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BLACK-BODY RADIATION IN TSALLIS STATISTICS

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Abstract

Some results for the black-body radiation obtained in the context of the q-thermostatistics are analyzed on both thermodynamical and statistical-mechanical levels. Since the thermodynamic potentials can be expressed in terms of Wright's special function a useful asymptotic expansion can be obtained. This expansion allows to consider thermodynamic properties away from the Boltzmann-Gibbs limit $q = 1$. The role of non-extensivity, $q < 1$, on the possible deviation from the Stefan-Boltzmann T^4 behavior is considered. The application of some approximation schemes widely used in the literature to analyze the cosmic radiation is discussed.

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I. INTRODUCTION

The nonextensive statistical mechanics (NSM) is based on the generalized entropy defined by:

$$S_q^T = -k\frac{1-\sum_{i=1}^{W} p_i^q}{1-q}, \tag{1}$$

where the index i labels the possible microstates of the system under consideration, $\{p_i\}$ is a set of normalized probabilities, the real parameter q characterizes the degree of non-extensivity and k is a positive constant. Notice that taking the limit $q \to 1$ leads to the popular Boltzmann-Gibbs statistics (BGS). For a recent review on the nonextensive thermostatistics and its current status see Refs. [1, 2].

The thermodynamics in the context of the (NSM) is investigated by generalizing the Gibbs canonical ensemble to the case $q \neq 1$. This is achieved by maximizing the entropy defined by Eq. (1) under the constrains: (i) normalization of the probabilities, and (ii) knowledge of the expectation value of the energy. The expectation values that lie in basis of the thermostatistical considerations are usually computed using two different approaches. The first one is the so-called 'unnormalized' approach proposed in Ref. [3]. Within this approach, for a given observable O with an eigenvalue O_i in the microstate i one has

$$\langle O\rangle = \sum_{i=1}^{W} p_i^q O_i. \tag{2}$$

This approach shows several difficulties in describing the thermodynamics (see e.g. [2]). It is unable to preserve many of the thermodynamic properties. To overcome these inconveniences the 'normalized approach' has been advanced in Ref. [4], where the expectation values are given by

$$\langle O\rangle = \frac{\sum_{i=1}^{W} p_i^q O_i}{\sum_{i=1}^{W} p_i^q}. \tag{3}$$

The normalized treatment seems to provide one with a natural bridge that connects the NSM to the thermodynamics [2]. The normalized approach has been in turn improved by the so-called 'optimal Lagrange multipliers' (OLM) approach [5]. Nowadays it is believed that both approaches are the most appropriate choice for investigating the thermodynamics within the framework of the NSM regarding the nature of the Lagrange multiplier associated with the temperature. In these cases the non-extensivity is restricted just to the entropy, while the internal energy remains extensive as in the case of the BGS. The Lagrange multipliers preserve their traditional intensive character and can be identified with their thermodynamic counterparts. An important consequence of the success of both approaches is the unification of the Tsallis and Rényi variational formalism under a common context. This success originates from the fact that the Rényi's entropy is extensive[2]. Let us mention that the extensivity of the internal energy is true as long as we consider a system with a large number of particles or the thermodynamic limit under the condition $q < 1$ [6].

In the last few years many papers have been published on the application of NSM to the black-body radiation [7–15]. Our belief is that rigorous and exact results have an instructive role in the field. The exact expression for the corresponding partition function has been obtained in Refs. [9, 13, 14]. The studies presented in Refs. [7–12, 15] employ the unnormalized approach, while those in Refs. [13, 14] used the normalized one.

Irrespective of the used approach one can see that in the case $q \neq 1$ the derivation of exact results is a very complicated task. The final expressions one has at hand are too cumbersome and obscures the underlying physics. In this situation an approximation that tries to make the exact results more simple and transparent is preferable. However, the real benefit from the exact treatment without a well defined range of validity of the used approximation seems to be doubtful. Let us note that the more appropriate approximations are limited to simply computing $(1 - q)$ corrections (see Refs. [11] and [14] and references therein) since the Boltzmann-Gibbs limit $q \to 1$ leads to great simplifications. In this situation the possible strong deviations from the usual Boltzmann-Gibbs case are of significant interest.

A well estimated approximation in this field would be useful, since there is criticism concerning the physical validity of Tsallis statistics [16, 17]. The objections of Refs. [16, 17] in their major parts are concentrated on applications of q-thermostatistics to the black-body radiation. Recently this issue has been a matter of a debate in the literature [16–18].

The aim of the present study is to illustrate another possibility the for simplification of basic expressions in both approaches not related to the small value of $(1 - q)$. It is based on the fact that in both cases the intricate sums that appear in the theory may be presented [19] in terms of the Wright function with well studied analytical properties [20]. This is justified since we are considering a tremendous system. We hope this possibility will shed some light on the existing debate [16–18].

This paper is organized as follows: In Section II we discuss the thermodynamic derivation of the popular Stefan-Boltzmann law in terms of the Tsallis statistics. In Section III we introduce the mathematical background we need in our analysis. This is presented in Section IV. Section V is devoted to the discussion of our results.

II. SOME THERMODYNAMIC RELATIONS

First we shall introduce some basic notions. The radiation field in a large cavity can be considered to consist of a denumerably infinite set of electromagnetic oscillators corresponding to the various quantum states $\mathbf{k}$ in a d-dimensional box. The oscillator frequencies, $\omega_i = ck_i$, are related to the total energy E by $E = \Sigma_i n_{i,\epsilon} \hbar\omega_i$, where $n_{i,\epsilon}$ is the number of oscillator quanta with frequency ω_i and polarization ϵ, c is the light speed, $\hbar$ is the Planck constant and $k_i = |\mathbf{k}_i|$.

The Boltzmann-Gibbs partition function Z_1, for a large volume V, can be written as

$$Z_1(A_d) = \exp(A_d), \tag{4}$$

where

$$A_d = \frac{\Gamma(d)\zeta(d+1)2\tau_d}{(4\pi)^{d/2}\Gamma(d/2)} \left(\frac{k_B T}{\hbar c}\right)^d V. \tag{5}$$

In (5), $\tau_d = d-1$ is the number of linear-independent polarizations, k_B-Boltzmann constant, T-temperature, and $\Gamma(x)$ and $\zeta(x)$ are Gamma and Zeta functions, respectively.

The main obstacles related to the applicability of the q-thermostatistics to the black-body radiation may be considered in the context of the famous Stefan-Boltzmann law. In the Boltzmann-Gibbs thermodynamics the Stefan-Boltzmann law follows from the equation

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p \tag{6}$$

and the relation

$$p(T) = \frac{u(T)}{d}, \tag{7}$$

where $p \equiv p(T)$ is the pressure and $u(T) = U(T,V)/V$ - the internal energy per unit volume. Here the dependence on the temperature alone is crucial. As a result Eq. (6) reduces to an ordinary differential equation for $u(T)$ and its solution is $u(T) = \sigma T^{d+1}$, where σ is a constant that cannot be obtained on the macroscopic level.

Let us now consider the corresponding generalization of the Stefan-Boltzmann law to the NSM context. In the q-thermodynamics the following expressions for the internal energy $U_q(T,V)$ and the pressure $p_q(T,V)$ hold [9]:

$$U_q(T,V) = kT^2 \frac{\partial}{\partial T}\frac{[Z_q]^{1-q}-1}{(1-q)} \tag{8}$$

and

$$p_q(T,V) = kT\frac{\partial}{\partial V}\frac{[Z_q]^{1-q}-1}{(1-q)}, \tag{9}$$

where Z_q is the q-generalized partition function. Now, we shall give some thermodynamic relations for the black-body radiation using as input the definitions (8) and (9). Because of the simple dimensional arguments it is evident that $Z_q \equiv Z_q(A_d)$. If we introduce the convenient notation $\ln_q x = \frac{x^{q-1}-1}{1-q}$ the internal energy $U_q(T,V)$ may be expressed through $Z_q(A_d)$ and

$$U_q(T,V) = dkTA_d \frac{\mathrm{d}}{\mathrm{d}A_d} \ln_q Z_q(A_d). \tag{10}$$

Correspondingly for the pressure $p_q(T,V)$ we get

$$p_q(T,V)V = kTA_d \frac{\mathrm{d}}{\mathrm{d}A_d} \ln_q Z_q(A_d). \tag{11}$$

The q-generalization of the relation (7) immediately follows from Eqs. (10) and (11)

$$p_q(T,V)V = \frac{U_q(T,V)}{d}. \tag{12}$$

Relation (12), between the pressure and internal energy, is q-independent. As it should be. The violation of the relation between the pressure and the internal energy would compromise the theory since this can be established from pure electrodynamic reasoning. Eq. (12) was verified in Refs. [9] and [14] on the basis of the explicit expression of the partition function.

A necessary and sufficient condition for the internal energy $U_q(T,V)$ to be proportional to the volume V and to obey Eq. (10) is that $Z_q(A_d)$ must have the general form

$$Z_q(A_d) = e_q^{C_1(q)A_d} \tag{13}$$

where $e_q^x = [1 + (1-q)x]^{1/(1-q)}$ is the inverse function of $\ln_q(x)$ and $C_1(q)$ is an unknown, regular at $q = 1$, function. Indeed the relation (13) cannot be exact. It can be obtained only as an approximation and $C_1(q)$ depend upon the used approximation scheme. For example, within the framework of the factorization approximation used in Ref. [15], we have $C_1(q) = [(4-3q)(3-2q)(2-q)]^{-1}$. Now, though $q \neq 1$ the Stefan-Boltzmann law temperature behavior in its usual form is preserved. The constant $\sigma = \sigma(q)$ must be q-dependent (see e.g. Refs. [7, 8, 10, 15]).

The same relation as Eq. (6) between $p_q(T,V)$ and $U_q(T,V)$ exists in the general case of the q-thermodynamics [21]. Here however instead of Eq. (7) the more general relation (12) takes place and it is necessary to consider the following partial differential equation for $U_q(T,V)$

$$V\left(\frac{\partial U_q}{\partial V}\right)_T = \frac{T}{d}\left(\frac{\partial U_q}{\partial T}\right)_V - \frac{1}{d}U_q. \tag{14}$$

This equation has a solution of the type

$$U_q(T,V) = \sigma_q(T^d V)V^{C(q,d)/d}T^{1+C(q,d)}, \tag{15}$$

where the constant $C(q,d)$ (independent of T and V) and the unknown function $\sigma_q(x)$ can be obtained only at the microscopic level. This is the generalization of the Stefan-Boltzmann law that can be obtained without using the explicit expression for the partition function.

The result (15) means that in the considered case we loose the 'famous' T^{d+1} behavior of the internal energy as a function of the temperature. This is a strict consequence of the fact that $U_q(T,V)$ does not depend linearly on the volume V. This is in agreement with the findings of Ref. [16]. However a consideration on a pure thermodynamic level does not exclude a q-dependence of the proportionality coefficient of the T^4 law.

Without loss of generality let us consider a system in a cube with $V = L^d$. If we introduce the mean thermal wavelength of the black-body photons $l = l(T) \equiv \hbar c/kT$, Eq. (15) may be transformed into the following scaling forms

$$U_q(T,L) = \frac{\kappa}{l}\mathrm{g}_q\left(\frac{L}{l}\right), \tag{16}$$

where κ is a dimensionless constant and $\mathrm{g}_q(x)$ is a function, of which the explicit form depends on the way of writing the energy constraint (see Ref. [4]) *i.e.* its expression may be quite different as a function of the ratio L/l (see e.g. Eqs.(22) and (26)) depending on the used approach: unnormalized or normalized.

III. THE WRIGHT FUNCTION

In Section I we have advanced that for the investigation of the black-body radiation in the context of NSM different approaches have been used in the literature. Earlier, using the normalized approach the exact q counterpart of (4) is found to be [9]

$$Z_q(A_d) = \Gamma\left(\frac{2-q}{1-q}\right) \sum_{m=0}^{\infty} \frac{A_d^m}{(1-q)^{dm} m!} \frac{1}{\Gamma[(2-q)/(1-q)+dm]}. \tag{17}$$

Later, another expression for the partition function was obtained within the framework of the OLM and the normalized approaches. It is given [14] by the relation

$$\bar{Z}_q(U_q, A_d) = \Gamma\left(\frac{2-q}{1-q}\right) \sum_{m=0}^{\infty} \frac{A_d^m}{(1-q)^{dm} m!} \frac{[1+(1-q)kTU_q]^{dm+1/(1-q)}}{\Gamma[(2-q)/(1-q)+dm]}. \tag{18}$$

obtained under the cut-off-like condition $1 + (1-q)(kT)^{-1}U_q > 0$, otherwise we have $\bar{Z}_q(U_q, A_d) = 0$.

In spite of the fact that in the last case all the thermodynamic quantities, e.g. the internal energy, can be expressed exactly a complication arises. The corresponding expressions are *self-referential* [13, 14] in the sense that the thermodynamic functions are not expressed in a closed form. This fact leads to mathematical difficulties that make the problem for the most part only numerically tractable. Notice that Eqs. (17) and (18) are valid only for $q < 1$.

If one tries to apply the above results to the experimental data of the cosmic back-ground radiation the condition $A_d \gg 1$ is always satisfied since $\hbar c/(kT)$ is of the order of $1/10$ cm and V is of cosmological dimensions [16]. This physical fact will lead to a great simplification in the mathematical expressions given in Eqs. (17) and (18). Having this in mind, we take advantage of the fact that the series in the r.h.s of Eqs. (17) and (18) can be presented in terms of the entire function

$$\phi(\rho, \alpha; z) = \sum_{m=0}^{\infty} \frac{z^m}{m!\Gamma(\rho m + \alpha)}, \qquad \rho > 0, \alpha \in \mathbb{C}, \tag{19}$$

introduced in 1933 by E.M. Wright in the asymptotic theory of partitions. For analytical properties, some generalizations and applications of this function the interested reader may consult Ref. [20]. We note here a useful mathematical result concerning the behavior of Wright's function $\phi(\rho, \alpha; z)$. If $\rho > 0$, for a large real z, we have the asymptotic expansion [20]:

$$\phi(\rho,\alpha;z) = (\rho z)^{\frac{(1-2\alpha)}{(2+2\rho)}} \sqrt{\frac{2\pi}{\rho+1}} \exp[(1+\rho^{-1})(\rho z)^{\frac{1}{(1+\rho)}}]$$

$$\times \left[1 + \sum_{m=1}^{M} \frac{(-1)^m a_m(\rho,\alpha)}{(\rho z)^{\frac{m}{(1+\rho)}}} + O((\rho z)^{-\frac{M+1}{(1+\rho)}})\right], \qquad (20)$$

i.e. the asymptotic behavior of the Wright function is presented in terms of elementary functions. This result permits to obtain the different thermodynamic functions of the black-body radiation in a more simple form in some particular cases. The constants $a_m(\rho,\alpha)$ can be exactly evaluated [20]. For our analysis below we need

$$a_1(\rho,\alpha) = \frac{1}{\rho+1}\left[\frac{\alpha}{2}(\alpha-\rho-1) + \frac{1}{24}(2+\rho)(1+2\rho)\right]$$

and

$$a_2(\rho,\alpha) = \frac{1}{(1+\rho)^2}\left[\frac{\alpha}{48}(\alpha-\rho-1)[6\alpha^2+\alpha(2-14\rho)+\rho(6\rho-7)-2] + \frac{7}{1152}(2+\rho)^2[103+4\rho(7+\rho)]\right].$$

IV. THE STEFAN-BOLTZMANN LAW

In order to define the unknown function and constants in the thermodynamic relations discussed in Section II and to obtain the q-generalization of the Stefan-Boltzmann law one must use the concrete expression for the partition functions: (17) for $Z_q(A_d)$, obtained using the unnormalized approach, or (18) for $\bar{Z}_q(U_q, A_d)$, which is a result of the normalized approach. This motivates us to consider below both approaches separately.

A. Unnormalized approach

Within this approach the thermodynamic quantities are computed using the so-called unnormalized expectation values introduced in Eq. (2). In this case the generalization of the Stefan-Boltzmann law in terms of the Wright function is given by the following expression for the internal energy

$$U_q(T,V) = \frac{dkTA_d}{(1-q)^d}\left[\Gamma\left(\frac{2-q}{1-q}\right)\right]^{1-q} \frac{\phi\left(d, \frac{2-q}{1-q}+d; \frac{A_d}{(1-q)^d}\right)}{\left[\phi\left(d, \frac{2-q}{1-q}; \frac{A_d}{(1-q)^d}\right)\right]^q}. \qquad (21)$$

Now, let us consider the physically interesting case $d=3$. In the limit $q \to 1$ the asymptotic expansion (20) fails. Then if $(1-q)$ is fixed, for $A_3 \gg 1$, using Eq. (20) (up to the zeroth order in small values of z^{-1}) we get

$$U_q(T,V) = \frac{kT}{(8\pi)^{(1-q)/2}}\left[\Gamma\left(\frac{2-q}{1-q}\right)\right]^{1-q}\left[\frac{3A_3}{(1-q)^3}\right]^{-(1-q)/8} \exp\left\{\frac{4}{3}\left[3A_3(1-q)\right]^{1/4}\right\}. \qquad (22)$$

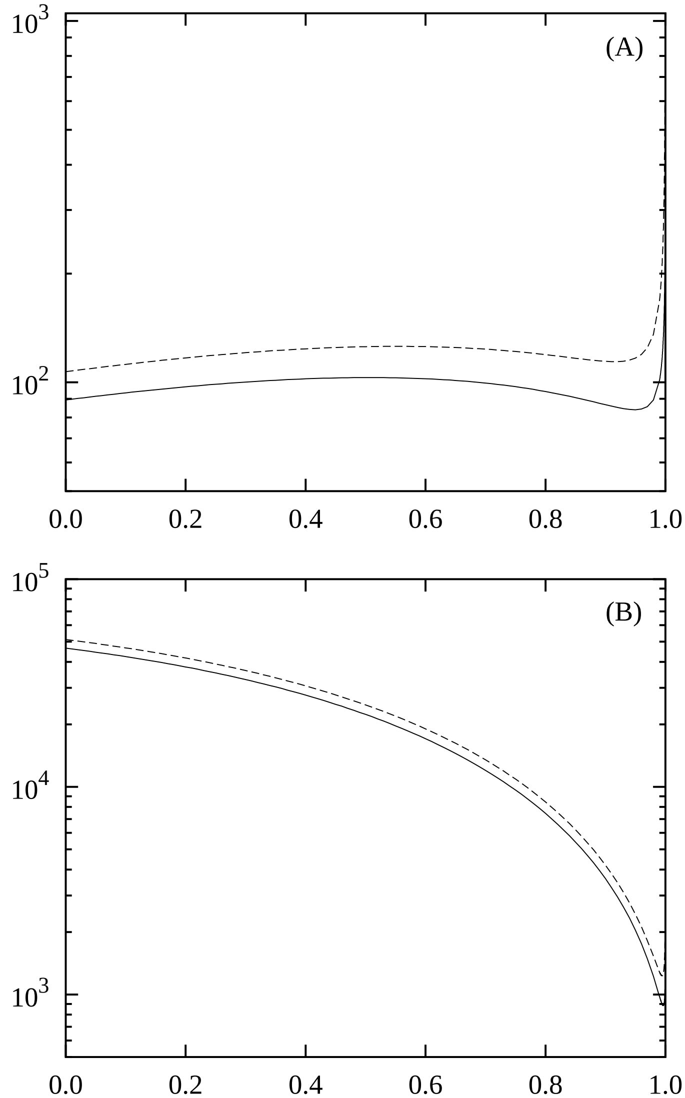

FIG. 1: Behavior of q-dependence of $U_q(T,V)/3kT$ from Eqs. (21) (solid line) and (22) (dashed line) for (A) $A_3 = 500$, (B) $A_3 = 5000$. Notice that we used the logarithmic scale along the vertical axis.

The last equation is in full consistency with relation (15) if for the constant $C(q,3)$ we take the value $-\frac{3}{8}(1-q)$ and the function $\sigma_q(T^3V)$ is equal to the exponential function in the r.h.s. of Eq. (22) with the corresponding factor.

In FIG. 1 we present the comparison of the behaviors of $U_q(T,V)$ from Eqs. (21) and (22) at large A_3. It shows that for large A_3, the expression (22) for the internal energy is a good approximation of the exact one given by (21). To our knowledge such kind of approximations is presented for the first time for the black-body radiation problem, while expansions around $q=1$ investigating deviations from the BGS are known [11].

As a conclusion we find that our result (22) shows that in the case of the unnormalized approach the Stefan-Boltzmann law is not preserved. Remark that in this case the internal energy is not proportional to the volume. This is in agreement with the discussion of Section II.

B. Normalized approach

Within this framework the expectation values are computed using Eq. (3). This approach has the advantage of reproducing the traditional thermodynamic relations. The internal energy $U_q \equiv U_q(T, V)$ is found to obey a nonlinear equation that can be expressed in terms of the Wright function. It has the form:

$$U_q = kT \frac{dA_d[1+(1-q)(kT)^{-1}U_q]^{d+1}}{(1-q)^{d+1}} \frac{\phi\left(d, \frac{2-q}{1-q}+d; \frac{A_d[1+(1-q)(kT)^{-1}U_q]^d}{(1-q)^d}\right)}{\phi\left(d, \frac{1}{1-q}; \frac{A_d[1+(1-q)(kT)^{-1}U_q]^d}{(1-q)^d}\right)}. \tag{23}$$

For fixed $(1-q)$ and

$$\frac{A_d[1+(1-q)(kT)^{-1}U_q]^d}{(1-q)^d} \gg 1, \tag{24}$$

using Eq. (20) (up to the first order in small values of z^{-1}) after some algebra Eq. (23) reads

$$\begin{aligned}(1-q)(kT)^{-1}U_q &= [1+(1-q)(kT)^{-1}U_q] \\ &\times \left[1 - \frac{1}{1-q}\left(\frac{dA_d}{(1-q)^d}\left[1+(1-q)(kT)^{-1}U_q\right]^d\right)^{-\frac{1}{1+d}}\right].\end{aligned} \tag{25}$$

The solution of this equation is surprisingly simple. The result is

$$U_q = \sigma V T^{d+1} - \frac{1}{1-q}kT, \tag{26}$$

where

$$\sigma = \frac{\Gamma(d)\zeta(d+1)2d(d-1)}{(4\pi)^{d/2}\Gamma(d/2)} \frac{k^{d+1}}{(\hbar c)^d}$$

is the *usual* Stefan-Boltzmann constant. Since it is not possible to take the limit $q \to 1$ in Eq. (26) we cannot recover the Stefan-Boltzamnn any more. Inserting now Eq. (26) into Eq. (24) we obtain the condition

$$A_d(dA_d)^d \gg 1, \tag{27}$$

restricting the range of the validity of the obtained solution for U_q. In order to improve our result we calculated the next term of the internal energy (26). To this end we used the expansion of r.h.s of (23) to the second term i.e. in small values of z^{-2}. The complicated ensuing equation could be solved using an iteration method, which leads to the additional term $[1-(1-q)d][2d(1-q)]^{-1}kT$ in the solution (26). Note here that in spite of taking into account the next order in our calculations we see that our results are not improving in the sense that we cannot take the limit $q \to 1$.

Remark that the internal energy (26) is non-extensive for relatively small volumes of the system. In the thermodynamic limit it becomes extensive, confirming the conclusions of Ref. [6].

The entropy, S, follows from the thermodynamic relation $(\partial S/\partial V)_T = (\partial p/\partial T)_V$. It has been demonstrated that in the case of the NSM the traditional thermodynamic relations remain valid only if one uses the Rényi entropy, S^R, instead of its Tsallis counterpart S_q^T [22]. The first one has the remarkable property of being extensive. In the thermodynamic limit we have in the case under consideration

$$S^R = V\sigma \frac{d}{d+1} T^d. \tag{28}$$

The Tsallis entropy can be deduced through the relation [2]

$$S_q^T = \frac{k}{1-q}(\exp\left[(1-q)S^R\right] - 1). \tag{29}$$

This is a sign that the Stefan-Boltzmann law remains valid in the NSM context as well.

V. DISCUSSION

Let us discuss the most important case $d = 3$. In our consideration the crucial point is the condition

$$A_3 = \frac{\pi^2}{45}\left(\frac{\hbar c}{kT}\right)^3 V \gg 1 \tag{30}$$

that was used to truncate the asymptotic expansion (20) for obtaining the results given by Eq. (22) and Eq. (26). From a physical point of view the condition (30) is always satisfied when considering the cosmic background radiation since $\hbar c/kT$ is of order of 0.1 cm and V is of cosmological dimensions.

Our considerations shows that the application of the thermodynamical concepts of the NSM may lead to the T^4 Stefan-Boltzmann law. However this takes place if the partition function has the form (13). This form would be a result of some approximations (see e.g. [15]) and the T^4 behavior is a strict consequence of a linear on V dependence of $U(T, V)$. What is important to note is that the inequality (30) permits to use quite different approximation, e.g. based on the used asymptotic expansion (20), and so prohibits the use of any approximation formula of the type of Eq. (13) to the *cosmic back-ground* radiation.

In the context of NSM two different formalisms have been suggested to investigate the thermodynamics of physical systems: unnormalized and normalized. In the following we discuss in more detail both approaches separately taking advantage of the validity of the condition (30).

In the case of the unnormalized approach our investigation results in the formula (22). If the inequality (30) is fulfilled we loose the Stefan-Boltzmann's T^4 behavior. Furthermore the *internal energy density*, Eq. (22), does not depend linearly on the volume of the system, which is unacceptable. These findings are in agreement with the conclusions of Ref. [16].

Free of such a defect would be a theory based on the normalized approach [4, 14, 18]. In this case one can immediately see that the condition (27) is the relaxed version of (30). This means that if one tries to apply the normalized approach to the analysis of the cosmic back-ground radiation the expression (26) has to be used. This result is consistent with the thermodynamic relations (14) and (15). Indeed the first term in Eq. (26) is the usual Stefan-Boltzman law. The question is: how to interpret the last one? The wisdom of the standard statistical mechanics is that such terms are to be omitted since they are of the order of $O(1/V)$ and do not contribute in the thermodynamic limit. On the other hand, this term diverges with $q \to 1^-$ and may be considered as a sign that in NSM the Boltzman-Gibbs limit $q \to 1$ and the thermodynamic limit do not commute with each other. This is in agreement with the results obtained in the framework of a classical gas [23].

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