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ENERGY AND MORSE INDEX OF SOLUTIONS OF YAMABE TYPE PROBLEMS ON THIN ANNULI

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Abstract

In this paper we consider the following Yamabe type family of problem $(P_\varepsilon): \quad -\Delta u_\varepsilon = u_\varepsilon^{\frac{n+2}{n-2}}$, $u_\varepsilon > 0$ in A_ε, $u_\varepsilon = 0$ on ∂A_ε, where A_ε is an annulus-shaped domain of $\mathbb{R}^n$, $n \geq 3$, which becomes thinner when $\varepsilon \to 0$. We show that for every solution u_ε, the energy $\int_{A_\varepsilon} |\nabla u_\varepsilon|^2$, as well as the Morse index tends to infinity as $\varepsilon \to 0$. Such a result is proved through a fine blow-up analysis of some appropriate scalings of solutions whose limiting profiles are regular as well as singular solutions of some elliptic problem on $\mathbb{R}^n$, a half space or an infinite strip. Our argument involves also a Liouville-type theorem for regular solutions on the infinite strip.

1 Introduction and Main Results

In this paper we consider the following Yamabe type family of problem:

$$(P_\varepsilon) \qquad \begin{cases} -\Delta u_\varepsilon = u_\varepsilon^{\frac{n+2}{n-2}} & \text{in } A_\varepsilon \\ u_\varepsilon > 0 & \text{in } A_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial A_\varepsilon, \end{cases}$$

where A_ε is an annulus-shaped open domain of $\mathbb{R}^n$, $n \geq 3$ and ε is a small positive parameter. The domain A_ε becomes thinner as $\varepsilon \to 0$ (see the precise definition of A_ε below).

We define on $H_0^1(A_\varepsilon)$ the functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{A_\varepsilon} |\nabla u|^2 - \frac{n-2}{2n}\int_{A_\varepsilon} |u|^{\frac{2n}{n-2}} \tag{1.1}$$

whose positive critical points are solutions of (P_ε).

We denote by $m(u_\varepsilon)$ the Morse index of u_ε as a critical point of the functional J_ε, that is the number of negative eigenvalues of the linearized operator $-\Delta - \frac{n+2}{n-2}u_\varepsilon^{\frac{4}{n-2}}$ in $H_0^1(A_\varepsilon) \cap H^2(A_\varepsilon)$.

We are mainly concerned with what happens to the energy and the Morse index of u_ε when ε tends to zero. Our main motivation for investigating such a behavior of the solutions comes from the fact that information about the energy and or spectral properties is closely related to the existence and multiplicity of solutions of nonlinear equations having variational structure. It is also related to the geometric properties of the solutions in PDE problems. For details please see works of Bahri [2], Bahri-Lions [4], De Figueiredo-Yang [10], Lazer-Solimini [16], Pacella [18], Ramos-Terracini-Troestler [19], Solimini [24] and Yang [26], [27].

In [5], Bahri and Lions have shown that given a sequence of solutions of some superlinear and subcritical elliptic equations with Dirichlet boundary conditions in a fixed smooth and bounded domain of $\mathbb{R}^n$, $m(u_k) \to +\infty$ if and only if $||u_k||_{L^\infty} \to +\infty$, provided that the nonlinearity has a prescribed behavior at infinity. Later Harrabi-Rebhi-Selmi [14], Yang [27], and Aubin-Bahri [1] extended this result to more general subcritical nonlinearities.

In the critical case, Bénichou and Pomet [8] proved that for radial solutions on standard thin annulus, the energy and the Morse index tend to infinity. Our goal in this paper is to prove that this result holds true for all solutions, and also on nonstandard annuli.

To be more precise, we need to introduce some notations.
Let f be any smooth function

$$f : \mathbb{R}^{n-1} \longrightarrow [1,2]\ , (\theta_1, ..., \theta_{n-1}) \mapsto f(\theta_1, ..., \theta_{n-1})$$

which is periodic of period π with respect to $\theta_1, ..., \theta_{n-2}$ and of period 2π with respect to θ_{n-1}.
We set

$$S_1(f) = \{x \in \mathbb{R}^n / r = f(\theta_1, ..., \theta_{n-1})\}\,,$$

where $(r, \theta_1, ..., \theta_{n-1})$ are the polar coordinates of x.
For ε positive small enough, we introduce the following map

$$g_\varepsilon : S_1(f) \longrightarrow g_\varepsilon(S_1(f)) = S_2(f), \;\; x \longmapsto g_\varepsilon(x) = x + \varepsilon n_x,$$

where n_x is the outward normal to $S_1(f)$ at x. We denote by $(A_\varepsilon)_{\varepsilon>0}$ the family of annulus shaped open sets in $\mathbb{R}^n$ such that $\partial A_\varepsilon = S_1(f) \cup S_2(f)$.

Our main result is the following.

Theorem 1.1 *Let u_ε be any solution of (P_ε). We then have*

$$\begin{array}{llll} (i) & \displaystyle\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \to +\infty, & \textit{when} & \varepsilon \to 0 \\ (ii) & m(u_\varepsilon) \to +\infty, & \textit{when} & \varepsilon \to 0, \end{array}$$

where $m(u_\varepsilon)$ is the Morse index of u_ε as a critical point of the functional J_ε defined by (1.1).

Remark 1.2 *Statement (i) of Theorem 1.1 has been already proved in [6] and [7], using different arguments. However our argument, which is drastically different from theirs, proves at the same time the two statements displaying a deep connection between the energy and the spectral properties of the solutions.*

During the process to prove Theorem 1.1 we perform some blow up and find limit equations on $\mathbb{R}^n$ or a half space or an infinite strip, and it turns out that the following Liouville-type theorem that we prove in Section 4 is useful.

Theorem 1.3 *Let $u \in C^2(\Omega)$ be a positive bounded solution of*

$$(I) \qquad \left\{ \begin{array}{rcll} -\Delta u & = & u^{\frac{n+2}{n-2}} & \textit{in } \Omega \\ u & = & 0 & \textit{on } \partial\Omega \\ m(u) & < & \infty, & \end{array} \right.$$

where $m(u)$ is the number of negative eigenvalues of $-\Delta - \frac{n+2}{n-2} u^{\frac{4}{n-2}}$ in $H_0^1(\Omega) \cap H^2(\Omega)$ and where Ω is the strip defined by

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} / a < x_n < b\}, \quad a, b \in \mathbb{R}.$$

Then $u \equiv 0$ in Ω.

Our proof, which is by contradiction, relies on a careful analysis of successive scalings of the solutions. Such scalings give rise to singular solutions of limiting equations as well as regular ones. The analysis of the regular solutions is based on the above Liouville type Theorem, while the analysis of the singular case uses in a crucial way the blow up analysis introduced by R. Schoen, and studied extensively by Y.Y. Li. In particular, the isolated simple properties of the blow up points in the Yamabe equation on locally conformally flat manifolds, is a cornerstone in our analysis as well as the extensive use of Pohozaev identity. However, our analysis bears new features which are not present in the above mentioned works. A drastic difference is the fact that, unlike them our domain changes, and a big source of worry is that it may become degenerate during the blowing up process. Therefore our first aim is to scale in such a way that the limit domain does not degenerate.

Another main ingredient of the proof of Theorem 1.1 is to show that if the Morse index of the solutions is a positive integer m then also the number of blow up points of the solutions remains bounded by m. This is similar to what happens in other asymptotical critical problems described by El Mehdi-Pacella [11].

The organization of the remainder of the present paper is outlined as follows. In Section 2 we start our blowing up scheme, blowing up first at the global maximum of u_ε, then finding another point which escapes the first one under appropriate scaling, and conclude that each of them contribute to the total energy by at least a fixed amount. Section 3, devoted to the proof of Theorem 1.1, shows that the process started in section 2 does not stop after finitely many steps, and that each point contributes by at least one to the total index of u_ε, proving that both the energy and the Morse index must be infinite. In section 4 we prove Theorem 1.3, while in the Appendix we recall some well known facts about the blow up analysis of Yamabe type equations.

2 The Blowing up process

To prove Theorem 1.1, we argue by contradiction, that is, we suppose that (P_ε) has a solution u_ε which satisfies

$$(H_1) \qquad \int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \leq C_1 \quad \text{or} \qquad (H_2) \qquad m(u_\varepsilon) \leq C_2,$$

where C_1 and C_2 are given positive constants independent of ε.
We first recall the following result

Lemma 2.1 *[6] The following holds true*

1. $\displaystyle\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \not\longrightarrow 0$, *when* $\varepsilon \longrightarrow 0$.
2. $M_{1,\varepsilon} \longrightarrow +\infty$, *when* $\varepsilon \longrightarrow 0$, *where* $M_{1,\varepsilon} = ||u_\varepsilon||_{L^\infty(A_\varepsilon)}$.
3. $\exists\, c > 0$ *such that for* ε *small enough, we have* $\varepsilon M_{1,\varepsilon}^{\frac{2}{n-2}} \geq c$.

Now let $A_{1,\varepsilon} = M_{1,\varepsilon}^{\frac{2}{n-2}}(A_\varepsilon - a_{1,\varepsilon})$, where $a_{1,\varepsilon} \in A_\varepsilon$ such that $M_{1,\varepsilon} = u_\varepsilon(a_{1,\varepsilon})$, and we denote by v_ε the function defined on $A_{1,\varepsilon}$ by

$$v_\varepsilon(X) = M_{1,\varepsilon}^{-1} u_\varepsilon(a_{1,\varepsilon} + M_{1,\varepsilon}^{\frac{-2}{n-2}} X). \tag{2.1}$$

It is easy to see that v_ε satisfies

$$\begin{cases} -\Delta v_\varepsilon &= v_\varepsilon^{\frac{n+2}{n-2}}, \quad 0 < v_\varepsilon \leq 1 \qquad \text{in } A_{1,\varepsilon} \\ v_\varepsilon(0) &= 1, \qquad v_\varepsilon = 0 \qquad \text{on } \partial A_{1.\varepsilon}. \end{cases} \tag{2.2}$$

Due to Liouville type Theorems and Pohozaev identity on the limit domain, we have the following lemma:

Lemma 2.2 *There holds*

$$M_{1,\varepsilon}^{\frac{2}{n-2}} d(a_{1,\varepsilon}, \partial A_\varepsilon) \to +\infty, \quad when \ \varepsilon \to 0,$$

where $d(a_{1,\varepsilon}, \partial A_\varepsilon)$ denotes the distance of $a_{1,\varepsilon}$ to the boundary of A_ε.

Proof. Let $l = \lim_{\varepsilon\to 0} M_{1,\varepsilon}^{2/(n-2)} d(a_{1,\varepsilon}, \partial A_\varepsilon)$. According to the proof of Lemma 2.3 of [6], we have that $l > 0$. Arguing by contradiction, we suppose that $l < \infty$. Then it follows from (2.2) and standard elliptic theories that there exists some positive function v, such that (after passing to a subsequence), $v_\varepsilon \to v$ in $C^1_{loc}(\Omega)$, where Ω is a half space or a strip of $\mathbb{R}^n$, and v satisfies

$$\begin{cases} -\Delta v = v^{\frac{n+2}{n-2}}, & 0 < v \le 1 \quad \text{in} \quad \Omega \\ v(0) = 1, & v = 0 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

But if Ω is a half space, by [12], then v must vanish identically and thus we derive a contradiction. If Ω is a strip of $\mathbb{R}^n$ and condition (H_1) is satisfied, by Pohozaev Identity (see e.g. Theorem III.1.3 [24]), then $v \equiv 0$ and thus we also obtain a contradiction in this case. Lastly, if Ω is a strip of $\mathbb{R}^n$ and condition (H_2) is satisfied, by Theorem 1.3, we also find a contradiction. Thus our lemma follows. □

From Lemma 2.2, we derive that there exists some positive function v, such that (after passing to a subsequence), $v_\varepsilon \longrightarrow v$ in $C^1_{loc}(\mathbb{R}^n)$, and v satisfies

$$\begin{cases} -\Delta v = v^{\frac{n+2}{n-2}}, & v > 0 \quad \text{in} \quad \mathbb{R}^n \\ v(0) = 1, & \nabla v(0) = 0\,. \end{cases} \tag{2.3}$$

It follows from [9], that

$$v(X) = \delta_{(0,\alpha_n)}(X),$$

where $\alpha_n = (n(n-2))^{-1/2}$ and where, for $a \in \mathbb{R}^n$ and $\lambda > 0$, $\delta_{(a,\lambda)}$ denotes the function

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{\frac{n-2}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{n-2}{2}}}, \quad \text{with } c_0 = (n(n-2))^{\frac{n-2}{4}}. \tag{2.4}$$

We recall that $\delta_{(a,\lambda)}$ are the only minimizers for the Sobolev inequality

$$S = \inf\{||\nabla u||^2_{L^2(\mathbb{R}^n)} ||u||^{-2}_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}, \ s.t.\, |\nabla u| \in L^2, u \in L^{\frac{2n}{n-2}}, u \neq 0\}. \tag{2.5}$$

We note that, by the above arguments, we have for any $R > 0$

$$\int_{B(a_{1,\varepsilon}, \frac{R}{\lambda_{1,\varepsilon}})} u_\varepsilon^{\frac{2n}{n-2}}(x)dx \to \int_{B(0,R)} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}}(x)dx \qquad \text{as } \ \varepsilon \to 0, \tag{2.6}$$

where $\lambda_{1,\varepsilon} = M_{1,\varepsilon}^{2/(n-2)}$.

To proceed further, we introduce the following function

$$\tilde{u}_\varepsilon(X) = d_{1,\varepsilon}^{\frac{n-2}{2}} u_\varepsilon(a_{1,\varepsilon} + d_{1,\varepsilon} X), \quad X \in \tilde{A}_\varepsilon := d_{1,\varepsilon}^{-1}(A_\varepsilon - a_{1,\varepsilon}), \tag{2.7}$$

where $d_{1,\varepsilon} = d(a_{1,\varepsilon}, \partial A_\varepsilon)$.

Notice that from Lemma 2.2, we know that:

$$\tilde{u}_\varepsilon(0) = d_{1,\varepsilon}^{\frac{n-2}{2}} u_\varepsilon(a_{1,\varepsilon}) \to +\infty, \qquad \text{as } \ \varepsilon \to 0.$$

We observe that the limit domain of $\tilde{A}_\varepsilon$ is a strip or a half space of $\mathbb{R}^n$, we denote it by Π in both cases.

As a first step of our blowing up process, we prove the following proposition.

Proposition 2.3 *We have that*

$$h_\varepsilon := \max_{x \in A_\varepsilon} \left(|x - a_{1,\varepsilon}|^{\frac{n-2}{2}} u_\varepsilon(x) \right) \to +\infty, \quad \textit{as } \ \varepsilon \to 0.$$

Proof. Arguing by contradiction, we suppose that

$$h_\varepsilon \leq C, \quad \text{with } C \text{ is a positive constant independent of } \varepsilon.$$

Thus, we have

$$|X|^{(n-2)/2} \tilde{u}_\varepsilon(X) \leq C, \quad \forall X \in \tilde{A}_\varepsilon.$$

In particular, we have

$$\begin{cases} \tilde{u}_\varepsilon(X) \leq C|X|^{(2-n)/2}, & \forall X \in B(0,1/2) \setminus \{0\} \\ \tilde{u_\varepsilon}(0) \to +\infty\,. \end{cases}$$

Therefore 0 is an isolated blow up point of $\tilde{u_\varepsilon}$ (see the Appendix for definition). Then it follows from Proposition 5.6 that 0 is an isolated simple blow up (see Appendix for definition) in $B(0,1/2)$. Applying now Proposition 5.5 of the Appendix we derive that there exist positive constants c_1 and c_2 such that

$$c_1 \tilde{u}_\varepsilon(0)^{-1} |y|^{2-n} \leq \tilde{u}_\varepsilon(y) \leq c_2 \tilde{u}_\varepsilon(0)^{-1} |y|^{2-n}, \quad \text{for } \ |y| \leq (1/4).$$

Considering now the linear equation

$$\Delta u + V u = 0, \qquad \text{with } \ V = \tilde{u}_\varepsilon^{4/(n-2)},$$

we deduce from Lemma 5.3 and Harnack inequality(see [13]) that

$$\tilde{u}_\varepsilon(y) \leq c_2 \tilde{u}_\varepsilon(0)^{-1} |y|^{2-n}, \quad \forall y \in K, \tag{2.8}$$

where K is any compact set of $\tilde{A}_\varepsilon$ which does not contain 0.

Now we set

$$\tilde{v}_\varepsilon(X) = \tilde{u}_\varepsilon(0) \tilde{u}_\varepsilon(X).$$

It is easy to check that $\tilde{v}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \tilde{v}_\varepsilon & = & \tilde{u}_\varepsilon(0)^{\frac{-4}{n-2}} \tilde{v}_\varepsilon^{\frac{n+2}{n-2}}, \quad \tilde{v}_\varepsilon > 0 & \text{in} & \tilde{A}_\varepsilon \\ \tilde{v}_\varepsilon & = & 0, & \text{on} & \partial \tilde{A}_\varepsilon \end{cases}$$

and

$$\tilde{v}_\varepsilon(0) \to +\infty, \quad \text{as } \varepsilon \to 0$$
$$c_1|y|^{2-n} \le \tilde{v}_\varepsilon(y) \le c_2|y|^{2-n} \quad \forall y \in K,$$

where K is any compact set of $\tilde{A}_\varepsilon \setminus \{0\}$.
It follows from standard elliptic theories that

$$\tilde{v}_\varepsilon \to \alpha G_\Pi(0,.) \quad \text{in } \ C^2_{loc}(\Pi),$$

where $G_\Pi(0,.)$ is the Green's function of Laplacian operator with Dirichlet boundary condition defined on the limit domain Π (half space or strip) and where α is a positive constant.
Such a Green's function can be written as

$$G_\Pi(0,x) = |x|^{2-n} - H(0,x),$$

where by the Maximum principle $H(0,x) > 0$.
We now observe that $\tilde{u}_\varepsilon$ satisfies

$$-\Delta \tilde{u}_\varepsilon = \tilde{u}_\varepsilon^{\frac{n+2}{n-2}} \quad \text{in } \ B_r := B(0,r) \ \text{ for any } \ r < 1/2.$$

Applying Pohozaev Identity, see for example Corollary 1.1 of [17], we derive that

$$-r\frac{n-2}{2n}\int_{\partial B_r} \tilde{u}_\varepsilon^{\frac{2n}{n-2}} = \int_{\partial B_r} B(r,x,\tilde{u}_\varepsilon,\nabla \tilde{u}_\varepsilon), \tag{2.9}$$

where

$$B(r,x,\tilde{u}_\varepsilon,\nabla\tilde{u}_\varepsilon) = \frac{n-2}{2}\tilde{u}_\varepsilon\frac{\partial \tilde{u}_\varepsilon}{\partial \nu} - \frac{r}{2}|\nabla\tilde{u}_\varepsilon|^2 + r(\frac{\partial \tilde{u}_\varepsilon}{\partial \nu})^2.$$

On one hand, using (2.8), we obtain

$$r\frac{n-2}{2n}\int_{\partial B_r} \tilde{u}_\varepsilon^{\frac{2n}{n-2}} \le c\frac{\tilde{u}_\varepsilon(0)^{\frac{-2n}{n-2}}}{r^n}.$$

Multiplying (2.9) by $\tilde{u}_\varepsilon(0)^2$, we derive that

$$\tilde{u}_\varepsilon(0)^2\int_{\partial B_r} B(r,x,\tilde{u}_\varepsilon,\nabla\tilde{u}_\varepsilon) = O\left(\frac{\tilde{u}_\varepsilon(0)^{\frac{-4}{n-2}}}{r^n}\right).$$

Using the homogeneity of the operator B, we obtain

$$\int_{\partial B_r} B(r,x,\tilde{v}_\varepsilon,\nabla\tilde{v}_\varepsilon) = O\left(\frac{\tilde{u}_\varepsilon(0)^{\frac{-4}{n-2}}}{r^n}\right).$$

In particular, we conclude that

$$\lim_{\varepsilon\to 0}\int_{\partial B_r} B(r,x,\tilde{v}_\varepsilon,\nabla\tilde{v}_\varepsilon) = 0, \ \text{for } 0 < r < 1/2. \tag{2.10}$$

On the other hand, we have

$$\tilde{v}_\varepsilon \to \alpha G_\Pi(0,.) \qquad \text{in } C^2(\partial B_r), \text{ for } 0 < r < 1/2$$

and for r small enough

$$G_\Pi(0,x) = |x|^{2-n} - H(0,0) + o(|x|), \quad \text{with } |x| = r.$$

Thus we have

$$\lim_{\varepsilon\to 0, r\to 0} \int_{\partial B_r} B(r,x,\tilde{v}_\varepsilon,\nabla\tilde{v}_\varepsilon) = \frac{(n-2)^2}{2} H(0,0)\alpha^2 |S^{n-1}| > 0$$

which contradicts (2.10) and then our proposition follows. □

Let $a_{2,\varepsilon} \in A_\varepsilon$ such that

$$h_\varepsilon = |a_{2,\varepsilon} - a_{1,\varepsilon}|^{\frac{n-2}{2}} u_\varepsilon(a_{2,\varepsilon}),$$

where h_ε is defined in Proposition 2.3. Now if we blow up at the point $a_{2,\varepsilon}$, Proposition 2.3 implies that the image under the new scaling, of the first point $a_{1,\varepsilon}$ will escape to infinity, a fact that we express loosely by saying that these points *ignore themselves*. However the domain may become degenerate, that is its width becomes thinner and thinner along the blowing up process. The following Lemma rules out such a situation.

Lemma 2.4 *There exists $\delta > 0$ such that for every ε, we have that:*

$$\lambda_{2,\varepsilon}\varepsilon \geq \delta,$$

where $\lambda_{2,\varepsilon} = u_\varepsilon(a_{2,\varepsilon})^{2/(n-2)}$.

Proof. For $X \in B(0, \frac{\lambda_{2,\varepsilon}}{2}|a_{1,\varepsilon} - a_{2,\varepsilon}|) \cap D_\varepsilon$, we set

$$w_\varepsilon(X) = \lambda_{2,\varepsilon}^{\frac{2-n}{2}} u_\varepsilon(a_{2,\varepsilon} + \lambda_{2,\varepsilon}^{-1} X), \quad \text{with } D_\varepsilon = \lambda_{2,\varepsilon}(A_\varepsilon - a_{2,\varepsilon}). \tag{2.11}$$

Recall that, for any $x \in A_\varepsilon$, we have

$$|x - a_{1,\varepsilon}|^{\frac{n-2}{2}} u_\varepsilon(x) \leq |a_{2,\varepsilon} - a_{1,\varepsilon}|^{\frac{n-2}{2}} u_\varepsilon(a_{2,\varepsilon}) = |a_{2,\varepsilon} - a_{1,\varepsilon}|^{\frac{n-2}{2}} \lambda_{2,\varepsilon}^{\frac{n-2}{2}}.$$

Thus, for any $x \in A_\varepsilon$, we obtain

$$\frac{u_\varepsilon(x)}{\lambda_{2,\varepsilon}^{\frac{n-2}{2}}} \leq \frac{|a_{2,\varepsilon} - a_{1,\varepsilon}|^{\frac{n-2}{2}}}{|x - a_{1,\varepsilon}|^{\frac{n-2}{2}}}.$$

But, for $x \in B(a_{2,\varepsilon}, \frac{|a_{2,\varepsilon} - a_{1,\varepsilon}|}{2})$, we have

$$|x - a_{1,\varepsilon}| \geq \frac{|a_{2,\varepsilon} - a_{1,\varepsilon}|}{2}.$$

Hence

$$\frac{u_\varepsilon(x)}{\lambda_{2,\varepsilon}^{\frac{n-2}{2}}} \leq 2^{(n-2)/2}, \qquad \text{for any} \quad x \in B(a_{2,\varepsilon}, \frac{|a_{2,\varepsilon} - a_{1,\varepsilon}|}{2}).$$

Thus we obtain

$$w_\varepsilon(X) \leq 2^{\frac{n-2}{2}}, \quad \forall\, X \in B(0, \frac{\lambda_{2,\varepsilon}}{2}|a_{1,\varepsilon} - a_{2,\varepsilon}|) \cap D_\varepsilon.$$

Arguing by contradiction, we suppose that

$$\lambda_{2,\varepsilon}\varepsilon \to 0 \qquad \text{as} \ \ \varepsilon \to 0.$$

Let $\bar{a}_{2,\varepsilon} \in \partial D_\varepsilon$ such that $|\bar{a}_{2,\varepsilon}| = d(0, \partial D_\varepsilon)$. We may assume without loss of generality that the unit outward normal to ∂D_ε at $\bar{a}_{2,\varepsilon}$ is e_n, where e_n is the nth element of the canonical basis of $\mathbb{R}^n$.

Let

$$B(\bar{a}'_{2,\varepsilon}, 1) = \{x' \in \mathbb{R}^{n-1} / |x' - \bar{a}'_{2,\varepsilon}| < 1\},$$

where

$$\bar{a}_{2,\varepsilon} = (\bar{a}'_{2,\varepsilon}, \bar{a}^n_{2,\varepsilon}) \in \mathbb{R}^{n-1} \times \mathbb{R} \quad \text{and} \quad x = (x', x^n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Let

$$T_\varepsilon = \big(B(\bar{a}'_{2,\varepsilon}, 1) \times [-1, 1]\big) \cap D_\varepsilon$$

and

$$\partial T^1_\varepsilon = \partial(T_\varepsilon) \cap \partial D_\varepsilon \quad \text{and} \quad \partial T^2_\varepsilon = \partial(T_\varepsilon) \cap D_\varepsilon.$$

We denote by G_{T_ε} the Green's function of Laplace operator with Dirichlet boundary condition defined on T_ε. Let $X \in T_\varepsilon$ such that $X = \beta_\varepsilon e_n$, with $-1 \leq \beta_\varepsilon \leq 1$.

By easy computations, one can check that

$$\int_{T_\varepsilon} G_{T_\varepsilon}(X, y)dy \leq \int_{T_\varepsilon} \frac{dy}{|X - y|^{n-2}} = O\left(\lambda_{2,\varepsilon}\varepsilon\right).$$

Now we observe that

$$\begin{aligned} w_\varepsilon(X) &= c_n \left(\int_{T_\varepsilon} G_{T_\varepsilon}(X, y) w_\varepsilon^{\frac{n+2}{n-2}} dy - \int_{\partial T_\varepsilon} \frac{\partial G_{T_\varepsilon}}{\partial \nu}(X, y) w_\varepsilon(y) dy \right) \\ &= c_n \left(\int_{T_\varepsilon} G_{T_\varepsilon}(X, y) w_\varepsilon^{\frac{n+2}{n-2}} dy - \int_{\partial T^2_\varepsilon} \frac{\partial G_{T_\varepsilon}}{\partial \nu}(X, y) w_\varepsilon(y) dy \right), \end{aligned}$$

where c_n is a positive constant.

But, since $X = \beta_\varepsilon e_n$, we have

$$\frac{\partial G_{T_\varepsilon}}{\partial \nu}(X, y) \leq c, \quad \forall y \in \partial T^2_\varepsilon.$$

Since $w_\varepsilon \leq 2^{(n-2)/2}$, we derive that

$$\int_{T_\varepsilon} G_{T_\varepsilon}(X, y) w_\varepsilon^{\frac{n+2}{n-2}} dy = O\left(\lambda_{2,\varepsilon}\varepsilon\right) \quad \text{and} \quad \int_{\partial T^2_\varepsilon} \frac{\partial G_{T_\varepsilon}}{\partial \nu}(X, y) w_\varepsilon(y) dy = O\left(\lambda_{2,\varepsilon}\varepsilon\right).$$

Thus we obtain

$$w_\varepsilon(X) = O\left(\lambda_{2,\varepsilon}\varepsilon\right),$$

and in particular $w_\varepsilon(0) = 1 \leq c\lambda_{2,\varepsilon}\varepsilon$. Thus we derive a contradiction and therefore our lemma follows. □

Now, since $\lambda_{2,\varepsilon}\varepsilon \not\to 0$ as $\varepsilon \to 0$, we can prove, as in Lemma 2.2, that

$$\lambda_{2,\varepsilon} d(a_{2,\varepsilon}, \partial A_\varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to 0$$

and therefore there exist $b \in \mathbb{R}^n$ and $\mu > 0$ such that the function w_ε defined by (2.11) converges in $C^1_{loc}(\mathbb{R}^n)$ to $\delta_{(b,\mu)}$. Thus we have found a second blow up point $\bar{a}_{2,\varepsilon}$ of u_ε with the concentration $\bar{\lambda}_{2,\varepsilon}$ defined by

$$\bar{a}_{2,\varepsilon} = a_{2,\varepsilon} + \frac{b}{\lambda_{2,\varepsilon}}, \quad \text{and} \quad \bar{\lambda}_{2,\varepsilon} = \mu\lambda_{2,\varepsilon}.$$

Observe that $\bar{\lambda}_{2,\varepsilon}\,\varepsilon = \mu\lambda_{2,\varepsilon}\varepsilon \not\to 0$ as $\varepsilon \to 0$, and therefore as above we have that

$$\bar{\lambda}_{2,\varepsilon} d(\bar{a}_{2,\varepsilon}, \partial A_\varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to 0.$$

Summarizing, we have built two points $a_{1,\varepsilon}$, $\bar{a}_{2,\varepsilon}$ with concentrations $\lambda_{1,\varepsilon}$ and $\bar{\lambda}_{2,\varepsilon}$ such that

$$\lambda_{1,\varepsilon} d(a_{1,\varepsilon}, \partial A_\varepsilon) \to +\infty,\ \bar{\lambda}_{2,\varepsilon} d(\bar{a}_{2,\varepsilon}, \partial A_\varepsilon) \to +\infty, \quad \text{as} \quad \varepsilon \to 0, \tag{2.12}$$

$$\forall R > 0 \int_{B(a_{1,\varepsilon}, \frac{R}{\lambda_{1,\varepsilon}})} u_\varepsilon^{\frac{2n}{n-2}}(x)dx \to \int_{B(0,R)} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}}(x)dx \text{ as } \varepsilon \to 0, \tag{2.13}$$

$$\forall R > 0 \int_{B(\bar{a}_{2,\varepsilon}, \frac{R}{\lambda_{2,\varepsilon}})} u_\varepsilon^{\frac{2n}{n-2}}(x)dx \to \int_{B(b,\frac{R}{\mu})} \delta_{(b,\mu)}^{\frac{2n}{n-2}}(x)dx \text{ as } \varepsilon \to 0, \tag{2.14}$$

$$|a_{1,\varepsilon} - \bar{a}_{2,\varepsilon}|\lambda_{1,\varepsilon} \to +\infty,\ |a_{1,\varepsilon} - \bar{a}_{2,\varepsilon}|\bar{\lambda}_{2,\varepsilon} \to +\infty, \quad \text{as} \quad \varepsilon \to 0. \tag{2.15}$$

In this section , we have started a blowing up process, producing blow up points which *ignore each other*, and therefore contribute to the total energy by at least a fixed amount. Our goal now is to prove that such a process does not stop after finitely many steps. Such a fact is a key argument in the proof of Theorem 1.1. See Proposition 3.1, in the next section for a quantitative statement of this fact.

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To this aim we first prove that the process started in section 2 does not stop after finitely many steps, actually we have:

Proposition 3.1 *Let $\mathcal{S} = \{x_1^\varepsilon, \cdots, x_p^\varepsilon\}$, $p \geq 2$, be such that*

$$d(x_i^\varepsilon, \partial A_\varepsilon)^{\frac{n-2}{2}} u_\varepsilon(x_i^\varepsilon) \to \infty \text{ as } \varepsilon \to 0 \text{ for } 1 \leq i \leq p \tag{3.1}$$

$$|x_i^\varepsilon - x_j^\varepsilon|^{\frac{n-2}{2}} u_\varepsilon(x_j^\varepsilon) \to \infty \text{ as } \varepsilon \to 0 \text{ for } i \neq j,\ 1 \leq i, j \leq p. \tag{3.2}$$

Then

$$\max_{x \in A_\varepsilon} d(x, \mathcal{S})^{\frac{n-2}{2}} u_\varepsilon(x) \to \infty, \quad \text{as } \varepsilon \to 0.$$

Proof. Arguing by contradiction, we assume that:

$$\text{There exists } C>0, \text{ such that } d(x,\mathcal{S})^{\frac{n-2}{2}} u_\varepsilon(x) < C \quad \forall\, x \in A_\varepsilon.$$

Without loss of generality, we may assume that:

$$d(x_p^\varepsilon, \partial A_\varepsilon) = min_{1\le i\le p}\, d(x_i^\varepsilon, \partial A_\varepsilon)\,.$$

We set

$$d_p^\varepsilon = d(x_p^\varepsilon, \partial A_\varepsilon);\ w_\varepsilon(X) = d_p^{\varepsilon\,\frac{n-2}{2}}\, u_\varepsilon(d_p^\varepsilon X + x_p^\varepsilon);\ X_j^\varepsilon = \frac{x_j^\varepsilon - x_p^\varepsilon}{d_p^\varepsilon}.$$

Observe that $X_p^\varepsilon = 0$. We distinguish two cases:

- **1st case:** $min_{1\le i\le p-1}|X_i^\varepsilon| \le min_{i\ne j}\,|X_i^\varepsilon - X_j^\varepsilon|$.

  In this case we prove the following lemma:

  Lemma 3.2 *There exists $\delta>0$ such that*

  $$|X_i^\varepsilon| \ge \delta, \text{ for } i \in \{1,\cdots,p-1\}.$$

  Proof. Without loss of generality we may assume that $|X_1^\varepsilon| = min_{1\le i\le p-1}\,|X_i^\varepsilon|$. Arguing by contradiction, we assume that $\tau := |X_1^\varepsilon| \to 0$. Consider:

  $$\tilde{w_\varepsilon}(\tilde{X}) := \tau^{\frac{n-2}{2}}\, w_\varepsilon(\tau\tilde{X}), \text{where } \tilde{X} = \frac{X}{\tau}, \quad \text{so that } |\tilde{X_1^\varepsilon}| = 1.$$

  Observe that

  $$d(X,\mathcal{S}_1)^{\frac{n-2}{2}}\, w_\varepsilon(X) \le C, \quad \text{where } \mathcal{S}_1 = \{0, X_1^\varepsilon, \cdots, X_{p-1}^\varepsilon\}$$

  implies that:

  $$\begin{cases} |y|^{\frac{n-2}{2}}\, w_\varepsilon(y)\ \le C \text{ for all } |y|\ \le \frac{1}{2}\tau, \\ |y - X_1^\varepsilon|^{\frac{n-2}{2}}\, w_\varepsilon(y)\ \le C \text{ for all } |y - X_1^\varepsilon|\ \le \frac{1}{2}\tau. \end{cases}$$

  It follows that:

  $$\begin{cases} |y|^{\frac{n-2}{2}}\, \tilde{w_\varepsilon}(y)\ \le C \text{ for all } |y|\ \le \frac{1}{2}, \\ |y - \tilde{X_1^\varepsilon}|^{\frac{n-2}{2}}\, \tilde{w_\varepsilon}(y)\ \le C \text{ for all } |y - \tilde{X_1^\varepsilon}|\ \le \frac{1}{2}. \end{cases}$$

  Notice that:

  $$\tilde{w_\varepsilon}(0) = \tau^{\frac{n-2}{2}} w_\varepsilon(0) = |x_1^\varepsilon - x_p^\varepsilon|^{\frac{n-2}{2}}\, u_\varepsilon(x_p^\varepsilon) \to \infty$$
  $$\tilde{w_\varepsilon}(\tilde{X_1^\varepsilon}) = |x_1^\varepsilon - x_p^\varepsilon|^{\frac{n-2}{2}}\, u_\varepsilon(x_1^\varepsilon) \to \infty.$$

  It follows that 0 and $\tilde{X_1} := \lim_{\varepsilon\to 0} \tilde{X_1^\varepsilon}$ are isolated simple blow up, see the Appendix. Now it follows from standard elliptic theories and properties of isolated simple blow up, that

  $$\begin{cases} \lim_{\varepsilon\to 0}\ \tilde{w_\varepsilon}(0)\tilde{w_\varepsilon}(y) = h(y) \quad \text{in } C^0_{loc}(\Pi\setminus\tilde{\mathcal{S}_2}) \\ h(y) > 0, \quad y \in \Pi\setminus\tilde{\mathcal{S}_2}\,, \end{cases}$$

where Π is the limit domain after blowing up, h is harmonic outside its *singular set* $\tilde{\mathcal{S}}_2 \subset \mathcal{S}_2$, and $\mathcal{S}_2 = \{0, \tilde{X}_1, ..., \tilde{X}_{p-1}\}$, with $\tilde{X}_i = \lim_{\varepsilon \to 0} \tilde{X}_i^\varepsilon$ for $1 \le i \le p-1$.
Observing that $(0, \tilde{X}_1) \in \tilde{\mathcal{S}}_2$, we then deduce from Böcher's Theorem (see e.g. [15]) and the maximum principle that there exist some nonnegative function $b(y)$ and some positive constants M_1, M_2 such that:

$$\begin{cases} b(y) \ge 0, & y \in \Pi \setminus \{\tilde{\mathcal{S}}_2 \setminus \{0, \tilde{X}_1\}\} \\ \Delta\, b(y) = 0 & y \in \Pi \setminus \{\tilde{\mathcal{S}}_2 \setminus \{0, \tilde{X}_1\}\} \\ h(y) = M_1\, |y|^{2-n} \;+\; M_2\, |y - \tilde{X}_1|^{2-n} + b(y) & y \in \Pi \setminus \{\tilde{\mathcal{S}}_2 \setminus \{0, \tilde{X}_1\}\}. \end{cases}$$

Therefore for some constant $A > 0$, there holds:

$$h(y) = M_1\, |y|^{2-n} \;+ A + O(|y|) \quad \text{for } \; y \text{ close to } 0.$$

As usual we derive a contradiction like in the proof of Proposition 2.3. The proof of Lemma 3.2 is thereby completed. □

- **2nd case:** $min_{1 \le i \le p-1} |X_i^\varepsilon| > min_{i \ne j}\, |X_i^\varepsilon - X_j^\varepsilon|$.

Lemma 3.3 *There exists* $\delta > 0$, *such that*

$$min_{i \ne j} |X_i^\varepsilon - X_j^\varepsilon| \;\ge \delta \; for\; i\,, j \;\in\; \{1, \cdots, p-1\}.$$

Proof. Without loss of generality, we may assume that

$$|X_1^\varepsilon - X_2^\varepsilon| \;=\; min_{i \ne j}\, |X_i^\varepsilon - X_j^\varepsilon|.$$

Suppose by contradiction that:

$$\sigma_\varepsilon := |X_1^\varepsilon - X_2^\varepsilon| \to 0$$

and set

$$v_\varepsilon(y) = \sigma^{\frac{n-2}{2}}\; w_\varepsilon(\sigma_\varepsilon y + X_1^\varepsilon).$$

It follows that v_ε satisfies:

$$\begin{cases} -\,\Delta v_\varepsilon = v_\varepsilon^{\frac{n+2}{n-2}} \;\text{ in } |y| \le \frac{1}{\sigma_\varepsilon} \\ v_\varepsilon > 0 \text{ in } |y| \le \frac{1}{\sigma_\varepsilon}. \end{cases}$$

Setting Y_2^ε such that $X_2^\varepsilon = \sigma_\varepsilon\, Y_2^\varepsilon \;+ X_1^\varepsilon$, it is easy to see that:

$$\begin{cases} v_\varepsilon(y) \le C\, |y|^{-\frac{n-2}{2}}, \;\text{ for all } |y| \le \frac{1}{2}, & v_\varepsilon(0) \to \infty. \\ v_\varepsilon(y) \le C\, |y - Y_2^\varepsilon|^{-\frac{n-2}{2}}, \;\text{ for all } |y - Y_2^\varepsilon| \le \frac{1}{2}, & v_\varepsilon(Y_2^\varepsilon) \to \infty. \end{cases}$$

It follows that 0 and $Y_2 := \lim_{\varepsilon \to 0}\, Y_2^\varepsilon$ are isolated simple blow up, therefore arguing as in the first case, we derive a contradiction. □

Coming back to the proof of Proposition 3.1, we see that, from Lemmas 3.2 and 3.3, there exists $\delta > 0$, which does not depend on ε, such that:

$$\begin{cases} |X|^{\frac{n-2}{2}}\, w_\varepsilon(X) \le C & \text{for every } X \in B(0, \frac{\delta}{2}) \\ |X - X_1^\varepsilon|^{\frac{n-2}{2}}\, w_\varepsilon(X) \le C & \text{for every } X \in B(X_1^\varepsilon, \frac{\delta}{2}) \\ w_\varepsilon(0) = (d_p^\varepsilon)^{\frac{n-2}{2}}\, u_\varepsilon(x_p^\varepsilon) \to \infty. \end{cases}$$

We distinguish two cases:

- $$|X_1^\varepsilon| \to \infty \text{ as } \ \varepsilon \to 0.$$

  In this case 0 is the only isolated blow up point of w_ε and thus, arguing as in the proof of Proposition 2.3, we derive a contradiction.

- There exists a constant $C > 0$ such that $|X_1^\varepsilon| \le C$.

  In this case we argue as in the proof of Lemma 3.2 or Lemma 3.3 to derive a contradiction.

The proof of Proposition 3.1 is thereby completed. □

In the next proposition, we prove that at each blow up point constructed by our blowing up scheme, the projection on $H_0^1(A_\varepsilon)$ of the bubble concentrating there, contributes at least by one to the total Morse index of u_ε.

Proposition 3.4 *Let $\{x_1^\varepsilon, \cdots, x_p^\varepsilon\}$, $p \ge 2$, be such that:*

$$d(x_i^\varepsilon, \partial A_\varepsilon)^{\frac{n-2}{2}}\, u_\varepsilon(x_i^\varepsilon) \to \infty \ \text{ as } \varepsilon \to 0 \text{ for } 1 \le i \le p \tag{3.3}$$

$$|x_i^\varepsilon - x_j^\varepsilon|^{\frac{n-2}{2}}\, u_\varepsilon(x_j^\varepsilon) \to \infty \ \text{ as } \varepsilon \to 0 \text{ for } i \ne j,\ 1 \le i, j \le p. \tag{3.4}$$

Then $m(u_\varepsilon) \ge p$.

Proof. We begin by introducing some notation.
We denote by q the quadratic form associated to the linearized operator $-\Delta - \frac{n+2}{n-2} u_\varepsilon^{\frac{4}{n-2}}$ defined on $H_0^1(A_\varepsilon) \cap H^2(A_\varepsilon)$. Thus, for $v \in H_0^1(A_\varepsilon) \cap H^2(A_\varepsilon)$, we have

$$q(v) = \int_{A_\varepsilon} |\nabla v|^2 - \frac{n+2}{n-2} \int_{A_\varepsilon} u_\varepsilon^{\frac{4}{n-2}} v^2.$$

For $a \in A_\varepsilon$ and $\lambda > 0$, we denote by $P_\varepsilon \delta_{(a,\lambda)}$ the projection on $H_0^1(A_\varepsilon)$ of the function $\delta_{(a,\lambda)}$ defined in (2.4), that is

$$\Delta P_\varepsilon \delta_{(a,\lambda)} = \Delta \delta_{(a,\lambda)} \quad \text{in} \quad A_\varepsilon \quad \text{and} \quad P_\varepsilon \delta_{(a,\lambda)} = 0 \quad \text{on} \quad \partial A_\varepsilon.$$

In order to prove our proposition, it is sufficient to prove the following, for ε small,

$$q\left(\sum_{i=1}^{p} \alpha_i P_\varepsilon \delta_{(x_i^\varepsilon, \lambda_i^\varepsilon)} \right) < 0 \quad \forall \alpha_i \in \mathbb{R}, \tag{3.5}$$

where $\lambda_i^\varepsilon = (u_\varepsilon(x_i^\varepsilon))^{2/(n-2)}$.

To simplify our notation we will write, in the sequel, $P_\varepsilon\delta_i$ and δ_i instead of $P_\varepsilon\delta_{(x_i^\varepsilon,\lambda_i^\varepsilon)}$ and $\delta_{(x_i^\varepsilon,\lambda_i^\varepsilon)}$ respectively.

Now, we observe that

$$\begin{aligned} q\left(\sum_{i=1}^p \alpha_i P_\varepsilon\delta_i\right) &= \sum_{i=1}^p \alpha_i^2 \int_{A_\varepsilon} |\nabla P_\varepsilon\delta_i|^2 + \sum_{i\neq j} \alpha_i\alpha_j \int_{A_\varepsilon} \nabla P_\varepsilon\delta_i \nabla P_\varepsilon\delta_j \\ &- \frac{n+2}{n-2}\int_{A_\varepsilon} u_\varepsilon^{\frac{4}{n-2}} \left(\sum_{i=1}^p \alpha_i^2 P_\varepsilon\delta_i^2 + \sum_{i\neq j}\alpha_i\alpha_j P_\varepsilon\delta_i P_\varepsilon\delta_j\right) \\ &= \sum_{i=1}^p \alpha_i^2 q(P_\varepsilon\delta_i) + \sum_{i\neq j}\alpha_i\alpha_j \int_{A_\varepsilon} \nabla P_\varepsilon\delta_i\nabla P_\varepsilon\delta_j \\ &- \frac{n+2}{n-2}\sum_{i\neq j}\alpha_i\alpha_j \int_{A_\varepsilon} u_\varepsilon^{\frac{4}{n-2}} P_\varepsilon\delta_i P_\varepsilon\delta_j \\ &\leq \sum_{i=1}^p \alpha_i^2 q(P_\varepsilon\delta_i) + \sum_{i\neq j}\alpha_i\alpha_j \int_{A_\varepsilon} \nabla P_\varepsilon\delta_i\nabla P_\varepsilon\delta_j. \end{aligned} \tag{3.6}$$

But, on one hand, one can check that (see [3])

$$\int_{A_\varepsilon} \nabla P_\varepsilon\delta_i \nabla P_\varepsilon\delta_j = O\left(\left(\frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} + \frac{\lambda_j^\varepsilon}{\lambda_i^\varepsilon} + \lambda_i^\varepsilon\lambda_j^\varepsilon|x_i^\varepsilon - x_j^\varepsilon|^2\right)^{\frac{-(n-2)}{2}}\right) \qquad \forall i\neq j$$

and therefore, using assumption (3.4), we derive that

$$\int_{A_\varepsilon} \nabla P_\varepsilon\delta_i\nabla P_\varepsilon\delta_j = o(1), \quad \forall i \neq j. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} q(P_\varepsilon\delta_i) &= \int_{A_\varepsilon} |\nabla P_\varepsilon\delta_i|^2 - \frac{n+2}{n-2}\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} u_\varepsilon^{\frac{4}{n-2}} P_\varepsilon\delta_i^2 - \frac{n+2}{n-2}\int_{A_\varepsilon\setminus B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} u_\varepsilon^{\frac{4}{n-2}} P_\varepsilon\delta_i^2 \\ &\leq \int_{A_\varepsilon} |\nabla P_\varepsilon\delta_i|^2 - \frac{n+2}{n-2}\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} u_\varepsilon^{\frac{4}{n-2}} P_\varepsilon\delta_i^2, \end{aligned}$$

where R is a large positive constant such that $\int_{\mathbb{R}^n\setminus B(0,R)} \delta_{0,\alpha_n}^{\frac{2n}{n-2}} = o(1)$, here $\alpha_n = (n(n-2))^{-1/2}$.

Notice that

$$\int_{A_\varepsilon} |\nabla P_\varepsilon\delta_i|^2 = \int_{A_\varepsilon} \delta_i^{\frac{n+2}{n-2}} P_\varepsilon\delta_i = \int_{A_\varepsilon}\delta_i^{\frac{2n}{n-2}} - \int_{A_\varepsilon} \delta_i^{\frac{n+2}{n-2}}\left(\delta_i - P_\varepsilon\delta_i\right).$$

For the second integral, we have

$$\int_{A_\varepsilon} \delta_i^{\frac{n+2}{n-2}}\left(\delta_i - P_\varepsilon\delta_i\right) \leq c|\delta_i - P_\varepsilon\delta_i|_{L^{\frac{2n}{n-2}}(A_\varepsilon)} \leq c\left(\lambda_i^\varepsilon d(x_i^\varepsilon, \partial A_\varepsilon)\right)^{\frac{2-n}{2}} \to 0, \quad \text{as } \varepsilon \to 0,$$

where we have used in the last inequality the assumption (3.3) and Proposition 1 of [20].

Thus we have

$$\int_{A_\varepsilon} |\nabla P_\varepsilon\delta_i|^2 = \int_{A_\varepsilon} \delta_i^{\frac{2n}{n-2}} + o(1).$$

We also have

$$\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} u_\varepsilon^{\frac{4}{n-2}} P_\varepsilon \delta_i^2 = \int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} \delta_i^{\frac{4}{n-2}} P_\varepsilon \delta_i^2 + O\left(\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} |u_\varepsilon - \delta_i|^{\frac{4}{n-2}} P_\varepsilon \delta_i^2\right).$$

Thus, using the following

$$\begin{aligned}\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} \delta_i^{\frac{4}{n-2}} P_\varepsilon \delta_i^2 &= \int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} \delta_i^{\frac{2n}{n-2}} + \int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} \delta_i^{\frac{4}{n-2}} (\delta_i - P_\varepsilon \delta_i)^2 \\ &\quad - 2\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} \delta_i^{\frac{n+2}{n-2}} (\delta_i - P_\varepsilon \delta_i),\end{aligned}$$

$$\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} \delta_i^{\frac{4}{n-2}} (\delta_i - P_\varepsilon \delta_i)^2 \leq c|\delta_i - P_\varepsilon \delta_i|^2_{L^{\frac{2n}{n-2}}(A_\varepsilon)} = o(1),$$

we derive that

$$\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} u_\varepsilon^{\frac{4}{n-2}} P_\varepsilon \delta_i^2 = \int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} \delta_i^{\frac{2n}{n-2}} + o(1) + O\left(\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} |u_\varepsilon - \delta_i|^{\frac{4}{n-2}} P_\varepsilon \delta_i^2\right).$$

Therefore we obtain

$$q(P_\varepsilon \delta_i) \leq \int_{A_\varepsilon} \delta_i^{\frac{2n}{n-2}} - \frac{n+2}{n-2}\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} \delta_i^{\frac{2n}{n-2}} + O\left(\int_{B\left(x_i^\varepsilon,\frac{R}{\lambda_i^\varepsilon}\right)} |u_\varepsilon - \delta_i|^{\frac{4}{n-2}} \delta_i^2\right) + o(1). \tag{3.8}$$

Now, letting $A_\varepsilon^i = \lambda_i^\varepsilon (A_\varepsilon - x_i^\varepsilon)$ and setting, for $X \in A_\varepsilon^i$,

$$v_i^\varepsilon(X) = \frac{1}{(\lambda_i^\varepsilon)^{\frac{n-2}{2}}} u_\varepsilon(x_i^\varepsilon + \frac{X}{\lambda_i^\varepsilon}),$$

we know that $v_i^\varepsilon \to \delta_{0,\alpha_n}$ in $C^1_{loc}(\mathbb{R}^n)$. Thus (3.8) becomes

$$\begin{aligned} q(P_\varepsilon \delta_i) &\leq \int_{A_\varepsilon^i} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}} - \frac{n+2}{n-2}\int_{B(0,R)} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}} + O\left(\int_{B(0,R)} |v_i^\varepsilon - \delta_{(0,\alpha_n)}|^{\frac{4}{n-2}} \delta^2_{(0,\alpha_n)}\right) + o(1) \\ &= \frac{-4}{n-2}\int_{\mathbb{R}^n} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}} - \int_{\mathbb{R}^n \setminus A_\varepsilon^i} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}} + \frac{n+2}{n-2}\int_{\mathbb{R}^n \setminus B(0,R)} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}} + o(1).\end{aligned}$$

Since $A_\varepsilon^i \to \mathbb{R}^n$, we deduce that

$$q(P_\varepsilon \delta_i) \leq \frac{-4}{n-2} S^{n/2} + o(1), \tag{3.9}$$

where S is the Sobolev constant defined by (2.5).

Clearly, (3.6),...,(3.9) give (3.5) and therefore our result follows. □

Proof of Theorem 1.1 Arguing by contradiction, we assume that either the energy is uniformly bounded (H_1), or the Morse index is uniformly bounded (H_2). Using the results of Section 2, we start a blowing up process, which enables us to gain at each step at least a fixed amount of energy, and at least one in the Morse index. Namely at the k-th step, we have constructed k points $(a_{1,\varepsilon}, \cdots, a_{k,\varepsilon})$ with concentrations $(\lambda_{1,\varepsilon}, \cdots, \lambda_{k,\varepsilon})$ satisfying

$$\forall i \in \{1, \cdots, k\}, \quad \lambda_{i,\varepsilon} d(a_{i,\varepsilon}, \partial A_\varepsilon) \to +\infty, \quad \text{as} \quad \varepsilon \to 0, \tag{3.10}$$

$$\forall R > 0 \int_{B(a_{1,\varepsilon}, \frac{R}{\lambda_{1,\varepsilon}})} u_\varepsilon^{\frac{2n}{n-2}}(x)dx \to \int_{B(0,R)} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}}(x)dx \text{ as } \varepsilon \to 0, \tag{3.11}$$

$$\forall i \neq 1 \ \forall R > 0 \int_{B(a_{i,\varepsilon}, \frac{R}{\lambda_{i,\varepsilon}})} u_\varepsilon^{\frac{2n}{n-2}}(x)dx \to \int_{B(b_i, \frac{R}{\mu_i})} \delta_{(b_i,\mu_i)}^{\frac{2n}{n-2}}(x)dx \text{ as } \varepsilon \to 0, \tag{3.12}$$

$$\forall i \neq j \in \{1, \cdots, k\}, \ |a_{i,\varepsilon} - a_{j,\varepsilon}|\lambda_{i,\varepsilon} \to +\infty, \quad \text{as} \quad \varepsilon \to 0. \tag{3.13}$$

Therefore we derive that:

$$\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \geq kS^{\frac{n}{2}} \qquad \text{and } m(u_\varepsilon) \geq k.$$

Then using Propositions 3.1 and 3.4, we derive that such a process does not stop after finitely many steps, contradicting our assumption (H_1), respectively (H_2). Our Theorem follows. □

4 A Liouville type Theorem

This section is devoted to prove the Liouville type Theorem, Theorem 1.3 stated in the introduction.

The main idea is to use the spectral information to gain more integrability of the solution, and this is the content of the next two lemmas.

Lemma 4.1 *Let u be a positive bounded solution of (I). We then have*

$$\int_\Omega u^{\frac{2n}{n-2}}(x)dx < +\infty.$$

Proof. Without loss of generality, we may translate the origin in such a way that

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}/0 < x_n < k\}, \quad (k \text{ is a fixed real}).$$

We denote by q the quadratic form associated to the linearized operator $(-\Delta - \frac{n+2}{n-2} u^{\frac{4}{n-2}})$ defined on $H_0^1(\Omega) \cap H^2(\Omega)$.

For $h \in H_0^1(\Omega) \cap H^2(\Omega)$, we have

$$q(h) = \int_\Omega |\nabla h|^2 - \frac{n+2}{n-2} \int_\Omega u^{\frac{4}{n-2}} h^2.$$

Let $d_0 > 0$, and for $R > 2d_0$, we set

$$\Omega_R = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}/|x'| < R, 0 < x_n < k\}.$$

Now we introduce the following function

$$\varphi_{d_0,R}(x) = \begin{cases} 0 & \text{if} \quad r \le d_0 \\ \frac{r-d_0}{d_0} & \text{if} \quad d_0 \le r \le 2d_0 \\ 1 & \text{if} \quad 2d_0 \le r \le R \\ \frac{2R-r}{R} & \text{if} \quad R \le r \le 2R \\ 0 & \text{if} \quad r \ge 2R, \end{cases}$$

where $r = |x'|$.

We distinguish two cases:

Case i : $\forall R > 2d_0 \quad \forall \alpha \in (0,1) \quad q(\varphi_{d_0,R}u^{1+\alpha}) \ge 0.$

Case ii : $\exists R_1 > 2d_0 \quad \exists \alpha_1 \in (0,1)$ such that $q(\varphi_{d_0,R_1}u^{1+\alpha_1}) < 0.$

Now, we study the first case, that is

$$\forall\, R > 2d_0 \quad \forall \alpha \in (0,1) \quad q(\varphi_{d_0,R}u^{1+\alpha}) \ge 0.$$

Expanding $q(\varphi_{d_0,R}u^{1+\alpha})$ and letting α tend to zero, we obtain

$$\int_{\Omega_{2R}} |\nabla u|^2 \varphi_{d_0,R}^2 - \frac{n+2}{n-2}\int_{\Omega_{2R}} u^{\frac{2n}{n-2}} \varphi_{d_0,R}^2 \ge \int_{\Omega_{2R}} u^2 (\Delta \varphi_{d_0,R})\varphi_{d_0,R}. \tag{4.1}$$

Now, multiplying the equation $-\Delta u = u^{\frac{n+2}{n-2}}$ by $u^{1+\alpha}\varphi_{d_0,R}^2$ and integrating by parts on Ω_{2R} and letting α tend to zero, we find that

$$\int_{\Omega_{2R}} |\nabla u|^2 \varphi_{d_0,R}^2 - \int_{\Omega_{2R}} u^{\frac{2n}{n-2}} \varphi_{d_0,R}^2 = \frac{1}{2}\int_{\Omega_{2R}} u^2 \Delta(\varphi_{d_0,R}^2). \tag{4.2}$$

From (4.1) and (4.2), we derive that

$$\frac{4}{n-2}\int_{\Omega_{2R}} u^{\frac{2n}{n-2}} \varphi_{d_0,R}^2 \le \int_{\Omega_{2R}} u^2 \left(\frac{1}{2}\Delta(\varphi_{d_0,R}^2) - \Delta\varphi_{d_0,R}.\varphi_{d_0,R}\right).$$

Since

$$\Delta(\varphi_{d_0,R}^2) = 2\varphi_{d_0,R}\Delta\varphi_{d_0,R} + 2|\nabla\varphi_{d_0,R}|^2,$$

we derive that

$$\frac{4}{n-2}\int_{\Omega_{2R}} u^{\frac{2n}{n-2}} \varphi_{d_0,R}^2 \le \int_{\Omega_{2R}} u^2 |\nabla\varphi_{d_0,R}|^2.$$

Thus

$$\frac{4}{n-2}\int_{2d_0<r<R} u^{\frac{2n}{n-2}} \le \int_{\Omega_{2R}} u^2 |\nabla\varphi_{d_0,R}|^2.$$

We now observe that

$$\text{for} \quad R \le r \le 2R, \text{ we have} \qquad \frac{\partial\varphi_{d_0,R}(x)}{\partial x_i} = -\frac{x_i}{rR}, \qquad \text{for} \quad 1 \le i \le n-1$$

$$\text{for} \quad d_0 \le r \le 2d_0, \text{ we have} \qquad \frac{\partial\varphi_{d_0,R}(x)}{\partial x_i} = \frac{x_i}{rd_0}, \qquad \text{for} \quad 1 \le i \le n-1,$$

and therefore

$$|\nabla\varphi_{d_0,R}(x)|^2 = \frac{1}{R^2} \qquad \text{for} \quad R \le r \le 2R$$
$$|\nabla\varphi_{d_0,R}(x)|^2 = \frac{1}{d_0^2} \qquad \text{for} \quad d_0 \le r \le 2d_0.$$

Thus

$$\int_{\Omega_R} u^{\frac{2n}{n-2}} \le \frac{1}{R^2}\int_{\Omega_{2R}} u^2 + c(d_0),$$

where $c(d_0)$ is a positive constant depending only on d_0 and n.
Using Hölder's inequality, we find that

$$\int_{\Omega_R} u^{\frac{2n}{n-2}} \le \frac{c}{R^2}\left(\int_{\Omega_{2R}} u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} R^{\frac{2(n-1)}{n}} + c(d_0).$$

That is,

$$\int_{\Omega_R} u^{\frac{2n}{n-2}} \le \frac{c}{R^{2/n}}\left(\int_{\Omega_{2R}} u^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + c(d_0). \tag{4.3}$$

Since $0 \le u \le c$ on Ω, we deduce that

$$\int_{\Omega_{2R}} u^{\frac{2n}{n-2}} \le cR^{n-1}.$$

Therefore by (4.3), we have

$$\int_{\Omega_R} u^{\frac{2n}{n-2}} \le cR^{n-3} + c(d_0).$$

We insert this bound in (4.3) and iterate this argument, we obtain that

$$\int_{\Omega_R} u^{\frac{2n}{n-2}} \le cR^{\alpha_p} + c'(d_0)$$

with $\alpha_0 = n-3$, $\alpha_{p+1} = \frac{n-2}{n}\alpha_p - \frac{2}{n}$ and $c'(d_0)$ is a positive constant depending only on d_0 and n.
It is easy to see that α_p converges to -1 when p tends to ∞. Taking p_0 be such that $\alpha_{p_0} < 0$, we then derive

$$\int_{\Omega} u^{\frac{2n}{n-2}} < \infty.$$

in the first case.
In the second case, we have

$$\exists R_1 > 2d_0 \quad \exists \alpha_1 \in (0,1) \qquad \text{such that} \qquad q(\varphi_{d_0,R_1} u^{1+\alpha_1}) < 0.$$

That is, the Morse index of u is at least 1.
Now we consider $d_1 > 2R_1$. Then either we have $q(\varphi_{d_1,R} u^{1+\alpha}) \ge 0$ for all $R > 2d_1$ and for all $\alpha \in (0,1)$ (as in the first case we prove that $\int_\Omega u^{\frac{2n}{n-2}} < \infty$) or there exist $R_2 > 2d_1$, $\alpha_2 \in (0,1)$ such that $q(\varphi_{d_1,R_2} u^{1+\alpha_2}) < 0$. Since $d_1 > 2R_1$, the supports of φ_{d_0,R_1} and φ_{d_1,R_2} are disjoints

and therefore the Morse index of u is larger than or equal to 2. We iterate again this argument. Since $m(u) < \infty$, there exists $d > 0$ such that

$$q\left(\varphi_{d,R}u^{1+\alpha}\right) \geq 0, \quad \forall \alpha \in (0,1), \quad \forall R > 2d. \tag{4.4}$$

Then, as in the first case, we prove (4.4) implies

$$\int_\Omega u^{\frac{2n}{n-2}} < \infty.$$

Therefore our lemma follows. □

Lemma 4.2 *Let u be a positive bounded solution of (I). We then have*

$$\int_\Omega |\nabla u|^2 dx < \infty.$$

Proof. For $\varepsilon > 0$ small let $h = h_\varepsilon \in C_c^1(\Omega)$ be a cut-of function such that

$$0 \leq h \leq 1, \; h(x) = 1 \; \text{if} \, x \in \Omega_{\frac{1}{\varepsilon}}, \; h(x) = 0 \; \text{in} \, \Omega \setminus \Omega_{\frac{2}{\varepsilon}} \; \text{and} \, |\nabla h| \leq 2\varepsilon \; \text{in} \, \Omega_{\frac{2}{\varepsilon}} \setminus \Omega_{\frac{1}{\varepsilon}},$$

where, for $l > 0$ Ω_l is the set of Ω defined by

$$\Omega_l = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} / |x'| < l \text{ and } 0 < x_n < k\}.$$

(We recall that after translation we may suppose that $\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}/0 < x_n < k\}$ k is a fixed real.)
We then test the equation

$$-\Delta u = u^{\frac{n+2}{n-2}}$$

with the function $\varphi = \varphi_\varepsilon = uh^2$ to obtain estimates for the function $\psi = \psi_\varepsilon = uh$.
Observe that

$$\begin{aligned}
\nabla \varphi &= h^2 \nabla u + 2uh\nabla h \\
\nabla u \nabla \varphi &= h^2 |\nabla u|^2 + 2uh\nabla h.\nabla u \\
|\nabla \psi|^2 &= h^2|\nabla u|^2 + 2uh\nabla u \nabla h + u^2|\nabla h|^2 \\
&= \nabla u \nabla \varphi + u^2 |\nabla h|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_\Omega |\nabla \psi|^2 &= \int_\Omega \nabla u \nabla \varphi + \int_\Omega u^2 |\nabla h|^2 \\
&= \int_\Omega u^{\frac{2n}{n-2}} h^2 + \int_\Omega u^2 |\nabla h|^2.
\end{aligned}$$

Using Lemma 4.1, Hölder's inequality and the fact that

$$|\nabla h| \leq 2\varepsilon \quad \text{in} \quad \Omega_{\frac{2}{\varepsilon}} \setminus \Omega_{\frac{1}{\varepsilon}},$$

we derive that

$$\int_\Omega |\nabla\psi|^2 \leq \int_\Omega u^{\frac{2n}{n-2}} h^2 + c.$$

Letting ε tend to zero, we derive our lemma. $\square$

Proof of Theorem 1.3

Using Lemma 4.1 and Lemma 4.2 and Pohozaev identity, we derive that u vanishes identically (see e.g Theorem 1.3, p 156 [25]). $\square$

5 Appendix : blow up analysis

In this appendix, we give the definitions, and recall basic properties of isolated and isolated simple blow-up, which were first introduced by R. Schoen [21], [22], [23] and extensively studied by Y.Y. Li [17].

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded smooth domain. Consider the family of equations

$$-\Delta u_i = u_i^{\frac{n+2}{n-2}}, \quad u_i > 0 \quad \text{in} \quad \Omega. \tag{5.1}$$

The aim of the blow up analysis is to describe the behavior of u_i when i tends to infinity. It follows from standard elliptic regularity that if $\{u_i\}_i$ remains bounded in $L^\infty_{loc}(\Omega)$, then for any $\alpha \in (0,1)$ $u_i \to u$ in $C^{2,\alpha}_{loc}(\Omega)$ along some subsequence. Otherwise, we say that $\{u_i\}_i$ *blows up.* Let $B_r(x) = \{y \in \mathbb{R}^n \,:\, |y-x| < r\}$.

Definition 5.1 *Suppose that $\{u_i\}_i$ satisfy* (5.1). *A point $\overline{y} \in \Omega$ is called a blow up point for $\{u_i\}_i$ if there exists $y_i \to \overline{y}$, such that $u_i(y_i) \to +\infty$.*

In the sequel, if $\overline{y}$ is a blow up point for $\{u_i\}_i$, writing $y_i \to \overline{y}$ we mean that, $y_i \to \overline{y}$ and $u_i(y_i) \to +\infty$ as $i \to +\infty$.

Definition 5.2 *Assume that $y_i \to \overline{y}$ is a blow up point for $\{u_i\}_i$. The point $\overline{y} \in \Omega$ is called an* isolated blow up point *if there exist $\overline{r} \in (0, d(\overline{y}, \partial\Omega))$ and $\overline{C} > 0$ such that*

$$u_i(y) \leq \overline{C}\,|y - y_i|^{-\frac{n-2}{2}}, \qquad \textit{for all } y \in B_{\overline{r}}(y_i) \cap \Omega. \tag{5.2}$$

Isolated blow up enjoys nice properties, such as a Harnack inequality around singular points:

Lemma 5.3 *[17] Let u_i satisfy* (5.1) *and $y_i \to \bar{y} \in B_3$ be an isolated blow-up of $\{u_i\}_i$. Then for any $0 < r < \bar{r}$, we have*

$$\max_{B_{2r}(y_i)\setminus B_{r/2}(y_i)} u_i \leq C_3 \min_{B_{2r}(y_i)\setminus B_{r/2}(y_i)} u_i,$$

where C_3 is some positive constant independent of i and r.

The property of being isolated prevents accumulation of blow up points, however it does not prevent the superposition of bubbles over bubbles. For this we need the notion of isolated simple blow up. Let $y_i \to \overline{y}$ be an isolated blow up point for $\{u_i\}_i$, we define $\overline{u}_i(r)$ to be (here $|\partial B_r|$ is the $n-1$-dimensional volume of ∂B_r)

$$\overline{u}_i(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(y_i)} u_i, \quad r \in (0, d(y_i, \partial\Omega)), \tag{5.3}$$

and

$$\hat{u}_i(r) = r^{\frac{n-2}{2}} \overline{u}_i(r), \quad r \in (0, d(y_i, \partial\Omega)).$$

Definition 5.4 *An isolated blow up point $\overline{y} \in \Omega$ for $\{u_i\}_i$ is called an* isolated simple blow up point *if there exists some $\varrho \in (0, \overline{r})$, independent of i, such that $\hat{u}_i(r)$ has precisely one critical point in $(0, \varrho)$ for large i.*

The property of being isolated simple blow up means that in a ball of fixed radius around the blow up point, the solution is upper bounded and lower bounded by a constant times the bubble. In the following lemma, we give a quantitative statement of this fact.

Proposition 5.5 *[17] Assume that $\{u_i\}_i$ satisfies (5.1) with $\Omega = B_2$, and let $y_i \to \overline{y} \in \Omega$ be an isolated simple blow up point for $\{u_i\}_i$, which for some positive constant M satisfies*

$$|y - y_i|^{\frac{n-2}{2}} u_i(y) \leq M, \quad \forall y \in B_2. \tag{5.4}$$

Then there exists some positive constant $C = C(n, M, \varrho)$ (ϱ being given in the definition of isolated simple blow up point) such that for $0 < |y - y_i| \leq 1$

$$C^{-1} u_i(y_i)^{-1} |y - y_i|^{2-n} \leq u_i(y) \leq C\, u_i(y_i)^{-1} |y - y_i|^{2-n}. \tag{5.5}$$

The main result of the blow up analysis of Yamabe type equation on locally conformally flat manifold is that all isolated blow up are actually isolated simple blow up. This is what we recall in the following proposition

Proposition 5.6 *[17] Assume that $\{u_i\}_i$ satisfies equation (5.1) on $\Omega = B_2 \subset \mathbb{R}^n (n \geq 3)$ and let $\overline{y}$ be an isolated blow up point for $\{u_i\}_i$. Then $\overline{y}$ is an isolated simple blow up point.*

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