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ON PARAMETRIC DOMAIN FOR ASYMPTOTIC STABILITY WITH PROBABILITY ONE OF ZERO SOLUTION OF LINEAR ITO STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

We describe a pratical implementation for finding parametric domain for asymptotic stability with probability one of zero solution of linear Ito stochastic differential equations based on Korenevskij and Mitropolskij's sufficient condition and our sufficient conditions. Numerical results show that all of these sufficient conditions are crucial in the implementation.

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1 Introduction

Consider the asymptotic stability with probability one of the zero solution of the following linear Ito stochastic differential equation

$$Ddx^{\epsilon}(t) = Ax^{\epsilon}(t)dt + B(\epsilon)x^{\epsilon}(t)dw(t) \tag{1}$$

($t \geq 0,\ x^{\epsilon}(0) = x_0^{\epsilon}$), where $x^{\epsilon}(t), x_0^{\epsilon} \in \mathbb{R}^n, A$ and D are constant $n \times n$-matrices, D is nonsingular, w is a standard one dimensional Wiener process on a stochastic base $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, P)$, perturbation matrix $B(\epsilon)$ is dependent analytically on a parameter ϵ and $B(0) = 0$ (see [2], [3] and [6]).

Definition 1.1 (see [6]): *The stability with probability one of the zero solution $x^{\epsilon}(t) \equiv 0$ of equation (1) is said to be asymptotically stable in the Lyapunov sense with probability one, if there exists a number $\delta > 0$ such that*

$$P\Big\{\lim_{T\to\infty} \sup_{t_0+T\leq t} \|x^{\epsilon}(t)\| = 0 \mid x^{\epsilon}(0) = x_0^{\epsilon}, \|x_0^{\epsilon}\| \leq \delta\Big\} = 1.$$

Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Assumption 1.1: *The unperturbed equation*

$$Ddx(t) = Ax(t)dt$$

is asymptotically stable in the Lyapunov sense, which means that all solutions $\lambda_i (i = 1, 2, \ldots, n)$ of the equation $\det(A - \lambda D) = 0$ *have negative real parts, i.e.,*

$$\mathrm{Re}\lambda_i < 0, i = 1, 2, \ldots, n. \tag{2}$$

Note that λ_i are eigenvalues of the matrix $D^{-1}A$.

We now consider equation (1) satisfying Assumption 1.1. Some conditions that are difficult to verify for asymptotical stability with probability one of zero solution of equation (1) have been established previously (see, for example, [6], Chapter 6). There arises two problems:

Problem 1: For what easily verified algebraic conditions on the matrices A, D and $B(\epsilon)$, zero solution of equation (1) is asymptotically stable with probability one?

Problem 2: Under the algebraic conditions on the matrices A, D and $B(\epsilon)$ given in Problem 1, what is the domain of ϵ? Does it contain only one interval?

Problem 1 was first solved in [2] and [3]. By using the Lyapunov function technique, a criterion for the asymptotic stability with probability one of the zero solution was given in form of a Sylvester matrix equation

$$A^T HD + D^T HA + B(\epsilon)^T HB(\epsilon) = -G,$$

where the solution H is negative definite for each symmetric positive definite matrix G. Consequently, a sufficient condition for the asymptotic stability with probability one of the zero solution was given (see [2] and [3]). But, in view of computation, solving the Sylvester matrix equation is not efficient (see [4] and [10]). On the other hand, examples given in [2] and [3] show that the domain of ϵ contains only one interval.

In this paper, Korenevskij and Mitropolskij's criterion is revised in the form of a Lyapunov matrix equation (Theorem 2.1), which can be solved efficiently (see [8]). Some sufficient conditions for asymptotic stability are given in Section 2 (Corollaries 2.1, 2.3-2.5). To solve Problem 2, an implementation algorithm for finding parametric intervals of ϵ based on Corollaries 2.1-2.5 is given. Finally, numerical results are described. One of them indicates that the parametric domain contains some intervals (Example 4.1 and Table 1).

2 Sufficient Conditions for Asymptotic Stability with Probability One of Zero Solution

We recall the following known results.

Proposition 2.1 (see Corollary 1, p.168 [6]): *If for equation (1), there exists a twice differentiable positive definite function of the phase variables (a stochastic Lyapunov function) $V(x^\epsilon)$ such that $V(0) = 0$ and the expectation $E\{dV/dt\}$ of its total time derivative taken on the basis of (1) is negative, then the zero solution of (1) is asymptotically stable with probability one.*

Proposition 2.2 (see [1]): *If Assumption 1.1 is satisfied, then for any symmetric positive definite $n \times n$-matrix G there exists a unique symmetric positive definite $n \times n$-matrix H solving the Lyapunov matrix equation*

$$A^T HD + D^T HA = -G.$$

Korenevskij and Mitropolskij's criterion given in [3] can be revised as follows.

Theorem 2.1 ([3]): *Suppose that Assumption 1.1 is satisfied and some $n \times n$-matrix G is symmetric positive definite. Then the zero solution of equation (1) is asymptotically stable with probability one, if and only if the matrix $B^T(\epsilon)H_0B(\epsilon) - G$ is negative definite where H_0 is the solution of the Lyapunov matrix equation*

$$A^T H_0D + D^T H_0A = -G. \tag{3}$$

Proof: The theorem can be seen directly from Korenevskij and Mitropolskij's criterion [3]. We present a proof (which repeats the proof in [3]) because a part of the proof is needed for the last section.

Suppose that H_0 is the solution of the Lyapunov matrix equation (3). By Proposition 2.1, H_0 is symmetric positive definite. Choose a suitable Lyapunov function for the perturbed equation

(1), being the quadratic form $V(x^\epsilon)$ of the phase variable, as follows

$$V(x^\epsilon) = (x^\epsilon)^T D^T H_0 D x^\epsilon. \tag{4}$$

Hence,

$$\begin{aligned} dV(x^\epsilon) &= (x^\epsilon)^T \Big[(A^T H_0 D + D^T H_0 A + B^T(\epsilon) H_0 B(\epsilon)) dt \\ &\quad + (B^T(\epsilon) H_0 D + D^T H_0 B(\epsilon)) dw(t) \Big] x^\epsilon. \end{aligned}$$

Then, the expectation $E\{dV/dt\}$ of the total time derivative of (4) on the basis of (1) is equal to

$$E\Big\{ \frac{dV(x^\epsilon(t))}{dt} |_{x^\epsilon \equiv x} \Big\} = x^T (A^T H_0 D + D^T H_0 A + B^T(\epsilon) H_0 B(\epsilon)) x.$$

It follows from (3) that the matrix

$$A^T H_0 D + D^T H_0 A + B^T(\epsilon) H_0 B(\epsilon)$$

is negative definite. Therefore,

$$E\Big\{ \frac{dV(x^\epsilon(t))}{dt} |_{x^\epsilon \equiv x} \Big\} < 0.$$

From Proposition 2.1, the requirement ensuring asymptotic stability is satisfied.

Inversely, suppose that the zero solution of equation (1) is asymptotically stable with probability one and G is a symmetric positive definite. Since Assumption 1.1 is satisfied, equation (3) has a symmetric positive definite matrix solution H_0. It follows that $B^T(\epsilon)H_0B(\epsilon) - G$ is negative definite. Because $B(0) = 0$, $B^T(0)H_0B(0) - G$ is negative definite, too. Since $B(\epsilon)$ is dependent analytically on ϵ, there exists $\alpha > 0$ such that $B^T(\epsilon)H_0B(\epsilon) - G$ is negative definite for $0 < \epsilon < \alpha$. □

It follows directly from Theorem 2.1 that:

Corollary 2.1: *Suppose that Assumption 1.1 is satisfied. Then the zero solution of equation (1) is asymptotically stable with probability one, if the matrix $B^T(\epsilon)H_1B(\epsilon) - I$ is negative definite where H_1 is the solution of the Lyapunov matrix equation*

$$A^T H_1 D + D^T H_1 A = -I, \tag{5}$$

and I is the identity matrix.

Since D is nonsingular, $G := D^T D$ is symmetric positive definite. If the perturbation matrix $B(\epsilon)$ (A, respectively) in (1) is assumed to be nonsingular then $G := B(\epsilon)^T B(\epsilon)$ ($G := A^T A$, respectively) is symmetric positive definite and $B^T(\epsilon)H_0B(\epsilon) - G = B^T(\epsilon)(H_0 - I)B(\epsilon)$. It follows from Theorem 2.1 that:

Corollary 2.2 (see [2] and [3]): *Suppose that Assumption 1.1 is satisfied and the perturbation matrix $B(\epsilon)$ is nonsingular. Then the zero solution of equation (1) is asymptotically stable with probability one, if the matrix $H_2 - I$ is negative definite where H_2 is the solution of the Lyapunov matrix equation*

$$A^T H_2 D + D^T H_2 A = -B(\epsilon)^T B(\epsilon). \tag{6}$$

Corollary 2.3: *Suppose that Assumption 1.1 is satisfied and A is nonsingular. Then the zero solution of equation (1) is asymptotically stable with probability one, if the matrix $B^T(\epsilon)H_3B(\epsilon) - A^TA$ is negative definite where H_3 is the solution of the Lyapunov matrix equation*

$$A^T H_3 D + D^T H_3 A = -A^T A. \tag{7}$$

Corollary 2.4: *Suppose that Assumption 1.1 is satisfied. Then the zero solution of equation (1) is asymptotically stable with probability one, if the matrix $B^T(\epsilon)H_4B(\epsilon) - D^TD$ is negative definite where H_4 is the solution of the Lyapunov matrix equation*

$$A^T H_4 D + D^T H_4 A = -D^T D. \tag{8}$$

If A and D are assumed to be symmetric, then $D^{-1}A$ is symmetric, too. Hence, by (2), the matrix $-D^{-1}A$ is positive definite (see Theorem 7.2.1 [7]). Hence, the symmetric matrix $G := -\frac{1}{2}(D^{-1}A + (D^{-1}A)^T)$ is positive definite, too. Therefore, by Theorem 2.1, we conclude that

Corollary 2.5: *Suppose that Assumption 1.1 is satisfied, A and D are symmetric. Then the zero solution of equation (1) is asymptotically stable with probability one, if the matrix $B^T(\epsilon)H_5B(\epsilon) + \frac{1}{2}(D^{-1}A + (D^{-1}A)^T)$ is negative definite where H_5 is the solution of the Lyapunov matrix equation*

$$A^T H_5 D + D^T H_5 A = \frac{1}{2}(D^{-1}A + (D^{-1}A)^T). \tag{9}$$

3 Parametric Domain for Asymptotic Stability with Probability One of Zero Solution

Based on Corollaries 2.1-2.5, we now present an algorithm for finding parametric domain of all positive parameters ϵ bounded from above by a number $\alpha > 0$ such that the zero solution of equation (1) is asymptotically stable with probability one.

Algorithm

Given: a stepsize d and $\alpha > 0$, constant $n \times n$-matrices A and nonsingular D satisfying Assumption 1.1, perturbation $n \times n$-matrix $B(\epsilon)$ satisying $B(0) = 0$. For simplicity we assume that D is symmetric positive definite, $n \times n$-matrix $B(\epsilon)$ is of the form

$$B(\epsilon) = (\phi(\epsilon) b_{ij})_{n \times n}$$

where $\phi(\epsilon)$ is a univariate polynomial satisfying $\phi(0) = 0$ and $B = (b_{ij})_{n \times n}$.

Task: Find $\epsilon \in [0, \alpha]$ such that zero solution of equation (1) is asymptotically stable with probability one.

Step 1. Solve equations (5) and (8). If A is nonsingular (symmetric, respectively), solve equation (7) ((9), respectively). Suppose that symmetric positive definite matrix H_1 (H_3, H_4, H_5, respectively) is a solution of equation (5) ((7), (8), (9), respectively) (Proposition 2.2 ensures the existence of such H_1, H_3, H_4 and H_5). Set $\epsilon = d$.

Step 2. If $\epsilon - d > \alpha$, set $\epsilon = \alpha$, $[0, \alpha]$ is the required domain. STOP.

Else, check whether $B^T(\epsilon)H_1B(\epsilon) - I$ is negative definite. If so, by Corollary 2.1, the zero solution of equation (1) is asymptotically stable with probability one, then set $\epsilon = \epsilon + d$, go to Step 2.

Step 3. Check whether $B^T(\epsilon)H_4B(\epsilon) - D^TD$ is negative definite. If so, by Corollary 2.4, the zero solution of equation (1) is asymptotically stable with probability one, then set $\epsilon = \epsilon + d$, go to Step 2.

Step 4. If A is nonsingular, check whether $B^T(\epsilon)H_3B(\epsilon) - A^TA$ is negative definite. If so, by Corollary 2.3, the zero solution of equation (1) is asymptotically stable with probability one, then set $\epsilon = \epsilon + d$, go to Step 2.

Step 5. If A is symmetric, then check whether $B^T(\epsilon)H_5B(\epsilon) + \frac{1}{2}(D^{-1}A + (D^{-1}A)^T)$ is negative definite. If so, by Corollary 2.5, the zero solution of equation (1) is asymptotically stable with probability one, then set $\epsilon = \epsilon + d$, go to Step 2.

Step 6. If $B = (b_{ij})_{n\times n}$ is nonsingular, solve equation (6) and suppose that symmetric positive definite matrix H_2 is a solution of equation (6) (Proposition 2.2 ensures the existence of such H_2). Check whether $H_2 - I$ is negative definite. If so, by Corollary 2.2, the zero solution of equation (1) is asymptotically stable with probability one, then set $\epsilon = \epsilon + d$, go to Step 2.

Else, $[0, \epsilon[$ is the required inverval. STOP.

Since the set of parametrics ϵ is bounded, the algorithm stops after a finite number of steps.

This algorithm is implemented by a Fortran code. It is based on the LAPACK library [5] and NAG library [9] for checking whether the matrices $B^T(\epsilon)H_1B(\epsilon) - I$, $H_2 - I$, $B^T(\epsilon)H_3B(\epsilon) - A^TA$, $B^T(\epsilon)H_3B(\epsilon) - D^TD$ or $B^T(\epsilon)H_3B(\epsilon) + \frac{1}{2}(D^{-1}A + (D^{-1}A)^T)$ are negative definite and the Fortran-77 software package [8] for solving (5), (6), (7), (8) or (9). It can be seen that using the Fortran-77 software package [8] more often, the computing time increases. Hence, in the algorithm, to check whether the points $\epsilon = id$ with $i = 1, 2, \ldots$ belong to the required parametric domain, Corollary 2.2 is the last corollary to be used (at step 6) after either corollaries.

Moreover, we use the Fortran code for the case $n \leq 6$ since the Fortran-77 software package [8] is effective for $n \leq 6$. This code is also designed for finding many intervals of the parametric domain in $[0, \alpha]$.

On the other hand, all of Corollaries 2.1-2.5 are crucial in the algorithm. Indeed, having an algorithm basing only one of Corollaries 2.1-2.5, we can accordingly modify this algorithm. But no single according algorithm is best for finding the largest parametric domain. This is illustrated in Table 2.

• Using the Fortran code

For a given application, the user provides the files:

- *inname*: gives all needed data: the stepsize d, the boundary above α of the parametric domain, the dimension n of matrices A, D, and B as well as their values.

- *functeps.f*: gives the univariate polynomial $\phi(\epsilon) = \sum_{i=1}^{k} a_i \epsilon^i$ in the matrix $B(\epsilon)$ ($a_i \in I\!R$).

4 Examples and Numerical Results

In this section, our Fortran code is compiled by Intel(R) Fortran Compiler 7.0 under Mandrake Linux 9.2 and is executed on a Pentium IV processor. The examples are considered with the stepsize $d = 10^{-5}$ and $\alpha = 4$.

Example 4.1 (given in [2]):

$$\mathbf{A} = \begin{pmatrix} -1 & 10 & 0 \\ 0 & -1 & 10 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$\phi(\epsilon) := \epsilon$. Then, the zero solution of equation (1) is asymptotically stable with probability one with $\epsilon \in [0, 0.02283[$. This domain is also the largest (see [2]).

If $\phi(\epsilon) = \epsilon$ is replaced by some $\phi(\cdot)$, the parametric domain may contain some intervals in $[0, \alpha]$. Table 1 below gives some results in cases $\phi(\epsilon) := \epsilon$, $\phi(\epsilon) := -\epsilon^2 + \epsilon$, $\phi(\epsilon) := -\epsilon(\epsilon^2 - 1)$, and

$$\phi(\epsilon) := -\epsilon^3(\epsilon - 0.3)(\epsilon - 0.8)(\epsilon - 1.2)(\epsilon - 1.7)$$

respectively.

Table 1: Performance Results of Example 4.1 with some $\phi(\cdot)$

$\phi(\cdot)$	`Parametric Domain in` $[0,4]$
ϵ	$[0, 0.02283[$
$-\epsilon(\epsilon^2-1)$	$[0, 0.02284[\cup]0.98838, 1.01123[$
$-\epsilon^3(\epsilon-0.3)(\epsilon-0.8)(\epsilon-1.2)(\epsilon-1.7)$	$[0, 1.25894[\cup]1.69230, 1.70709[$

Example 4.2:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -3.5 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 59 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$$

$\phi(\epsilon) := \epsilon$. Then, the zero solution of equation (1) is asymptotically stable with probability one with $\epsilon \in [0, 2.20687[$.

Example 4.3:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$$

$\phi(\epsilon) := \epsilon$, Then, the zero solution of equation (1) is asymptotically stable with probability one with $\epsilon \in [0, 0.51660[$.

Example 4.4:

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -1 & -3.1 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & -12 & -16.9 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

$\phi(\epsilon) := \epsilon$. Then, zero of equation (1) is asymptotically stable with probability one with $\epsilon \in [0, 0.35080[$.

Example 4.5:

$$\mathbf{A} = \begin{pmatrix} -8 & -1 & 3 \\ -1 & -4 & -1 \\ 3 & -1 & -3.1 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & -912 & -0.1 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

$\phi(\epsilon) := \epsilon$. Then, the zero solution of equation (1) is asymptotically stable with probability one with $\epsilon \in [0, 0.00862[$.

Table 2: Parametric Domains Corresponding to Corollaries (with $\phi(\epsilon) = \epsilon$)

Example	Algorithm based only on	Parametric Domain in $[0, 4]$
4.1	Corollary 2.1	$[\mathbf{0}, \mathbf{0.02283}[$
	Corollary 2.2	$[\mathbf{0}, \mathbf{0.02283}[$
	Corollary 2.3	$[0, 0.00132[$
	Corollary 2.4	$[\mathbf{0}, \mathbf{0.02283}[$
4.2	Corollary 2.1	$[\mathbf{0}, \mathbf{2.20687}[$
	Corollary 2.2	$[0, 1.46934[$
	Corollary 2.3	$[0, 1.06409[$
	Corollary 2.4	$[0, 0.19511[$
	Corollary 2.5	$[0, 0.08342[$
4.3	Corollary 2.1	$[0, 0.28767[$
	Corollary 2.2	$[\mathbf{0}, \mathbf{0.51660}[$
	Corollary 2.3	$[0, 0.29390[$
	Corollary 2.4	$[0, 0.28767[$
	Corollary 2.5	$[0, 0.48320[$
4.4	Corollary 2.1	$[0, 0.08118[$
	Corollary 2.2	$[0, 0.13938[$
	Corollary 2.3	$[\mathbf{0}, \mathbf{0.35080}[$
	Corollary 2.4	$[0, 0.04166[$
	Corollary 2.5	$[0, 0.27008[$
4.5	Corollary 2.1	$[0, 0.00484[$
	Corollary 2.2	$[0, 0.00291[$
	Corollary 2.3	$[0, 0.00282[$
	Corollary 2.4	$[0, 0.00171[$
	Corollary 2.5	$[\mathbf{0}, \mathbf{0.00862}[$

5 Concluding Remarks

The algorithm presented in Section 3 shows that computations at points $\epsilon = id$ with $i = 1, 2, \dots$ are completely decoupled and hence can be computated in parallel on appropriate subintervals of $[0, \alpha]$. It folllows from the proof of Theorem 2.1 that if the equation $B(\epsilon) = 0$ has positive solutions then the parametric domain contains some open neighbourhoods of these solutions (this is illustrated in Table 1). Therefore, appropriate subintervals can be determined by these positive solutions.

Finally, to find the largest parametric domain such that the zero solution of equation (1) is asymptotically stable with probability one is still an open problem.

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