1^{j-1,\ell-1}, \\ X_\beta(D_2^{j,\ell}) &= \frac{(\ell-j)(-r+v+2\ell-2j+2)}{2\ell(\ell+1)} X_{-\alpha}(D_2^{j,\ell+1}) + \frac{(r-v+2j-2)}{2} D_2^{j-1,\ell-1}, \\ X_\beta(D_3^{j,\ell}) &= \frac{(\ell-j+1)(-r+v+2\ell-2j+2)}{2\ell(\ell+1)} X_{-\alpha}(D_3^{j,\ell+1}) + \frac{(r-v+2j+2)}{2} D_3^{j-1,\ell-1}. \end{aligned}$$

**Lemma 3.16.** *We have*

$$\begin{aligned} X_{-\beta}(X_{-\alpha}^{\ell-1}(D_1^{j,\ell})) \\ = -\frac{j(r+v+2j)}{2\ell(\ell+1)} X_{-\alpha}^{\ell-1}(D_1^{j+1,\ell+1}) + \frac{(\ell-1)(r+v-2\ell+2j)}{2} X_{-\alpha}^{\ell-2}(D_1^{j,\ell-1}), \end{aligned}$$

$$\begin{aligned}
& X_{-\beta}(X_{-\alpha}^{\ell-1}(D_2^{j,\ell})) \\
&= -\frac{(j+1)(r+v+2j)}{2\ell(\ell+1)}X_{-\alpha}^{\ell-1}(D_2^{j+1,\ell+1}) + \frac{(\ell-1)(r+v-2\ell+2j)}{2}X_{-\alpha}^{\ell-2}(D_2^{j,\ell-1}), \\
& X_{-\beta}(X_{-\alpha}^{\ell-1}(D_3^{j,\ell})) \\
&= -\frac{j(r+v+2j+2)}{2\ell(\ell+1)}X_{-\alpha}^{\ell-1}(D_3^{j+1,\ell+1}) + \frac{(\ell-1)(r+v-2\ell+2j+2)}{2}X_{-\alpha}^{\ell-2}(D_3^{j,\ell-1}).
\end{aligned}$$

**Proposition 3.17.** *If  $v(v+r)(v-r) \neq 0$  then*

$$Y_1 \simeq Y^{r+1,v-1}, \quad Y_2 \simeq Y^{r-2,v}, \quad Y_3 \simeq Y^{r+1,v+1}$$

as  $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules.

*Proof.* From (11) and Theorem 3.13 we have

$$Y^{r+1,v-1} = \bigoplus_{0 \leq j \leq \ell} \langle v_0^{j,\ell} \rangle_K, \quad \text{and} \quad Y_1 = \bigoplus_{0 < j \leq \ell} \langle D_1^{j,\ell} \rangle_K,$$

where  $v_0^{j,\ell}$  and  $D_1^{j,\ell}$  are  $K$ -dominant vectors of highest weights  $\mu^{j+1,\ell+1}$  and  $\mu^{j,\ell}$ , respectively.

Thus an isomorphism of  $K$ -modules  $\eta : Y^{r+1,v-1} \longrightarrow Y_1$  is characterized by

$$\eta(v_0^{j,\ell}) = c_{j,\ell} D_1^{j+1,\ell+1}, \quad c_{j,\ell} \neq 0, \quad 0 \leq j \leq \ell.$$

Since  $\mathfrak{g}_{\mathbb{C}}$  is generated by  $\mathfrak{k}_{\mathbb{C}}$  and  $X_{\pm\beta}$ , to say that  $\eta$  is a  $\mathfrak{g}_{\mathbb{C}}$ -morphism is equivalent to require that  $\eta$  commutes with  $X_{\pm\beta}$ . This in turn is equivalent to

$$X_{\beta}(\eta(v_0^{j,\ell})) = \eta(X_{\beta}(v_0^{j,\ell})), \quad X_{-\beta}(\eta(v_{\ell}^{j,\ell})) = \eta(X_{-\beta}(v_{\ell}^{j,\ell})),$$

for  $0 \leq j \leq \ell$ , because  $[X_{\alpha}, X_{-\beta}] = [X_{-\alpha}, \beta] = 0$ .

From Lemma 3.15 we obtain

$$\begin{aligned}
X_{\beta}(\eta(v_0^{j,\ell})) &= c_{j,\ell} X_{\beta}(D_1^{j+1,\ell+1}) \\
&= c_{j,\ell} \frac{(\ell-j+1)(-r+v+2\ell-2j)}{2(\ell+1)(\ell+2)} X_{-\alpha}(D_1^{j+1,\ell+2}) + c_{j,\ell} \frac{(r-v+2j+2)}{2} D_1^{j,\ell}.
\end{aligned}$$

On the other hand from Proposition 3.9, changing  $r$  by  $r+1$  and  $v$  by  $v-1$ , and using (13) we get

$$\begin{aligned}
X_{\beta} v_0^{j,\ell} &= \frac{(\ell-j+1)(-r+v+2\ell-2j)}{2(\ell+1)} v_1^{j,\ell+1} + \frac{\ell(r-v+2j+2)}{2(\ell+1)} v_0^{j-1,\ell-1} \\
&= \frac{(\ell-j+1)(-r+v+2\ell-2j)}{2(\ell+1)^2} X_{-\alpha}(v_0^{j,\ell+1}) + \frac{\ell(r-v+2j+2)}{2(\ell+1)} v_0^{j-1,\ell-1}.
\end{aligned}$$

Then

$$\begin{aligned}
& \eta(X_{\beta}(v_0^{j,\ell})) \\
&= c_{j,\ell+1} \frac{(\ell-j+1)(-r+v+2\ell-2j)}{2(\ell+1)^2} X_{-\alpha}(D_1^{j+1,\ell+2}) + c_{j-1,\ell-1} \frac{\ell(r-v+2j+2)}{2(\ell+1)} D_1^{j,\ell}.
\end{aligned}$$

Now  $X_{\beta}(\eta(v_0^{j,\ell})) = \eta(X_{\beta}(v_0^{j,\ell}))$  if and only if

$$(18) \quad (\ell+1)c_{j,\ell} = (\ell+2)c_{j,\ell+1} = \ell c_{j-1,\ell-1},$$

since  $X_{-\alpha}(D_1^{j+1,\ell+2})$  and  $D_1^{j,\ell}$  are linearly independent.



Similarly one gets that  $X_{-\beta}(\eta(v_\ell^{j,\ell})) = \eta(X_{-\beta}(v_\ell^{j,\ell}))$  if and only if

$$(19) \quad (\ell + 1)c_{j,\ell} = (\ell + 2)c_{j+1,\ell+1} = \ell c_{j,\ell-1}.$$

From (18) and (19) we get

$$c_{j,\ell+1} = c_{j+1,\ell+1} \quad \text{and} \quad (\ell + 1)c_{j,\ell} = (\ell + 2)c_{j,\ell+1}.$$

Now it follows that  $c_{j,\ell} = c_{0,0}/(\ell + 1)$ . This proves that

$$(20) \quad \eta_1 : Y^{r+1,v-1} \longrightarrow Y_1, \quad \eta_1(v_0^{j,\ell}) = \frac{1}{\ell+1} D_1^{j+1,\ell+1},$$

is the unique isomorphism of  $(\mathfrak{g}_{\mathbb{C}}, K)$ -module, up to a nonzero constant, from  $Y^{r+1,v-1}$  onto  $Y_1$ .

In a similar way we can prove that, up to a non zero constant,

$$(21) \quad \eta_2 : Y^{r-2,v} \longrightarrow Y_2, \quad \eta_2(v_0^{j,\ell}) = \frac{1}{\ell+1} D_2^{j,\ell+1}$$

and

$$(22) \quad \eta_3 : Y^{r+1,v+1} \longrightarrow Y_3, \quad \eta_3(v_0^{j,\ell}) = \frac{1}{\ell+1} D_3^{j+1,\ell+1}$$

are, respectively the unique isomorphisms of  $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules.  $\square$

*Remark 3.18.* If  $v(r+v)(r-v) \neq 0$  the  $(\mathfrak{g}_{\mathbb{C}}, K)$  isomorphisms  $\eta_1$  and  $\eta_2$  defined in (20) and (21) are not continuous while  $\eta_3$  defined in (22) is bicontinuous.

In fact, by using Lemmas 3.20, 3.22 it follows that

$$\begin{aligned} \frac{\|\eta_1(v_0^{j,\ell})\|^2}{\|v_0^{j,\ell}\|^2} &= \frac{j!(\ell-j)!}{(\ell+1)!} \|D_1^{j+1,\ell+1}\|^2 \\ &= |r-v+2j+2|^2 |r-v-2\ell+2j|^2 \\ &\quad + |r-v+2j+2|^2 |r+v-2\ell+2j|^2 \\ &\quad + 8 \operatorname{Re}(v) (|v|^2 - (r+2j+2)^2 + 2(\ell+1)(r+2j+2)), \end{aligned}$$

and

$$\frac{\|\eta_2(v_0^{j,\ell})\|^2}{\|v_0^{j,\ell}\|^2} = \frac{j!(\ell-j)!}{(\ell+1)!} \|D_2^{j,\ell+1}\|^2 = |r+v|^2 + 8j(\ell-j+1),$$

which implies that the left hand sides go to infinite when  $\ell \rightarrow \infty$ . But  $\eta_3$  is bicontinuous since on one hand we have

$$\begin{aligned} \frac{\|\eta_3(X_{-\alpha}^s v_0^{j,\ell})\|^2}{\|X_{-\alpha}^s v_0^{j,\ell}\|^2} &= \frac{\|X_{-\alpha}^s(\eta_3(v_0^{j,\ell}))\|^2}{\|X_{-\alpha}^s v_0^{j,\ell}\|^2} = \frac{\|\eta_3(v_0^{j,\ell})\|^2}{\|v_0^{j,\ell}\|^2} \\ &= \frac{j!(\ell-j)!}{(\ell+1)!} \|D_3^{j+1,\ell+1}\|^2 = 2, \end{aligned}$$

and on the other hand  $\{X_{-\alpha}^s v_0^{j,\ell}\}$  and  $\{\eta_3(X_{-\alpha}^s v_0^{j,\ell})\}$  are, respectively, orthogonal basis of  $Y^{r+1,v+1}$  and  $Y_3$ . Therefore  $\|\eta_3\| = \sqrt{2}$ .

*Remark 3.19.* The pairs of  $G$ -modules  $U^{r+1,v-1}$  and  $\overline{Y}_1$  and  $U^{r-2,v}$  and  $\overline{Y}_2$  are infinitesimally equivalent, but not equivalent, while  $U^{r+1,v+1}$  and  $\overline{Y}_3$  are equivalent.

This is a consequence of the following facts proved in Proposition 3.17

$$\dim \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(Y^{r+1, v-1}, Y_1) = \dim \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(Y^{r-2, v}, Y_2) = 1.$$

We close this section with the following lemmas that were used in the previous remarks.

**Lemma 3.20.** *If  $v_s^{j, \ell}$ ,  $0 \leq s \leq \ell$ , are the functions defined in (12) then we have*

$$\|v_s^{j, \ell}\|^2 = \frac{1}{\ell + 1} \binom{\ell}{s}^{-1} \binom{\ell}{j}.$$

*Proof.* If  $V$  is an irreducible unitary  $K$ -module, then the following orthogonality relations are well known:

$$\int_K \langle k \cdot v, v' \rangle \overline{\langle k \cdot w, w' \rangle} dk = \frac{1}{\dim V} \langle v', w' \rangle \overline{\langle v, w \rangle}.$$

for all  $v, v', w, w' \in V$ . From (10) we have

$$v_s^{j, \ell}(k) = \lambda_j^{j, \ell}(k^{-1} a_s^{j, \ell}) = \frac{\langle k^{-1} \cdot a_s^{j, \ell}, a_j^{j, \ell} \rangle}{\|a_j^{j, \ell}\|^4}.$$

Then

$$\begin{aligned} \|v_s^{j, \ell}\|^2 &= \frac{1}{\|a_j^{j, \ell}\|^4} \int_K \langle k \cdot a_s^{j, \ell}, a_j^{j, \ell} \rangle \overline{\langle k \cdot a_s^{j, \ell}, a_j^{j, \ell} \rangle} dk \\ &= \frac{\|a_s^{j, \ell}\|^2}{(\ell + 1) \|a_j^{j, \ell}\|^2} = \frac{1}{\ell + 1} \binom{\ell}{s}^{-1} \binom{\ell}{j}, \end{aligned}$$

where we used Lemma 2.3 □

The following lemma is a direct consequence of the definitions given in Lemma 3.5 and the lemma above.

**Lemma 3.21.** *We have*

$$\|u_0^{j, \ell}\|^2 = \frac{1}{\ell} \binom{\ell}{j}, \quad \|z_0^{j-1, \ell-1}\|^2 = \frac{1}{\ell} \binom{\ell-1}{j-1}, \quad \|w_0^{j-1, \ell-2}\|^2 = \frac{1}{\ell-1} \binom{\ell-2}{j-1}.$$

The next lemma follows from the definitions (17), (15) and (16) and Lemma 3.21.

**Lemma 3.22.** *We have*

$$\begin{aligned} \|D_1^{j, \ell}\|^2 &= \ell \binom{\ell-1}{j-1} \left( |r - v + 2j|^2 |r - v - 2\ell + 2j|^2 \right. \\ &\quad \left. + |r - v + 2j|^2 |r + v - 2\ell + 2j|^2 \right. \\ &\quad \left. + 8j \operatorname{Re}(v) (|v|^2 - (r + 2j)^2 + 2\ell(r + 2j)) \right), \\ \|D_2^{j, \ell}\|^2 &= (\ell - j) \binom{\ell}{j} (|r + v|^2 + 8j(\ell - j)), \\ \|D_3^{j, \ell}\|^2 &= 2\ell \binom{\ell-1}{j-1}. \end{aligned}$$

#### 4. MULTIPLICATION FORMULAS

In this section we shall establish two three term multiplication formulas, one dual of the other, for matrix valued irreducible spherical functions, obtained from Theorem 3.13.

**Proposition 4.1.** *Let  $v(r+v)(r-v) \neq 0$ . Then*

$$v_0^{j,\ell} \otimes e_3 = \frac{1}{2(\ell+1)v(r-v)} D_1^{j+1,\ell+1} - \frac{2(j+1)}{(\ell+1)(r+v)(r-v)} D_2^{j+1,\ell+1} + \frac{(r+v+2j+2)(r+v-2\ell+2j)}{2(\ell+1)v(r+v)} D_3^{j+1,\ell+1}.$$

*Proof.* Since  $v_0^{j,\ell} \otimes e_3 = z_0^{j,\ell}$  is a  $K$ -dominant vector of weight  $\mu^{j+1,\ell+1}$  it is a linear combination of  $D_1^{j+1,\ell+1}$ ,  $D_2^{j+1,\ell+1}$  and  $D_3^{j+1,\ell+1}$ . The elements  $u_0^{j+1,\ell+1}$ ,  $z_0^{j,\ell}$  and  $w_0^{j,\ell-1}$  are linear independent, hence it is straightforward to verify, using (15),(16) and (17) that the identity in the statement follows.  $\square$

We recall that (see (9)) given an irreducible  $K$ -module  $V$  of type  $(n, \ell)$  there exists a basis  $\{v_k\}_{k=0}^\ell$  of  $V$ , unique up to a multiplicative constant, such that

$$(23) \quad \begin{aligned} H_\alpha v_k &= (\ell - 2k) v_k, & Z v_k &= (2n + \ell) v_k \\ X_\alpha v_k &= k v_{k-1}, & X_{-\alpha} v_k &= (\ell - k) v_{k+1}. \end{aligned}$$

Therefore the matrix of a linear map  $T : V \rightarrow V$  associated to any basis of  $V$  satisfying (23) is the same.

Let  $V$  be a  $K$ -isotypic component of a representation of  $G$  in a Hilbert space  $U$  and let  $P : U \rightarrow V$  be the orthogonal projection. Let  $\{v_i\}_{i=0}^\ell$  be a basis of  $V$  satisfying (23), and let  $\{\lambda_i\}_{i=0}^\ell$  be its dual basis. Then the matrix coefficients of the spherical function  $\Phi$  associated to  $(U, V)$  in the basis  $\{v_i\}_{i=0}^\ell$  are given by

$$(24) \quad \Phi(g)_{ik} = \lambda_i(P(g v_k)).$$

**Lemma 4.2.** *Let  $v(r+v)(r-v) \neq 0$  and let  $\{v_i^{j,\ell}\}_{i=0}^\ell$  be a basis of the  $K$ -module  $V_{(-r+\ell-3j,\ell)} \subset U^{r,v}$  such that (23) holds. Let*

$$v_i^{j,\ell} \otimes e_3 = w_i^{(1)} + w_i^{(2)} + w_i^{(3)} \in \bar{Y}_1 \oplus \bar{Y}_2 \oplus \bar{Y}_3.$$

*For  $p = 1, 2, 3$ , if  $w_0^{(p)} \neq 0$  then  $\{w_i^{(p)}\}_{i=0}^\ell$  is a basis of the  $K$  irreducible module of  $\bar{Y}_p$  of type  $(n, \ell) = (-r + \ell - 3j - 1, \ell)$ , such that (23) holds.*

*Proof.* We have that each  $w_0^{(p)}$  is a  $K$  dominant vector of weight  $\mu^{j+1,\ell+1}$ . On the other hand, for  $0 \leq i \leq \ell$  we have

$$\begin{aligned} X_{-\alpha}^i(w_0^{(1)}) + X_{-\alpha}^i(w_0^{(2)}) + X_{-\alpha}^i(w_0^{(3)}) &= X_{-\alpha}^i(v_0 \otimes e_3) = \frac{\ell!}{(\ell-i)!} v_i \otimes e_3 \\ &= \frac{\ell!}{(\ell-i)!} \left( w_i^{(1)} + w_i^{(2)} + w_i^{(3)} \right). \end{aligned}$$

Therefore  $X_{-\alpha}^i(w_0^{(p)}) = \frac{\ell!}{(\ell-i)!} w_i^{(p)}$  for  $p = 1, 2, 3$ . This completes the proof of the lemma.  $\square$

*Remark 4.3.* When  $v(r+v)(r-v) \neq 0$ , from Proposition 4.1, and the definitions (15), (16) and (17) it follows that

$$w_0^{(1)} = \frac{1}{2(\ell+1)v(r-v)} D_1^{j+1, \ell+1} \neq 0,$$

$$w_0^{(2)} = \frac{-2(j+1)}{(\ell+1)(r+v)(r-v)} D_2^{j+1, \ell+1} \neq 0, \quad \text{if } j \neq \ell,$$

and if  $(r+v+2j+2)(r+v-2\ell+2j) \neq 0$ , we have

$$w_0^{(3)} = \frac{(r+v+2j+2)(r+v-2\ell+2j)}{2(\ell+1)v(r+v)} D_3^{j+1, \ell+1} \neq 0.$$

**Theorem 4.4.** *Let  $v(r+v)(r-v) \neq 0$  and let  $\Phi_{(-r+\ell-3j, \ell)}^{r, v}$  be the irreducible matrix valued spherical function associated to the  $G$ -module  $U^{r, v}$  and to the  $K$ -submodule  $V_{(-r+\ell-3j, \ell)}$ . Let  $\phi$  be the spherical function of type  $(-1, 0)$  associated to the  $G$  module  $W$ . Then*

$$(25) \quad \begin{aligned} \phi(g) \Phi_{(-r+\ell-3j, \ell)}^{r, v}(g) &= a_1 \Phi_{(-r+\ell-3j-1, \ell)}^{r+1, v-1}(g) + a_2 \Phi_{(-r+\ell-3j-1, \ell)}^{r-2, v}(g) \\ &\quad + a_3 \Phi_{(-r+\ell-3j-1, \ell)}^{r+1, v+1}(g), \end{aligned}$$

where

$$(26) \quad \begin{aligned} a_1 &= a_1(r, v, j, \ell) = \frac{(r-v+2j+2)(-r+v+2\ell-2j)}{2v(r-v)} \\ a_2 &= a_2(r, v, j, \ell) = -\frac{4(j+1)(\ell-j)}{(r+v)(r-v)} \\ a_3 &= a_3(r, v, j, \ell) = \frac{(r+v+2j+2)(r+v-2\ell+2j)}{2v(r+v)} \end{aligned}$$

*Proof.* Let us consider the basis  $\{v_i^{j, \ell}\}_{i=0}^{\ell}$  of  $V_{(-r+\ell-3j, \ell)} \subset U^{r, v}$  introduced in (10) and let  $\{\lambda_i\}_{i=0}^{\ell}$  be its dual basis. Also let  $\{\mu_i\}_{i=1}^3$  be the dual basis of the canonical basis  $\{e_i\}_{i=1}^3$  of  $\mathbb{C}^3$ .

Let  $D_p = \langle D_p^{j+1, \ell+1} \rangle_K$  for  $p = 1, 2, 3$ . From Lemma 4.2 we have

$$v_i^{j, \ell} \otimes e_3 = w_i^{(1)} + w_i^{(2)} + w_i^{(3)} \in D_1 \oplus D_2 \oplus D_3,$$

where  $\{w_i^{(p)}\}_{i=0}^{\ell}$  is a basis of  $D_p$  satisfying (23). As it was pointed out in Remark 4.3,  $w_0^{(p)}$  may be zero. In such a case  $D_p = 0$  and the corresponding coefficient  $a_p = 0$ , and everything is all right.

Let  $\{\chi_i^{(p)}\}_{i=0}^{\ell}$  be the dual basis of  $\{w_i^{(p)}\}_{i=0}^{\ell}$ . We consider  $\chi_i^{(p)} \in (D_1 \oplus D_2 \oplus D_3)^*$  by using the canonical isomorphism

$$(D_1 \oplus D_2 \oplus D_3)^* \simeq D_1^* \oplus D_2^* \oplus D_3^*.$$

We also consider  $\lambda_i \otimes \mu_3 \in (D_1 \oplus D_2 \oplus D_3)^*$  by setting  $\lambda_i \otimes \mu_3$  equal to zero on the orthogonal complement of  $V_{(-r+\ell-3j, \ell)} \otimes \mathbb{C}e_3$  in  $D_1 \oplus D_2 \oplus D_3$ . Now let

$$(27) \quad \lambda_i \otimes \mu_3 = \nu_i^{(1)} + \nu_i^{(2)} + \nu_i^{(3)} \in D_1^* \oplus D_2^* \oplus D_3^*.$$

Since  $\lambda_i \otimes \mu_3$  is of weight  $-(\ell - 2i)$  with respect to  $H_\alpha$  it follows that  $\nu_i^{(p)} = a_i^{(p)} \chi_i^{(p)}$  where

$$(28) \quad a_i^{(p)} = \nu_i^{(p)}(w_i^{(p)}) = (\lambda_i \otimes \mu_3)(w_i^{(p)}) \quad \text{for } p = 1, 2, 3.$$

Now we observe that  $a_i^{(p)}$  does not depend on  $i$ , that is

$$(29) \quad (\lambda_i \otimes \mu_3)(w_i^{(p)}) = (\lambda_0 \otimes \mu_3)(w_0^{(p)}).$$

Because  $\{v_i^{j,\ell}\}_{i=0}^\ell$  satisfies (23) we have  $v_i^{j,\ell} = \frac{(\ell-i)!}{\ell!} X_{-\alpha}^i(v_0^{j,\ell})$ , which implies that  $X_{-\alpha}^i(\lambda_i) = \frac{(-1)^i \ell!}{(\ell-i)!} \lambda_0$ . Also from Lemma 4.2 we get that  $w_i^{(p)} = \frac{(\ell-i)!}{\ell!} X_{-\alpha}^i(w_0^{(p)})$ . Then

$$\begin{aligned} (\lambda_i \otimes \mu_3)(w_i^{(p)}) &= \frac{(\ell-i)!}{\ell!} (\lambda_i \otimes \mu_3)(X_{-\alpha}^i w_0^{(p)}) \\ &= \frac{(-1)^i (\ell-i)!}{\ell!} (X_{-\alpha}^i(\lambda_i) \otimes \mu_3)(w_0^{(p)}) \\ &= (\lambda_0 \otimes \mu_3)(w_0^{(p)}). \end{aligned}$$

Thus from (27) we obtain

$$(30) \quad \lambda_i \otimes \mu_3 = a_1 \chi_i^{(1)} + a_2 \chi_i^{(2)} + a_3 \chi_i^{(3)}$$

where  $a_p = (\lambda_0 \otimes \mu_3)(w_0^{(p)})$ .

From Proposition 4.1 we have  $v_0^{j,\ell} \otimes e_3 = w_0^{(1)} + w_0^{(2)} + w_0^{(3)}$ , with

$$\begin{aligned} w_0^{(1)} &= \frac{1}{2(\ell+1)v(r-v)} D_1^{j+1,\ell+1}, & w_0^{(2)} &= -\frac{2(j+1)}{(\ell+1)(r+v)(r-v)} D_2^{j+1,\ell+1} \\ w_0^{(3)} &= \frac{(r+v+2j+2)(r+v-2\ell+2j)}{2(\ell+1)v(r+v)} D_3^{j+1,\ell+1}. \end{aligned}$$

From (15), (16), (17) and recalling that  $u_0^{j+1,\ell+1} = v_1^{j+1,\ell+1} \otimes e_1 - v_0^{j+1,\ell+1} \otimes e_2$ ,  $z_0^{j,\ell} = v_0^{j,\ell} \otimes e_3$ ,  $w_0^{j,\ell-1} = v_0^{j,\ell-1} \otimes e_1$ , we get

$$\begin{aligned} a_1 &= (\lambda_0 \otimes \mu_3)(w_0^{(1)}) = \frac{1}{2(\ell+1)v(r-v)} (\lambda_0 \otimes \mu_3)(D_1^{j+1,\ell+1}) \\ &= \frac{(r-v+2j+2)(-r+v+2\ell-2j)}{2v(r-v)} \\ a_2 &= (\lambda_0 \otimes \mu_3)(w_0^{(2)}) = -\frac{2(j+1)}{(\ell+1)(r+v)(r-v)} (\lambda_0 \otimes \mu_3)(D_2^{j+1,\ell+1}) \\ &= -\frac{4(j+1)(\ell-j)}{(r+v)(r-v)} \\ a_3 &= (\lambda_0 \otimes \mu_3)(w_0^{(3)}) = \frac{(r+v+2j+2)(r+v-2\ell+2j)}{2(\ell+1)v(r+v)} (\lambda_0 \otimes \mu_3)(D_3^{j+1,\ell+1}) \\ &= \frac{(r+v+2j+2)(r+v-2\ell+2j)}{2v(r+v)}. \end{aligned}$$

Let  $Q$  be the orthogonal projection of  $U^{r,v}$  onto  $V_{(-r+\ell-3j,\ell)}$ , the isotypic component of type  $(-r+\ell-3j,\ell)$ , and let  $Q_3$  be the orthogonal projection of  $W$  onto  $\mathbb{C}e_3$ . Also let  $P$  be the orthogonal projection of  $U^{r,v} \otimes W$  onto  $D_1 \oplus D_2 \oplus D_3$ , the  $K$ -isotypic component of type  $(-r+\ell-3j-1,\ell)$ .

We claim that

$$(31) \quad (\lambda_i \otimes \mu_3) \circ (Q \otimes Q_3) = (\lambda_i \otimes \mu_3) \circ P.$$

Since  $\text{Im}(Q \otimes Q_3) = V_{(-r+\ell-3j,\ell)} \otimes \mathbb{C}e_3 \subset D_1 \oplus D_2 \oplus D_3 = \text{Im}(P)$  it follows that  $\ker(Q \otimes Q_3) \supset \ker P$ . Therefore to prove (31) it is enough to see that

$$(32) \quad ((\lambda_i \otimes \mu_3) \circ (Q \otimes Q_3))|_{D_1 \oplus D_2 \oplus D_3} = (\lambda_i \otimes \mu_3)|_{D_1 \oplus D_2 \oplus D_3}.$$

We notice that  $D_1 \oplus D_2 \oplus D_3 = \langle u_0^{j+1,\ell+1} \rangle_K \oplus \langle z_0^{j,\ell} \rangle_K \oplus \langle w_0^{j,\ell-1} \rangle_K$  (see (15), (16) and (17)).

From Lemma 3.5 it is clear that

$$\begin{aligned} \ker \left( (Q \otimes Q_3) |_{D_1 \oplus D_2 \oplus D_3} \right) &= \langle u_0^{j+1, \ell+1} \rangle_K \oplus \langle u_0^{j, \ell-1} \rangle_K \\ &= \ker \left( (\lambda_i \otimes \mu_3) |_{D_1 \oplus D_2 \oplus D_3} \right). \end{aligned}$$

Moreover, since  $\langle z_0^{j, \ell} \rangle_K = \text{Im}(Q \otimes Q_3)$  the identity (32) follows, hence (31) is proved.

Now we are ready to prove the theorem. For  $g \in G$  we apply the left hand side of (31) to  $g(v_k^{j, \ell} \otimes e_3)$  and using (24) we obtain

$$\begin{aligned} (\lambda_i \otimes \mu_3) \left( (Q \otimes Q_3)(g(v_k^{j, \ell} \otimes e_3)) \right) &= (\lambda_i \otimes \mu_3) \left( Q(g v_k^{j, \ell}) \otimes Q_3(g e_3) \right) \\ &= \lambda_i(Q(g v_k^{j, \ell})) \mu_3(Q_3(g e_3)) \\ &= \phi(g) (\Phi(g))_{ik}, \end{aligned}$$

where  $\Phi$  is the spherical function of type  $(-r + \ell - 3j, \ell)$  associated to the  $G$ -module  $U^{r, v}$ .

Let  $P_i$  be the orthogonal projection of  $\overline{Y}_i$  onto  $D_i$ . Then  $P|_{\overline{Y}_1 \oplus \overline{Y}_2 \oplus \overline{Y}_3} = P_1 \oplus P_2 \oplus P_3$ . In fact  $P_i(D_i^\perp \cap \overline{Y}_i) = 0$  because  $P$  is the projection of  $U^{r, v} \otimes W$  onto the isotypic component  $D_1 \oplus D_2 \oplus D_3$  and  $D_i^\perp \cap \overline{Y}_i$  does not contain such  $K$ -type. Thus  $P|_{\overline{Y}_i} = P_i$ .

Now we apply the right hand side of (31) to  $g(v_k^{j, \ell} \otimes e_3)$  and using Lemma 4.2 and (30) we obtain

$$\begin{aligned} (\lambda_i \otimes \mu_3) P(g(v_k^{j, \ell} \otimes e_3)) &= (\lambda_i \otimes \mu_3) P(g w_k^{(1)} + g w_k^{(2)} + g w_k^{(3)}) \\ &= (\lambda_i \otimes \mu_3) (P_1(g w_k^{(1)}) + P_2(g w_k^{(2)}) + P_3(g w_k^{(3)})) \\ &= a_1 \chi_i^{(1)}(P_1(g w_k^{(1)})) + a_2 \chi_i^{(2)}(P_2(g w_k^{(2)})) + a_3 \chi_i^{(3)}(P_3(g w_k^{(3)})) \\ &= a_1 (\Phi_1(g))_{ik} + a_2 (\Phi_2(g))_{ik} + a_3 (\Phi_3(g))_{ik} \end{aligned}$$

where  $\Phi_p$  is the spherical function of type  $(-r + \ell - 3j - 1, \ell)$  associated to the  $G$ -module  $\overline{Y}_p$ .

Since  $\overline{Y}_1$  and  $U^{r+1, v-1}$ ,  $\overline{Y}_2$  and  $U^{r-2, v}$ ,  $\overline{Y}_3$  and  $U^{r+1, v+1}$  are infinitesimally equivalent (see Proposition 3.17) and taking into account the remark below Proposition 1.4 the theorem follows.  $\square$

If  $\Phi : G \longrightarrow \text{End}(V_\pi)$  is an irreducible spherical function of type  $\delta \in \hat{K}$  then the function  $\Phi^* : G \longrightarrow \text{End}(V_\pi^*)$  defined by  $\Phi^*(g) = \Phi(g^{-1})^t$  is a spherical function of type  $\delta^*$ , where  $\delta^*$  denotes the equivalence class of the contragredient representation  $\pi^*$  (see [GPT1]). If  $\pi$  is of type  $(n, \ell)$  then it is easy to verify that  $\pi^*$  is of type  $(-n - \ell, \ell)$ .

We recall that  $(U^{r,v})^* \simeq U^{-r,-v}$ , see the proof of Theorem 3.4. Therefore if  $\Phi_{(-r+\ell-3j,\ell)}^{r,v}$  denotes the irreducible spherical function associated to the  $G$ -module  $U^{r,v}$  of type  $(-r+\ell-3j,\ell)$ , then

$$(33) \quad \left( \Phi_{(-r+\ell-3j,\ell)}^{r,v} \right)^* = \Phi_{(r+\ell-3(\ell-j),\ell)}^{-r,-v}.$$

**Theorem 4.5.** *Let  $v(r+v)(r-v) \neq 0$  and let  $\Phi_{(-r+\ell-3j,\ell)}^{r,v}$  be the irreducible matrix valued spherical function associated to the  $G$ -module  $U^{r,v}$  and to the  $K$ -submodule  $V_{(-r+\ell-3j,\ell)}$ . Let  $\psi$  be the spherical function of type  $(1,0)$  associated to the  $G$  module  $W^*$ . Then*

$$(34) \quad \begin{aligned} \psi(g)\Phi_{(-r+\ell-3j,\ell)}^{r,v}(g) &= b_1 \Phi_{(-r+\ell-3j+1,\ell)}^{r-1,v+1}(g) + b_2 \Phi_{(-r+\ell-3j+1,\ell)}^{r+2,v}(g) \\ &\quad + b_3 \Phi_{(-r+\ell-3j+1,\ell)}^{r-1,v-1}(g), \end{aligned}$$

where

$$(35) \quad \begin{aligned} b_1 &= b_1(r, v, j, \ell) = \frac{(r-v+2j)(-r+v+2\ell-2j+2)}{2v(r-v)} \\ b_2 &= b_2(r, v, j, \ell) = -\frac{4j(\ell-j+1)}{(r+v)(r-v)} \\ b_3 &= b_3(r, v, j, \ell) = \frac{(r+v+2j)(r+v-2\ell+2j-2)}{2v(r+v)} \end{aligned}$$

*Proof.* We start from the following identity established in Theorem 4.4

$$(36) \quad \begin{aligned} \phi(g)\Phi_{(-r+\ell-3j,\ell)}^{r,v}(g) &= a_1 \Phi_{(-r+\ell-3j-1,\ell)}^{r+1,v-1}(g) + a_2 \Phi_{(-r+\ell-3j-1,\ell)}^{r-2,v}(g) \\ &\quad + a_3 \Phi_{(-r+\ell-3j-1,\ell)}^{r+1,v+1}(g), \end{aligned}$$

If we take  $*$  on both sides of (36), and we use (33) we obtain

$$\begin{aligned} \phi(g)^* \Phi_{(r+\ell-3(\ell-j),\ell)}^{-r,-v}(g) &= a_1 \Phi_{(r+\ell-3(\ell-j)+1,\ell)}^{-r-1,-v+1}(g) + a_2 \Phi_{(r+\ell-3(\ell-j)+1,\ell)}^{-r+2,-v}(g) \\ &\quad + a_3 \Phi_{(r+\ell-3(\ell-j)+1,\ell)}^{-r-1,-v-1}(g). \end{aligned}$$

We note that  $\psi(g) = \phi^*(g)$ , because both spherical functions are associated to the same  $G$  module and are of the same  $K$ - type. Now if we change  $r$  by  $-r$ ,  $v$  by  $-v$ , and  $j$  by  $\ell - j$ , then we obtain that

$$\begin{aligned} \psi(g)\Phi_{(-r+\ell-3j,\ell)}^{r,v}(g) &= b_1 \Phi_{(-r+\ell-3j+1,\ell)}^{r-1,v+1}(g) + b_2 \Phi_{(-r+\ell-3j+1,\ell)}^{r+2,v}(g) \\ &\quad + b_3 \Phi_{(-r+\ell-3j+1,\ell)}^{r-1,v-1}(g), \end{aligned}$$

with

$$b_i = b_i(r, v, j, \ell) = a_i(-r, -v, \ell - j, \ell).$$

Now the explicit expressions for  $b_j(r, v, j, \ell)$  follow from Theorem 4.4 and this completes the proof of the theorem.  $\square$

### 5. THREE TERM RECURSION RELATION

Let  $(v+i)(r+v+2i)(r-v+2i) \neq 0$  for  $i = -1, 0, 1$ . If we apply successively Theorems 4.4 and 4.5 and we set  $n = -r + \ell - 3j$  we get

$$\begin{aligned}
 \psi(g)\varphi(g)\Phi_{(n,\ell)}^{r,v}(g) &= \psi(g) \left( a_1\Phi_{(n-1,\ell)}^{r+1,v-1}(g) + a_2\Phi_{(n-1,\ell)}^{r-2,v}(g) + a_3\Phi_{(n-1,\ell)}^{r+1,v+1}(g) \right) \\
 &= a_1 \left( b_1(r+1, v-1, j, \ell) \Phi_{(n,\ell)}^{r,v}(g) + b_2(r+1, v-1, j, \ell) \Phi_{(n,\ell)}^{r+3,v-1}(g) \right. \\
 &\quad \left. + b_3(r+1, v-1, j, \ell) \Phi_{(n,\ell)}^{r,v-2}(g) \right) \\
 (37) \quad &+ a_2 \left( b_1(r-2, v, j+1, \ell) \Phi_{(n,\ell)}^{r-3,v+1}(g) + b_2(r-2, v, j+1, \ell) \Phi_{(n,\ell)}^{r,v}(g) \right. \\
 &\quad \left. + b_3(r-2, v, j+1, \ell) \Phi_{(n,\ell)}^{r-3,v-1}(g) \right) \\
 &+ a_3 \left( b_1(r+1, v+1, j, \ell) \Phi_{(n,\ell)}^{r,v+2}(g) + b_2(r+1, v+1, j, \ell) \Phi_{(n,\ell)}^{r+3,v+1}(g) \right. \\
 &\quad \left. + b_3(r+1, v+1, j, \ell) \Phi_{(n,\ell)}^{r,v}(g) \right).
 \end{aligned}$$

Now we want to *package* these spherical functions in larger matrices to obtain in such a way a three term recursion relation for these matrices.

We fix the type  $(n, \ell)$  and we take  $r = \ell - n$  and  $v \in \mathbb{C}$ . We define the  $(\ell + 1)^2 \times (\ell + 1)$  matrix valued function  $\tilde{\Phi}(g; v)$  of  $\ell + 1$  spherical functions of type  $(n, \ell)$  as follows

$$\tilde{\Phi}(g; v) = \tilde{\Phi}_{(n,\ell)}(g; v) = \left( \Phi_{(n,\ell)}^{r,v}(g), \dots, \Phi_{(n,\ell)}^{r-3j,v+j}(g), \dots, \Phi_{(n,\ell)}^{r-3\ell,v+\ell}(g) \right)^t.$$

If we write (37) replacing  $r$  by  $r - 3j = \ell - n - 3j$  and  $v$  by  $v - j$  with  $0 \leq j \leq \ell$  we obtain

$$\begin{aligned}
 \psi(g)\varphi(g)\Phi_{(n,\ell)}^{r-3j,v-j}(g) &= B_{j,j} \Phi_{(n,\ell)}^{r-3j,v-j}(g) + A_{j,j-1} \Phi_{(n,\ell)}^{r-3j+3,v-j-1}(g) \\
 (38) \quad &+ A_{j,j} \Phi_{(n,\ell)}^{r-3j,v-j-2}(g) + C_{j,j+1} \Phi_{(n,\ell)}^{r-3j-3,v-j+1}(g) \\
 &+ B_{j,j+1} \Phi_{(n,\ell)}^{r-3j-3,v-j-1}(g) + C_{j,j} \Phi_{(n,\ell)}^{r-3j,v-j+2}(g) \\
 &+ B_{j,j-1} \Phi_{(n,\ell)}^{r-3j+3,v-j+1}(g),
 \end{aligned}$$

where

$$\begin{aligned}
 A_{j,j-1} &= a_1(r-3j, v-j, j, \ell)b_2(r-3j+1, v-j-1, j, \ell), \\
 A_{j,j} &= a_1(r-3j, v-j, j, \ell)b_3(r-3j+1, v-j-1, j, \ell), \\
 B_{j,j-1} &= a_3(r-3j, v-j, j, \ell)b_2(r-3j+1, v-j+1, j, \ell), \\
 B_{j,j} &= a_1(r-3j, v-j, j, \ell)b_1(r-3j+1, v-j-1, j, \ell), \\
 &\quad + a_2(r-3j, v-j, j, \ell)b_2(r-3j-2, v-j, j+1, \ell), \\
 &\quad + a_3(r-3j, v-j, j, \ell)b_3(r-3j+1, v-j+1, j, \ell), \\
 B_{j,j+1} &= a_2(r-3j, v-j, j, \ell)b_3(r-3j-2, v-j, j+1, \ell), \\
 C_{j,j} &= a_3(r-3j, v-j, j, \ell)b_1(r-3j+1, v-j+1, j, \ell), \\
 C_{j,j+1} &= a_2(r-3j, v-j, j, \ell)b_1(r-3j-2, v-j, j+1, \ell).
 \end{aligned}$$



Taking into account the expressions of the coefficients  $a_i$  and  $b_i$  given in (26) and (35) we get

$$\begin{aligned}
A_{j,j-1} &= \frac{2j(\ell-j+1)(r-v+2)(r-v-2\ell)}{(v-j)(r+v-4j)(r-v-2j)(r-v-2j+2)}, \\
A_{j,j} &= -\frac{(r+v-2j)(r+v-2\ell-2j-2)(r-v+2)(r-v-2\ell)}{4(v-j)(v-j-1)(r+v-4j)(r-v-2j)}, \\
B_{j,j-1} &= -\frac{2j(\ell-j+1)(r+v-2j+2)(r+v-2\ell-2j)}{(v-j)(r+v-4j)(r+v-4j+2)(r-v-2j)}, \\
B_{j,j} &= \frac{(r-v+2)^2(r-v-2\ell)^2}{4(v-j)(v-j-1)(r-v-2j)(r-v-2j+2)} \\
&\quad + \frac{16(j+1)^2(\ell-j)^2}{(r+v-4j)(r+v-4j-2)(r-v-2j)(r-v-2j-2)} \\
&\quad + \frac{(r+v-2j+2)^2(r+v-2\ell-2j)^2}{4(v-j)(v-j+1)(r+v-4j)(r+v-4j+2)}, \\
B_{j,j+1} &= -\frac{2(j+1)(\ell-j)(r+v-2j)(r+v-2\ell-2j-2)}{(v-j)(r+v-4j)(r+v-4j-2)(r-v-2j)}, \\
C_{j,j} &= -\frac{(r-v)(r+v-2j+2)(r+v-2\ell-2j)(r-v-2\ell-2)}{4(v-j)(v-j+1)(r+v-4j)(r-v-2j)}, \\
C_{j,j+1} &= \frac{2(j+1)(\ell-j)(r-v)(r-v-2\ell-2)}{4(v-j)(r+v-4j)(r-v-2j)(r-v-2j-2)}.
\end{aligned}$$

Now we consider the following  $(\ell+1) \times (\ell+1)$  matrices

$$\begin{aligned}
(39) \quad A_v &= \sum_{k=1}^{\ell} A_{k,k-1} E_{k,k-1} + \sum_{k=0}^{\ell} A_{k,k} E_{k,k}, \\
B_v &= \sum_{k=1}^{\ell} B_{k,k-1} E_{k,k-1} + \sum_{k=0}^{\ell} B_{k,k} E_{k,k} + \sum_{k=0}^{\ell-1} B_{k,k+1} E_{k,k+1}, \\
C_v &= \sum_{k=0}^{\ell} C_{k,k} E_{k,k} + \sum_{k=0}^{\ell-1} C_{k,k+1} E_{k,k+1}.
\end{aligned}$$

We recall that given two square matrices  $M$  and  $P$  the tensor product matrix  $M \otimes P$  is the matrix obtained by blowing up each entry  $M_{ij}$  of  $M$  to the matrix  $M_{ij}P$ . Let

$$\tilde{A}_v = A_v \otimes I, \quad \tilde{B}_v = B_v \otimes I, \quad \tilde{C}_v = C_v \otimes I,$$

where  $I$  denotes the  $(\ell+1) \times (\ell+1)$  identity matrix.

**Theorem 5.1.** *For each  $K$ -type  $(n, \ell)$  if  $(v-k)(\ell-n-v-2k) \neq 0$  for  $-1 \leq k \leq \ell+1$  and  $\ell-n+v-2k \neq 0$  for  $-1 \leq k \leq 2\ell-1$ , then the matrix valued function*

$$\tilde{\Phi}(g; v) = \tilde{\Phi}_{(n, \ell)}(g; v) = \left( \Phi_{(n, \ell)}^{r, v}(g), \dots, \Phi_{(n, \ell)}^{r-3j, v-j}(g), \dots, \Phi_{(n, \ell)}^{r-3\ell, v-\ell}(g) \right)^t,$$

with  $r = \ell - n$ , satisfies the following three term recursion relation

$$(40) \quad \phi(g)\psi(g)\tilde{\Phi}(g; v) = \tilde{A}_v\tilde{\Phi}(g; v-2) + \tilde{B}_v\tilde{\Phi}(g; v) + \tilde{C}_v\tilde{\Phi}(g; v+2),$$

for all  $g \in G$ .

*Remark.* If instead of the parameter  $v \in \mathbb{C}$  we use  $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ , see (8), then (40) can be written as

$$\phi(g)\psi(g)\tilde{\Phi}(g; \nu) = \tilde{A}_{\nu}\tilde{\Phi}(g; \nu - \rho) + \tilde{B}_{\nu}\tilde{\Phi}(g; \nu) + \tilde{C}_{\nu}\tilde{\Phi}(g; \nu + \rho).$$

*Proof.* An  $(\ell + 1)^2 \times (\ell + 1)$  matrix  $V$  will be seen as an  $(\ell + 1)$ -column vector  $V = (V_0, \dots, V_{\ell})^t$  of  $(\ell + 1) \times (\ell + 1)$  matrices  $V_k$ .

If  $M$  is an  $(\ell + 1) \times (\ell + 1)$  matrix and  $\tilde{M} = M \otimes I$ , then

$$(\tilde{M}\tilde{\Phi}(g; v))_k = \sum_{j=0}^{\ell} M_{kj}\Phi_{(n,\ell)}^{r-3j, v-j}(g).$$

Thus

$$\begin{aligned} (\tilde{A}_v\tilde{\Phi}(g; v - 2))_k &= A_{k,k-1}\Phi_{(n,\ell)}^{r-3k+3, v-k-1}(g) + A_{k,k}\Phi_{(n,\ell)}^{r-3k, v-k-2}(g), \\ (\tilde{B}_v\tilde{\Phi}(g; v))_k &= B_{k,k-1}\Phi_{(n,\ell)}^{r-3k+3, v-k+1}(g) + B_{k,k}\Phi_{(n,\ell)}^{r-3k, v-k}(g) \\ &\quad + B_{k,k+1}\Phi_{(n,\ell)}^{r-3k-3, v-k-1}(g), \\ (\tilde{C}_v\tilde{\Phi}(g; v + 2))_k &= C_{k,k}\Phi_{(n,\ell)}^{r-3k, v-k+2}(g) + C_{k,k+1}\Phi_{(n,\ell)}^{r-3k-3, v-k+1}(g). \end{aligned}$$

Therefore, from the identities (38) with  $j = k$  and  $0 \leq k \leq \ell$ , we obtain

$$\phi(g)\psi(g)\tilde{\Phi}(g; v)_k = (\tilde{A}_v\tilde{\Phi}(g; v - 2))_k + (\tilde{B}_v\tilde{\Phi}(g; v))_k + (\tilde{C}_v\tilde{\Phi}(g; v + 2))_k,$$

which proves the theorem.  $\square$

## 6. REDUCTION TO ONE VARIABLE

It is of interest to see how we can write the above theorem when we restrict the spherical functions to the abelian subgroup  $A$  of  $G$  of all matrices of the form

$$a_s = \begin{pmatrix} \cosh s & 0 & \sinh s \\ 0 & 1 & 0 \\ \sinh s & 0 & \cosh s \end{pmatrix},$$

for any  $s \in \mathbb{R}$ . Recall that the centralizer of  $A$  in  $K$  is the subgroup  $M$  of all elements of the form

$$m_{\theta} = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix},$$

for any  $\theta \in \mathbb{R}$ .

If  $\Phi$  is a spherical function on  $G$  of type  $\pi = \pi_{(n,\ell)} \in \hat{K}$  then  $\Phi(a_s)$  commutes with  $\pi(m)$  for all  $m \in M$ . On the other hand we observed that in a basis  $\{v_i^{(n,\ell)}\}_{i=0}^{\ell}$  satisfying (9) we have  $m_{\theta} v_k^{(n,\ell)} = e^{i\theta(\ell-2k-n)} v_k$ ,  $k = 0, \dots, \ell$ . Therefore we have that  $\Phi(a_s)$  diagonalizes in such a basis for each  $s \in \mathbb{R}$ . We denote by  $\Phi_k(a_s)$  the  $k$ -th diagonal entry of the matrix  $\Phi(a_s)$ .

In the open subset  $\{a(s) \in A : s > 0\}$  of  $A$  we introduce the coordinate  $t = \cosh^2(s)$  and define the vector valued function

$$F(t) = (\Phi_0(a_s), \dots, \Phi_{\ell}(a_s))$$

associated to the spherical function  $\Phi$ . If  $\Phi(g) = \Phi_{(n,\ell)}^{r,v}(g)$  then we shall also denote  $F(t) = F_{(n,\ell)}^{r,v}(t)$ . In a similar way, corresponding to the function  $\tilde{\Phi}(g; v)$  we consider the  $(\ell + 1) \times (\ell + 1)$

matrix valued function  $\tilde{F}(t; v)$  whose  $j$ -th row is given by the vector  $F_{(n, \ell)}^{r-3j, v-j}(t)$ , with  $r = \ell - n$ , for  $0 \leq j \leq \ell$ . More explicitly

$$\tilde{F}(t; v) = (F_{jk}(t; v)) \quad \text{with} \quad F_{jk}(t; v) = \left( \Phi_{(n, \ell)}^{-n+\ell-3j, v-j}(a_s) \right)_k.$$

**Proposition 6.1.** *For each  $K$ -type  $(n, \ell)$  if  $(v-k)(\ell-n-v-2k) \neq 0$  for  $-1 \leq k \leq \ell+1$  and  $\ell-n+v-2k \neq 0$  for  $-1 \leq k \leq 2\ell-1$ , then the matrix valued function  $\tilde{F}(t; v)$  satisfies the following three term recursion relation*

$$(41) \quad t\tilde{F}(t; v) = A_v\tilde{F}(t; v-2) + B_v\tilde{F}(t; v) + C_v\tilde{F}(t; v+2),$$

for all  $t \geq 1$ .

*Proof.* We recall that  $\phi(g)$  is the spherical function of type  $(-1, 0)$  associated to the  $G$ -module  $W = \mathbb{C}^3$  and that  $\psi(g)$  is the spherical function of type  $(1, 0)$  associated to  $W^*$ . Then a direct computation gives

$$\phi(a(s)) = \cosh s = t^{1/2} \quad \text{and} \quad \psi(a(s)) = \phi^*(a(s)) = t^{1/2}.$$

Therefore from the identity (38), for  $g = a_s$ , we obtain the following vector identity

$$\begin{aligned} tF_{n, \ell}^{r-3j, v+j}(t) &= A_{j, j-1}F_{n, \ell}^{r-3j+3, v-j-1}(t) + A_{j, j}F_{n, \ell}^{r-3j, v-j-2}(t) \\ &+ B_{j, j-1}F_{n, \ell}^{r-3j+3, v-j+1}(t) + B_{j, j}F_{n, \ell}^{r-3j, v-j}(t) + B_{j, j+1}F_{n, \ell}^{r-3j-3, v-j-1}(t) \\ &+ C_{j, j}F_{n, \ell}^{r-3j, v-j+2}(t) + C_{j, j+1}F_{n, \ell}^{r-3j-3, v-j+1}(t). \end{aligned}$$

This is nothing else than the equality of the  $j$ -th rows of the identity (41).

In the introduction we consider the function  $H(g) = \Phi(g)\Phi_\pi(g)^{-1}$  (see (2)), associated to a spherical function  $\Phi(g)$  of type  $\pi = \pi_{n, \ell} \in \hat{K}$ , and its restriction  $H(t) = H(a(s))$  where  $t = \cosh^2(s)$ . We view the diagonal matrix  $H(t)$  as a column vector. Then it is easy to verify that the functions  $H$  and  $F$  are related by the identity

$$F(t) = t^{n/2}H(t)^t \begin{pmatrix} t^{1/2} & 0 \\ 0 & 1 \end{pmatrix}^\ell,$$

where the exponent  $\ell$  denotes the  $\ell$ -th symmetric power of the matrix. Explicitly  $\begin{pmatrix} t^{1/2} & 0 \\ 0 & 1 \end{pmatrix}^\ell$  is a diagonal matrix whose  $j$ -th entry is  $t^{(\ell-j)/2}$ , with  $0 \leq j \leq \ell$ .

If  $\Phi(g) = \Phi_{(n, \ell)}^{\ell-n-3j, v}(g)$  we denote  $H(t) = H(t; v, j)$ . Corresponding to the function  $\tilde{F}(t; v)$ , we also consider the  $(\ell+1) \times (\ell+1)$  matrix valued function  $\tilde{H}(t; v)$  whose  $j$ -th row is the vector  $H(t; v-j, j)$ , for  $0 \leq j \leq \ell$ . Then

$$\tilde{F}(t; v) = t^{n/2}\tilde{H}(t; v) \begin{pmatrix} t^{1/2} & 0 \\ 0 & 1 \end{pmatrix}^\ell.$$

Then from (41) we obtain

**Proposition 6.2.** *For each  $K$ -type  $(n, \ell)$  if  $(v-k)(\ell-n-v-2k) \neq 0$  for  $-1 \leq k \leq \ell+1$  and  $\ell-n+v-2k \neq 0$  for  $-1 \leq k \leq 2\ell-1$ , then the matrix valued function  $\tilde{H}(t; v)$  satisfies the following three term recursion relation*

$$(42) \quad t\tilde{H}(t; v) = A_v\tilde{H}(t; v-2) + B_v\tilde{H}(t; v) + C_v\tilde{H}(t; v+2),$$

where the matrices  $A_v, B_v$  and  $C_v$  are defined in (39).

*Remark 6.3.* If we fix  $(n, \ell) \in \hat{K}$ , then the map  $(v, j) \mapsto \Phi_{(n, \ell)}^{-n+\ell-3j, v}$  is a surjective map from  $\{(v, j) : v \in \mathbb{C}, 0 \leq j \leq \ell\}$  onto the set of all equivalence classes of matrix valued spherical functions of type  $(n, \ell)$ . In order to related (42) with the three term recursion relation established at the end of Section 5 in [PT], we introduce a new parameter  $w = w(v, j)$  by

$$w = \frac{-(v+n+\ell+2+j)}{2},$$

and we define the function  $h(t; w, j) = H(t; v, j)$ . The matrix function  $\tilde{H}(t; v)$  whose  $j$ -th row is  $H(t; v-j, j)$  corresponds to the function  $\tilde{h}(t; w)$  whose  $j$ -th row is  $h(t; w(v-j, j), j)$ . Observe that  $w(v-j, j) = w(v, 0)$ .

Given  $v \in \mathbb{C}$  we set  $w = \frac{-(v+n+\ell+2)}{2}$ , then the function  $\tilde{H}(t; v)$  associated to the matrix function

$$\tilde{\Phi}(g; v) = \left( \Phi_{(n, \ell)}^{r, v}(g), \dots, \Phi_{(n, \ell)}^{r-3j, v-j}(g), \dots, \Phi_{(n, \ell)}^{r-3\ell, v-\ell}(g) \right)^t,$$

corresponds, for the parameter  $w$ , to the matrix function

$$\tilde{h}(t; w) = (h(t; w, 0), \dots, h(t; w, j), \dots, h(t; w, \ell))^t.$$

Then we have that the function  $\tilde{h}(t; w)$  satisfies the following three term recursion relation

$$t \tilde{h}(t; w) = A'_w \tilde{h}(t; w-1) + B'_w \tilde{h}(t; w) + C'_w \tilde{h}(t; w+1),$$

with

$$A'_w = C_v, \quad B'_w = B_v \quad \text{and} \quad C'_w = A_v,$$

because the functions  $\tilde{H}(t; v-2)$  and  $\tilde{H}(t; v+2)$  correspond with  $\tilde{h}(t, w+1)$  and  $\tilde{h}(t, w-1)$ , respectively.

With this choice of the parameter  $w$  we get exactly the same three term recursion relation that the one obtained in [PT], when  $w$  is an integer such that  $w \geq 0$  and  $w+n \geq 0$ . This can be explain in the following way: the functions  $h(t; w)$  associated to the spherical functions of  $(\text{SU}(3), \text{U}(2))$ , are polynomial eigenfunctions of  $D$  and  $E$  (see (3)) and  $h(1; w) = (1, \dots, 1)^t$ . Therefore  $H(t; v) = h(t; w)$  is the function associated to a spherical function of  $(\text{SU}(2, 1), \text{U}(2))$ .

**6.1. The case  $\ell = 0$ .** In this subsection we shall display the results obtained in Proposition 6.2 when  $\ell = 0$  and  $n$  arbitrary. In these cases the spherical functions are complex valued, and for  $n = 0$  they are the zonal or classical spherical functions.

As we mentioned in the introduction of this paper, the irreducible spherical function  $\Phi_{(n, 0)}^{-n, v}$  of type  $(n, 0)$  corresponds precisely to the complex valued functions  $h(t) = h(t; v)$  which are: eigenfunctions of  $D$ , analytic in the interval  $[1, \infty)$  and  $h(1) = 1$ . (In this case the differential operator  $E$  is a multiple of  $D$ ). Then  $h(t)$  satisfies

$$t(1-t)h''(t) + (n+1-(n+3))h'(t) - \lambda h(t) = 0,$$

with  $\lambda = \frac{1}{4}(n+2+v)(n+2-v)$ . This is a hypergeometric equation with

$$a = \frac{n+2+v}{2}, \quad b = \frac{n+2-v}{2}, \quad c = n+1.$$

Therefore the analytic solution on  $[1, \infty)$  with  $h(1) = 1$  is

$$(43) \quad h(t) = t^{-(n+2+v)/2} {}_2F_1 \left( \frac{1}{2}(n+2+v), \frac{1}{2}(-n+2+v); 1 - \frac{1}{t} \right).$$

The matrices  $A_v$ ,  $B_v$  and  $C_v$  in this case are the following numbers

$$a_v = \frac{(v+n-2)(v-n-2)}{4v(v-1)}, \quad b_v = \frac{n^2+v^2-2}{2(v-1)(v+1)}, \quad c_v = \frac{(v+n+2)(v-n+2)}{4v(v+1)}.$$

In the parameter  $w = \frac{-(v+n+2)}{2}$  introduced before, we have

$$h(t) = h(t; w) = t^w {}_2F_1 \left( -w, \frac{-w-n}{2}; 1 - \frac{1}{t} \right) \quad t > 1,$$

and

$$\begin{aligned} a'_w &= \frac{w(w+n)}{(2w+n+2)(2w+n+1)}, & c'_w &= \frac{(w+2)(w+n+2)}{(2w+n+2)(2w+n+3)}, \\ b'_w &= \frac{(w+1)(w+2)}{(2w+n+2)(2w+n+3)} + \frac{(w+n)(w+n+1)}{(2w+n+2)(2w+n+1)} \\ &= \frac{2w(w+n+2)+(n+1)^2}{(2w+n+1)(2w+n+3)}. \end{aligned}$$

By using the Pfaff's identity we have that

$$h(t) = t^w {}_2F_1 \left( -w, \frac{-w-n}{2}; 1 - \frac{1}{t} \right) = {}_2F_1 \left( -w, \frac{w+n+2}{2}; 1 - t \right).$$

We observe that  $h(t)$  is a polynomial precisely when  $w$  is an integer and  $w \geq 0$  or  $w \leq -n-2$ .

Then Proposition 6.2 for  $\ell = 0$  gives

$$\begin{aligned} t {}_2F_1 \left( -w, \frac{w+n+2}{2}; 1 - t \right) &= a'_w {}_2F_1 \left( -w+1, \frac{w+n+1}{2}; 1 - t \right) \\ &+ b'_w {}_2F_1 \left( -w, \frac{w+n+2}{2}; 1 - t \right) + c'_w {}_2F_1 \left( -w-1, \frac{w+n+3}{2}; 1 - t \right). \end{aligned}$$

By making the change of variables  $x = 1 - t$  we obtain the three term recursion relation, in the spectral parameter  $w$ , for the Jacobi functions

$$\begin{aligned} x {}_2F_1 \left( -w, \frac{w+n+2}{2}; x \right) &= \tilde{a}_w {}_2F_1 \left( -w+1, \frac{w+n+1}{2}; x \right) \\ &+ \tilde{b}_w {}_2F_1 \left( -w, \frac{w+n+2}{2}; x \right) + \tilde{c}_w {}_2F_1 \left( -w-1, \frac{w+n+3}{2}; x \right), \end{aligned}$$

where  $\tilde{a}_w = -a'_w$ ,  $\tilde{b}_w = 1 - b'_w$ ,  $\tilde{c}_w = -c'_w$ .

**6.2. The case  $\ell = 1$ .** We have two disjoint families of spherical functions of type  $(n, 1)$ :  $\{\Phi_{(n,1)}^{1-n,v}\}_{v \in \mathbb{C}}$  and  $\{\Phi_{(n,1)}^{-2-n,v-1}\}_{v \in \mathbb{C}}$ . Correspondingly we let  $H = H(t; v, j)$  to be the functions associated to the first family when  $j = 0$  and to the second one when  $j = 1$ . We also set

$$H(t; v, j) = (H_0(t; v, j), H_1(t; v, j))^t$$

and let  $\lambda_j$  and  $\mu_j$  denote the corresponding eigenvalues of  $D$  and  $E$ , see (3). Then in [PRT] we prove that for  $j = 0, 1$

$$\begin{aligned} \lambda_j &= \frac{1}{4}(n+3-j+v)(n+3-j-v) + j(2-j) \\ \mu_j &= \lambda_j(n+3j-1) - 3j(2-j)(n+j+1), \end{aligned}$$

and that

$$\begin{aligned}
H_0(t; v, 0) &= t^{-(n+3+v)/2} {}_3F_2 \left( \begin{matrix} \frac{(n+3+v)}{2}, \frac{(-n+1+v)}{2}, \frac{2(n-v)}{n+1-v} \\ 3, \frac{n-1-v}{n+1-v} \end{matrix}; 1 - \frac{1}{t} \right), \\
H_1(t; v, 0) &= t^{-(n+3+v)/2} {}_2F_1 \left( \begin{matrix} \frac{(n+3+v)}{2}, \frac{(-n+3+v)}{2} \\ 3 \end{matrix}; 1 - \frac{1}{t} \right), \\
H_0(t; v, 1) &= t^{-(n+4+v)/2} {}_2F_1 \left( \begin{matrix} \frac{(n+4+v)}{2}, \frac{(-n+2+v)}{2} \\ 3 \end{matrix}; 1 - \frac{1}{t} \right), \\
H_1(t; v, 1) &= t^{-(n+2+v)/2} {}_3F_2 \left( \begin{matrix} \frac{(n+2+v)}{2}, \frac{(-n+2+v)}{2}, \frac{2(n+v+1)}{n+v} \\ 3, \frac{n+2+v}{n+v} \end{matrix}; 1 - \frac{1}{t} \right).
\end{aligned}$$

The first row of the matrix function  $\tilde{H}(t; v)$  is  $H(t; v, 0)$  and the second one is  $H(t; v - 1, 1)$ .

Then

$$\tilde{H}(t; v) = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

with

$$\begin{aligned}
F_{11} &= t^{-(n+3+v)/2} {}_3F_2 \left( \begin{matrix} \frac{(n+3+v)}{2}, \frac{(-n+1+v)}{2}, \frac{2(n-v)}{n+1-v} \\ 3, \frac{n-1-v}{n+1-v} \end{matrix}; 1 - \frac{1}{t} \right), \\
F_{12} &= t^{-(n+3+v)/2} {}_2F_1 \left( \begin{matrix} \frac{(n+3+v)}{2}, \frac{(-n+3+v)}{2} \\ 3 \end{matrix}; 1 - \frac{1}{t} \right), \\
F_{21} &= t^{-(n+3+v)/2} {}_2F_1 \left( \begin{matrix} \frac{(n+3+v)}{2}, \frac{(-n+1+v)}{2} \\ 3 \end{matrix}; 1 - \frac{1}{t} \right), \\
F_{22} &= t^{-(n+1+v)/2} {}_3F_2 \left( \begin{matrix} \frac{(n+1+v)}{2}, \frac{(-n+1+v)}{2}, \frac{2(n+v)}{n-1+v} \\ 3, \frac{n+1+v}{n-1+v} \end{matrix}; 1 - \frac{1}{t} \right).
\end{aligned}$$

The matrices  $A_v$ ,  $B_v$  and  $C_v$  are

$$\begin{aligned}
A_v &= \begin{pmatrix} \frac{(v+n+1)(v+n-3)(v-n-3)}{4v(v-1)(v+n-1)} & 0 \\ \frac{2(v+n-3)}{(v-1)(v+n-1)(v-n-3)} & \frac{(v+n-3)(v+n-1)(v-n-5)}{4(v-1)(v-2)(v-n-3)} \end{pmatrix}, \\
B_v &= \begin{pmatrix} \frac{v^4-n^4-4n^3+16n-6v^2+21}{2(v+1)(v-1)(v+n-1)(v-n-1)} & \frac{2(v-n-3)}{v(v+n-1)(v-n-1)} \\ \frac{2(v-n-1)}{(v-1)(v+n+1)(v-n-1)} & \frac{v^4-4v^3+8v+3+6n^2-n^4-8n}{2v(v-2)(v+n-1)(v-n-1)} \end{pmatrix}, \\
C_v &= \begin{pmatrix} \frac{(v+n+3)(v-n-3)(v-n-3)}{4v(v+1)(v-n+1)} & \frac{2(v+n+3)}{v(v+n+1)(v-n-1)} \\ 0 & \frac{(v+n-1)(v+n+3)(v-n-1)}{4v(v-1)(v+n+1)} \end{pmatrix}.
\end{aligned}$$

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