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ON P-ADIC λ-MODEL ON THE CAYLEY TREE

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Abstract

We consider a nearest-neighbour p-adic λ-model with spin values ± 1 on the Cayley tree of order $k \geq 1$. We prove that a p-adic Gibbs measure is unique for $p \geq 3$. If $p = 2$ then we find a condition which guarantees uniqueness of p-adic Gibbs measure. Besides, the results are applied to the p-adic Ising model.

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1 Introduction

The p-adic numbers were first introduced by the German mathematician K. Hensel. For about a century after the discovery of p-adic numbers, they were mainly considered objects of pure mathematics. However, numerous applications of these numbers to theoretical physics have been proposed in papers [1],[4],[6],[7],[13],[18]. It is known [7] that number of p-adic models in physics cannot be described using ordinary probability theory based on the Kolmogorov axioms [12]. New probability models - p-adic probability models were investigated in [7],[8]. This is non-Kolmogorovean model, since probabilities take values in fields of p-adic numbers.

In [9],[10] the theory of stochastic processes with values in p-adic and more general non-Archimedean fields having probability distributions with non-Archimedean values, has been developed. The non-Archimedean analog of the Kolmogorov theorem that gives the possibility to construct wide classes of stochastic processes by using finite dimensional probability distributions, was proved.

It is known that the theory of statistical mechanics lies in the base of is the theory of probability and stochastic processes. Since the theory of probabilities and stochastic processes in a non-Archimedean setting has been introduced, it is natural to begin the study and initiate further the development of the problems of statistical mechanics in the context of the p-adic theory of probability.

One of the central problems in the theory of Gibbs measures is to describe infinite-volume Gibbs measures corresponding to a given Hamiltonian. However, a complete analysis of the set of Gibbs measures for a specific Hamiltonian is often a difficult problem. If for a given Hamiltonian there are at least two Gibbs measures then it is said that a *phase transition* occurs for the model.

The existence of a phase transition for the Ising model (real case) on the Cayley tree of order $k \geq 2$ was established by Katsura and Takisawa [5]. The analysis of the Cayley tree Ising model can be extended in several directions (see [14],[15],[3]).

In this paper we develop the p-adic probability theory approaches to study of some statistical mechanics models on a Cayley tree in the field of p-adic numbers. In [2] we have proved the existence of the phase transition for the homogeneous p-adic Potts model with $q \geq 2$ spin variables on the set of integers $\mathbb{Z}$. The present paper deals with a nonhomogeneous p-adic λ-model on the Cayley tree of order k, $k \geq 1$. The aim of this paper is to show the uniqueness of Gibbs measures for the considered model.

2 Definitions and preliminary results

2.1 p-adic numbers and measures

Let $\mathbb{Q}$ be the field of rational numbers. Every rational number $x \neq 0$ can be represented in the form $x = p^r \dfrac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$ and p is a fixed prime number. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

It satisfies the following properties:

1) $|xy|_p = |x|_p|y|_p$,

2) the strong triangle inequality

$$|x + y|_p \le \max\{|x|_p, |y|_p\},$$

this is a non-Archimedean norm.

The completion of $\mathbb{Q}$ with respect to p-adic norm is called p-adic field which is denoted by $\mathbb{Q}_p$.

The well-known Ostrovsky's theorem asserts that norms $|x|_\infty = |x|$ and $|x|_p$, $p = 2, 3, 5...$ exhaust all nonequivalent norms on $\mathbb{Q}$ (see [11]). Any p-adic number $x \ne 0$ can be uniquely represented in the canonical series:

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \le x_j \le p - 1$, $x_0 > 0$, $j = 0, 1, 2, ...$ (for more details see [11],[17]). In this case $|x|_p = p^{-\gamma(x)}$.

Let $B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p \le r\}$, where $a \in \mathbb{Q}_p$, $r > 0$. The p-adic logarithm is defined by series

$$\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n},$$

which converges for $x \in B(1, 1)$. And p-adic exponential is defined by

$$\exp_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Lemma 2.1.[11],[17] *Let* $x \in B(0, p^{-1/(p-1)})$ *then we have*

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p < 1, \quad |\log_p(1 + x)|_p = |x|_p < p^{-1/(p-1)}$$

and

$$\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1 + x)) = 1 + x.$$

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a p-adic measure if for any $A_1, ..., A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ $(i \ne j)$ the equality holds

$$\mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu(A_j).$$

A p-adic measure is called a probability measure if $\mu(X) = 1$.

For more detailed information about p-adic measures please refer to [7],[8].

2.2 The Cayley tree

The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that each vertex of which lies on $k+1$ edges. Let $\Gamma^k = (V, \Lambda)$, where V is the set of vertices of Γ^k, Λ is the set of edges of Γ^k. The vertices x and y are called *nearest neighbours*, which is denoted by $l = <x, y>$ if there exists an edge connecting them. A collection of the pairs $<x, x_1>, ..., <x_{d-1}, y>$ is called *path* from the point x to the point y. The distance $d(x, y), x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y.

We set

$$W_n = \{x \in V | d(x, x^0) = n\},$$

$$V_n = \cup_{m=1}^n W_m = \{x \in V | d(x, x^0) \leq n\},$$

$$L_n = \{l = <x, y> \in L | x, y \in V_n\},$$

for an arbitrary point $x^0 \in V$.

Denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\} \ \ x \in W_n,$$

this set is called the set *direct successors* of x. Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$.

2.3 The p-adic λ-model

We consider the p-adic λ-model, where the spin takes values in the set $\Phi = \{-1, 1\} \subset \mathbb{Q}_p$ and is assigned to the vertices of the tree. A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in \Phi$; in a similar fashion one defines a configuration on V_n and W_n respectively. The set of all configurations on V (resp. V_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_n = \Phi^{V_n}$). The Hamiltonian $H_n : \Omega_n \to \mathbb{Q}_p$ of the p-adic inhomogeneous λ-model has the form:

$$H_n(\sigma) = \sum_{<x,y> \in L_n} \lambda_{x,y}(\sigma(x), \sigma(y)), \ \ n \in \mathbb{N} \tag{2.1}$$

where the sum is taken over all pairs of neighbouring vertices $<x, y>$, $\sigma \in \Omega$. Here and below $\lambda : \Phi \times \Phi \to \mathbb{Q}_p$ is some given function such that $|\lambda_{x,y}(u, v)|_p < p^{-1/(p-1)}$ for all $u, v \in \Phi$, $x, y \in V$ and p is a fixed prime number.

We say that (2.1) is *homogeneous λ-model* if $\lambda_{xy}(u, v) = \lambda(u, v), \ \ \forall <x, y> \in L$.

We note that λ-model of this type were firstly considered in [15].

3 Construction of Gibbs measures

In this subsection we give a construction of a special class of Gibbs measures for p-adic λ-model on the Cayley tree.

To define Gibbs measure we have need in the following

Lemma 3.1. *Let $h_x, x \in V$ be a $\mathbb{Q}_p$-valued function such that $h_x \in B(0, p^{-1/(p-1)})$ for all $x \in V$ and $|\lambda_{x,y}(u,v)|_p < p^{-1/(p-1)}$ for all $u, v \in \Phi$. Then the relation*

$$H_n(\sigma) + \sum_{x \in W_n} h_x \sigma(x) \in B(0, p^{-1/(p-1)})$$

is valid for any $n \in \mathbb{N}$.

The proof easily follows from the strong triangle inequality for the norm $|\cdot|_p$.

Let $h : x \in V \to h_x \in \mathbb{Q}_p$ be a function of $x \in V$ such that $|h_x|_p < p^{-1/(p-1)}$ for all $x \in V$. Given $n = 1, 2, ...$ consider a p-adic probability measure $\mu^{(n)}$ on Φ^{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp_p\{H_n(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\}, \tag{3.1}$$

Here, as before, $\sigma_n : x \in V_n \to \sigma_n(x)$ and Z_n is the corresponding partition function:

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega_{V_n}} \exp_p\{H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_x \tilde{\sigma}(x)\}.$$

Note that according to Lemma 3.1 the measures $\mu^{(n)}$ exist.

The compatibility conditions for $\mu^{(n)}(\sigma_n), n \geq 1$ are given by the equality

$$\sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}), \tag{3.2}$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

We note that an analog of the Kolmogorov extension theorem for distributions can be proved for p-adic distributions given by (3.1) (see [10]). Then according to the Kolmogorov theorem there exists a unique p-adic measure μ_h on $\Omega = \Phi^V$ such that for every $n = 1, 2, ...$ and $\sigma_n \in \Phi^{V_n}$ the equality holds

$$\mu\left(\{\sigma|_{V_n} = \sigma_n\}\right) = \mu^{(n)}(\sigma_n),$$

which will be called *p-adic Gibbs measure* for the considered λ-model. It is clear that the measure μ_h depends on function h_x, so if the Gibbs measure for a given Hamiltonian non unique then we say that for this model there is *a phase transition*.

The following statement describes conditions on h_x guaranteeing the compatibility condition of measures $\mu^{(n)}(\sigma_n)$.

Theorem 3.2. *The measures $\mu^{(n)}(\sigma_n), \; n = 1, 2, ...$ satisfy the compatibility condition (3.2) if and only if for any $x \in V$ the following equation holds:*

$$h_x = \sum_{y \in S(x)} F_{x,y}(h_y; \lambda) \tag{3.3}$$

where $S(x)$ is the set of all direct successors of $x \in V$ and

$$F_{x,y}(h, \lambda) = \frac{1}{2} \log_p \left(\frac{\exp_p(\lambda_{x,y}(1,1)) \exp_p(2h) + \exp_p(\lambda_{x,y}(1,-1))}{\exp_p(\lambda_{x,y}(-1,1)) \exp_p(2h) + \exp_p(\lambda_{x,y}(-1,-1))} \right).$$

Proof. *Necessity.* According to the compatibility condition (3.2) we have

$$Z_n^{-1} \sum_{\sigma^{(n)}} \exp_p \left[\sum_{<x,y>\in L_n} \lambda_{x,y}(\sigma(x),\sigma(y)) + \sum_{x\in W_n} h_x\sigma(x) \right] =$$

$$Z_{n-1}^{-1} \exp_p \left[\sum_{<x,y>\in L_{n-1}} \lambda_{x,y}(\sigma(x),\sigma(y)) + \sum_{x\in W_{n-1}} h_x\sigma(x) \right]. \tag{3.4}$$

It yields

$$\frac{Z_{n-1}}{Z_n} \sum_{\sigma(n)} \exp_p \left[\sum_{x\in W_{n-1}} \sum_{y\in S(x)} \lambda_{x,y}(\sigma(x),\sigma(y)) + \sum_{x\in W_{n-1}} \sum_{y\in S(x)} h_y\sigma(y) \right] = \prod_{x\in W_{n-1}} \exp_p \left(h_x\sigma(x) \right). \tag{3.5}$$

From this equality we find

$$\frac{Z_{n-1}}{Z_n} \prod_{x\in W_{n-1}} \prod_{y\in S(x)} \sum_{\sigma(y)\in\Phi} \exp_p \left(\lambda_{x,y}(\sigma(x),\sigma(y)) + h_y\sigma(y) \right) = \prod_{x\in W_{n-1}} \exp_p \left(h_x\sigma(x) \right). \tag{3.6}$$

Now fix $x \in W_{n-1}$ and dividing the equalities (3.6) with $\sigma(x) = 1$ and $\sigma(x) = -1$ we obtain

$$\prod_{y\in S(x)} \frac{\sum_{\sigma(y)\in\Phi} \exp_p \left(\lambda_{x,y}(1,\sigma(y)) + h_y\sigma(y) \right)}{\sum_{\sigma(y)\in\Phi} \exp_p \left(\lambda_{x,y}(-1,\sigma(y)) + h_y\sigma(y) \right)} = \exp_p(2h_x), \tag{3.7}$$

hence we get

$$\prod_{y\in S(x)} \frac{\exp_p(\lambda_{x,y}(1,1)) \exp_p(2h_y) + \exp_p(\lambda_{x,y}(1,-1))}{\exp_p(\lambda_{x,y}(-1,1)) \exp_p(2h_y) + \exp_p(\lambda_{x,y}(-1,-1))} = \exp_p(2h_x), \tag{3.8}$$

which implies (3.3).

Sufficiency. Now assume that (3.3) is valid, then it implies (3.8), and hence (3.7). From (3.7) we obtain the following equality

$$a(x) \exp_p \left(h_x\sigma \right) = \prod_{y\in S(x)} \sum_{\sigma(y)\in\Phi} \exp_p \left(\lambda_{x,y}(\sigma,\sigma(y)) + h_y\sigma(y) \right), \quad \sigma \in \{-1,1\},$$

this equality implies

$$\prod_{x\in W_{n-1}} a(x) \exp_p \left(h_x\sigma(x) \right) = \prod_{x\in W_{n-1}} \prod_{y\in S(x)} \sum_{\sigma(y)\in\Phi} \exp_p \left(\lambda_{x,y}(\sigma(x),\sigma(y)) + h_y\sigma(y) \right), \tag{3.9}$$

where

$$\sigma(z) = \begin{cases} \sigma, & z = x \\ \sigma(z), & z \neq z \end{cases} \quad \sigma \in \{-1,1\}.$$

Denoting $A_n(x) = \prod_{x\in W_n} a(x)$ from (3.9) and (3.2) we find

$$Z_{n-1}A_{n-1}\mu^{(n-1)}(\sigma_{n-1}) = Z_n \sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1},\sigma^{(n)}).$$

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Since each $\mu^{(n)}$, $n \geq 1$ measure is a p-adic probability measure, so we should have

$$\sum_{\sigma_{n-1}} \sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = 1, \quad \sum_{\sigma_{n-1}} \mu^{(n-1)}(\sigma_{n-1}) = 1.$$

Therefore, from these equalities we find $Z_{n-1}A_{n-1} = Z_n$ which means that (3.2) is valid.

Observe that according to this Theorem the problem of describing of p-adic Gibbs measures is reduced to the description of solutions of functional equation (3.3).

4 The uniqueness of Gibbs measure for the p-adic λ-model

In this section we will show that the phase transition does not occur for the p-adic λ-model.

Put

$$\Xi = \{h = (h_x, x \in V) : \ h_x \text{ satisfies the eqation (3.3)}\}.$$

According to Theorem 3.2 the description of Gibbs measures is reduced to the description of elements of the set Ξ.

4.1 Nonhomogeneous case

In this subsection we will consider nonhomogeneous λ-model. We claim that the function $\lambda_{x,y}$ satisfies the following condition: for all nearest-neighbor vertices $x, y \in V$ the equality

$$\exp_p(\lambda_{x,y}(1,1)) + \exp_p(\lambda_{x,y}(1,-1)) = \exp_p(\lambda_{x,y}(-1,1)) + \exp_p(\lambda_{x,y}(-1,-1)) \tag{4.1}$$

is valid.

This condition implies that the function $h_x = 0, \forall x \in V$ is a solution of (3.3).

Let $S(x) = \{x_1, ..., x_k\}$, here as before $S(x)$ is the set of direct successors of x. Then the equation (3.3) can be rewritten as follows

$$z_x = \prod_{i=1}^{k} \alpha_{x,i}, \tag{4.2}$$

where $z_x = \exp_p(h_x)$, $z_{x_i} = \exp_p(h_{x_i})$,

$$\alpha_{x,i} = \frac{a_{x,x_i} z_{x_i} + b_{x,x_i}}{c_{x,x_i} z_{x_i} + d_{x,x_i}},$$

$$\left. \begin{array}{cc} a_{x,x_i} = \exp_p(\lambda_{x,x_i}(1,1)), & b_{x,x_i} = \exp_p(\lambda_{x,x_i}(1,-1)), \\ c_{x,x_i} = \exp_p(\lambda_{x,x_i}(-1,1)), & d_{x,x_i} = \exp_p(\lambda_{x,x_i}(-1,-1)) \end{array} \right\} \tag{4.3}$$

for every $i = 1, ..., k$, here as before $|h_x|_p \leq \dfrac{1}{p}$ for all $x \in V$.

Lemma 4.2. *If $|a_i - 1|_p \leq M$ and $|a_i|_p = 1$, $i = 1, ..., n$, then*

$$\left| \prod_{i=1}^{n} a_i - 1 \right|_p \leq M. \tag{4.4}$$

Proof. We prove by induction on n. The case $n = 1$ is the condition of lemma. Suppose that (4.4) is valid at $n = m$. Now let $n = m + 1$. Then we have

$$\left| \prod_{i=1}^{m+1} a_i - 1 \right|_p = \left| \prod_{i=1}^{m+1} a_i - \prod_{i=1}^{m} a_i + \prod_{i=1}^{m} a_i - 1 \right|_p \leq$$

$$\leq \max \left\{ \left| \prod_{i=1}^{n} a_i (a_{n+1} - 1) \right|_p, \left| \prod_{i=1}^{n} a_i - 1 \right|_p \right\} \leq M$$

This completes the proof.

Lemma 4.3. *For every $x \in V$ the following inequality holds*

$$|h_x|_p \leq \frac{1}{p} \max_{1 \leq i \leq k} \{|h_{x_i}|_p\}.$$

Proof. For every $m \in \{1, 2, ..., k\}$ we have

$$|\alpha_{x,m} - 1|_p = \left| \frac{(a_{x,x_m} - c_{x,x_m})(z_{x_m} - 1)}{c_{x,x_m} z_{x_m} + d_{x,x_m}} \right|_p \leq \frac{1}{p} |h_{x_m}|_p.$$

Here we have used (4.1) and the following relations: for $p \geq 3$

$$|a_{x,x_m} - c_{x,x_m}|_p \leq \frac{1}{p}, \quad |c_{x,x_m} z_{x_m} + d_{x,x_m}|_p = 1,$$

for $p = 2$

$$|a_{x,x_m} - c_{x,x_m}|_p \leq \frac{1}{2^2}, \quad |c_{x,x_m} z_{x_m} + d_{x,x_m}|_p = \frac{1}{2},$$

which follow from (4.3) and the equality $|\exp_p(x) - 1|_p = |x|_p$ (see Lemma 2.1). Then according to Lemma 4.2 and (4.2) we obtain

$$|h_x|_p = |z_x - 1|_p = \frac{1}{p} \max_{1 \leq i \leq k} \{|h_{x_i}|_p\}.$$

Lemma is proved.

Theorem 4.4. *Let $k \geq 1$, $|\lambda_{x,y}(u, v)|_p \leq \frac{1}{p}$ for all $< x, y >\in L$, $u, v \in \Phi$ and (4.1) be satisfied. Then for the p-adic nonhomogeneous λ-model (2.1) on the Cayley tree of order k there is no phase transition for any prime p.*

Proof. To obtain the proof it is enough to show that $\Xi = \{h_x \equiv 0\}$. In order to do so it is enough to show that for arbitrary $\varepsilon > 0$ and every $x \in V$ the inequality $\|h_x\|_p < \varepsilon$ is valid. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{p^{n_0}} < \varepsilon$. According to Lemma 4.3 we have

$$|h_x|_p \leq \frac{1}{p} |h_{x_{i_0}}|_p \leq \frac{1}{p^2} |h_{x_{i_0,i_1}}|_p \leq \cdots$$

$$\leq \frac{1}{p^{n_0 - 1}} |h_{x_{i_0,...,i_{n_0-2}}}|_p \leq \frac{1}{p^{n_0}} < \varepsilon,$$

here $x_{i_0,...,i_n,j}, j = \overline{1, k}$ are direct successors of $x_{i_0,...,i_n}$, where

$$|h_{x_{i_0,...,i_m}}|_p = \max_{1 \leq j \leq k} \{|h_{x_{i_0,...,i_{m-1},j}}|_p\}.$$

This completes the proof.

4.2 Homogeneous case

In this subsection we will consider the homogeneous λ-model, i.e. $\lambda_{xy}(u,v) = \lambda(u,v)$, $\forall < x,y > \in L$.

In this subsection at first we restrict ourselves to the description of translation - invariant ($h_x = h \in \mathbb{Q}_p, \forall x \in V$) elements of Ξ.

Let $h_x = h$ for all $x \in V$. Then (3.3) implies

$$\left(\frac{\exp_p(\lambda(1,1))\exp_p(2h) + \exp_p(\lambda(1,-1))}{\exp_p(\lambda(-1,1))\exp_p(2h) + \exp_p(\lambda(-1,-1))} \right)^k = \exp_p(2h). \tag{4.5}$$

Denoting

$$\left. \begin{array}{ll} z = \exp_p(2h), & a = \exp_p(\lambda(1,1)), \quad b = \exp_p(\lambda(1,-1)), \\ c = \exp_p(\lambda(-1,1)), & d = \exp_p(\lambda(-1,-1)), \end{array} \right\} \tag{4.6}$$

from (4.5) we obtain

$$\left(\frac{az+b}{cz+d} \right)^k = z. \tag{4.7}$$

Denote

$$f(x) = \left(\frac{ax+b}{cx+d} \right)^k.$$

Let $S_1 = \{x \in \mathbb{Q}_p : |x|_p = 1\}$. Then it is clear that $f(S_1) \subset S_1$. Using this fact for every $x \in S_1$ we find

$$|f(x) - 1|_p = \left| \frac{(a-c)x + b - d}{cx+d} \right|_p \left| \sum_{m=0} k - 1 \left(\frac{ax+b}{cx+d} \right)^m \right|_p \le \frac{1}{p}, \tag{4.8}$$

here we have used (4.6) and Lemma 2.1.

Let $x, y \in S_1$, then

$$|f(x) - f(y)|_p = \left| \frac{ax+b}{cx+d} - \frac{ay+b}{cy+d} \right|_p \left| \sum_{m=0}^{k-1} \left(\frac{ay+b}{cy+d} \right)^m \left(\frac{ax+b}{cx+d} \right)^{k-m-1} \right|_p \le$$

$$\le \frac{|ad - bc|_p |x - y|_p}{|cx+d|_p |cy+d|_p}. \tag{4.9}$$

Now consider two different cases with respect to p.

Let us assume that $p \ge 3$. In this case we have

$$|ad - bc|_p \le \frac{1}{p}, \quad |cx+d|_p = 1, \quad |cy+d|_p = 1,$$

which are obtained from (4.6) and Lemma 2.1. Using these equalities from (4.9) it can be found

$$|f(x) - f(y)|_p \le \frac{1}{p}|x - y|_p. \tag{4.10}$$

Now suppose $p = 2$. Then

$$|ad - bc|_p \le \frac{1}{2^2}, \quad |cx+d|_p = \frac{1}{2}, \quad |cy+d|_p = \frac{1}{2}.$$

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We claim that $|ad - bc|_p \leq \dfrac{1}{2^3}$ is satisfied. It follows from (4.9) that

$$|f(x) - f(y)|_2 \leq \frac{1}{2}|x - y|_2. \tag{4.11}$$

Thus the equalities (4.10) and (4.11) imply that f is a contraction of S_1, hence f has a unique fixed point $\zeta \in S_1$ such that $|\zeta - 1|_p \leq \frac{1}{p}$ (see (4.8)). So we have proved the following

Proposition 4.5. *(i) Let $p \geq 3$ and $|\lambda(u, v)|_p \leq \dfrac{1}{p}$ for all $< x, y > \in L$, $u, v \in \Phi$. Then for the p-adic homogeneous λ-model (2.1) on the Cayley tree of order $k(k \geq 1)$ the equation (4.5) has a unique solution.*

(ii) Let $p = 2$, $|\lambda(u, v)|_p \leq \dfrac{1}{p}$ for all $< x, y > \in L$, $u, v \in \Phi$ and the following condition be satisfied

$$|\exp_p(\lambda(1, 1)) \exp_p(\lambda(-1, -1)) - \exp_p(\lambda(-1, 1)) \exp_p(\lambda(1, -1))|_2 \leq \frac{1}{2^3} \tag{4.12}$$

Then for the 2-adic homogeneous λ-model (2.1) on the Cayley tree of order $k(k \geq 1)$ the equation (4.5) has a unique solution.

Theorem 4.6. *Let the condition of the previous Proposition be satisfied. Then for the p-adic homogeneous λ-model (2.1) on the Cayley tree of order k there is no phase transition for any prime p.*

Proof. In the homogeneous model (4.2) is written as

$$z_x = \prod_{i=1}^{k} \alpha_{x,i}, \tag{4.13}$$

here

$$\alpha_{x,i} = \frac{az_{x_i} + b}{cz_{x_i} + d},$$

where as before $z_x = \exp_p(h_x)$, $z_{x_i} = \exp_p(h_{x_i})$, and the coefficients a, b, c, d are defined by (4.6).

Let ζ be a solution of (4.7). Then using (4.13) we have

$$|z_x - \zeta|_p = \left| \prod_{i=1}^{k} \left(\frac{az_{x_i} + b}{cz_{x_i} + d} \right) - \left(\frac{a\zeta + b}{c\zeta + d} \right)^k \right|_p =$$

$$\left| \prod_{i=1}^{k} \left(\frac{az_{x_i} + b}{cz_{x_i} + d} \right) - \left(\frac{az_{x_k} + b}{cz_{x_k} + d} \right) \left(\frac{a\zeta + b}{c\zeta + d} \right)^{k-1} + \right.$$

$$\left. + \left(\frac{az_{x_k} + b}{cz_{x_k} + d} \right) \left(\frac{a\zeta + b}{c\zeta + d} \right)^{k-1} - \left(\frac{a\zeta + b}{c\zeta + d} \right)^k \right|_p \leq$$

$$\leq \max \left\{ \left| \frac{az_{x_k} + b}{cz_{x_k} + d} \right|_p \left| \prod_{i=1}^{k-1} \left(\frac{az_{x_i} + b}{cz_{x_i} + d} \right) - \left(\frac{a\zeta + b}{c\zeta + d} \right)^{k-1} \right|_p, \right.$$

$$\left. \left| \frac{a\zeta + b}{c\zeta + d} \right|_p^{k-1} \left| \frac{az_{x_k} + b}{cz_{x_k} + d} - \frac{a\zeta + b}{c\zeta + d} \right|_p \right\} =$$

$$= \max \left\{ \left| \frac{az_{x_k} + b}{cz_{x_k} + d} \right|_p \left| \prod_{i=1}^{k-1} \left(\frac{az_{x_i} + b}{cz_{x_i} + d} \right) - \left(\frac{a\zeta + b}{c\zeta + d} \right)^{k-1} \right|_p, \frac{|z_{x_k} - \zeta|_p |ad - bc|_p}{|cz_{x_k} + d|_p |c\zeta + d|_p} \right\} \leq \cdots$$

$$\leq \max \left\{ \frac{|z_{x_k} - \zeta|_p |ad - bc|_p}{|cz_{x_k} + d|_p |c\zeta + d|_p} \right\} \leq$$

$$\leq \frac{1}{p} \max_{1 \leq m \leq k} \{|z_{x_m} - \zeta|_p\}.$$

Now repeating the argument of the proof of Theorem 4.4 we obtain $z_x = \zeta$ for all $x \in V$. This completes the proof.

5 Applications to p-adic Ising model

In this section we will show that the phase transition does not occur for the p-adic Ising model.

Recall the p-adic Ising model. This model is a particular case of λ-model, namely it corresponds to the function:

$$\lambda_{x,y}(u, v) = J_{x,y}uv + \eta(u + v), \tag{5.1}$$

here $|J_{x,y}| \leq p^{-1/(p-1)}$, $|\eta|_p \leq p^{-1/(p-1)}$ and $< x, y >\in L$, $u, v \in \{-1, 1\}$.

First consider the case $\eta = 0$, this corresponds to the inhomogeneous p-adic Ising model without external field. For the considered model it is easy to see that the condition (4.1) is satisfied. So according to Theorem 4.4 we infer that the following

Theorem 5.1. *Let $k \geq 1$, $|J_{x,y}|_p \leq p^{-1/(p-1)}$ for all $< x, y >\in L$. Then for the p-adic inhomogeneous Ising model on the Cayley tree of order k there is no phase transition for any prime p.*

Now consider a case $J_{x,y} = J$ for all $< x, y >\in L$ and $\eta \neq 0$. This corresponds to the homogeneous p-adic Ising model with an external field.

Let $p = 2$, then the condition (4.12) can be written as follows

$$|\exp_p(J + 2\eta)\exp_p(J - 2\eta) - \exp_p(-J)\exp_p(-J)|_2 = |\exp_p(4J) - 1|_2 =$$

$$= |4J|_2 \leq \frac{1}{2^4},$$

here we have used Lemma 2.1. Hence (4.12) is satisfied. So we can formulate the following

Theorem 5.2. *Let $k \geq 1$, $|\eta|_p \leq p^{-1/(p-1)}$ and $|J|_p \leq p^{-1/(p-1)}$ for all $< x, y >\in L$. Then for the p-adic homogeneous Ising model on the Cayley tree of order k there is no phase transition for any prime p.*

Remark. It is known [5],[16] that for the Ising model on the Cayley tree of order $k \geq 2$ over $\mathbb{R}$ on some condition upon parameter $J_{x,y}$ there is a phase transition. Theorems 5.1 and 5.2 show the difference between real the Ising model and the considered p-adic one.

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