United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON THE LERAY-HIRSCH THEOREM FOR THE LICHNEROWICZ COHOMOLOGY

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Abstract

The purpose of this paper is to prove the Leray-Hirsch theorem for the Lichnerowicz cohomology with respect to basic and vertical closed 1-forms. This is a generalization of the Künneth theorem to fiber bundles.

MIRAMARE – TRIESTE
March 2004

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1 Introduction

The Lichnerowicz cohomology, $H^*_\omega(M)$ (i.e. cohomology of the complex of differential forms on a smooth manifold with the de Rham differential operator deformed by a closed 1-form ω) was initiated by A. Lichnerowicz [7] . Since its introduction, Lichnerowicz cohomology has attracted a lot of interest, see for instance [1, 2, 3, 8, 10]. Its importance comes from the fact that Lichnerowicz cohomology is well adapted to locally conformal symplectic (lcs) geometry. Recall that a lcs manifold consists of manifold M, together with a non-degenerate 2-form Ω on M and a closed 1-form ω such that $d\Omega = -\omega \wedge \Omega$. See for example [1, 2, 4]. It was observed in [1] that $H^*_\omega(M)$ is an invariant of the lcs structure. This cohomology is very different from the de Rham cohomology (see proposition 2.2, theorem 2.2).

In this note, we show that many properties of the de Rham cohomology still have their analogues within the Lichnerowicz cohomology.

In the second section, we first recall the definitions and some basic properties: Poincaré duality and Künneth formula hold for the Lichnerowicz cohomology.

In the last section, we consider a generalization of Künneth formula; namely the Leray-Hirsch Theorem for basic and vertical forms.

2 Preliminaries

In this section, we define the Lichnerowicz cohomology and give some basic theorems.
Let M be a differentiable manifold and ω a closed 1-form on M.

The Lichnerowicz cohomology is the cohomology of differential forms on M with the differential operator d^ω (see [7]) defined by

$$d^\omega = d + e(\omega),$$

d being the usual exterior differential and $e(\omega)$ the operator given by

$$e(\omega)\alpha = \omega \wedge \alpha \text{ for all } \alpha \in \Omega^*(M).$$

Obviously we have $d^\omega \circ d^\omega = 0$. Denote by $H^*_\omega(M)$ the cohomology of the complex $(\Omega^*(M), d_\omega)$.
Similarly we define the Lichnerowicz cohomology with compact support $H^*_\omega(M)_c$.
We consider 0 as a 1-form and define $H^*_0(M)$ to be the de Rham cohomology of M.

For two closed 1-forms ω_1, ω_2 and $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$ an easy calculation shows that

$$d^{\omega_1+\omega_2}(\alpha \wedge \beta) = d^{\omega_1}\alpha \wedge \beta + (-1)^p \alpha \wedge d^{\omega_2}\beta.$$

Hence the wedge product induces a bilinear mapping

$$\begin{array}{rcl} \wedge : H^p_{\omega_1}(M) \times H^q_{\omega_2}(M) & \longrightarrow & H^{p+q}_{\omega_1+\omega_2}(M) \\ ([\alpha],[\beta]) & \longmapsto & [\alpha \wedge \beta]. \end{array}$$

Proposition 2.1 *(see [8])*
Let M be a differentiable manifold and ω a closed 1-form on M.
(a) If ω is exact then $H^*_\omega(M) \cong H^*_{dR}(M)$;
(b) Suppose $[\omega'] = [\omega] \in H^1(M)$: there exists a smooth positive function f such that $\omega' = \omega + d(lnf)$. Then the following application

$$\begin{array}{rcl} \varphi : H^*_{\omega'}(M) & \longrightarrow & H^*_\omega(M) \\ [\alpha] & \longmapsto & [f\alpha] \end{array}$$

is an isomorphism.

The spaces $H^p_\omega(M)$ can also be obtained as the cohomology of M with coefficients in a sheaf. In fact, let us denote by $\mathcal{F}_\omega(M)$ the sheaf of germs of differentiable functions $f : M \longrightarrow I\!R$ which are d^ω-closed, i.e.

$$d^\omega f = df + f\omega = 0.$$

Using the fact that d^ω satisfies a Poincaré lemma and that

$$0 \longrightarrow \mathcal{F}_\omega(M) \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{A}^1 \longrightarrow ...$$

is a fine resolution of $\mathcal{F}_\omega(M)$, where $\mathcal{A}^p$ is the sheaf of germs of differentiable p-forms on M, one can prove

Proposition 2.2 *[14]*
For every manifold M and a closed 1-form ω on M, one has an isomorphism

$$H^p(M, \mathcal{F}_\omega(M) \approx H^p_\omega(M).$$

Let us recall another interpretation of the Lichnerowicz cohomology given by Banyaga in [1]. Let $\pi : \tilde{M} \longrightarrow M$ be the minimum regular cover over which the 1-form ω is pulled back to an exact 1-form and let $\lambda : \tilde{M} \longrightarrow I\!R$ be a positive function on $\tilde{M}$ such that

$$\pi^*\omega = d\lambda.$$

It is well known that the group of automorphisms $\mathcal{A}$ of the covering $\tilde{M}$, is isomorphic to the group of periods of ω [6].

Lemma 2.1

For any $\tau \in \mathcal{A}$, the function

$$\frac{(\lambda \circ \tau)}{\lambda}$$

is a constant, we denote c_τ, independent of the choice of λ.

The set $\mathcal{F}^*_{c\mathcal{A}}(M)$ of all differential forms α on $\tilde{M}$ such that $\tau^*\alpha = c_\tau \alpha$ for all $\tau \in \mathcal{A}$, is a subcomplex of the de Rham complex of $\tilde{M}$. We denote its cohomology by $H^*_{c\mathcal{A}}(M)$. The following theorem is proved in [2]

Theorem 2.1

$H^*_{c\mathcal{A}}(M)$ is isomorphic with $H^*_\omega(M)$.

Example 2.1

Let $M = S^1$ and ω be a generator of its first de Rham cohomology. We claim that $H^0_\omega(S^1) = 0$. So let $f \in \Omega^0(S^1)$ such that $df = 0$ and $\tau^* f = c_\tau f$. The condition $df = 0$ means that $f = k$ is a constant. Then, the second condition $k = f o \tau = c_\tau k$ translates to $(1 - c_\tau)k = 0$, but if ω is non-exact, $c_\tau \neq 1$, hence $k = 0$. So $H^0_\omega(S^1) = 0$ and by Poincaré duality $H^1_\omega(S^1) = 0$.

The following lemma proves the homotopy invariance of the Lichnerowicz cohomology. For a smooth $g : M \longrightarrow N$ we have an induced mapping $g^* : H^*_\omega(N) \longrightarrow H^*_{g^*\omega}(M)$.

Lemma 2.2 (Homotopy invariance)

Let ω be a closed 1-form on differentiable manifold N and let $g : M \times I \longrightarrow N$ be a smooth homotopy. Define $f \in C^\infty(M \times I, I\!R)$ by $f_t := exp(\int_0^t \phi_s^* i_T g^* \omega ds)$ where
$\phi_s : M \longrightarrow M \times I; x \longmapsto (x, s)$ and i_T is the contraction by the vector field $T = \frac{\partial}{\partial t}$.
Then $f_1 g_1^* = f_0 g_0^* : H^*_\omega(N) \longrightarrow H^*_{g_0^*\omega}(M)$.
If g is proper the same holds with compact supports.

For the proof, see [8].

2.1 Hodge Theory

Suppose that M is a compact differentiable manifold of dimension n, that ω is a closed 1-form on M and that g is a Riemannian metric. Consider the vector field T on M given by $\omega(X) = g(X, T)$, for all $X \in \mathcal{X}(M)$. Denote by δ the codifferential operator and by i_T the contraction by the vector field T (see for example [16]). Then one can define the operator $\delta^\omega : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$ by

$$\delta^\omega = \delta + i_T.$$

Now, one can consider the standard scalar product on the space $\Omega^*(M)$ given by
$\langle \alpha, \beta \rangle = \int \alpha \wedge *\beta$, for all $\alpha, \beta \in \Omega^p(M)$ [13]. Since M is compact and the operator δ_ω is elliptic,

the Hodge decomposition into orthogonal parts holds:

$$\Omega^p(M) = \mathcal{H}^p_\omega(M) \oplus d^\omega(\Omega^{p-1}(M)) \oplus \delta^\omega(\Omega^{p+1}(M)),$$

where $\mathcal{H}^p_\omega(M) = \{\alpha \in \Omega^p(M)/d^\omega(\alpha = 0, \delta^\omega(\alpha) = 0\}$. Hence,

$$H^p_\omega(M) \cong \mathcal{H}^p_\omega(M).$$

Using the Hodge theory, we get:

Theorem 2.2 *(Guédira and Lichnerowicz)[7].*
Let M a n-dimensional manifold and ω a closed 1-form not d^ω-exact on M. All n-forms on M are d^ω-exact i.e. $H^n_\omega(M) = \{0\}$.

Remark 2.1
It is shown in [10] that if ω is everywhere non zero and parallel with respect to some Riemannian metric on M, then $H^*_\omega(M)$ is trivial.

2.2 Poincaré duality

Let M be an oriented n-dimensional manifold and ω a closed 1-form on M. For any integer $p \leq n$, we may define a pairing

$$\langle .,. \rangle_\omega : H^p_{-\omega}(X) \times H^{n-p}_\omega(M)_c \xrightarrow{\wedge} H^n_c(M) \xrightarrow{\int_M} \mathbb{R}$$

by

$$\langle [\sigma], [\tau] \rangle_\omega = \int_M [\sigma] \wedge [\tau] = \int_M [\sigma \wedge \tau],$$

which can also be expressed in the guise of the linear maps;

$$\begin{array}{rcl} D^p_\omega : H^p_{-\omega}(M) & \longrightarrow & (H^{n-p}_\omega(M)_c)^* \\ [\sigma] & \longrightarrow & D^p_\omega([\sigma])([\tau]) := \langle [\sigma], [\tau] \rangle_\omega \end{array}$$

Theorem 2.3 (Poincaré Duality)
For any n-dimensional oriented differentiable manifold M and any integer $p \leq n$, the linear map

$$D^p_\omega : H^p_{-\omega}(M) \longrightarrow (H^{n-p}_\omega(M)_c)^*$$

is an isomorphism.

3 Künneth formula

In this section, we will prove the Leray-Hirsch Theorem which is the generalization of the Künneth formula.
The purpose of Künneth formula is the computation of the cohomology of the Cartesian product

when the cohomologies of the factors are known.

Suppose we have two differentiable manifolds M, N and two closed 1-forms ω_1 and ω_2 on M and N respectively. Then $\omega = pr_1^*\omega_1 + pr_2^*\omega_2$ is a closed 1-form in $M \times N$ where pr_1 and pr_2 are the canonical projections.

Let α be a p-form on M and β a q-form on N. We write $\alpha\bar{\wedge}\beta =: pr_1^*\alpha \wedge pr_2^*\beta \in \Omega^{(p+q)}(M \times N)$. It is obvious that

$$d^\omega(\alpha\bar{\wedge}\beta) = d^{\omega_1}\alpha\bar{\wedge}\beta + (-1)^p\alpha\bar{\wedge}d^{\omega_2}\beta$$

and hence we have an induced mapping in cohomology given by

$$\begin{array}{rcl} \cup : H^p_{\omega_1}(M) \otimes H^q_{\omega_2}(N) & \longrightarrow & H^{p+q}_\omega(M \times N) \\ ([\alpha], [\beta]) & \longmapsto & [\alpha \wedge \beta] =: \alpha \cup \beta. \end{array}$$

Definition 3.1 *[5]*
A covering $\mathcal{U}$ of manifold M is called good if for all $m \in I\!N$ and $U_1, ..., U_m \in \mathcal{U}$ the intersection $U_1 \cap ... \cap U_m$ is either empty or contractible.

Remark 3.1
Every Riemannian manifold admits a good covering and these coverings are cofinal in the set of all coverings [15].

Theorem 3.1 (Künneth formula)
Suppose that M and N have good covers and let ω_1 respectively ω_2 be a closed 1-form on M respectively N, then the map

$$\cup : H^*_{\omega_1}(M) \otimes H^*_{\omega_2}(N) \longrightarrow H^*_\omega(M \times N)$$

is an isomorphism.

Example 3.1
Let (M, ω_1) be a smooth manifold equipped with the closed 1-form ω_1 and consider ζ the generator of the first de Rham cohomology of S^1. Denote $\omega = pr_1^*\omega_1 + pr_2^*\zeta$. The Künneth formula shows that

$$H^r_\omega(M \times S^1) = \oplus_{p+q=r} H^p_{\omega_1}(M) \otimes H^q_\zeta(S^1).$$

Example 2.1 shows that $H^0_\zeta(S^1) = 0$ and hence $H^1_\zeta(S^1) = 0$.
Thus $H^r_\omega(M \times S^1) = 0$.

We generalize the above example as follows.

Example 3.2
Let G be a Lie group with Lie algebra $\mathcal{G}$, $E \longrightarrow M$ a principal G-bundle on M, and ω a connection 1-form on E. Thus $\omega \in \Omega^1(E, \mathcal{G})$, and its curvature is given by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(E, \mathcal{G}).$$

A connection ω with $\Omega = 0$ is a flat connection and a bundle with a flat connection is said to be flat.

If E is an S^1-flat bundle, then $[\omega, \omega] = 0$ and therefore $\Omega = d\omega = 0$. So ω is an ordinary closed 1-form.

Suppose that M has a good cover $\mathcal{U}$ and let $U \in \mathcal{U}$. Take df an exact 1-form on U, then;

$$\begin{aligned} H^*_{pr_2^* df + \omega_{|U \times S^1}}(U \times S^1) &\cong H^*_{df}(U) \otimes H^*_{\omega_{|S^1}}(S^1) \\ &\cong H^*(U) \otimes H^*_{\omega_{|S^1}}(S^1) \\ &\cong 0 \end{aligned}$$

since $H^*_{\omega_{|S^1}}(S^1) = 0$.

Similarly, for $V \in \mathcal{U}$, we get

$$\begin{aligned} H^*_{pr_2^* df + \omega_{|V \times S^1}}(V \times S^1) &\cong H^*_{df}(V) \otimes H^*_{\omega_{|S^1}}(S^1) \\ &\cong H^*(V) \otimes H^*_{\omega_{|S^1}}(S^1) \\ &\cong 0 \end{aligned}$$

and

$$H^*_{pr_2^* df + \omega_{|U \cap V \times S^1}}(U \cap V \times S^1) \cong 0.$$

Using the Mayer-Vietoris sequence inductively, we immediately obtain

$$H^*_\omega(E) \cong 0.$$

This is a particular case of the Leray-Hirsch theorem for vertical one form (see theorem 3.3).

Theorem 3.2

Let (E, M, F, π) be a fiber bundle over M. Suppose that M has a finite good cover. If there are global cohomology classes $(e_1, e_2, ..., e_r)$ on E which when restricted to each fiber freely generate the cohomology of the fiber, then $H^*_{\pi^*\omega}(E)$ is a free module over $H^*_\omega(M)$ with basis $(e_1, e_2, ..., e_r)$, i.e.

$$H^*_{\pi^*\omega}(E) \cong H^*_\omega(M) \otimes I\!R\{e_1, ..., e_r\} \cong H^*_\omega(M) \otimes H^*(F).$$

Proof

The assumption that M has a finite good cover is necessary for the induction argument.

Remark also that the assumption that there are global cohomology classes $(e_1, e_2, ..., e_r)$ on E which when restricted to each fiber, freely generate the cohomology of the fiber, is necessary because all fibers have the same cohomology but their generators may be different. For example take x, y in M and put $E_x = \pi^{-1}(x)$ and $E_y = \pi^{-1}(y)$. Denote by $(u_1, u_2, ..., u_r)$ resp.

$(v_1, v_2, ..., v_r)$ the generators of de Rham cohomology of E_x resp. E_y. We have $H^*(E_x) \cong H^*(E_y)$, but (u_i) may have no relation with (v_j). The assumption means that the generators of the cohomology of the fibers over intersection agree since they are restrictions of e_i.
Let U and V be open contractible sets in M.
Since U is contractible, there exists a smooth function f nowhere vanish such that $\omega_{|U} = df$ and then $H^*_\omega(U) \cong H^*_{DR}(U)$.
Using this and the classical Leray-Hirsch theorem [5] we show that

$$\begin{aligned} H^*_{\pi^*\omega}(E|_U) \cong H^*(E|_U) &\cong H^*(U) \otimes I\!R\{e_1|_U, ..., e_r|_U\} \\ &\cong H^*(U) \otimes H^*(F) \\ &\cong H^*_\omega(U) \otimes H^*(F). \end{aligned}$$

Similarly, we get

$$H^*_{\pi^*\omega}(E|_V) \cong H^*_\omega(V) \otimes I\!R\{e_1|_V, ..., e_r|_V\} \cong H^*_\omega(V) \otimes H^*(F)$$

and

$$H^*_{\pi^*\omega}(E|_{U\cap V}) \cong H^*_\omega(U\cap V) \otimes I\!R\{e_1|_{U\cap V}, ..., e_r|_{U\cap V}\} \cong H^*_\omega(U\cap V) \otimes H^*(F).$$

So the Leray-Hirsch theorem holds for U, V and $U \cap V$.
Suppose M is the union of two open subsets U, V and define

$$\begin{aligned} r : \Omega^p(M) &\longrightarrow \Omega^p(U) \oplus \Omega^p(V), \\ \theta &\longmapsto (\theta|_U, \theta|_V); \end{aligned}$$

$$\begin{aligned} s : \Omega^p(U) \oplus \Omega^p(V) &\longrightarrow \Omega^p(U\cap V) \\ (\alpha, \beta) &\longmapsto \alpha|_{U\cap V} - \beta|_{U\cap V}. \end{aligned}$$

Then the following is a short exact sequence of cochain complexes

$$0 \longrightarrow \Omega^p(M) \xrightarrow{r} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{s} \Omega^p(U\cap V), \longrightarrow 0.$$

So we obtain the following Mayer-Vietoris sequence;

$$\cdots \longrightarrow H^p_\omega(U\cup V) \longrightarrow H^p_{\omega_{|U}}(U) \oplus H^p_{\omega_{|V}}(V) \longrightarrow H^p_{\omega_{|U\cap V}}(U\cap V) \longrightarrow H^{p+1}_\omega(U\cup V) \longrightarrow \cdots$$

and then, we get an exact sequence by tensoring with $H^{n-p}(F)$

$$\begin{aligned} \cdots \longrightarrow H^p_\omega(U\cup V) \otimes H^{n-p}(F) &\longrightarrow (H^p_{\omega_{|U}}(U) \otimes H^{n-p}(F)) \oplus (H^p_{\omega_{|V}}(V) \otimes H^{n-p}(F)) \longrightarrow \\ &\longrightarrow H^p_{\omega_{|U\cap V}}(U\cap V) \otimes H^{n-p}(F) \longrightarrow H^{p+1}_\omega(U\cup V) \otimes H^{n-p}(F) \longrightarrow \cdots \end{aligned}$$

since tensoring with a vector space preserve exactness.
Summing over $p = 0, ..., n$, yields the exact sequence;

$$\begin{aligned} \cdots \longrightarrow \oplus_0^n H^p_\omega(U\cup V) \otimes H^{n-p}(F) &\longrightarrow \oplus_0^n (H^p_{\omega_{|U}}(U) \otimes H^{n-p}(F)) \oplus (H^p_{\omega_{|V}}(V) \otimes H^{n-p}(F)) \longrightarrow \\ &\longrightarrow \oplus_0^n H^p_{\omega_{|U\cap V}}(U\cap V) \otimes H^{n-p}(F) \longrightarrow \oplus_0^n H^{p+1}_\omega(U\cup V) \otimes H^{n-p}(F) \longrightarrow \cdots \end{aligned}$$

It was checked in [5] page 49-50, that the last square of the following diagram is commutative for the de Rham cohomology. Since the Lichnerowicz cohomology and the de Rham cohomology for contractible sets are isomorphic (see proposition 2.1(b)), the following diagram is commutative;

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & \oplus_0^n H_\omega^p(U\cup V)\otimes H^{n-p}(F) & \longrightarrow & \oplus_0^n (H^p_{\omega_{|U}}(U)\otimes H^{n-p}(F)) \oplus (H^p_{\omega_{|V}}(V)\otimes H^{n-p}(F)) & \longrightarrow \\
\circlearrowleft & \downarrow \psi & \circlearrowleft & \downarrow \psi_{|U}\oplus\psi_{|V} & \circlearrowleft \\
\cdots \longrightarrow & H^n_{\pi^*\omega}(E) & \longrightarrow & H^n_{\pi^*\omega}(E_{|U}) \oplus H^n_{\pi^*\omega}(E_{|V}) & \longrightarrow
\end{array}
$$

$$
\begin{array}{ccc}
\longrightarrow & \oplus_0^n H^p_{\omega_{|U\cap V}}(U\cap V)\otimes H^{n-p}(F) & \longrightarrow \cdots \\
\circlearrowleft & \downarrow \psi_{|U\cap V} & \circlearrowleft \\
\longrightarrow & H^n_{\pi^*\omega}(E_{|U\cap V}) & \longrightarrow \cdots
\end{array}
$$

So if the Leray-Hirsch theorem holds for U, V and $U\cap V$ it also holds for $U\cup V$ by the "five lemma". Finally, one chooses a good covering $\mathcal{U}$ such that every $U\in\mathcal{U}$ does only intersect finitely many other sets of $\mathcal{U}$. Then we can write $M = W_1\cup ...\cup W_n$ where every W_i is a disjoint union of open balls in $\mathcal{U}$. Since the Leray-Hirsch theorem holds for W_i, W_j and $W_i\cap W_j$ (the latter is also an disjoint union of open balls) it holds also for $W_i\cup W_j$. Proceeding inductively we get the result $\boxdot$

In the next paragraph, we will prove the Leray-Hirsch theorem for vertical forms.

Theorem 3.3
Let (E, M, F, π) be a fiber bundle over M and ω a closed 1-form on E. Suppose that M has a finite good cover. If there are global cohomology classes $(e_2, ..., e_r)$ on E such that $([\omega] = e_1, e_2, ..., e_r)$ restricted to each fiber freely generate the cohomology of the fiber, then

$$H^*_\omega(E) \cong H^*(M)\otimes H^*_{i^*\omega}(F)$$

where the application $i : F \longrightarrow E$ is the natural induction.

Proof
Let U and V be open sets in M. The following diagram is commutative

$$
\begin{array}{ccc}
U\times F & \xrightarrow{pr_2} & F \\
& \searrow^{\phi} \quad {}^{i}\swarrow & \\
& E|_U &
\end{array}
$$

where $E_{|_U} = \pi^{-1}(U)$ and $\phi : U\times F \longrightarrow E_{|_U}$ is a smooth chart of the fiberation.
Let $\omega_{|_U}$ be the restriction of ω to $E_{|_U}$ and $\omega_1 = i^*\omega_{|_U}$.
Set $\tilde{\omega} =: pr_1^* df + pr_2^*\omega_1$ where f is a non zero function on U, then $[\tilde{\omega}] = [pr_2^*\omega_1]$.

The diffeomorphism ϕ induces an isomorphism $\phi^* : H^*_{\omega_{|U}}(E_{|U}) \cong H^*_{\phi^*\omega_{|U}}(U \times F)$. By the previous commutative diagram, $pr_2^* i^*(\omega_{|U}) = \phi^*(\omega_{|U})$. Therefore

$$H^*_{\phi^*\omega_{|U}}(U \times F) \cong H^*_{pr_2^* i^*\omega}(U \times F),$$

then by proposition 2.1 (b)

$$H^*_{pr_2^* i^*\omega}(U \times F) \cong H^*_{pr_1^* df + pr_2^* i^*\omega}(U \times F)$$

and hence by Künneth formula

$$H^*_{pr_1^* df + pr_2^* i^*\omega}(U \times F) \cong H^*_{df}(U) \otimes H^*_{i^*\omega}(F) \cong H^*(U) \otimes H^*_{i^*\omega}(F).$$

So we have shown

$$H^*_{\omega_{|U}}(E_{|U}) \cong H^*(U) \otimes H^*_{\omega_1}(F)$$

for all open set in M.

Similarly, we get $H^*_{\omega_{|V}}(E_{|V}) \cong H^*(V) \otimes H^*_{\omega_1}(F)$ and $H^*_{\omega_{|U \cap V}}(E_{|U \cap V}) \cong H^*(U \cap V) \otimes H^*_{\omega_1}(F)$.

Using the Mayer-Vietoris sequences and the "five lemma", the result holds for $U \cup V$. Again, one can choose a good covering and proceeding inductively we get the result $\boxdot$

Acknowledgments

I would like to thank the Abdus Salam International Centre for Theoretical Physics for its warm hospitality and support during the preparation of the last phase of this work. Also many thanks to Augustin Banyaga and Jean-Pierre Ezin for their permanent encouragement, many helpful discussions, advice and interest in this work.

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