Available at: http://www.ictp.trieste.it/~pub_off IC/2004/60

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON GENERALIZED CO-COHEN-MACAULAY AND CO-BUCHSBAUM MODULES OVER COMMUTATIVE RINGS

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Abstract

We study two classes of Artinian modules over commutative Noetherian rings called co-Buchsbaum modules and generalized co-Cohen-Macaulay modules. Some properties on $\mathfrak{q}-$weak co-sequences, co-standard sequences, multiplicity, local homology modules, localization, ... , of these modules are presented.

MIRAMARE – TRIESTE
July 2004

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1 Introduction

Let R be a commutative Noetherian (not necesarily local) ring and A an Artinian $R-$module. Then $\operatorname{Supp} A$ is a finite set which contains only maximal ideals. Let $\mathfrak{m}$ be the intersection of all elements in $\operatorname{Supp} A$. Note that if R is local then this $\mathfrak{m}$ is exactly the unique maximal ideal of R. For each system of parameter (s.o.p for short) $\underline{x}$ of A contained in $\mathfrak{m}$ we set

$$I(\underline{x};A)=\ell_R(0:_A \underline{x}R)-e_R(\underline{x};A) \text{ and } I(A)=\sup_{\underline{x}} I(\underline{x};A),$$

where $e_R(\underline{x};A)$ is the multiplicity of A with respect to $\underline{x}$, see [6] for the definition, and the supremum takes over all s.o.p $\underline{x}$ of A. Recall that A is *co-Cohen-Macaulay* if the Noetherian dimension of A is equal to the width of A in $\mathfrak{m}$, see [18], [6]. The structure of co-Cohen-Macaulay modules is known by the properties of system of parameters, attached primes, local homology modules, ... Especially, A is co-Cohen-Macaulay if and only if $I(A)=0$.

The purpose of this paper is to study two classes of Artinian modules over commutative Noetherian rings called co-Buchsbaum modules and generalized co-Cohen-Macaulay modules: A is called *co-Buchsbaum* if $I(\underline{x};A)$ is a constance not depending on $\underline{x}$, and A is called *generalized co-Cohen-Macaulay* if $I(A)<\infty$.

The main tools for our work are the theories of secondary representation [12], local homology modules [2], multiplicity [6] for Artinian modules over commutative rings, which are respectively in some sense dual to the well known theories of primary decomposition, local cohomology modules, multiplicity for Noetherian modules over local rings. Especially, the results presented in the method of studying Artinian modules of R.Y. Sharp [16] are applied time to time in this paper.

It should be mentioned that, even when $(R,\mathfrak{m})$ is a Noetherian local ring, the local homology module $H_i^{\mathfrak{m}}(A)$ is not necessarily a Noetherian $R-$module, while the local cohomology module $H^i_{\mathfrak{m}}(M)$ is always an Artinian $R-$module for every Noetherian $R-$module M and every integer $i\geq 0$. Moreover, although the Noetherian dimension $\operatorname{N-dim} A$ seems suitable to define the notions of system of parameters and the multiplicity, see [6], and sensible for the local homology modules of A (for example, $H_i^I(A)=0$ for all ideal I and all $i>\operatorname{N-dim} A$, and $\operatorname{N-dim} A$ is the biggest integer i such that $H_i^{\mathfrak{m}}(A)$ is not vanishing), see [2], but Noetherian dimension is not sensible for attached primes in some sense. Concretely, while $\dim M=\max_{\mathfrak{p}\in \operatorname{Ass} M}\dim R/\mathfrak{p}$ for any Noetherian $R-$module M, Noetherian dimension $\operatorname{N-dim} A$ of A is in general less strict than $\max_{\mathfrak{p}\in \operatorname{Att}_R A}\dim R/\mathfrak{p}$, see [5, 4.1]. Therefore it is complex in using attached primes of A to compute $\operatorname{N-dim} A$. Finally, for each ideal I of R, by using the exact, closed property of the localization functors between the categories of Noetherian modules, we can imply that $\operatorname{Rad}(\operatorname{Ann}(M/IM))=\operatorname{Rad}(I+\operatorname{Ann} M)$ for every Noetherian module M, but the similar equality $\operatorname{Rad}(\operatorname{Ann}(0:_A I))=\operatorname{Rad}(I+\operatorname{Ann} A)$ does not hold for all Artinian modules A, see [5, 4.3]. The reason is that, in general, we do not have so-called "co-localization" functors $F_{\mathfrak{p}}(-)$, with respect to prime ideals $\mathfrak{p}$ of R, which are exact and closed from the category of Artinian $R-$modules to the category of Artinian $R_{\mathfrak{p}}-$modules such that $F_{\mathfrak{p}}(A) \neq 0$ if and only if $\mathfrak{p} \supseteq \operatorname{Ann} A$, see [13], [14]. These facts explain some difficulties in our work.

This paper is divided into 5 sections. In the next section we give some preliminaries of Artinian modules which are often used later. Some properties of local homology modules and $\mathfrak{q}-$weak co-sequences will be presented in Section 3. In Section 4, we study generalized co-Cohen-Macaulay modules. The properties of co-Buchsbaum modules will be given in Section 5.

2 Preliminaries

Throughout this paper we use the following assumptions and notations: R is a commutative Noetherian ring (not necessarily local), $A \neq 0$ an Artinian $R-$module. Note that $\operatorname{Supp} A$ is a finite set which contains only maximal ideals, see [16]. So, we can write $\operatorname{Supp} A = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$. Set

$$\mathfrak{m} = \bigcup_{j=1}^{t} \mathfrak{m}_j; \qquad A_j = \bigcup_{n \geq 0} (0 :_A \mathfrak{m}_j^n), \quad j = 1, \ldots, t.$$

Then $A_j \neq 0$ for all $j \leq t$ and $A = A_1 \oplus \ldots \oplus A_t$. Set $R_j = R_{\mathfrak{m}_j}$, $\widehat{R_j}$ the $\mathfrak{m}_j R_j-$adic completion of R_j for all $j \leq t$, and $\widehat{R}$ the $\mathfrak{m}-$adic completion of R.

a) Secondary representation and Noetherian dimension

The theory of secondary representation introduced by I. G. Macdonald in [12] is in some sense dual to the more known theory of primary decomposition. Note that every Artinian $R-$module A has a secondary representation $A = B_1 + \ldots + B_n$ of $\mathfrak{p}_i-$secondary submodules B_i. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the minimal secondary representation of A and denoted by $\operatorname{Att} A$. On the other hand, R. N. Roberts [15] introduced the concept of Krull dimension (Kdim) for Artinian modules. Then D. Kirby [11] changed the terminology of Roberts and referred to Noetherian dimension (N-dim) to avoid any confusion. Here we use the terminology of Kirby [11]. There are some good properties of Noetherian dimension for Artinian modules which are in some sense dual to that of Krull dimension for Noetherian modules. For example,

$$\text{N-dim}\, A = \inf\{t : \exists x_1, \ldots, x_t \in \mathfrak{m} \text{ such that } \ell(0 :_A (x_1, \ldots, x_t)R) < \infty\},$$

and if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of Artinian $R-$modules then $\text{N-dim}\, A = \max\{\text{N-dim}\, A', \text{N-dim}\, A''\}$. Unfortunately, while $\dim M = \max_{\mathfrak{p} \in \operatorname{Ass} M} \dim R/\mathfrak{p}$ for any Noetherian $R-$module M, Noetherian dimension $\text{N-dim}\, A$ of A is not equal to $\max_{\mathfrak{p} \in \operatorname{Att}_R A} \dim R/\mathfrak{p}$ in general, see [5, 4.1]. Moreover A may have, see the example below, a minimal secondary representation $A = \sum B_k$ of $\mathfrak{p}_k-$secondary submodules B_k such that $\text{N-dim}\, B_k = \text{N-dim}\, B_{k'}$ but

$\mathfrak{p}_k \subset \mathfrak{p}_{k'}$ for some $k \neq k'$. These facts make some difficulties in using attached primes of A to compute N-dim A.

Example. Let $(R, \mathfrak{m})$ be the local domain constructed by D. Ferrand and M. Raynaund [8] such that $\dim R = 2$ and $\dim \widehat{R}/\widehat{\mathfrak{q}} = 1$ for some $\widehat{\mathfrak{q}} \in \operatorname{Ass} \widehat{R}$. Let $B = H^1_{\mathfrak{m}}(R)$. Then $\operatorname{Att}_R B = \{0\}$ and N-dim $B = 1$, see [5, 4.1]. Let $a \in \mathfrak{m}$, $a \neq 0$. Put $C = H^1_{\mathfrak{m}}(R/aR)$. Then $\operatorname{Att} C = \{\mathfrak{p} \in \operatorname{Ass} R/aR, \dim R/\mathfrak{p} = 1\}$. Let B' be a secondary component of C. Then $\operatorname{Att} B' = \{\mathfrak{p}\}$ for some $\mathfrak{p} \neq 0$. Set $A = B \oplus B'$. Then N-dim $B =$ N-dim $B' = 1$ and $0 \subset \mathfrak{p}$, $\mathfrak{p} \neq 0$. Note that $A = B \oplus B'$ is a minimal secondary representation of A.

The following elementary properties, see [16], are useful for our work.

Lemma 2.1. *For each $j \leq t$, A_j has a natural structure as an Artinian R_j-module and with this structure each subset of A_j is a $R-$submodule if and only if it is an R_j-submodule. Therefore* N-dim$_R A_j =$ N-dim$_{R_j} A_j$.

Lemma 2.2. *Assume that R is local. Then A has a natural structure as an Artinian $\widehat{R}-$module and with this structure each subset of A is a $R-$submodule if and only if it is an $\widehat{R}-$submodule. Moreover,*

$$\text{N-dim}_R A = \text{N-dim}_{\widehat{R}} A \text{ and } \operatorname{Att}_R(A) = \{\mathfrak{q} \cap R : \mathfrak{q} \in \operatorname{Att}_{\widehat{R}}(A)\}.$$

b) System of parameters and the multiplicity

From now on, we always assume that N-dim $A = d > 0$. There exists by [6] a system $\underline{x}$ contained in $\mathfrak{m}$ such that $\ell_R(0 :_A \underline{x}R) < \infty$. Such a system is called a *system of parameters* (s.o.p for short) of A. An element $x \in \mathfrak{m}$ is called a *parameter element* of A if N-dim$(0 :_A x) =$ N-dim $A - 1$.

By modifying the proof of [18, 2.14], we can show the following result.

Lemma 2.3. *An element $x \in \mathfrak{m}$ is a parameter element of A if and only if $x \notin \mathfrak{p}_i$ for all i satisfying* N-dim $B_i = d$, *where $A = B_1 + \ldots + B_s$ is a minimal secondary representation of A, with B_i is $\mathfrak{p}_i-$secondary for $i \leq s$.*

Recall that a system $\underline{x} = (x_1, \ldots, x_t)$ contained in $\mathfrak{m}$ is called a *multiplicative system* of A if $\ell_R(0 :_A \underline{x}R) < \infty$. The *multiplicity* $e(\underline{x}; A)$ of A with respect to the multiplicative system $\underline{x}$ is defined by the obvious way, see [6]. Many properties of the multiplicity for Artinian modules, which are similar to that of multiplicity for Noetherian modules over local rings, have been shown in [6]. For example, $0 \leq e(\underline{x}; A) \leq \ell(0 :_A \underline{x}R)$ and $e(x_1^{n_1}, \ldots, x_t^{n_t}; A) = n_1 \ldots n_t e(\underline{x}; A)$ for all integers $n_1, \ldots, n_t > 0$. Especially, if $\underline{x}$ is an s.o.p of A then $e(\underline{x}; A)/d!$ is exactly the leading coefficient of the Hilbert polynomial $\ell(0 :_A (\underline{x}R)^n)$ for $n \gg 0$.

The following result, see [6, 5.4], is used in the sequel.

Lemma 2.4. *Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p of A. For $i = 1, \ldots, d$ we set $C_i = 0 :_A (x_1, \ldots, x_{i-1})R$. Then*

$$\ell(0 :_A \underline{x}R) - e(\underline{x}; A) = \sum_{i=1}^{d} e(x_{i+1}, \ldots, x_d; C_i/x_iC_i).$$

3 Local homology module and weak co-sequence

The notion of local homology modules was defined by Cuong-Nam [2] as follows: Let I be an ideal of R and M an arbitrary $R-$module. The *$i-th$ local homology module* $H_i^I(M)$ of M with respect to I is defined as $\varprojlim_t Tor_i^R(R/I^t; M)$. This definition is dual to Grothendieck's definition of local cohomology modules for Noetherian modules over Noetherian rings, and it slightly differs from that of Greenlees-May of local homology groups [9]. However, both notions are the same for Artinian modules. It has been presented in [2], by elementary methods of homological and commutative algebra, many basis properties of local homology modules for Artinian modules, which show that this theory of local homology modules is in some sense dual to the well-known theory of local cohomology of A. Grothendieck for Noetherian modules.

The following facts from [2] will often be used in this paper.

Lemma 3.1. *(i) Let $f : R \longrightarrow R'$ be a homomorphism of Noetherian rings and I an ideal of R. Then there exists a isomorphism $H_i^I(A) \cong H_i^{IR'}(A)$ of $\Lambda_I(R)-$modules for all $i \geq 0$, where $\Lambda_I(-)$ is the $I-$adic completion functor.*

(ii) $H_i^I(A) = 0$ for all $i >$ N-dim A.

(iii) $\bigcap_{n\geq 0} I^n H_i^I(A) = 0$ for all ideal I of R and all $i \geq 0$.

Now we show a condition for local homology module being of finite length.

Lemma 3.2. *Let $\mathfrak{q}$ be an ideal of R such that* $\mathrm{Rad}(\mathfrak{q}) = \mathfrak{m}$, *and $i \geq 0$ an integer. If $\mathfrak{q}H_i^{\mathfrak{m}}(A) = 0$ then $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty$.*

Proof. By Lemma 3.1,(i), $H_i^{\mathfrak{m}}(A_j) \cong H_i^{\mathfrak{m}R_j}(A_j) = H_i^{\mathfrak{m}_jR_j}(A_j)$ for all $j \leq t$. Since $\mathfrak{q}H_i^{\mathfrak{m}}(A) = 0$, $\mathfrak{m}^s H_i^{\mathfrak{m}}(A) = 0$ for some $s > 0$. So, by Lemmas 2.1, 2.2,

$$\mathfrak{m}_j^s \widehat{R_j} H_i^{\mathfrak{m}R_j}(A_j) = \mathfrak{m}_j^s R_j H_i^{\mathfrak{m}_jR_j}(A_j) = \mathfrak{m}^s H_i^{\mathfrak{m}}(A_j) = 0$$

for all $j \leq t$. Since $H_i^{\mathfrak{m}_jR_j}(A_j)$ is a Noetherian $\widehat{R_j}-$module by [2, 4.6], it follows that $\ell_{\widehat{R_j}}\big(H_i^{\mathfrak{m}_jR_j}(A_j)\big) < \infty$, and hence $\ell_{R_j}\big(H_i^{\mathfrak{m}_jR_j}(A_j)\big) < \infty$ for all $j \leq t$. So $\ell_R\big(H_i^{\mathfrak{m}}(A)\big) = \sum_{j=1}^t \ell_{R_j}\big(H_i^{\mathfrak{m}_jR_j}(A_j)\big) < \infty$. □

For a subset T of $\mathrm{Spec}\, R$ we set $(T)_i = \{\mathfrak{p} \in T : \dim R/\mathfrak{p} = i\}$ for every integer $i \geq 0$.

Proposition 3.3. *Assume that* $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty$ *for all* $i < d$. *Let* $(x_1, \ldots, x_r)$ *be a part of a s.o.p of* A. *Let* $0 :_A (x_1, \ldots, x_r)R = B_{1,r} + \ldots + B_{n,r}$, *with* $B_{k,r}$ *is* $\mathfrak{p}_{k,r}-$*secondary, be a minimal secondary representation of* $0 :_A (x_1, \ldots, x_r)R$. *Then* $\text{N-dim}_R\, B_{k,r} = d - r$ *for all* k *satisfying* $\mathfrak{p}_{k,r} \notin \operatorname{Supp} A$.

Proof. We prove by induction on r. Let $r = 0$. Let $A = B_1 + \ldots + B_n$, B_k is $\mathfrak{p}_k-$secondary, be a minimal secondary representation of A. We first claim that

Claim. *Assume that* R *is local. If* $\mathfrak{p}_k \notin \operatorname{Supp} A$ *then* $\text{N-dim}\, B_k = d$.

Proof of the claim. Let $D(A)$ be the Matlis dual of A. Then

$$\big(\operatorname{Att}_{\widehat{R}}(A)\big)_i = \big(\operatorname{Ass}_{\widehat{R}}(D(A))\big)_i \subseteq \operatorname{Att}_{\widehat{R}}(H^i_{\mathfrak{m}\widehat{R}}(D(A))$$

by [1, 11.3.3]. For every $i < d$, we obtain by [2, 3.3] that

$$\big(\operatorname{Att}_{\widehat{R}}(A)\big)_i \subseteq \operatorname{Ass}_{\widehat{R}}\big(D(H^i_{\mathfrak{m}\widehat{R}}(D(A))\big) = \operatorname{Ass}_{\widehat{R}}\big(H_i^{\mathfrak{m}\widehat{R}}(A)\big) \subseteq \{\mathfrak{m}\widehat{R}\}.$$

Therefore $\big(\operatorname{Att}_{\widehat{R}}(A)\big)_i = \emptyset$ for all $i = 1, \ldots, d-1$. For each $k \leq n$, let $B_k = \sum_{u=1}^{n_k} C_{u,k}$, $C_{u,k}$ is $\widehat{\mathfrak{p}}_{u,k}-$secondary, be a minimal secondary representation of $\widehat{R}-$module B_k. Then $\widehat{\mathfrak{p}}_{u,k} \cap R = \mathfrak{p}_k$ for all $u \leq n_k$. So, if $1 \leq k \neq k' \leq n$ then $\widehat{\mathfrak{p}}_{u,k} \neq \widehat{\mathfrak{p}}_{v,k'}$ for all $u \leq n_k, v \leq n_{k'}$. Therefore we can reduce the secondary representation $A = \sum_{k \leq n, u \leq n_k} C_{u,k}$ of $\widehat{R}-$module A into a minimal one, say $A = \sum_{h=1}^{m} C_h$, by cancelling all the redundant components $C_{u,k}$. Assume that C_h is $\widehat{\mathfrak{p}}_h-$secondary for $h \leq m$. Since $\big(\operatorname{Att}_{\widehat{R}}(A)\big)_i = \emptyset$ for all $0 < i < d$, we get by [5, 4.7] that $\text{N-dim}_R\, C_h = \dim \widehat{R}/\widehat{\mathfrak{p}}_h = d$ whenever $\text{N-dim}\, C_h > 0$. For each $k \leq n$, the component B_k is not redundant in the representation $A = B_1 + \ldots + B_n$. Therefore there exists at least an integer $u \leq n_k$ such that $C_{u,k} = C_h$ for some $h \leq m$. Note that $\mathfrak{p}_k \notin \operatorname{Supp} A$ if and only if $\text{N-dim}\, C_{u,k} > 0$. Hence $\text{N-dim}\, B_k = d$ if $\mathfrak{p}_k \notin \operatorname{Supp} A$, and the claim is proved.

Now we prove the proposition for the case $r = 0$ and R is not local. For $k \leq n$, let $B_{j,k} = \bigcup_{u \geq 0}(0 :_{B_k} \mathfrak{m}_j^u)$ for all $j \leq t$. Then $B_k = B_{1,k} \oplus \ldots \oplus B_{t,k}$. Let $j \leq t$. It is easily seen that $B_{j,k}$ is $\mathfrak{p}_k-$secondary for all $k \leq n$ satisfying $B_{j,k} \neq 0$, and $A_j = \sum_{k=1}^{n} B_{j,k}$ is a secondary representation of A_j. We can reduce this representation of A_j into a minimal one by cancelling all redundant components $B_{j,k}$. Let $B'_k = \sum_j B_{j,k}$, where the sum takes over all integers $j \leq t$ such that $B_{j,k}$ appears in the above minimal secondary representation of A_j. Then $B'_k \subseteq B_k$ for all $k \leq n$ and $A = \sum_{k=1}^{n} B'_k$ is a minimal secondary representation of A with B'_k is $\mathfrak{p}_k-$secondary for all $k \leq n$. Suppose that $\mathfrak{p}_k \notin \operatorname{Supp} A$. Since $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty$, $\ell_{R_j}(H_i^{\mathfrak{m}_j R_j}(A_j)) < \infty$ for all $j \leq t$. Therefore we get by the claim that $\text{N-dim}\, B_{j,k} = d$ for all $B_{j,k}$ appearing in the above minimal representation of A_j for all $j \leq t$. Therefore $\text{N-dim}\, B'_k = d$ and hence $\text{N-dim}\, B_k = d$. Thus, the result is true for $r = 0$.

Let $r > 0$. It follows by Lemma 2.3 and by the above fact that $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}_R A \setminus \operatorname{Supp} A$. So $\ell_R(A/x_1A) < \infty$. Therefore from the exact sequences

$$0 \longrightarrow x_1A \longrightarrow A \longrightarrow A/x_1A \longrightarrow 0$$

$$0 \longrightarrow 0 :_A x_1 \xrightarrow{x_1} A \xrightarrow{x_1} x_1A \longrightarrow 0$$

with notice that $H_i^{\mathfrak{m}}(A/x_1A) = 0$ for all $i > 0$ by Lemma 3.1,(ii), we get the long exact sequence for $i = 1, \ldots, d-1$,

$$H_i^{\mathfrak{m}}(A) \xrightarrow{x_1} H_i^{\mathfrak{m}}(A) \xrightarrow{\Delta_i} H_{i-1}^{\mathfrak{m}}(0_A : x_1) \longrightarrow H_{i-1}^{\mathfrak{m}}(A) \xrightarrow{x_1} H_{i-1}^{\mathfrak{m}}(A) \qquad (*)$$

Since $\ell\big(H_i^{\mathfrak{m}}(A)\big) < \infty$ for $i \leq d-1$, $\ell\big(H_i^{\mathfrak{m}}(0_A : x_1)\big) < \infty$ for $i \leq d-2$. Moreover, $\operatorname{Supp}(0 :_A x_1) \subseteq \operatorname{Supp} A$. So, we can apply the induction hypothesis for the part of the s.o.p $(x_2, \ldots, x_r)$ of $(0 :_A x_1)$, and we get the result.

Lemma 3.4. *For every s.o.p.* $\underline{x}$ *of* A *we have*

$$\ell_R(0 :_A \underline{x}R) - e(\underline{x}; A) \leq \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^{\mathfrak{m}}(A)).$$

Moreover, if $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty$ *for all* $i < d$ *then there exists an ideal* $\mathfrak{q}$ *with* $\operatorname{Rad}(\mathfrak{q}) = \mathfrak{m}$ *such that the equality holds for every s.o.p* $\underline{x}$ *contained in* $\mathfrak{q}$.

Proof: If $\ell(H_i^{\mathfrak{m}}(A)) = \infty$ for some $i < d$ then it is trivial. Let $\ell(H_i^{\mathfrak{m}}(A)) < \infty$ for all $i < d$. Then $\ell(H_i^{\mathfrak{m}}(A_j)) < \infty$ for all $j \leq t, i < d$. So, there exists an integer $n_{ij} > 0$ such that $\mathfrak{m}_j^{n_{ij}} H_i^{\mathfrak{m}}(A_j) = 0$ for all $j \leq t, i < d$. Therefore there exists an ideal $\mathfrak{p}$ such that $\operatorname{Rad}(\mathfrak{p}) = \mathfrak{m}$ and $\mathfrak{p} H_i^{\mathfrak{m}}(A) = 0$ for all $i < d$. Now, by using Proposition 3.3 and modifying the proof of [17, Lemma 15, Appendix], we get the result with the equality holding for every s.o.p contained in $\mathfrak{p}^{2^d}$. □

Next, we introduce the notion of $\mathfrak{q}-$weak co-sequence which is in some sense dual to the known concept of $\mathfrak{q}-$weak sequence, see [17].

Definition 3.5. Let $\mathfrak{q}$ be an ideal of R with $\operatorname{Rad}(\mathfrak{q}) = \mathfrak{m}$. A sequence $(x_1, \ldots, x_r)$ of elements in $\mathfrak{m}$ is called a $\mathfrak{q}-$*weak co-sequence of* A if

$$x_i(0 :_A (x_1, \ldots, x_{i-1})R) \supseteq \mathfrak{q}(0 :_A (x_1, \ldots, x_{i-1})R) \text{ for all } i = 1, \ldots, r,$$

where we mean $x_1A \supseteq \mathfrak{q}A$ when $i = 1$. A sequence $(x_1, \ldots, x_r) \subseteq \mathfrak{m}$ is called a *weak co-sequence* if it is an $\mathfrak{m}-$weak co-sequence.

Lemma 3.6. *Let* $\mathfrak{q}$ *be an ideal of* R *such that* $\operatorname{Rad}(\mathfrak{q}) = \mathfrak{m}$. *If there exists a s.o.p* $\underline{x} = (x_1, \ldots, x_d)$ *of* A *contained in* $\mathfrak{m}\mathfrak{q}$ *such that* $\underline{x}$ *is a* $\mathfrak{q}-$*weak co-sequence then* $\mathfrak{q}H_i^{\mathfrak{m}}(A) = 0$ *for all* $i < d$.

Proof. We prove by induction on d. Let $d = 1$. Since $x_1 \in \mathfrak{mq}$ and x_1 is a $\mathfrak{q}$- weak co-sequence of A, $\mathfrak{q}A \subseteq x_1A \subseteq \mathfrak{mq}A$. Hence $\mathfrak{q}A \subseteq \mathfrak{m}^n\mathfrak{q}A$ for all n. Since $\operatorname{Rad}(\mathfrak{q}) = \mathfrak{m}$, $\mathfrak{q}A = \bigcap\limits_{n\geq 0} \mathfrak{m}^n A$. So, $\mathfrak{q}H_0^{\mathfrak{m}}(A) = 0$. Let $d > 1$. As in the above proof, $\mathfrak{q}H_0^{\mathfrak{m}}(A) = 0$. Since $x_1A \supseteq \mathfrak{q}A$, $l(A/x_1A) < \infty$. Therefore we have the long exact sequence (*) as in the proof of Proposition 3.3. Since $(x_2, \ldots, x_d)$ is a $\mathfrak{q}$- weak co-sequence of $(0 :_A x_1)$, $\mathfrak{q}H_i^{\mathfrak{m}}(0_A : x_1) = 0$ for all $i < d-1$ by the induction assumption. Let $i \in \{1, \ldots, d-1\}$. Then $\mathfrak{q}(\operatorname{Im}\Delta_i) = 0$, where Δ_i is the map as in the exact sequence (*). Therefore $\mathfrak{q}H_i^{\mathfrak{m}}(A) \subseteq x_1H_i^{\mathfrak{m}}(A)$. Hence $\mathfrak{q}H_i^{\mathfrak{m}}(A) \subseteq x_1H_i^{\mathfrak{m}}(A) \subseteq \mathfrak{m}(\mathfrak{q}H_i^{\mathfrak{m}}(A))$. Thus, $\mathfrak{q}H_i^{\mathfrak{m}}(A) \subseteq \bigcap_{n\geq 0} \mathfrak{m}^n(H_i^{\mathfrak{m}}(A)) = 0$ by Lemma 3.1,(iii). $\square$

Proposition 3.7. *Let $\mathfrak{q}$ be an ideal of R such that* $\operatorname{Rad}(\mathfrak{q}) = \mathfrak{m}$. *Then the following statements are equivalent:*

(i) $\mathfrak{q}H_i^{\mathfrak{m}}(A) = 0$ for all $i \leq d-1$.

(ii) There exists a s.o.p $\underline{x} = (x_1, \ldots, x_d)$ of A contained in $\mathfrak{q}^2$ such that $\underline{x}$ is a $\mathfrak{q}-$weak co-sequence.

(iii) Every s.o.p $(y_1, \ldots, y_d)$ of A, which satisfies $y_i = x_i^{n_i}, i = 1, \ldots, d$, for some $x_i \in \mathfrak{q}$ and $n_i \geq 2$, is a $\mathfrak{q}-$weak co-sequence.

Proof. (ii)⇒(i) follows by Lemma 3.6. (iii)⇒(ii) is trivial.
(i)⇒ (iii). Let $\underline{y} = (y_1, \ldots, y_d)$ be a s.o.p of A such that $y_i = x_i^{n_i}, i = 1, \ldots, d$, for some $x_i \in \mathfrak{q}$ and $n_i \geq 2$. We prove by induction on d that $\underline{y}$ is a $\mathfrak{q}-$weak co-sequence of A. By Lemma 3.2, $\ell(H_i^{\mathfrak{m}}(A)) < \infty$ for all $i < d$. Let $d = 1$. By Proposition 3.3, $y_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}_R(A) \setminus \operatorname{Supp} A$. Hence $y_1A \supseteq \mathfrak{m}^nA$ for $n \gg 0$. Since $\mathfrak{q}H_0^{\mathfrak{m}}(A) = 0$, we have $\mathfrak{q}A \subseteq y_1A$. Let $d > 1$. By the above fact, $\mathfrak{q}A \subseteq y_1A$. Let $x \in \mathfrak{q}$ and $n \geq 2$ such that $y_1 = x^n$. Note that $\ell(A/xA) < \infty$. Therefore $H_i^{\mathfrak{m}}(x^nA) \cong H_i^{\mathfrak{m}}(y_1A) \cong H_i^{\mathfrak{m}}(A)$ for all $i \geq 1$. Since $x \in \mathfrak{q}$, $xH_i^{\mathfrak{m}}(A) = 0$ for all $i < d$. Therefore from the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0_A : x & \xrightarrow{j_x} & A & \xrightarrow{x} & xA & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j} & & \| & & \downarrow{\scriptstyle x^{n-1}} & & \\
0 & \longrightarrow & 0_A : x^n & \xrightarrow{j_{x^n}} & A & \xrightarrow{x^n} & x^nA & \longrightarrow & 0,
\end{array}$$

where j, j_{x^n}, j_x are the natural inclusions, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{i+1}^{\mathfrak{m}}(A) & \xrightarrow{\delta_1} & H_i^{\mathfrak{m}}(0_A : x) & \xrightarrow{j*_x} & H_i^{\mathfrak{m}}(A) & \longrightarrow & 0 \\
 & & {\scriptstyle x^{n-1}}\downarrow & & {\scriptstyle j^*}\downarrow & & \| & & \\
0 & \longrightarrow & H_{i+1}^{\mathfrak{m}}(A) & \xrightarrow{\delta_n} & H_i^{\mathfrak{m}}(0_A : x^n) & \xrightarrow{j*_{x^n}} & H_i^{\mathfrak{m}}(A) & \longrightarrow & 0,
\end{array}$$

where $j^*, j^*_{x^n}, j^*_x$ are the induced homomorphisms. In the second diagram, the multiplication by x^{n-1} on $H_{i+1}^{\mathfrak{m}}(A)$ is zero. So, $\delta_n x^{n-1} = j^*\delta_1 = 0$. Hence $\operatorname{Im}\delta_1 \subseteq \operatorname{Ker} j^*$. Therefore there exists a homomorphism $g_i : H_i^{\mathfrak{m}}(A) \longrightarrow H_i^{\mathfrak{m}}(0_A : x^n)$ such that $j^*_{x^n} g_i = Id_{H_i^{\mathfrak{m}}(A)}$. So the second row in the last diagram is split. Hence

$$H_i^{\mathfrak{m}}(0 :_A y_1) \cong H_{i+1}^{\mathfrak{m}}(A) \oplus H_i^{\mathfrak{m}}(A)$$

for all $i < d-1$. So, $\mathfrak{q} H_i^{\mathfrak{m}}(0 :_A y_1) = 0$ for all $i < d-1$. Now by applying the induction hypothesis to $0 :_A y_1$, we get the result. □

4 Generalized co-Cohen-Macaulay modules

From now on we set $I(\underline{x}; A) = \ell_R(0 :_A \underline{x}R) - e(\underline{x}; A)$ for every s.o.p $\underline{x}$ of A. Put $I(A) = \sup_{\underline{x}} I(\underline{x}; A)$, where $\underline{x}$ runs over all s.o.p of A.

Definition 4.1. We say that A is *generalized co-Cohen-Macaulay* (gCCM for short) if $I(A) < \infty$.

Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p of A and $\underline{n} = (n_1, \ldots, n_d)$ a $d-$tupe of positive integers. Set $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_d^{n_d})$. Let

$$I(\underline{x}(\underline{n}); A) = \ell_R(0 :_A \underline{x}(\underline{n})R) - n_1 \ldots n_d \, e(\underline{x}; A).$$

By using Lemma 2.4, we can show that $I(\underline{x}(\underline{n}); A) \geq I(\underline{x}(\underline{m}); A)$ whenever $n_i \geq m_i$ for $i = 1, \ldots, d$.

The notion of standard s.o.p, which is an important role in the study of generalized Cohen-Macaulay modules, was introduced by N. V. Trung [19]. Below we introduce the dual notion for Artinian modules.

Definition 4.2. A s.o.p $\underline{x} = (x_1, \ldots, x_d)$ of A is called *co-standard* if

$$I(\underline{x}; A) = I(x_1^2, \ldots, x_d^2; A).$$

To make use of co-standard s.o.p, we have the following result which can be proved by using Lemma 2.4 and modifying the proof of [17, Theorem and Definition 17, Appendix].

Lemma 4.3. *Let $\underline{x} = (x_1, \ldots, x_d)$ be a co-standard s.o.p of A. Then for all $n \geq 1$,*

$$I(x_1^n, \ldots, x_d^n; A) = I(\underline{x}; A).$$

The following theorem is the main result of this section.

Theorem 4.4. *The following statements are equivalent:*

(i) A is gCCM.

(ii) $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty$ for all $i \leq d-1$.

(iii) There exists a co-standard s.o.p of A.

(iv) There exists an s.o.p $(x_1, \ldots, x_d)$ of A and an ideal $\mathfrak{q}$ with $\mathrm{Rad}(\mathfrak{q}) = \mathfrak{m}$ such that $(x_1^n, \ldots x_d^n)$ is a $\mathfrak{q}-$weak co-sequence for all integers $n > 0$.

(v) There exists an ideal $\mathfrak{q}$ such that $\mathrm{Rad}(\mathfrak{q}) = \mathfrak{m}$ and every s.o.p of A is a $\mathfrak{q}-$weak co-sequence.

(vi) There exists an integer $s > 0$ and a s.o.p $(x_1, \ldots, x_d)$ satisfying the condition $I(x_1^n, \ldots, x_d^n; A) \leq s$ for all $n \geq 1$.

When A satisfies one of the above equivalent conditions, we have

$$I(A) = \sum_{i=1}^{d-1} \binom{d-1}{i} \ell_R(H_i^{\mathfrak{m}}(A)).$$

Proof. (ii)$\Rightarrow$ (i) follows by Lemma 3.4. (iv)$\Rightarrow$(ii) follows by Lemmas 3.2, 3.6. The inclusions (v) $\Rightarrow$(iv) and (i)$\Rightarrow$(vi) are trivial. Now we prove (vi)$\Rightarrow$(iv). Let $(x_1, \ldots, x_d)$ be the s.o.p of A satisfying (vi). Let $n \geq 1$ and $1 \leq i \leq d$. Set $B = 0 :_A (x_1^n, \ldots, x_{i-1}^n)R$ and $C_m = 0 :_B (x_{i+1}^m, \ldots, x_d^m)R$ for every $m \geq n$. By Lemma 2.4 and the assumption (vi),

$$\ell(C_m/x_i^n C_m) \leq I(\underline{x}(\underline{m}), A) \leq s$$

for any $m \geq n$. Therefore $\mathfrak{m}^s(C_m/x_i^n C_m) = 0$ for all $m \geq n$. Hence

$$\bigcup_{m \geq n} \mathfrak{m}^s(0 :_B (x_{i+1}^m, \ldots, x_d^m)R) \subseteq \bigcup_{m \geq n} x_i^n(0 :_B (x_{i+1}^m, \ldots, x_d^m)R).$$

Therefore we have $\mathfrak{m}^s B \subseteq x_i^n B$, see [11]. Now we choose $\mathfrak{q} = \mathfrak{m}^s$. Then $(x_1^n, \ldots, x_d^n)$ is a $\mathfrak{q}-$weak co-sequence for all n.

(i)$\Rightarrow$(v). This is similar to the proof of (vi)$\Rightarrow$(iv).

(i)$\Rightarrow$(iii). By the statement (i)$\Rightarrow$(ii), $\ell(H_i^{\mathfrak{m}}(A)) < \infty$ for all $i < d$. Therefore there exists by Lemma 3.4 an s.o.p $\underline{x} = (x_1, \ldots, x_d)$ of A such that

$$I(\underline{x}; A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H_i^{\mathfrak{m}}(A)) \geq I(x_1^2, \ldots, x_d^2; A) \geq I(\underline{x}; A).$$

Thus $\underline{x}$ is a co-standard.

(iii)$\Rightarrow$(i). Let $\underline{x} = (x_1, \ldots, x_d)$ be a co-standard s.o.p of A. By Lemma 4.3, $I(x_1^n, \ldots, x_d^n; A) = I(\underline{x}; A)$ for all $n \geq 1$. So, A is gCCM by (vi)$\Rightarrow$(i).

When A satisfies one of the above equivalent conditions, the equality follows immediately by Lemma 3.4. $\square$

Corollary 4.5. *If A is gCCM then* $\text{N-dim}\, A_j = d$ *or* $\text{N-dim}\, A_j = 0$ *for all* $j = 1, \ldots, t$.

Proof. Let $j \leq t$ and set $\text{N-dim}\, A_j = d_j$. Assume that $d_j > 0$. By [2, 4.10], $\dim \dfrac{\widehat{R}_j}{\text{Ann}(H_{d_j}^{\mathfrak{m}_j R_j}(A_j))} = d_j$. So, $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty$ for $i < d$ by Theorem 4.4. Hence $\ell_R(H_i^{\mathfrak{m}}(A_j)) < \infty$. Therefore, $\ell_{\widehat{R}_j}(H_i^{\mathfrak{m}R_j}(A_j)) < \infty$ for all $i < d$. Since $d_j > 0$, we have $d = d_j$. □

Corollary 4.6. *Suppose that* $\text{N-dim}\, A_j = 0$ *or* $\text{N-dim}\, A_j = d$ *for all* $j \leq t$. *Then A is gCCM if and only if $A_{\mathfrak{p}}$ is gCCM for all* $\mathfrak{p} \in \text{Supp}\, A$.

Proof. Assume that A is gCCM. Let $j \leq t$. It is clear that $A_{\mathfrak{m}_j} \cong A_j$. If $\text{N-dim}\, A_j = 0$ then $A_{\mathfrak{m}_j}$ is obviously gCCM. Suppose that $\text{N-dim}\, A_j = d$. It is clear that $I(A_j) \leq I(A) < \infty$. Thus $A_{\mathfrak{m}_j}$ is gCCM. Conversely, assume that $A_{\mathfrak{m}_j}$ is gCCM for all $j \leq t$. Then $I(A_j) < \infty$ for all $j \leq t$. Let $\underline{x} = (x_1, \ldots, x_d)$ be an s.o.p of A. Then $\underline{x}$ is an s.o.p of A_j for all j satisfying $\text{N-dim}\, A_j = d$. Therefore,

$$I(x_1^n, \ldots, x_d^n; A) \leq \sum_{\text{N-dim}\, A_j = d} I(A_j) + \sum_{\text{N-dim}\, A_j = 0} \ell(A_j) < \infty$$

for all integers $n > 0$. So, A is gCCM by Theorem 4.4,(vi). □

The following corollaries produce many examples of gCCM modules.

Corollary 4.7. *The following statements are true.*

(i) If A is gCCM and x is a parameter element of A then $0 :_A x$ is gCCM.

(ii) Let $B_1, \ldots, B_n$ be gCCM such that $\text{N-dim}\, B_i = d$ *or* $\text{N-dim}\, B_i = 0$ *for all* $i \leq n$. *Then* $A = \bigoplus_{i=1}^{n} B_i$ *is gCCM.*

Proof. (i). By Theorem 4.4, $\ell(H^{\mathfrak{m}}(A)) < \infty$ for all $i < d$. Therefore, as in the proof of Proposition 3.3, $\ell(H_i^{\mathfrak{m}}(0 :_A x)) < \infty$ for all $i < d - 1$. Hence $0 :_A x$ is gCCM by Theorem 4.4.
(ii). Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p of A. Then $\underline{x}$ is a s.o.p of B_i whenever $\text{N-dim}\, B_i = d$. Therefore

$$I(x_1^n, \ldots, x_d^n; A) \leq \sum_{\text{N-dim}\, B_i = 0} \ell(B_i) + \sum_{\text{N-dim}\, B_i = d} I(B_i) < \infty,$$

for all integers $n > 0$. So, A is gCCM by Theorem 4.4, (vi). □

Corollary 4.8. *Let $(R, \mathfrak{m})$ be a local ring. The following statements are true.*

(i) A is a gCCM $R-$module if and only if it is a gCCM $\widehat{R}-$module.

(ii) If M is generalized Cohen-Macaulay then the Matlis dual $D(M)$ of M is gCCM.

(iii) If M is generalized Cohen-Macaulay of dimension d then $H_{\mathfrak{m}}^d(M)$ is gCCM.

Proof. (i). Let $\underline{x} = (x_1, \ldots, x_d)$ be an s.o.p of $R-$module A. Then it is a s.o.p of $\widehat{R}-$module A. Set $\underline{x}(n) = (x_1^n, \ldots, x_d^n)$ for every integer $n > 0$. Then

$$\ell_{\widehat{R}}(0 :_A \underline{x}(n)\widehat{R}) - e_{\widehat{R}}(\underline{x}(n); A) = \ell_R(0 :_A \underline{x}(n))R - e_R(\underline{x}(n); A)$$

for all $n > 0$. Now the result follows by Theorem 4.4,(vi).
(ii). Let $\underline{x}$ be a s.o.p of $D(M)$. Then $\underline{x}$ is a s.o.p of M. Moreover, it is easy to check that $I(\underline{x}; D(M)) = \ell(M/\underline{x}M) - e(\underline{x}; M)$. Therefore $D(M)$ is gCCM.
(iii). Since M is generalized Cohen-Macaulay, the Matlis dual $D(H^d_{\mathfrak{m}}(M))$ of $H^d_{\mathfrak{m}}(M)$ is a generalized Cohen-Macaulay $\widehat{R}-$module by [4, 5.3]. Therefore, for every s.o.p $\underline{x}$ of $R-$module $H^d_{\mathfrak{m}}(M)$, $\underline{x}$ is a s.o.p of $\widehat{R}-$module $D(H^d_{\mathfrak{m}}(M))$ and

$$I(\underline{x}; H^d_{\mathfrak{m}}(M)) = \ell(D(H^d_{\mathfrak{m}}(M))/\underline{x}D(H^d_{\mathfrak{m}}(M))) - e(\underline{x}; D(H^d_{\mathfrak{m}}(M))) \leq C$$

for some constance C not depending on $\underline{x}$. Thus, $H^d_{\mathfrak{m}}(M)$ is gCCM. □

5 Co-Buchsbaum modules

We first introduce the notion of co-Buchsbaum module.

Definition 5.1. A is called *co-Buchsbaum* if $I(\underline{x}; A)$ *is a constance (not depending on* $\underline{x}$*) for all s.o.p* $\underline{x}$ *of* A.

Before giving a homological property of co-Buchsbaum modules, we need the following lemma.

Lemma 5.2. *Let* $\mathfrak{a} \subseteq \mathfrak{m}$ *be an ideal of* R *such that* $\ell(0 :_A \mathfrak{a}) < \infty$. *Then there exists a minimal system of generators* $(x_1, \ldots, x_k)$ *of* $\mathfrak{a}$ *such that* $(x_{i_1}, \ldots, x_{i_d})$ *is a s.o.p of* A *for all* $1 \leq i_1 < \ldots < i_d \leq k$.

Proof. We first claim that
Claim. *Let* $\mathfrak{p} \in \operatorname{Att} A$. *Then* $\operatorname{Ann}_R(0 :_A \mathfrak{p}) = \mathfrak{p}$. *Suppose in addition that the secondary component of* A *with respect to* $\mathfrak{p}$ *has Noetherian dimension* d. *Then* $\operatorname{N-dim}(0 :_A \mathfrak{p}) = d$.

Proof of the claim. By Lemma 2.2, $\mathfrak{p} = \widehat{\mathfrak{p}} \cap R$ for some $\widehat{\mathfrak{p}} \in \operatorname{Att}_{\widehat{R}} A$. It is clear that $\operatorname{Ann}_{\widehat{R}}(0 :_A \widehat{\mathfrak{p}}) = \widehat{\mathfrak{p}}$. Therefore

$$\mathfrak{p} = \widehat{\mathfrak{p}} \cap R \supseteq \operatorname{Ann}_R(0 :_A \widehat{\mathfrak{p}}) \supseteq \operatorname{Ann}_R(0 :_A \mathfrak{p}) \supseteq \mathfrak{p}.$$

Assume that the secondary component of A with respect to $\mathfrak{p}$ has Noetherian dimension d. Then there is no parameter element of A in $\mathfrak{p}$. Therefore, by [18, 2.10], $\operatorname{N-dim}(0 :_A \mathfrak{p}) = \operatorname{N-dim} A = d$ and the claim is proved.

Now we prove the lemma. Assume that $(y_1, \ldots, y_k)$ is a minimal system of generators of $\mathfrak{a}$. We prove by induction on m, $m = 1, \ldots, k$, that there exist $x_1, \ldots, x_m$ such that $x_1, \ldots, x_m, y_{m+1}, \ldots, y_k$ is a minimal system of generators of $\mathfrak{a}$ and $(x_{i_1}, \ldots, x_{i_j})$ is a p.s.o.p of A for all $j \leq \min\{d, m\}$ and all integers $1 \leq i_1 < i_2 < \ldots < i_j \leq m$. The case $m = 0$ is trivial. Let $m > 0$. Assume that there exists elements $x_1, \ldots, x_{m-1}$ such that $(x_1, \ldots, x_{m-1}, y_m, \ldots, y_k)$ is a minimal system of generators of $\mathfrak{a}$ and for any positive integer j such that $j \leq \min\{m-1, d\}$, each subset of j elements in $\{x_1, \ldots, x_{m-1}\}$ is a p.s.o.p A. For every $j = 1, \ldots, \min\{m-1, d-1\}$, we denote by L_j the union of all the sets of all attached prime ideals $\mathfrak{p}$ of $0 :_A (x_{i_1}, \ldots, x_{i_j})R$ such that the secondary component with respect to $\mathfrak{p}$ has Noetherian dimension $d - j$, where the union runs over all $j-$tupes of integers $(i_1, \ldots, i_j)$ with $1 \leq i_1 < \ldots < i_j \leq m-1$. Denote by L the union of all L_j for $j = 1, \ldots, \min\{m-1, d-1\}$. Then by Lemma 2.3, the result is proved if we can choose an element $x_m \in \mathfrak{a}$ such that $x_m \notin \mathfrak{p}$ for all $\mathfrak{p} \in L$ and $(x_1, \ldots, x_m, y_{m+1}, \ldots, y_k)$ is a minimal system of generators of $\mathfrak{a}$. Since $j \leq \min\{m-1, d-1\}$, we have by the above claim that $\text{N-dim}(0 :_A \mathfrak{p}) > 0$ for all $\mathfrak{p} \in L_j$ and hence $\text{N-dim}(0 :_A \mathfrak{p}) > 0$ for all $\mathfrak{p} \in L$. Since $\ell(0 :_A \mathfrak{a}) < \infty$, it follows that $\mathfrak{a} \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in L$ and hence $(x_1, \ldots, x_{m-1}, y_m, y_{m+1}, \ldots, y_t)R \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in L$. So, there exists by [10, Theorem 124] an element $r \in (x_1, \ldots, x_{m-1}, y_{m+1}, \ldots, y_k)R$ such that $y_m + r \notin \mathfrak{p}$ for all $\mathfrak{p} \in L$. Set $x_m = y_m + r$. It is clear that $(x_1, \ldots, x_m, y_{m+1}, \ldots, y_k)$ is a minimal system of generators of $\mathfrak{a}$. Thus the lemma is proved. □

Remark. The equality $\text{Ann}(0 :_A \mathfrak{p}) = \mathfrak{p}$ (as in the above claim) does not hold for all prime ideals $\mathfrak{p} \supseteq \text{Ann}\, A$, see [5, 4.3], while it is clear that $\text{Ann}(M/\mathfrak{p}M) = \mathfrak{p}$ for all Noetherian $R-$module M and all prime ideals $\mathfrak{p} \supseteq \text{Ann}\, M$.

Proposition 5.3. *If A is co-Buchsbaum then $\mathfrak{m}H_i^{\mathfrak{m}}(A) = 0$ for all $i < d$.*

Proof. We prove by induction on d. Let $d = 1$. There exists by Lemma 5.2 a minimal system of generators $(x_1, \ldots, x_k)$ of $\mathfrak{m}$ such that $\ell(0 :_A x_i) < \infty$ for all $i \leq k$. Since $I(x_i; A) = I(x_i^2; A)$, $\ell(A/x_iA) = \ell(A/x_i^2A)$. Hence $x_iA = x_i^nA$ for all $n > 0$. Hence $x_iH_0^{\mathfrak{m}}(A) = 0$, for all $i \leq k$, and hence $\mathfrak{m}H_0^{\mathfrak{m}}(A) = 0$. Let $d > 1$. Let x be a parameter element of A. Set $x_1 = x^2$ and $A' = 0 :_A x_1$. Let $(x_2, \ldots, x_d)$ be an arbitrary s.o.p of A'. Since $\ell(A/x_1A) < \infty$, we imply that $I(x_2, \ldots, x_d; A') = I(x_1, \ldots, x_d; A)$. Therefore A' is co-Buchsbaum. So, by induction assumption, $\mathfrak{m}H_i^{\mathfrak{m}}(0 :_A x_1) = 0$ for all $i < d-1$. As in the proof of Proposition 3.7, (i)⇒(iii), $H_i^{\mathfrak{m}}(0 :_A x_1) \cong H_i^{\mathfrak{m}}(A) \oplus H_{i+1}^{\mathfrak{m}}(A)$ for all $i < d-1$. Thus $\mathfrak{m}H_i^{\mathfrak{m}}(A) = 0$ for all $i < d$. □

From Proposition 5.3, we can give examples of gCCM modules, not co-Buchsbaum modules: Let k be a field and $R = k[x_1, \ldots, x_d]$ the polynomial ring. Let $B = k[x_1^{-1}, \ldots, x_d^{-1}]$ be the Artinian $R-$modules of inverse polynomials. Let $n > 1$ be an integer and set $A = B \oplus R/(x_1^n, x_2, \ldots, x_d)R$. Then we have $\text{Supp}\, A = \{\mathfrak{m}\}$, where $\mathfrak{m} = (x_1, \ldots, x_d)R$, and A is gCCM, but not co-Buchsbaum since $\mathfrak{m}H_0^{\mathfrak{m}}(A) \neq 0$.

Theorem 5.4. *The following statements are equivalent:*

(i) A is co-Buchsbaum.

(ii) Every s.o.p of A is a weak co-sequence.

(iii) Every s.o.p of A is co-standard.

Proof. (i)⇒(iii) is trivial. We prove (i) ⇒ (ii) by induction on d. Let $d = 1$. Let x_1 be a parameter element of A. Since $\mathfrak{m}H_0^{\mathfrak{m}}(A) = 0$ by Proposition 5.3, $\mathfrak{m}A = \bigcap_{n\geq 0}\mathfrak{m}^n A$. Note that $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att} A \setminus \operatorname{Supp} A$ by Lemma 2.3 and Proposition 3.3. So, $x_1 A \supseteq \mathfrak{m}A$. Hence x_1 is a weak co-sequence. Let $d > 1$. Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p of A. Set $A' = 0 :_A x_1$. Let $(y_2, \ldots, y_d)$ be an arbitrary s.o.p of A'. Since $\ell(A/x_1A) < \infty$ and A is co-Buchsbaum,

$$I(y_2, \ldots, y_d; A') = I(x_1, y_2, \ldots, y_d; A) = I(A).$$

So A' is co-Buchsbaum. By induction assumption, $(x_2, \ldots, x_d)$ is a weak co-sequence of A'. Hence $\underline{x}$ is a weak co-sequence of A.

(ii)⇒(i). Let $\underline{x} = (x_1, \ldots, x_d)$ be an arbitrary s.o.p of A. By Lemma 3.6, $\mathfrak{m}H_i^{\mathfrak{m}}(A) = 0$ for all $i < d$. Set $A' = 0 :_A x_1$. Since every s.o.p of A' is again a weak co-sequence, $\mathfrak{m}H_i^{\mathfrak{m}}(0 :_A x_1) = 0$ for all $i < d-1$. Continuing this process we have $\mathfrak{m}H_i^{\mathfrak{m}}(0 :_A (x_1, \ldots, x_j)R) = 0$ for all $j \leq d$ and all $i \leq d - j - 1$. Therefore by induction on d and by using the exact sequence (*) in the proof of Proposition 3.3 we can show that

$$I(\underline{x}; A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^{\mathfrak{m}}(A)) = I(A).$$

Thus A is co-Buchsbaum.

(iii)⇒(i). Let $\underline{x}$ be an s.o.p of A. Then $\underline{x}$ is a co-standard. By Lemma 4.3, $I(x_1^n, \ldots, x_d^n; A) = I(\underline{x}; A)$ for all integer $n > 0$. By Theorem 4.4,(vi)⇒(ii), $\ell(H_i^{\mathfrak{m}}(A)) < \infty$ for all $i < d$. Therefore

$$I(\underline{x}; A) = I(x_1^n, \ldots, x_d^n; A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^{\mathfrak{m}}(A))$$

for $n \gg 0$, by Lemma 3.4. Thus A is co-Buchsbaum. □

Corollary 5.5. *(i) If A is co-Buchsbaum then* $\operatorname{N-dim} A_j = d$ *or* $\mathfrak{m}A_j = 0$ *for all* $j \leq t$.
(ii) If $I(\underline{x}; A) = 0$ *for all s.o.p* $\underline{x}$ *of A then* $\operatorname{N-dim} A_j = d$ *for all* $j \leq t$.

Proof. (i). By Corollary 4.5, $\operatorname{N-dim} A_j = 0$ or $\operatorname{N-dim} A_j = d$ for all $j \leq t$. Let $j \leq t$ such that $\operatorname{N-dim} A_j = 0$, i.e $\ell(A_j) < \infty$. Then $\mathfrak{m}A_j = \mathfrak{m}H_0^{\mathfrak{m}}(A_j) = 0$ by Proposition 5.3.
(ii). Assume that $I(\underline{x}; A) = 0$ for some s.o.p $\underline{x}$ of A. If $\operatorname{N-dim} A_j = 0$ for some $j \leq t$ then

$$0 = I(\underline{x}; A) \geq \ell(0 :_{A_j} \underline{x}R) - e(\underline{x}; A_j) = \ell(0 :_{A_j} \underline{x}R) \neq 0,$$

a contradiction. □

Corollary 5.6. *(i) Suppose that* $\text{N-dim}\, A_j = d$ *or* $\mathfrak{m}A_j = 0$ *for all* $j \le t$. *If* $A_{\mathfrak{p}}$ *is co-Buchsbaum for all* $\mathfrak{p} \in \operatorname{Supp} A$ *then* A *is co-Buchsbaum.*
(ii) Suppose that $\text{N-dim}\, A_j = d$ *for all* $j \le t$. *Then* $I(A) = 0$ *if and only if* $I(A_{\mathfrak{p}}) = 0$ *for all* $\mathfrak{p} \in \operatorname{Supp} A$.

Proof. (i). Let $\underline{x}$ be an arbitrary s.o.p of A. Let $j \le t$. If $\text{N-dim}\, A_j = d$ then $\underline{x}$ is a s.o.p of A_j and $I(\underline{x}; A_j) = I(A_j)$ since $A_j \cong A_{\mathfrak{m}_j}$ is co-Buchsbaum. If $\text{N-dim}\, A_j = 0$ then $\ell(0 :_{A_j} \underline{x}R) - e(\underline{x}; A_j) = \ell(A_j)$ since $\mathfrak{m}A_j = 0$. Therefore

$$I(\underline{x}; A) = \sum_{\text{N-dim}\, A_j = d} I(A_j) + \sum_{\text{N-dim}\, A_j = 0} \ell(A_j).$$

Thus A is co-Buchsbaum.
(ii). It is clear that $I(A) = 0$ if and only if $I(A_j) = 0$ for all $j \le t$. □

The following corollaries give some examples of co-Buchsbaum modules.

Corollary 5.7. *The following statements are true.*

(i) If A *is co-Buchsbaum then* $0 :_A x$ *is co-Buchsbaum for all parameter elements* x *of* A.

(ii) If $B_1, \ldots, B_n$ *is co-Buchsbaum* $R-$*modules such that* $\text{N-dim}\, B_i = d$ *or* $\mathfrak{m}B_i = 0$ *for all* $i \le n$ *then* $A = \oplus_{i=1}^n B_i$ *is co-Buchsbaum.*

Proof. (i). We prove by induction on d. The case $d = 1$ is trivial. Let $d > 1$ and $\underline{x}' = (x_2, \ldots, x_d)$ be an arbitrary s.o.p of $0 :_A x$. Since $\ell(A/xA) < \infty$ and A is co-Buchsbaum, $I(\underline{x}'; 0 :_A x) = I(x, x_2, \ldots, x_d; A) = I(A)$. Therefore $0 :_A x$ is co-Buchsbaum.
(ii). It follows by the same arguments as in the proof of Corollary 4.7. □

Corollary 5.8. *Assume that* $(R, \mathfrak{m})$ *is a local ring. Then we have*

(i) A *is a co-Buchsbaum* $R-$*module if and only if* A *is a co-Buchsbaum* $\widehat{R}-$*module.*

(ii) If M *is Buchsbaum then the Matlis dual* $D(M)$ *of* M *is co-Buchsbaum.*

(iii) If M *is Buchsbaum of dimension* d *then* $H^d_{\mathfrak{m}}(M)$ *is co-Buchsbaum.*

Proof. The proof is similar to the proof of Corollary 4.8 with notice that if M is Buchsbaum, the Matlis dual $D(H^d_{\mathfrak{m}}(M))$ of $H^d_{\mathfrak{m}}(M)$ is a Buchsbaum $\widehat{R}-$module, see [3]. □

Acknowledgments

This work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. Financial support from the Swedish International Development Cooperation Agency is acknowledged.

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