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SELF-DUAL HOPF QUIVERS

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Abstract

We study pointed graded self-dual Hopf algebras with a help of the dual Gabriel theorem for pointed Hopf algebras [15]. Quivers of such Hopf algebras are said to be self-dual. An explicit classification of self-dual Hopf quivers is obtained. We also prove that finite dimensional coradically graded pointed self-dual Hopf algebras are generated by group-like and skew-primitive elements as associative algebras. This partially justifies a conjecture of Andruskiewitsch and Schneider [3] and may help to classify finite dimensional self-dual pointed Hopf algebras.

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1. Introduction

One can start with quivers to construct path algebras and their quotient algebras. This produces finite dimensional elementary algebras in an exhaustive way, due to a well-known theorem of Gabriel. See Auslander, Reiten, Smalø [1] and Ringel [16]. There is a dual analogue for coalgebras given by Chin and Montgomery [6], which is remarkable for removing the restriction of finite dimensionality. Namely, any pointed coalgebra is a large subcoalgebra of the path coalgebra of some unique quiver.

A Hopf algebra is simultaneously an algebra and a coalgebra in a compatible way. We say that a Hopf algebra is pointed if its underlying coalgebra is pointed, i.e., its irreducible comodules are 1-dimensional. By Chin and Montgomery's theorem, for a pointed Hopf algebra H, the underlying coalgebra can be presented as a large subcoalgebra of a path coalgebra kQ for some unique quiver Q. In [15], Oystaeyen and Zhang proved that such a quiver is a Hopf quiver in the sense of Cibils and Rosso [10], and moreover, the associated graded Hopf algebra $\mathrm{gr}\, H$ of H (arising from the coradical filtration) is a Hopf subalgebra of some graded Hopf structure on kQ. This motivates a quiver approach to construct and classify pointed Hopf algebras.

In this paper we study graded pointed self-dual Hopf algebras with a help of the dual Gabriel's theorem. For a positively graded Hopf algebra $H = \oplus_{n\geq 0} H^n$ (may be infinite dimensional) with finite dimensional homogeneous spaces, the graded dual $H^{gr} = \oplus_{n\geq 0} H^{n*}$ is also a positively graded Hopf algebra. We say that H is self-dual if there exists a graded Hopf isomorphism $H \cong H^{gr}$. We remark that self-dual Hopf algebras generated in degrees 0 and 1 were studied by Green and Marcos in [11], where descriptions of such Hopf algebras via the so-called self-dual Hopf bimodules were obtained.

We study the quivers of graded pointed self-dual Hopf algebras, which we name self-dual Hopf quivers. An explicit classification of such quivers is obtained. We also prove that self-dual graded pointed Hopf algebras are generated by group-like and skew-primitive elements. In the finite dimensional case, this partially justifies a well-known conjecture of Andruskiewitsch and Schneider [3]. This result may help to give a classification of graded pointed self-dual Hopf algebras via the self-dual Hopf quivers.

For simplicity of exposition, we assume throughout that k is an algebraically closed field of characteristic 0. All algebras and coalgebras are over k. For a finite dimensional vector space V, we denote its k-linear dual by V^*. The unendowed tensor product $\otimes$ is $\otimes_k$.

We begin by recalling general facts, due to Cibils and Rosso [10] and van Oystaeyen and Zhang [15], about constructing graded Hopf structures from path coalgebras and the dual Gabriel's theorem for pointed Hopf algebras.

2.1. Let Q be a quiver and kQ the k-space with as a basis all the paths of Q. Then kQ has a natural length gradation $kQ = \bigoplus_{n \geqslant 0} kQ_n$, where kQ_n is spanned by all the paths of length n. Note that Q_0 is the set of vertices and Q_1 is the set of arrows. For each nontrivial path $p = a_n \cdots a_2 a_1 \in Q_n$ (i.e., $n \geq 1$) we define its starting vertex $s(p)$ as the tail of arrow a_1 and terminating vertex $t(p)$ as the head of arrow a_n.

Given a quiver Q, the graded space kQ has a natural graded path coalgebra structure as follows

$$\begin{aligned}\Delta(g) &= g \otimes g, \quad \varepsilon(g) = 1 \text{ for each } g \in Q_0,\\ \Delta(p) &= t(p) \otimes p + a_n \otimes a_{n-1} \cdots a_1 + \cdots + a_n \cdots a_2 \otimes a_1 + p \otimes s(p), \quad \varepsilon(p) = 0\\ & \quad \text{for each nontrivial path } p = a_n \cdots a_1.\end{aligned}$$

It is obvious that kQ is pointed with set of group-like elements $G(kQ) = Q_0$, and has the following coradical filtration

$$kQ_0 \subseteq kQ_0 \oplus kQ_1 \subseteq kQ_0 \oplus kQ_1 \oplus kQ_2 \subseteq \cdots$$

Hence kQ is coradically graded in the sense of Chin and Musson [7]. We remark that the path coalgebra kQ has another presentation as the so-called cotensor coalgebra and hence enjoy a universal property [14] (see also [15]).

2.2. Let G be a group and $\mathcal{C}$ the set of its conjugacy classes. A ramification datum of the group G is a formal sum $R = \sum_{C \in \mathcal{C}} R_C C$ with non-negative integer coefficients. Recall that for each ramification datum R of G, the corresponding Hopf quiver $Q = Q(G, R)$ is defined as follows: the set of vertices Q_0 is G, and for each $x \in G$ and $c \in C$, there are R_C arrows from x to cx.

A vector space M is said to be a kG-Hopf bimodule if it is simultaneously a kG-bimodule and a kG-bicomodule such that the comodule structure maps are homomorphisms of kG-bimodules.

Hopf bimodules over kG were classified in [9], Proposition 3.3. We briefly recall this result for later application. For each $C \in \mathcal{C}$, fix an element $u(C) \in C$, and let Z_C be the centralizer of $u(C)$. There is an equivalence of categories

$$V : \mathrm{b}(kG) \longrightarrow \Pi_{C \in \mathcal{C}} \operatorname{mod}(kZ_C),$$

where $\mathrm{b}(kG)$ is the category of kG-Hopf bimodules and $\operatorname{mod}(kZ_C)$ the category of left kZ_C-modules. Given $M \in \mathrm{b}(kG)$, then $V(M) = ({}^{u(C)}M^1)_{C \in \mathcal{C}}$, where the left module structure on ${}^{u(C)}M^1$ is defined by the conjugate action: $g \cdot m = g.m.g^{-1}$. On the contrary, for any $(M_C)_{C\in\mathcal{C}} \in \Pi_{C\in\mathcal{C}}\, \mathrm{mod}(kZ_C)$, the corresponding kG-Hopf bimodule is $\oplus_{C\in\mathcal{C}} kG \otimes_{kZ_C} M_C \otimes kG$.

Given a kG-Hopf bimodule M with bicomodule maps δ_L and δ_R, we define the Hopf quiver $Q = Q(G, M)$ of M as follows: the set of vertices Q_0 is G, and for any $g, h \in G$, there are $\dim_k {}^hM^g$ arrows from g to h. Here by ${}^hM^g$ we mean the (h, g)-isotypic component

$$\{m \in M \mid \delta_L(m) = h \otimes m, \quad \delta_R(m) = m \otimes g\}.$$

The following lemma shows that the Hopf quivers arising from ramification data coincide with those from Hopf bimodules over a group, hence we may identify them by just saying Hopf quivers.

Lemma 2.1. *For any quiver $Q = Q(M, G)$, there exists a ramification datum R of G such that $Q = Q(G, R)$, and vice versa.*

Proof. Let M be a kG-Hopf bimodule with comodule structure maps δ_L and δ_R and $Q = Q(G, M)$. For any $f, g, h \in G$ and $m \in {}^hM^g$, by the definition of kG-Hopf bimodules we have

$$\delta_L(f.m) = fh \otimes f.m, \quad \delta_L(m.f) = hf \otimes m.f$$

and

$$\delta_R(f.m) = f.m \otimes fg, \quad \delta_R(m.f) = m.f \otimes gf.$$

It follows that

$$f.{}^hM^g \subseteq {}^{fh}M^{fg}, \quad {}^hM^g.f \subseteq {}^{hf}M^{gf}.$$

Note that f is invertible, hence actually we have

$$f.{}^hM^g = {}^{fh}M^{fg}, \quad {}^hM^g.f = {}^{hf}M^{gf}.$$

It follows that for $x, g, c \in G$,

$${}^{g^{-1}cgx}M^x = {}^{g^{-1}cg}M^1.x = g^{-1}.{}^cM^1.g.x\ .$$

Since the actions of group elements are invertible, it is clear that

$$\dim_k {}^{g^{-1}cgx}M^x = \dim_k {}^cM^1\ .$$

In other words, for any $x \in G$ and any $c' \in C$, where C is the conjugacy class containing c, there are $\dim_k {}^cM^1$ arrows from x to $c'x$ in Q. Let $\mathcal{C}$ be the set of the conjugacy classes of G. For each $C \in \mathcal{C}$, fix an element $c \in C$. Take a ramification data of G as

$$R = \sum_{C\in\mathcal{C}} R_C C$$

with $R_C = \dim_k {}^cM^1$. It is clear that $Q = Q(G, R)$.

On the contrary, let $Q = Q(G, R)$ for some $R = \sum_{C\in\mathcal{C}} R_C C$. Take $(M_C)_{C\in\mathcal{C}} \in \Pi_{C\in\mathcal{C}}\, \mathrm{mod}(kZ_C)$ such that $\dim_k M_C = R_C$. This is always possible. For example, take M_C as trivial kZ_C-module.

Let M be the associated kG-Hopf bimodule. By direct calculation of the isotypic components of M, we have that $Q = Q(G, M)$. □

2.3. Suppose that kQ can be endowed with a graded Hopf algebra structure with length gradation. Then kQ is pointed and kQ_0 is the coradical. Hence $kQ_0 \cong kG$ for some finite group G and we now identify Q_0 and G. The graded Hopf algebra structure induces naturally on kQ_1 a kG-Hopf bimodule structure and Q is of course the Hopf quiver of it. By Lemma 2.1, the quiver Q is the Hopf quiver $Q(G, R)$ of some ramification data R.

Given a Hopf quiver $Q = Q(G, R)$ for some group G and some ramification data R, then kQ_1 admits kQ_0-Hopf bimodule structures. Fix a kQ_0-Hopf bimodule $(kQ_1,\ m_L,\ m_R,\ \delta_L,\ \delta_R)$. By the universal property of kQ, the bimodule structure can be extended to an associative multiplication and kQ becomes a graded bialgebra. The existence of antipode is guaranteed by Takeuchi [18]. Hence kQ admits a graded Hopf structure.

Cibils and Rosso's results [10] can be summarized as follows.

Theorem 2.2. *Let Q be a quiver. Then Q is a Hopf quiver if and only if the path coalgebra kQ admits graded Hopf algebra structures. Moreover, if Q is a Hopf quiver, then the complete list of graded Hopf structures on kQ is in one-to-one correspondence with that of kQ_0-Hopf bimodule structures on kQ_1.*

2.4. Let C be a pointed coalgebra with $G = G(C)$, then the corresponding quiver $Q(C)$ is obtained in the following way. The set of vertices of $Q(C)$ is G. For $\forall\ x, y \in G$, the number of arrows from x to y is $\dim_k P_{x,y}(C) - 1$, where $P_{x,y}(C) = \Delta^{-1}(C \otimes x + y \otimes C)$. Chin and Montgomery's theorem says that C is a large subcoalgebra of the path coalgebra $kQ(C)$. Here "large" means that the subcoalgebra contains all the vertices and arrows of $Q(C)$. Of course in this case such a quiver is unique. We remark that, according to the definition, a pointed coalgebra and its associated graded coalgebra (induced by the coradical filtration) enjoy the same quiver.

Let H be a pointed Hopf algebra. The coradical filtration $\{H_n | n \geq 0\}$ is in fact a Hopf algebra filtration and hence the associated graded space

$$\mathrm{gr}\, H = \bigoplus_{n\geq 0} \mathrm{gr}\, H^n = \bigoplus_{n\geq 0} H_n/H_{n-1}$$

(with $H_{-1} = 0$) is a coradically graded Hopf algebra (see [12], Lemma 5.2.8). Consider the quiver $Q(H)$ of the underlying coalgebra of H. The following result can be regarded as the version of the Gabriel's theorem for Hopf algebras from the coalgebra aspect, see [15], Proposition 4.4 and Theorem 4.6.

Theorem 2.3. *Suppose that H is a pointed Hopf algebra and that $G = G(H)$. Then $Q(H)$ is a Hopf quiver and there exists a graded Hopf algebra embedding $\operatorname{gr} H \hookrightarrow kQ(H)$, where the Hopf structure on $kQ(H)$ is determined by the kG-Hopf bimodule structure on $\operatorname{gr} H^1$.*

3. Self-Dual Hopf Quivers

In this section we consider the quivers of coradically graded pointed self-dual Hopf algebras, which are called self-dual Hopf quivers. An explicit classification of such quivers is obtained.

3.1. Let $H = \oplus_{n \geq 0} H^n$ be a positively graded Hopf algebra with finite dimensional homogeneous spaces. Recall that H is said to be self-dual if there exists a graded Hopf isomorphism $H \cong H^{gr}$. The self-duality is very natural and general in common: for any graded Hopf algebra H, the tensor product $H \otimes H^{gr}$ is self-dual.

In this section we always assume that H is coradically graded with $H^0 = kG$ (hence pointed) for some finite group G. In this case, if H is self-dual, then the group G is abelian. In fact, let $f: H \longrightarrow H^{gr}$ be a graded isomorphism, then f_0, the restriction of f to degree 0, induces a Hopf isomorphism of kG and $(kG)^*$, hence G is abelian since $(kG)^*$ is commutative. Furthermore H^1 has a so-called self-dual kG-Hopf bimodule structure, see [11]. The self-duality of H^1 comes naturally from that of H. Namely, the isomorphism f_0 induces a kG-Hopf bimodule structure on the $(kG)^*$-Hopf bimodule H^{1*}; the restriction of f to degree 1 gives rise to an isomorphism of kG-Hopf bimodules $f_1: H(1) \longrightarrow H(1)^*$.

3.2. The classification of self-dual Hopf bimodules over a finite abelian group algebra was given in [11] using Cibils and Rosso's results on Hopf bimodules. We recall it here for application later on.

Let G be a finite abelian group. Write $G = G_1 \times G_2 \times \cdots \times G_t$, where $G_i =< \alpha_i >$. The general elements of G are written as $\alpha^e = \alpha_1^{e_1} \cdot \alpha_2^{e_2} \cdots \alpha_t^{e_t}$. Let $\omega = \{\omega_1, \omega_2, \cdots, \omega_t\}$ be a set of roots of unity such that $\operatorname{order} \omega_i = \operatorname{order} \alpha_i$. We define a map $\chi^\omega: \ kG \longrightarrow (kG)^*$ as follows: for any element $\alpha^e \in G$, let $\chi^\omega(\alpha^e) = \chi^\omega_{\alpha^e} \in (kG)^*$; for any $\alpha^f \in G$, let $\chi^\omega_{\alpha^e}(\alpha^f) = \omega_1^{e_1 f_1} \omega_2^{e_2 f_2} \cdots \omega_t^{e_t f_t}$. It is well-known that such a map χ^ω is a Hopf isomorphism and that $\{\chi^\omega_g\}_{g \in G}$ is a complete set of irreducible characters of G. Denote by S_g the irreducible module associated to the character χ^ω_g.

By Cibils and Rosso's classification of Hopf bimodules, there is an equivalence of categories

$$V : \mathrm{b}(kG) \longrightarrow \Pi_{g \in G} \operatorname{mod}(kG),$$

where $\mathrm{b}(kG)$ is the category of kG-Hopf bimodules and $\operatorname{mod}(kG)$ the category of left kG-modules. Given $M \in \mathrm{b}(kG)$, then $V(M) = ({}^g M^1)_{g \in G}$. Write ${}^g M^1 = \oplus_{h \in G} m_h(g) S_h$ as the sum of irreducible modules. Then the isomorphic classes of objects in $\mathrm{b}(kG)$ are in one-to-one correspondence with the set of matrices

$$\{(m_h(g))_{g,h\in G} | m_h(g) \text{ is a nonnegative integer}, \forall\ g, h \in G\}.$$

Identifying kG with $(kG)^*$ via χ^ω, then M^* is a kG-Hopf bimodule. By [9], Proposition 5.1, if M corresponds to the matrix $(m_h(g))_{g,h\in G}$, then M^* corresponds to the matrix $(m^*_h(g))_{g,h\in G}$, where $m^*_h(g) = m_{g^{-1}}(h^{-1})$.

Now it is clear that a kG-Hopf bimodule M is self-dual if and only if there exists an ω as in the previous argument such that the corresponding matrix $(m_h(g))_{g,h\in G}$ of M satisfying $m_h(g) = m_{g^{-1}}(h^{-1})$, for any $g,\ h \in G$.

3.3. We say that a Hopf quiver $Q = Q(G, M)$ is self-dual if the kG-Hopf bimodule M is self-dual. It is immediate that the quiver $Q(H)$ of self-dual Hopf algebra H is self-dual. Precisely, $Q = Q(G, H^1)$. By Theorem 2.3, H is a Hopf subalgebra of kQ. On the other hand, given a self-dual Hopf quiver $Q = Q(G, M)$, then self-dual Hopf algebra arises naturally. Let $kG[M]$ be the Hopf subalgebra of kQ generated in degrees 0 and 1. This is the so-called bialgebra of type one introduced by Nichols [14]. By [11], Theorem 2.4, $kG[M]$ is self-dual. We can summarize the above arguments as follows

Proposition 3.1. *If H is coradically graded pointed Hopf algebra, then $Q(H)$ is a self-dual Hopf quiver. Conversely if $Q = Q(G, M)$ is a self-dual Hopf quiver, then $kG[M]$ is a self-dual Hopf algebra.*

3.4. Now we consider what self-dual Hopf quivers look like. The following theorem shows that such quivers are very general, as corresponds exactly to the naturalness and generality of the self-duality.

Theorem 3.2. *Any quiver of form $Q = Q(G, R)$ with G abelian and R a ramification data is self-dual.*

Proof. Let G be an abelian group and $R = \sum_{g\in G} R_g g$ a ramification datum. Then by Lemma 2.1, $Q = Q(G, M)$ for any Hopf bimodule M such that $\dim_k {}^gM^1 = R_g$, $\forall\ g \in G$. We need to prove that there exists a self-dual Hopf bimodule satisfying such condition.

For this, we fix an ω as in subsection 3.2. Let M be the kG-Hopf bimodule corresponding to matrix $(m_h(g))_{g,h\in G}$ with entries $m_{g^{-1}}(g) = R_g$, $\forall\ g \in G$ and 0 otherwise. It is clear that such an M is self-dual, and hence Q is self-dual. □

3.5. In this subsection we consider the case of kQ itself being self-dual. First of all kQ_1 must be a self-dual kQ_0-Hopf bimodule.

Proposition 3.3. *Let H be a slef-dual Hopf structure on kQ. Then H is generated by group-like and skew-primitive elements as an associative algebra.*

Proof. Denote by $\{H_n \mid n \geq 0\}$ the coradical filtration of H. On one hand, the underlying coalgebra of H is the path coalgebra kQ, hence we have $H_n = \oplus_{i \leq n} kQ_i$. On the other hand, the underlying algebra structure of H is $\oplus_{n \geq 0} kQ_n^*$. The Hopf structure on $\oplus_{n \geq 0} kQ_n^*$ is also graded, hence we have

$$\Delta(kQ_n^*) \subseteq \oplus_{i+j=n} kQ_i^* \otimes kQ_j^*.$$

Note that $kQ_0^* = (kG)^*$, then it is semisimple and cosemisimple, and hence $kQ_0^* \subseteq H_0$. By comparing the dimensions, we get $kQ_0^* = H_0$. Using induction and comparing dimensions arguments, we have $H_n = \oplus_{i \leq n} kQ_i^*$. In particular, $H_1 = kQ_0^* \oplus kQ_1^*$. Note that the algebra kQ^* is generated in degrees 0 and 1, hence H is generated by group-like and skew-primitive elements. □

Remark 3.4. *If kQ itself is self-dual, then $kQ = kG[M]$, i.e., it is a bialgebra of type one.*

4. Finite-Dimensional Self-Dual Hopf Algebras

The main purpose of this section is to prove that finite-dimensional coradically graded pointed self-dual Hopf algebras are generated by group-like and skew-primitive elements.

4.1. In [2], Andruskiewitsch and Schneider proposed the so-called lifting method for classifying finite dimensional pointed Hopf algebras. The reader is referred to an up-to-date survey [4]. In the programme, a key step is to find the generators. Andruskiewitsch and Schneider conjectured that all finite dimensional pointed Hopf algebras over an algebraically closed field of characteristic 0 are generated by group-like and skew-primitive elements (see [3], Conjecture 1.4). By [2], Lemmas 2.2 and 2.3, it is enough to consider coradically graded Hopf algebras.

4.2. The following theorem shows that Andruskiewitsch and Schneider's conjecture is true for finite dimensional self-dual Hopf algebras.

Theorem 4.1. *Let $H = \oplus_{n \geq 0} H^n$ be a coradically graded pointed Hopf algebra. If H is finite dimensional and self-dual, then H is generated by group-like and skew-primitive elements.*

Proof. We may assume that $H^0 = kG$ for some finite abelian group G. Let $J = \oplus_{n \geq 1} H^n$. It is clear that J is an nilpotent (Hopf) ideal of H. Note that $H/J = H^0 = kG$, which is isomorphic to $k^{|G|}$ as an associative algebra. It follows that H is an elementary algebra and J is the Jacobson radical. It is clear that $J^2 \subseteq \oplus_{n \geq 2} H^n$, and hence $H^1 \subseteq J/J^2$.

On the other hand, by the duality of coradical filtration and Jacobson radical filtration (see e.g. [12], 5.2.9), we have $J^2(H) = C_1(H^*)^{\perp}$, where $C_1(H^*)$ is the first term of the coradical filtration of the dual Hopf algebra H^*. By the self-duality of H, $C_1(H^*) = H_1$, which is exactly $H^0 \oplus H^1$ since H is coradically graded. This implies that $\dim_k J^2 = \dim_k H - \dim_k H_1$.

By comparing the dimensions, we have $J^2 = \oplus_{n\geq 2} H^n$, and hence $H^1 = J/J^2$. It is well-known that (see e.g. [1], Theorem 1.9, p.65), as an associative algebra, H is generated by H/J and J/J^2. Now the theorem follows. □

Remark 4.2. *Let $H = \oplus_{n\geq 0} H^n$ be as in the theorem. By J we denote its Jacobson radical. Then by a similar argument of comparing dimensions, via the duality of coradical filtration and Jacobson radical filtration, we have $J^m = \oplus_{n\geq m} H^n$, for any integer $m \geq 1$.*

4.3. The theorem above may help to classify completely finite dimensional graded pointed self-dual Hopf algebras. The following is direct consequence of the theorem.

Corollary 4.3. *Any finite dimensional coradically graded pointed self-dual Hopf algebra is of the form $kG[M]$ for some finite abelian group G and some self-dual kG-Hopf bimodule M.*

4.4. Finally, we remark that there is not known necessary and sufficient condition for general self-dual kG-Hopf bimodule M such that $kG[M]$ is finite dimensional. However we work out the simplest case with a help of results in [5].

Let G be a cyclic group of order n generated by g. Firstly let $R = g$ be the simplest ramification datum. Then the Hopf quiver $Q = Q(G, R)$ is a basic cycle. Namely, Q has set of vertices $\{g^i | i = 0, 1, ..., n-1\}$ and set of arrows $\{a_i : g^i \longrightarrow g^{i+1} | i = 0, 1, ..., n-1\}$. Finite dimensional pointed Hopf structures on such quiver are completely classified in [5], Theorem 3.6. As a consequence we have

Proposition 4.4. *Let H be a finite dimensional pointed Hopf algebra with $Q(H)$ being a basic cycle. Then H is self-dual if and only if H is the Taft algebra.*

Proof. Recall that the Taft algebra T of dimension n^2 is generated by two elements h and x with relations

$$x^n = 0, \quad h^n = 1, \quad xh = qxh,$$

where q is an n-th primitive root of unity (see [17]). A concrete Hopf isomorphism map of T and T^* was given in [8].

We include a proof of the self-duality via our settings. Firstly note that $Q(T)$ is a basic cycle. Take $\omega = q$ as in subsection 3.2. Let M be a kG-Hopf bimodule corresponding to matrix $(m_{g^j}(g^i))$ with entries $m_{g^{-1}}(g) = 1$ and 0 otherwise. Then M is self-dual. It is not difficult to see that T is the Hopf subalgebra $kG[M]$ of $kQ(T)$, and hence self-dual by Proposition 3.1.

On the contrary, all the finite dimensional Hopf structures on a basic cycle is isomorphic to the Hopf algebra $A(n, d, \mu, q)$ presented by generators and relations as follows

$$h^n = 1, \quad x^d = \mu(1 - h^d), \quad xg = ugx,$$

with u a root of unity of order d and $\mu = 0$ or 1. By [5], Theorem 4.3, if $A(n, d, \mu, q)$ is self-dual, then $\mu = 0$, and $d = n$. That is, $A(n, d, \mu, q)$ must be exactly the Taft algebra. □

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