Available at: http://www.ictp.trieste.it/~pub_off IC/2004/9

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON SYMPLECTOMORPHISMS OF THE SYMPLECTISATION OF A COMPACT CONTACT MANIFOLD

Augustin Banyaga
Department of Mathematics, The Pennsylvania State University,
University Park, PA 16802, USA
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

Let (N, α) be a compact contact manifold and $(N \times \mathbb{R}, d(e^t\alpha))$ its symplectisation. We show that the group G which is the identity component in the group of symplectic diffeomorphisms ϕ of $(N \times \mathbb{R}, d(e^t\alpha))$ that cover diffeomorphisms $\underline{\phi}$ of $N \times S^1$ is simple, by showing that G is isomorphic to the kernel of the Calabi homomorphism of the associated locally conformal symplectic structure.

MIRAMARE - TRIESTE

March 2004

1. Introduction and statements of the results

The structure of the group of compactly supported symplectic diffeomorphisms of a symplectic manifold is well understood [1], see also [2]. For instance, if (M, Ω) is a compact symplectic manifold, the commutator subgroup $[Diff_\Omega(M)_0, Diff_\Omega(M)_0]$ of the identity component $Diff_\Omega(M)_0$ in the group of all symplectic diffeomorphisms, is the kernel of a homomorphism from $Diff_\Omega(M)_0$ to a quotient of $H^1(M, \mathbb{R})$ (the Calabi homomorphism) and it is a simple group.

Unfortunately, the structure of the group of symplectic diffeomorphisms of a non compact manifold, with unrestricted supports in largely unkown.

In this paper, we study the group $Diff_{\tilde{\Omega}}(N \times \mathbb{R})$ of symplectic diffeomorphisms of the symplectisation $(N \times \mathbb{R}, \tilde{\Omega} = d(e^t\alpha))$ of a compact contact manifold (N, α). Our main result is the following

Theorem 1.

Let G be the subgroup of $Diff_{\tilde{\Omega}}(N \times \mathbb{R})$ consisting of elements ϕ, isotopic to the identity through isotopies ϕ_t in $Diff_{\tilde{\Omega}}(N \times \mathbb{R})$, which cover isotopies $\underline{\phi_t}$ of $N \times S^1$. Then G is a simple group.

Recall that a group G is said to be a simple group if it has no non-trivial normal subgroup. In particular it is equal to its commutator subgroup $[G, G]$.

Let $Diff_{\tilde{\Omega}}(N \times \mathbb{R})_0$ be the subgroup consisting of elements isotopic to the identity in $Diff_{\tilde{\Omega}}(N \times \mathbb{R})$. For $\phi \in Diff_{\tilde{\Omega}}(N \times \mathbb{R})_0$, the 1-form

$$\tilde{C}(\phi) = \phi^*(e^t\alpha) - e^t\alpha$$

is closed. Let $C(\phi)$ denote its cohomology class in $H^1(N \times \mathbb{R}, \mathbb{R}) \approx H^1(N, \mathbb{R})$.

The map $\phi \mapsto C(\phi)$ is a surjective homomorphism

$$C : Diff_{\tilde{\Omega}}(N \times \mathbb{R})_0 \to H^1(N, \mathbb{R})$$

(the Calabi homomorphism, see [1]).

Corollary.

The group G is contained in the kernel of C.

Proof.

Since G is simple, the kernel of the restriction C_0 of C to G is either the trivial group $\{id\}$ or the whole group G. But $KerC_0$ contains $[G, G] \neq \{1d\}$. Hence $KerC_0 = G$. □

Theorem 1 follows from the study of the structure of the group of diffeomorphisms preserving a locally conformal symplectic structure. Each locally conformal symplectic manifold (M, Ω), is covered in a natural way by a symplectic manifold $(\tilde{M}, \tilde{\Omega})$. We analyse the group of symplectic diffeomorphisms of $\tilde{M}$, which cover diffeomorphisms of M (Theorem 2). Our results will be deducted from the fact that, if (N, α) is a contact manifold, then $N \times S^1$ has a locally conformal symplectic structure and the associated symplectic manifold covering $N \times S^1$ is precisely the symplectisation. We show that the group G is isomorphic to the kernel of the Calabi homomorphism for locally conformal symplectic geometry.

2. The structure of the group of diffeomorphisms covering locally conformal symplectic diffeomorphisms.

A locally conformal symplectic form on a smooth manifold M is a non-degenerate 2-form Ω such that there exists a closed 1-form ω satisfying:

$$d\Omega = -\omega \wedge \Omega.$$

The 1-form ω is uniquely determined by Ω and is called the Lee form of Ω. The couple (M, Ω) is called a locally conformal symplectic (lcs, for short) manifold, see [3], [7], [11].

The group $Diff(M, \Omega)$ of automorphisms of a lcs manifold (M, Ω) consists of diffeomorphisms ϕ of M such that $\phi^*\Omega = f\Omega$ for some non-zero function f. Here we will always assume that f is a positive function. Such a diffeomorphism is said to be a locally conformal symplectic diffeomorphism.

Let $\tilde{M}$ be the minimum regular cover of M over which the form ω pulls to an exact form: i.e. if $\pi : \tilde{M} \to M$ is the covering map, then

$$\pi^*\omega = d(ln\lambda).$$

If λ' is another function such that $\pi^*\omega = d(ln\lambda')$, then $\lambda' = a\lambda$ for some constant a.

On $\tilde{M}$, we consider the symplectic form

$$\tilde{\Omega} = \lambda\pi^*\Omega$$

The conformal class of $\tilde{\Omega}$ is independent of the choice of λ [4].

A diffeomorphism ϕ of $\tilde{M}$ is said to be fibered if there exists a diffeomorphism h of M such that $\pi \circ \phi = h \circ \pi$. We also say that ϕ covers h.

Proposition 1.

If a diffeomorphism ϕ of $\tilde{M}$ covers a diffeomorphism h of M, then ϕ is conformal symplectic iff h is locally conformal symplectic.

Proof.

Suppose $\phi : \tilde{M} \to \tilde{M}$ is conformal symplectic, and covers $h : M \to M$. Then $\phi^*(\tilde{\Omega}) = a\tilde{\Omega}$ for some number $a \in \mathbb{R}$. We have:

$$\pi^*(h^*\Omega) = \phi^*(\pi^*\Omega) = \phi^*((1/\lambda)\tilde{\Omega}) = (\frac{1}{\lambda} \circ \phi))a\tilde{\Omega} = a(\frac{1}{\lambda} \circ \phi)\lambda\pi^*\Omega.$$

Let τ be an automorphism of the covering $\tilde{M} \to M$, then

$$\tau^*\pi^*(h^*\Omega) = (\pi \circ \tau)^*(h^*\Omega) = \pi^*(h^*\Omega) = \tau^*[(a\frac{1}{\lambda} \circ \phi)\lambda]\tau^*\pi^*\Omega = \tau^*[(a\frac{1}{\lambda} \circ \phi)\lambda]\pi^*\Omega.$$

Therefore $\tau^*[(a\frac{1}{\lambda} \circ \phi)\lambda] = (a\frac{1}{\lambda} \circ \phi)\lambda)$ since $\pi^*\Omega$ is non-degenerate. Hence $(a\frac{1}{\lambda} \circ \phi)\lambda) = u \circ \phi$, where u is a basic function. We thus get $\pi^*(h^*\Omega) = \pi^*(u\Omega)$. Since π is a covering map, $h^*\Omega = u\Omega$.

Conversely if $h \in Diff(M, \Omega)$, i.e. $h^*\Omega = u\Omega$ for some function u on M, and ϕ is its lift on $\tilde{M}$, then: $\phi^*\tilde{\Omega} = \phi^*(\lambda\pi^*\Omega) = (\lambda\circ\phi)\phi^*\pi^*\Omega = (\lambda\circ\phi)(\pi\circ\phi)^*\Omega = (\lambda\circ\phi)(h\circ\pi)^*\Omega = (\lambda\circ\phi)\pi^*h^*\Omega = (\lambda \circ \phi)\pi^*(u\Omega) = (\frac{\lambda \circ h}{\lambda} u \circ \pi)\tilde{\Omega}$.

A theorem of Liberman (see [9] or [5]) asserts that if a diffeomorphism preserves a symplectic form up to a smooth function, then this function is a constant provided that the dimension of the manifold is at least 4. Hence ϕ is a conformal symplectic diffeomorphism.

□

Let $Diff_{\tilde{\Omega}}(\tilde{M})_C$ be the group of conformal symplectic of $\tilde{M}$: a diffeomorphism ϕ of $\tilde{M}$ belongs to this group if $\phi^*\tilde{\Omega} = a\tilde{\Omega}$ for some positive number a.

The group $Diff_{\tilde{\Omega}}(\tilde{M})$ of symplectic diffeomorphisms is the kernel of the homomorphism:

$$d : Diff_{\tilde{\Omega}}(\tilde{M})_C \to \mathbb{R}^+$$

sending ϕ to $a \in \mathbb{R}^+$ when $\phi^*\tilde{\Omega} = a\tilde{\Omega}$.

We consider the subgroups $Diff_{\tilde{\Omega}}(\tilde{M})^F_C$, resp. $Diff_{\tilde{\Omega}}(\tilde{M})^F$ of $Diff_{\tilde{\Omega}}(M)_C$, resp. of $Diff_{\tilde{\Omega}}(\tilde{M})$ consisting of fibered elements.

Finally, let G_C, resp. G be the subgroups of $Diff_{\tilde{\Omega}}(\tilde{M})^F_C$, resp. $Diff_{\tilde{\Omega}}(\tilde{M})^F$ consisting of elements that are isotopic to the identity through these respective groups. We denote by $Diff(M, \Omega)_0$ the identity component in the group $Diff(M, \Omega)$, endowed with the C^∞ topology.

By Proposition 1, we have a homomorphism $\rho : G_C \to Diff(M,\Omega)_0$. This homomorphism is surjective: indeed, any diffeomorphism isotopic to the identity lifts to a diffeomorphism of the covering space $\tilde{M}$. See for instance [6]. By Proposition 1, that lifting must be a conformal symplectic diffeomorphism.

Let $\mathcal{A}$ be the group of automorphisms of the covering $\pi : \tilde{M} \to M$. For any $\tau \in \mathcal{A}$, $(\lambda \circ \tau)/\lambda$ is a constant c_τ independent of λ and the map $\tau \mapsto c_\tau$ is a group homomorphism [5]

$$c : \mathcal{A} \to \mathbb{R}^+$$

Let us denote by $\Delta \subset \mathbb{R}^+$ the image of c and by $K \subset \mathcal{A}$ its kernel.

For $\tau \in \mathcal{A}$, we have: $\tau^*\tilde{\Omega} = \tau^*(\lambda\pi^*\Omega) = (\lambda\circ\tau)\tau^*\pi^*\Omega = (\lambda\circ\tau)\pi^*\Omega = ((\lambda\circ\tau)/\lambda)(\lambda\pi^*\Omega) = c_\tau\tilde{\Omega}$. This shows that

$$Ker\rho = \mathcal{A}.$$

Each element $h \in Diff(M,\Omega)_0$ lifts to an element $\phi \in G_C$ and two different liftings differ by an element of $\mathcal{A}$. Hence the mapping $h \mapsto d(\phi)$ is a well defined map

$$\mathcal{L} : Diff(M,\Omega)_0 \to \mathbb{R}/\Delta.$$

It is a homomorphism since a lift of $\phi\psi$ differs from the product of their lifts by an element of $\mathcal{A}$.

Let $\mathcal{L}(M,\Omega)$ be the Lie algebra of locally conformal symplectic vector fields, consisting of vector fields X such that $L_X\Omega = \mu_X\Omega$ for some function μ_X on M and L_X stands for the Lie derivative in the direction X.

Let Ω be a lcs form with Lee form ω on a manifold M. One verifies that for all $X \in \mathcal{L}(M,\Omega)$, then the function $l(X) = \omega(X) + \mu_X$ is a constant, and that the map

$$l : \mathcal{L}(M,\Omega) \to \mathbb{R}; X \mapsto l(X)$$

is a Lie algebra homomorphism, called the extended Lee homomorphism [1], see also [3], [5].

We need now to recall the definition of the Lichnerowicz cohomology [7]. This is the cohomology of the complex of differential forms $\Lambda(M)$ on a smooth manifold with the de Rham differential replaced by d_ω, $d_\omega\theta = d\theta + \omega \wedge \theta$, where ω is a closed 1-form on M. We denote this cohomology by: $H^*_\omega(M)$.

If (M,Ω) is a locally conformal symplectic form with Lee form ω, the equation $d\Omega = -\omega \wedge \Omega$ says that the 2-form Ω is d_ω closed, and hence defines a class $[\Omega] \in H^2_\omega(M)$.

Proposition.

Let Ω be a lcs form with Lee form ω on a smooth manifold M. The extended Lee homomorphism is surjective iff the Lichnerowicz cohomology class $[\Omega] \in H^2_\omega(M)$ is zero, i.e. iff Ω is d_ω - exact.

This proposition is essentially due to Guedira-Lichnerowicz [7] and Viasman [11] can also be found in several places [4], [5], [8].

Let ϕ_t be a smooth family of locally conformal symplectic diffeomorphisms with $\phi_0 = id_M$, and let X_t be the family of vector fields defined by:

$$X_t(\phi_t(x)) = \frac{d}{dt}(\phi_t(x)).$$

Then X_t is a family of locally conformal vector fields: there exists a smooth family of functions μ_{X_t} such that $L_{X_t}\Omega = \mu_{X_t}\Omega$.

The mapping:

$$\phi_t \mapsto \int_0^1 l(X_t))dt$$

induces a well defined homomorphism $\tilde{L}$ from the universal covering $\mathcal{U}(Diff(M,\Omega)_0)$ of $Diff(M,\Omega)_0$ to $\mathbb{R}$, and therefore induces a homomorphim

$$L : Diff(M,\Omega)_0 \to \mathbb{R}/\Gamma$$

where $\Gamma \subset \mathbb{R}$ is the image by $\tilde{L}$ of the fundamental group of $Diff(M,\Omega)_0$.

This integration of the extended Lee homomorphism $l : \mathcal{L}(M,\Omega) \to \mathbb{R}$ was considered in [8].

Another integration of the extended Lee homomorphism was constructed in [4], [5]. It is shown there that the subroups Δ and Γ of $\mathbb{R}$ below are the same and that the homomorphisms L and $\mathcal{L}$ above coincide.

We will need the following result of Haller and Rybicki [8]:

Theorem.

Let (M,Ω) be a compact lcs manifold with $[\Omega] = 0 \in H^2_\omega(M)$, where ω is the Lee form of Ω, then

1. $KerL = [Diff(M,\Omega)_0, Diff(M,\Omega)_0]$.

2. There is a surjective homomorphism S from $KerL$ to a quotient of $H^1_\omega(M)$ whose kernel is a simple group.

The homomorphism S is an analogue of the Calabi homomorphism [1], and the theorem above is a generalization to locally conformal symplectic manifolds of the results on symplectic manifolds in [1]. The definition of the homomorphism S is recalled in the appendix.

As a consequence of these constructions and results, we have the following

Theorem 2.

Let (M,Ω) be a compact lcs manifold with Lee form ω and such that $[\Omega] = 0 \in H^2_\omega(M)$. Then:

1. *d and $\mathcal{L}$ are surjective.*
2. *We have the following exact sequence:*

$$\{1\} \longrightarrow K \longrightarrow G \longrightarrow Ker\mathcal{L} \longrightarrow \{1\}$$

3. *$Ker\mathcal{L} \approx [Diff(M,\Omega)_0, Diff(M,\Omega)_0]$.*

Proof.

Let θ be a 1-form such that $\Omega = d_\omega\theta$ and let X be defined by $i_X\Omega = \theta$. Then $X \in \mathcal{L}(M,\omega)$ and $l(X) = 1$. Hence $\mathcal{L}$ is surjective. The horizontal lift $\tilde{X}$ of X to $\tilde{M}$ is a complete vector field, and if h is its time 1 flow, then $d(h) = 1$. Hence the mapping d is surjective.

Since $\mathcal{L}$ is equal to L, the point 3 is just a part of Haller-Rybicki theorem.

Let $h, g \in Diff(M,\Omega)_0$ and their lifts ϕ,ψ on $\tilde{M}$. Let $a, b \in \mathbb{R}$ such that $\phi^*\tilde{\Omega} = a\tilde{\Omega}$, $\psi^*\tilde{\Omega} = b\tilde{\Omega}$. Then the commutator hgh^-1g^{-1} lifts to $\phi\psi\phi^{-1}\psi^{-1}$,and $(\phi\psi\phi^{-1}\psi^{-1})^*\tilde{\Omega} = b^{-1}a^{-1}ba\tilde{\Omega} = \tilde{\Omega}$. Hence all of $Ker\mathcal{L}$ lifts to G since $Ker\mathcal{L} \approx [Diff(M,\omega)_0, Diff(M,\Omega)_0]$. This finishes the proof that the sequence 2 is exact. □

3. The symplectisation of a contact manifold

Let α be a contact form on a smooth manifold N. Let p_1, p_2 be the projections from $M = N \times S^1$ to the factors N, S^1. If μ is the canonical 1-form on S^1 such that $\int_{S^1}\mu = 1$, then $\Omega = d\theta + \omega\wedge\theta$, where $\theta = p_1^\alpha, \omega = p_2^*\mu$, is a lcs form on $M = N \times S^1$.

The hypothesis of Theorem 2 are satisfied for $M = N \times S^1$, where N is a compact contact manifold and $\Omega = d_\omega\theta$ as above.

The minimun cover $\tilde{M}$ is $N\times\mathbb{R}$, the projection $\pi : N\times\mathbb{R} \to N\times S^1$ is the standard projection: $\pi(x,t) = (x, e^{2\pi it})$, and $\pi^*\omega = dt$, $\lambda = e^t$. We have: $\tilde{\Omega} = \lambda\pi^*\Omega = e^t(d\alpha + dt\wedge\alpha) = d(e^t\alpha)$. Hence $(\tilde{M},\tilde{\Omega})$ is the symplectisation $(N \times \mathbb{R}, d(e^t\alpha))$.

Here $\mathcal{A}$ consists of maps $\gamma_n(x,t) = (x, n+t)$, for all $n \in \mathbb{Z}$. We have $\gamma_n^*\tilde{\Omega} = d(\gamma_n^*(e^t\alpha)) = d(e^{(t+n)}\alpha = e^n\tilde{\Omega}$. Hence $\gamma_n \in Kerc = K$ iff $n = 0$, i.e. $Kerc = \{id\}$. This and Theorem 2 (2) show that

$$G = Diff_{\tilde{\Omega}}(N \times \mathbb{R})_0^F \approx Ker\mathcal{L}$$

The last step is to show that $Ker\mathcal{L}$ is a simple group. The Calabi homomorphism S takes $Ker\mathcal{L}$ to a quotient of $H_\omega^1(N \times S^1)$, as one can see in the appendix.. But we know that:

$$H_\omega^*(N \times S^1) \approx 0$$

Indeed, take an exact 1-form σ on N and consider $\omega' = \omega + p_1^*\sigma$. Then $H_\omega^*(N \times S^1) \approx H_{\omega'}^*(N \times S^1)$ since ω and ω' are cohomologous. By the Kunneth formula for the Lichnerowicz cohomology, $H_{\omega'}^i(N \times S^1) \approx \oplus(H_\mu^j(S^1) \otimes H_\sigma^{i-j}(N)$. But is known that $H_\mu^j(S^1) = 0$ for all j [7], [8], [3]. Therefore $H_{\omega''}^*(N \times S^1) \approx H_\omega^*(N \times S^1) = \{0\}$.

Hence, $KerS = Ker\mathcal{L}$ is a simple group. This ends the proof of Theorem 1. $\square$

Appendix

For completeness, we recall briefly the Calabi homomorphism in lcs geometry[8]: an element $\tilde{\phi}$ of the universal covering of $KerL$ can be represented by an isotopy $\phi_t \in Diff(M, \Omega)$ with tangent vector fields $X_t \in Kerl$. Recall that X_t is defined by : $X_t(\phi_t(x)) = \frac{d}{dt}(\phi_t(x))$. This implies that $d_\omega(i(X_t)\Omega = 0$, since

$$d_\omega(i(X_t)\Omega) = d(i(X_t)\Omega) + \omega \wedge (i(X_t)\Omega) =$$

$$L_{X_t}\Omega - i(X_t)(-\omega \wedge \Omega) + \omega \wedge (i(X_t)\Omega)$$

$$= (\mu_{X_t} + \omega(X_t))\Omega = l(X_t)\Omega = 0.$$

One shows that

$$[\int_0^1 (i(X_t)\Omega)dt] \in H_\omega^1(M)$$

depends only on $\tilde{\phi}$, and that the correspondance

$$\tilde{\phi} \mapsto [\int_0^1 (i(X_t)\Omega)dt]$$

is a surjective homomorphism from the universal cover of $KerL$ to $H_\omega^1(M)$. This defines a surjective homomorphism $S : KerL \to H_\omega^1(M)/\Lambda$, where Λ is the image of the fundamental group of $KerL$.

Acknowledgments

I would like to thank the Abdus Salam International Centre for Theoretical Physics for its hospitality and support during the last phase of the preparation of this work.

REFERENCES

[1] A. Banyaga, *Sur la structure du groupe de diffeomorphismses qui preservent une forme symplectique*, Comment. Math. Helv. 53(1978), 174-227.

[2] A. Banyaga, *The structure of classical diffeomorphism groups*, Mathematics and Its Applications. Vol 400, Kluwer Academic Publisher, Dordrecht, The Netherlands, 1997.

[3] A. Banyaga, *Some properties of locally conformal symplectic structures*, Comment. Math. Helv. 77 (2002) 383-398

[4] A. Banyaga, *Quelques invariants des structures localement conformement symplectiques*, C.R.Acad. Sci. Paris t 332 , Serie 1 (2001) 29-32.

[5] A. Banyaga, *A geometric integration of the extended Lee homomorphism*, Journal of Geometry and Physics, 39(2001) 30-44.

[6] W.D. Curtis, *The automorphism group of a compact group action*, Trans. Amer. Math. Soc. Vol 203(1975) 45-54

[7] F. Guedira and A. Lichnerowicz, *Geometrie des algebres de Lie locales de Kirillov*, J.Math. Pures et Appl. 63(1984), 407-484.

[8] S. Haller and T. Rybicki, *On the group of diffeomorphisms preserving a locally conformal symplectic structure*. Ann. Global Anal. and Geom. 17 (1999) 475-502.

[9] S. Kobayashi, *Transformation groups in differential geometry*, Erg. Math. Grenzgeb. Vol 70, Springer, Berlin.

[10] H.C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus*. Amer. J. Math. 65(1943) pp 433-438.

[11] I. Vaisman, *Locally conformal symplectic manifolds*, Inter. J. Math. and Math. Sc. 8 no 3(1983), 521-536.