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TOPOLOGICAL SEMIVECTOR SPACES:
CONVEXITY AND FIXED POINT THEORY*

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0. INTRODUCTION

Without speaking too roughly, (topological) semivector spaces are to (topological) semigroups as (topological) vector spaces are to (topological) groups. Recalling J. L. Kelley's [1955, p. 110] remark indicating the importance of convexity arguments as the basis of results distinguishing the theory of topological vector spaces from that of topological groups, one may expect to see convexity playing a corresponding role more generally in distinguishing the theory of topological semivector spaces from that of topological semigroups. Meanwhile, the results presented here may be taken to illustrate that much of the power of convexity properties is preserved in less stringent contexts than that of a vector space structure.

The notion of a "semivector space" was first introduced in [Prakash & Sertel, 1970a]; some structural aspects of such spaces were examined in [Prakash & Sertel, 1971a]. [Prakash & Sertel, 1972] is devoted entirely to the structure of semivector spaces.

Finally, an application of some of the fixed point theory of the present paper in generalizing existence results for equilibria in various types of social systems, systems which include economies and games, will be found in [Prakash, 1971], [Sertel, 1971] and [Prakash & Sertel, 1971b].

In this paper we start by defining topological semivector spaces. After briefly considering some problems in "strengthening" their topology, we extend the notion of convexity to such spaces. We then identify a hierarchy of local convexity axioms for such spaces and present some simple facts about products of spaces having various local convexity properties. Next, we illustrate how the spaces of concern arise naturally as certain hyperspaces of topological (semi-) vector spaces. Finally, we establish a number of fixed point and minmax theorems for topological semivector spaces with various local convexity properties. The fixed point results which we obtain here will be seen to generalize central fixed point theorems (for topological vector spaces) due to S. Kakutani [1941], H. F. Bohnenblust & S. Karlin [1950] and K. Fan [1952], which, in turn, are generalizations of results due to L. E. J. Brouwer [1912], J. Schauder [1930] and A. Tychonoff [1935], respectively.

1. PRELIMINARIES

Into this section we compress a quick review of some basic notions introduced and discussed in detail elsewhere [e.g., see Prakash & Sertel, 1972].

In defining "semivector" spaces, the notion of a "left skew semifield" serves as a catalyst.* By a left skew semifield we mean a bimonoid $\langle \mathbb{Q}, +, \cdot \rangle$ in which $\langle \mathbb{Q}, \cdot \rangle$ is a group with zero 0 distinct from its identity 1, $\langle \mathbb{Q}, + \rangle$ is a commutative semigroup with identity 0, and the (unitary) left action of $\langle \mathbb{Q}, \cdot \rangle$ on $\langle \mathbb{Q}, + \rangle$ is homomorphic: $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$. A left skew semifield $\langle \mathbb{Q}, +, \cdot \rangle$ will be called a skew semifield iff the (unitary) right action, too, of $\langle \mathbb{Q}, \cdot \rangle$ on $\langle \mathbb{Q}, + \rangle$, is homomorphic: $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$. This certainly obtains if $\langle \mathbb{Q}, \cdot \rangle$ is commutative, in which case we term $\langle \mathbb{Q}, +, \cdot \rangle$ a semifield. Notably, the set R_+ of non-negative reals under the usual addition and multiplication provides a useful example of a semifield, one which we call the usual real semifield. Whenever considered as a topological space, the set of non-negative reals will be understood to carry the Euclidean topology.

1.0 Definition: Let \mathbb{Q} be a topological space, $\langle \mathbb{Q}, +, \cdot \rangle$ a left skew semifield, and $\langle S, \oplus \rangle$ a commutative topological semigroup (not necessarily Hausdorff). Let $\Psi: \mathbb{Q} \times S \rightarrow S$, where we denote

$\Psi(\lambda, s) = \lambda s$, be a continuous map satisfying

Axiom 1: $\lambda(\mu s) = (\lambda \cdot \mu)s$ (action)

Axiom 2: $s \in \mathbb{O}s$ (unitariness)

Axiom 3: $\lambda(s \oplus t) = \lambda s \oplus \lambda t$ (homomorphism)

for all $\lambda, \mu \in \mathbb{O}$ and $s, t \in S$. S will be called a topological semivector space over \mathbb{O} , convex iff \mathbb{O} contains the usual real semifield.

1.1 Remark: (1) Clearly, when a left skew semifield \mathbb{O} is equipped with a topology yielding its operations continuous (for example, the discrete topology), \mathbb{O} may be viewed as a topological semivector space over itself. Hence our comment, at the outset, anticipating the catalytic role of the notion of left skew semifields in defining topological semivector spaces.

(2) In view of the fact that $\langle \mathbb{O}, \cdot \rangle$ is a group with zero, it is easily seen that the second axiom of the definition above is equivalent to the requirement that $1s = s$ should obtain for each $s \in S$.

(3) At this stage it might seem strange that, although Ψ is an action merely of $\langle \mathbb{O}, \cdot \rangle$ on S , we invoke $\langle \mathbb{O}, + \rangle$ by defining S to be over a left skew semifield. We intend, however, to make full use of the operation $(+)$, e.g., in speaking of "convexity" of sets in topological semivector spaces.

(4) The reason for adopting the qualifier 'convex' for S when \mathbb{O} contains the usual real semifield will become apparent immediately after we define convexity in 3.1.

Given a topological semivector space S over \mathbb{O} , for each $\lambda \in \mathbb{O}$, the restriction Ψ^λ of Ψ to $\{\lambda\} \times S$ will be called the (λ -) transition of S . From Axiom 3 we see that each transition of S is an endomorphism of S , and from the continuity of Ψ it is immediate that each transition is also continuous. In fact, when $\lambda \neq 0$, Ψ^λ is an automorphism of S , and, writing $\mu = \frac{1}{\lambda}$, where $\frac{1}{\lambda}$ denotes the inverse of λ under multiplication (\cdot), we have Ψ^μ continuous, whereby Ψ^λ is both an open and a closed map. It follows that Ψ is an open map when restricted to $(\mathbb{O} \setminus \{0\}) \times S$.

2. STRENGTHENABILITY OF THE TOPOLOGY^{*}

If $\langle S, \oplus \rangle$ is a commutative Hausdorff topological semigroup with identity e , it is possible to strengthen the topology on S , without destroying the continuity of \oplus , in such a way that (i) the neighborhood (nbd) system of e is unaltered, while (ii) $U \oplus s$ is now open whenever U is open in S ($s \in S$) [Paalman-De Miranda, 1964; Theorem 3.2.13]. Given a Hausdorff topological semivector space S with identity e , by a "strengthened" or "strong" topology on S we will mean one which satisfies (i) and (ii) as just stated.

Given a Hausdorff (topological) semivector space S over \mathbb{C} , we may now ask whether--or when--there exists a strengthened topology on S under which S remains a topological semivector space. (Of course, in a topological vector space the topology is already a strengthened version of itself).

Having Paalman-De Miranda's result as stated above, the question clearly boils down to whether the continuity of Ψ can be preserved under a strengthened topology on S . Although we are unable to assert in general when this can be done and when it cannot, we recognize a research problem here and offer the following as an example of where it cannot be done even though the space whose topology is to be strengthened is, as the reader may check, a pointwise convex (see 3.2) topological semivector space with

identity and with a topology which is locally compact, metrizable, 3° locally convex (see 4.1 and 6.5), etc.

2.1 Example: Let F be the real field with the usual topology, and let R be the topological group of the reals (under the usual addition and with the usual topology). Equip $\mathcal{X}\mathcal{L}(R)$, the set of closed intervals of R , with the Hausdorff metric topology. Defining $[a, b] \oplus [c, d] = \{x + y \mid (x, y) \in [a, b] \times [c, d]\}$ and $\lambda[a, b] = [\lambda a, \lambda b]$ ($a, b, c, d \in R; \lambda \in F$), $\mathcal{X}\mathcal{L}(R)$ with these operations is a topological semivector space over F and has $\{0\} = e$ as its identity. Now strengthen the topology on $\mathcal{X}\mathcal{L}(R)$ by declaring the translates $\mathcal{U} \oplus P$ to be (basic) open for each $P \in \mathcal{X}\mathcal{L}(R)$ and for each "originally" open nbd \mathcal{U} of e . Fix attention to any non-singleton $P \in \mathcal{X}\mathcal{L}(R)$, and consider the restriction $\Psi_P: F \times \{P\} \rightarrow \mathcal{X}\mathcal{L}(R)$ of Ψ . The fact is that Ψ_P is not continuous under the strengthened topology on $\mathcal{X}\mathcal{L}(R)$. For let \mathcal{U} be an open nbd of e , and consider the (basic) open nbd $\mathcal{U} \oplus \lambda P$ of λP for some $\lambda > 0$. Now the inverse image of $\mathcal{U} \oplus \lambda P$ under Ψ_P contains (λ, P) , but it contains no (μ, P) such that $0 < \mu < \lambda$. (For suppose that $0 < \mu < \lambda$ and that $\mu P = \lambda P \oplus U$ for some $U \in \mathcal{U}$. Then $P = \frac{\lambda}{\mu} P \oplus \frac{1}{\mu} U$. But this is impossible, since $\text{diam}(\frac{\lambda}{\mu} P \oplus \frac{1}{\mu} U) \geq \text{diam}(\frac{\lambda}{\mu} P) > \text{diam}(P)$.) This shows that Ψ_P is not continuous. Thus, the topological semivector space just considered, despite all its properties, does not remain a topological semivector space when its topology is strengthened in the fashion sought.

3. CONVEXITY & POINTWISE CONVEXITY

The familiar notion of convexity for (topological) vector spaces extends naturally and usefully to the case of topological semivector spaces. In the latter spaces, a property which we term "pointwise convexity" begins to assume an important role in its own right. This property, though automatically present in convex vector spaces, needs to be assumed separately in the general case of convex (topological semivector) spaces. In fact, it is when the two properties (convexity and pointwise convexity) are combined that they become specially useful.

3.0 Standing Notation: We denote the simplex $\{(\lambda_0, \dots, \lambda_m) \in \mathbb{R}_+^{m+1} \mid \sum_{i=0}^m \lambda_i = 1\}$ by Λ_m ($m = 0, 1, \dots$).

3.1 Definition: Let S be a convex topological semivector space (i.e., a topological semivector space over \mathbb{O} with \mathbb{O} containing the usual real semifield). Given any two points $x, x' \in S$, their segment $[x: x']$ is defined to be $\{s = \lambda x \oplus \lambda' x' \mid (\lambda, \lambda') \in \Lambda_1\}$. A subset $T \subset S$ will be called convex iff $[x: x'] \subset T$ whenever $x, x' \in T$.

Thus, what we call a convex topological semivector space (see 1.0), indeed checks to be convex according to the above definition.

The following are plain: if A is convex in a topological semivector space S , then $\mu A = \{\mu a \mid a \in A\}$ also is convex ($\mu \in \mathbb{R}$); if B , too, is convex in S , then so are $A \oplus B = \{a \oplus b \mid a \in A, b \in B\}$ and all convex combinations $\lambda A \oplus \lambda' B$ ($(\lambda, \lambda') \in \Lambda_1$).

It is important to note that, unlike in topological vector spaces, in topological semivector spaces there is no guarantee that x or x' belongs to $[x:x']$ or even that $x \in [x:x]$. For example, give the discrete topology to the set $[R]$ of all non-empty subsets $A, B \subset R$, where R stands for the set of reals, and obtain a topological semivector space (with identity element $e = \{0\}$) over the real field, with the understanding, as usual, that $A \oplus B = \{a + b \mid a \in A, b \in B\}$, while setting $\lambda A = \{\lambda a \mid a \in A\}$ if $\lambda \neq 0$ and $\lambda A = R$ otherwise ($A, B \in [R]; \lambda \in \mathbb{R}$). In this space the only element belonging to its own segment is $R \in [R]$. Possibilities such as the above motivate the following definition.

- 3.2 Definition: Let S be a convex topological semivector space, and let $T \subset S$. T will be said to be pointwise convex iff each $\{x\} \subset T$ is convex.
- 3.3 Exercise: A convex topological semivector space S is pointwise convex iff $(\alpha + \beta)s = \alpha s \oplus \beta s$ for all $\alpha, \beta \in \mathbb{R}_+$ and $s \in S$ [Prakash & Sertel, 1972].

Given a convex topological semivector space over a semifield \mathbb{O} , the largest pointwise convex subset of S forms a (convex) topological semivector subspace, i.e., it is a topological semivector space over \mathbb{O} when considered under the restrictions of the operations of S [Prakash & Sertel, 1972].

We close this section by defining the notion of convex hull and recording two intuitively pleasing facts concerning convexity in topological semivector spaces. (See also the last sentence preceding 5.1.)

3.4 Definition: Let S be a convex topological semivector space, and let $T \subset S$. The convex hull \hat{T} of T is defined to be the intersection of all convex subsets of S containing T .

3.5 Exercise: Let S and T be as above. Denote the set of finite subsets of T by $\mathcal{F}(T)$, and, for each $F = (t_0, \dots, t_m) \in \mathcal{F}(T)$, define (the "open simplex") $\sigma(F) = \{(\lambda_0 t_0 \oplus \dots \oplus \lambda_m t_m) \mid (\lambda_0, \dots, \lambda_m) \in \Lambda_m; \lambda_i > 0, i = 0, \dots, m\}$. Then $\hat{T} = \cup\{\sigma(F) \mid F \in \mathcal{F}(T)\}$, if S is pointwise convex [Prakash & Sertel, 1972].

3.6 Proposition: In convex topological semivector spaces, topological closure (Cl) preserves convexity.

Proof: Let Q be convex in S , a convex topological semivector

space. If $Q = \emptyset$ there is nothing to prove, so let q, q' be adherent points of Q . Suppose $\lambda q \oplus \lambda' q' = \bar{q} \notin \text{Cl}(Q)$ for some $(\lambda, \lambda') \in \Lambda_1$. Then there exists a nbd V of \bar{q} disjoint from $\text{Cl}(Q)$. The map $\Omega: S \times S \rightarrow S$, defined by $\Omega(x, x') = \lambda x \oplus \lambda' x'$, being continuous, there is a nbd $U \times U'$ of (q, q') such that $\Omega(U \times U') \subset V$. Since q and q' are adherent points of Q , there exists $(y, y') \in U \times U'$ such that $y, y' \in Q$. Then, by convexity of Q , $\Omega(y, y') \in Q$, a contradiction. Hence, $\bar{q} \in \text{Cl}(Q)$ and $\text{Cl}(Q)$ is convex, as to be shown.

#

4. LOCAL CONVEXITY

Apart from preparation for their use in the fixed point theory of Section 7, our motivation for stating the following "axioms of local convexity" derives from the fact that, although for a topological subspace X of a (Hausdorff) topological vector space the first three are always equivalent and all four are equivalent when X is convex, we are able to assert only weaker relationships between them in the case of topological semivector spaces. Given a subset X in a convex topological semivector space, we consider the following alternative

4.1 Axioms:

0. For any $x \in X$ and any nbd V of x , in the subspace topology of X there exists a convex nbd U of x such that $U \subset V$.
1. There exists a quasi-uniformity $\mathcal{B} = \{E_\alpha \subset X \times X \mid \alpha \in A\}$ of X inducing its subspace topology, such that, for each $E_\alpha \in \mathcal{B}$, there exists a closed $E_\beta \in \mathcal{B}$ with $E_\beta \subset E_\alpha$ and $E_\beta(x)$ convex for each $x \in X$.
2. There exists a quasi-uniformity $\mathcal{B} = \{E_\alpha \subset X \times X \mid \alpha \in A\}$ of X inducing its subspace topology, such that, for each $E_\alpha \in \mathcal{B}$, there exists a closed $E_\beta \in \mathcal{B}$ with $E_\beta \subset E_\alpha$ and $E_\beta(K)$ convex for each compact and convex subset $K \subset X$.

3. X is convex and there exists a uniformity $\mathcal{G} = \{E_\alpha \subset X \times X \mid \alpha \in A\}$ of X inducing its subspace topology, such that, for each $E_\alpha \in \mathcal{G}$, there exists a convex $E_\beta \in \mathcal{G}$ with $E_\beta \subset E_\alpha$.

X will be called $0^\circ/1^\circ/2^\circ/3^\circ$ locally convex (l.c.) accordingly as it satisfies $0/1/2/3$ among these axioms. Thus, 0° local convexity is the familiar local convexity.

4.2 Proposition: Given a subset X of a convex topological semivector space,

- (1) If X is 1° l.c., then it is 0° l.c.;
- (2) If X is 2° l.c. and pointwise convex, then it is 1° l.c.; and
- (3) If X is 3° l.c., then it is 2° l.c.

4.3 Proposition: Every T_1 space which is 0° l.c. is pointwise convex.

Proof: Let X be a 0° l.c. T_1 space, and suppose $x \in X$. As X is 0° l.c., there is a local base $\mathcal{B} = \{B_\alpha \mid \alpha \in A\}$ at x consisting of convex nbds. Thus, $x \in B = \bigcap_A B_\alpha$, and B is convex. In fact, $B = \{x\}$. For, supposing $y \in B$ for some $y \neq x$, as X is T_1 , there exists a nbd U of x to which y does not belong, whereby $y \notin B_\alpha \subset U$ for some $B_\alpha \in \mathcal{B}$,

contradicting that $y \in B$. Thus, $\{x\}$ is convex. This shows that X is pointwise convex, completing the proof.

#

Of course, all the local convexity properties 0°-3° are inherited by relative topologies on convex subsets. In the next section we turn to some basic facts relating local convexity properties of Cartesian products with those of their factor spaces.

5. CARTESIAN PRODUCTS OF TOPOLOGICAL SEMIVECTOR SPACES

Given a family $\{S_\alpha \mid \alpha \in A\}$ of topological semivector spaces over \mathcal{O} , we equip $S = \prod_A S_\alpha$ with the product topology and define its operations coordinatewise as follows:

$$\begin{aligned} \{s_\alpha\}_{\alpha \in A} \oplus \{t_\alpha\}_{\alpha \in A} &= \{s_\alpha \oplus_\alpha t_\alpha\}_{\alpha \in A}, \\ \lambda \{s_\alpha\}_{\alpha \in A} &= \{\lambda s_\alpha\}_{\alpha \in A}, \end{aligned}$$

where \oplus_α stands for the semigroup operation of S_α and $s_\alpha, t_\alpha \in S_\alpha$ are generic ($\alpha \in A$). Clearly, S is then a topological semivector space over \mathcal{O} . We call it the product of $\{S_\alpha \mid \alpha \in A\}$. Evidently, a set $X \subset S = \prod_A S_\alpha$ is convex/pointwise convex iff each projection $X_\alpha = \pi_{S_\alpha}(X) \subset S_\alpha$ is so.

5.1 Lemma: Let $\{X_\alpha \mid \alpha \in A\}$ be a family of 2° l.c. spaces of which all but finitely many are convex, and let \mathcal{O} be a quasi-uniformity inducing the product topology on $X = \prod_A X_\alpha$. Then, for every $F \in \mathcal{O}$, there exists a closed $E \in \mathcal{O}$ such that $E \subset F$ and $E(K)$ is convex whenever K is the product $K = \prod_A K_\alpha$ of compact and convex subsets $K_\alpha \subset X_\alpha$.

Proof: Contained in F , find a basic $H \in \mathcal{O}$ which restricts a finite set $N \subset A$ of coordinates, including (w.l.g.) the set $M \subset A$ of indices m for which X_m is not convex. Now

$$H = \prod_N H_n \times \prod_{A \setminus N} (X_\alpha \times X_\alpha),$$

where H_n belongs to the quasi-uniformity \mathcal{G}_n of X_n ($n \in N$).

For each $n \in N$, using the 2° l.c. of X_n , find a closed

$E_n \in \mathcal{G}_n$ such that $E_n \subset H_n$ with $E_n(K_n)$ convex for each

compact and convex $K_n \subset X_n$. Write $E = \prod_N E_n \times \prod_{A \setminus N} (X_\alpha \times X_\alpha)$.

#

- 5.2 Lemma: The product of a family of 1° l.c. spaces is 1° l.c., if all but a finite number of the factor spaces are convex.

Proof: Imitate the last proof.

#

- 5.3 Exercise: Let $S = \prod_A S_\alpha$ be the product of a family of convex topological semivector spaces over \mathbb{C} , and let $X \subset S$ be compact Hausdorff. If the projection $X_\alpha = \pi_\alpha(X)$ of X into S_α is Hausdorff, then X_α is 1°/2° l.c., accordingly as X is.

- 5.4 Exercise: The product of a family of spaces is 3° l.c. iff each of the factor spaces is 3° l.c.

6. HYPERSPACES AS EXAMPLES

In this section we show some natural ways in which topological semivector spaces arise as certain hyperspaces of topological semivector spaces, e.g., of topological vector spaces. In fact, this is how, at first, we came to define topological semivector spaces: as an abstraction from hyperspaces of topological vector spaces. Our motivation derived from a need to be able to deal with certain socio-economic adjustment processes in which some of the variables were set-valued in topological vector spaces [see Prakash, 1971; Sertel, 1971; Prakash & Sertel, 1971b]. This abstraction not only enables one to eliminate cumbersome details in the study of the above mentioned adjustment processes, but it also turns out to afford many interesting examples besides those discussed here [see Prakash & Sertel, 1972]. The hyperspace examples given here, however, form a unified collection, illustrating essentially all the salient aspects of (topological) semivector spaces discussed in the previous sections.

In topologising hyperspaces, we use the upper semifinite, finite or, when applicable, uniform topology, regarding all of which we adopt E. Michael [1951] as standard reference. N.B.: When a topology is unmentioned, it is to be understood as discrete.

6.0 Standing Notation: Given a set X , $[X]$ will denote the set of non-empty subsets of X . If X is a topological space, $\mathcal{C}(X)$, $\mathcal{O}(X)$ and $\mathcal{K}(X)$ will denote the set of non-empty subsets of X which are closed, open and compact, respectively. If X lies in a convex topological semivector space, $\mathcal{L}(X)$ will denote the set of non-empty convex subsets of X . Finally, we will denote $\mathcal{C}\mathcal{L}(X) = \mathcal{C}(X) \cap \mathcal{L}(X)$, $\mathcal{O}\mathcal{L}(X) = \mathcal{O}(X) \cap \mathcal{L}(X)$ and $\mathcal{K}\mathcal{L}(X) = \mathcal{K}(X) \cap \mathcal{L}(X)$.

6.1 Definition: Let X be a topological space. The upper semifinite (u.s.f.) topology on $[X]$ is the one generated by taking as a basis for open collections in $[X]$ all collections of the form $\langle U \rangle = \{A \in [X] \mid A \subset U\}$, and the lower semifinite (l.s.f.) topology on $[X]$ is the one generated by taking as a sub-basis for open collections in $[X]$ all collections of the form $\langle U \rangle^- = \{A \in [X] \mid A \cap U \neq \emptyset\}$, where U is an open subset in X . The finite topology on $[X]$ is the one generated by taking as a basis for open collections in $[X]$ all collections of the form $\langle U_1, \dots, U_n \rangle = \{A \in [X] \mid A \subset \bigcup_{i=1}^n U_i, A \cap U_i \neq \emptyset \text{ for } i = 1, \dots, n\}$ with U_1, \dots, U_n open in X . Furthermore, given a topological space Y , a mapping $f: Y \rightarrow [X]$ is called upper semi-continuous (u.s.c.) [resp. lower semi-continuous (l.s.c.)] iff it is continuous with respect to the u.s.f. [resp. l.s.f.] topology on $[X]$. (It follows that f is

continuous with the finite topology on $[X]$ iff it is both u.s.c. and l.s.c.)

Let S be a semivector space over \mathcal{O} and, for any $A, B \subset S$, define $A \oplus B = \{a \oplus b \mid a \in A, b \in B\}$ and $\lambda A = \{\lambda a \mid a \in A\}$ ($\lambda \in \mathcal{O}$). Then $[S]$ is a semivector space over \mathcal{O} , and if S is convex, then $\mathcal{L}(S)$ is a semivector subspace of $[S]$. Furthermore, if S is convex, then S may be embedded as a topological semivector subspace into $\mathcal{L}(S)$ iff S is pointwise convex; and S is pointwise convex only if $\mathcal{L}(S)$ is so.

Given a topological semivector space S , equip its hyperspaces with their respective finite topologies. $[S]$ is then a topological semivector space and $\mathcal{X}(S)$ is a topological semivector subspace of $[S]$, while S is embeddable as a topological semivector subspace into $\mathcal{X}(S)$. Furthermore, $\mathcal{X}(S)$ is Hausdorff iff S is so. If S is a topological semivector space with a "strong" topology (cf. Section 2), i.e., a topology in which $U \oplus s$ is open, ($s \in S$), whenever $U \subset S$ is open (such as in topological vector spaces), then $\mathcal{O}(S)$ is a topological semivector subspace of $[S]$.

It follows that $\mathcal{L}(S)$ and $\mathcal{X}\mathcal{L}(S)$ are topological semivector subspaces of $[S]$ whenever $[S]$ is a topological semivector space, and $\mathcal{O}\mathcal{L}(S)$ is a (topological) semivector space whenever $\mathcal{O}(S)$ is a (topological) semivector space.

6.2 Proposition: Let S be a convex topological semivector space with identity e , and let $X \subset S$ be convex. Then $\mathcal{X}\mathcal{L}(X)$ is convex. Assume that S has a strong topology and equip $\mathcal{X}\mathcal{L}(X)$ with the upper semifinite topology. If X is 0° l.c., then so is $\mathcal{X}\mathcal{L}(X)$ —although it need not be Hausdorff even if X is Hausdorff. Furthermore, if X is pointwise convex (so that $\mathcal{X}\mathcal{L}(X)$, too, is pointwise convex), then $\mathcal{X}\mathcal{L}(X)$ is 0° l.c. only if X is 0° l.c.

Proof: The rest being clear, we only prove that $\mathcal{X}\mathcal{L}(X)$ with the upper semifinite topology is 0° l.c. when X is so. Let $A \in \mathcal{X}\mathcal{L}(X)$, and let $\mathcal{N} \subset \mathcal{X}\mathcal{L}(X)$ be a nbd of A . Find a basic nbd $\langle V \rangle$ of A such that $\langle V \rangle \subset \mathcal{N}$. Then $V \subset X$ is a nbd of $A \subset X$. By continuity of \oplus , for each $a \in A$ there exist open nbds U_a of e and W_a of a such that $U_a \oplus W_a \subset V$, while the 0° local convexity of S allows us to assume each U_a to be convex and the strong topology assures us that each $U_a \oplus W_a$ is open. $\{U_a \oplus W_a \mid a \in A\}$ thus being an open cover of the compact A , it has a finite subcover $\{U_{a_i} \oplus W_{a_i} \mid i \in I\}$. Denoting $U = \bigcap_{i \in I} U_{a_i}$ and $W = \bigcup_{i \in I} W_{a_i}$, we see that $A \subset U \oplus A \subset U \oplus W \subset V$ and that $U \oplus A$ is convex. Furthermore, $U \oplus A$ is open in the strong topology, so that $\langle U \oplus A \rangle = \mathcal{X}\mathcal{L}(U \oplus A)$ is an open convex nbd of $A \in \mathcal{X}\mathcal{L}(X)$, while $\langle U \oplus A \rangle \subset \langle V \rangle \subset \mathcal{N}$, as desired.

6.3 Corollary: If X is convex in a (0°) locally convex topological vector space (not necessarily Hausdorff), then $\mathcal{X}\mathcal{L}(X)$ is convex, pointwise convex and, with the upper semifinite topology, 0° l.c. as well.

Proof: The topology of a linear topological space being strong, the last proposition applies.

#

6.4 Corollary: Let X be convex and T_1 in a topological semivector space with strong topology. Then X is 0° l.c. and $\mathcal{X}\mathcal{L}(X)$ is pointwise convex iff $\mathcal{X}\mathcal{L}(X)$ with the upper semifinite topology is 0° l.c. and X is pointwise convex.

Proof: Use 4.3 and 6.2.

#

6.5 Proposition: Let L be a (0°) locally convex (not necessarily Hausdorff) topological vector space. Then $\mathcal{L}(L)$, with the uniform topology induced by that of L , is 3° l.c.

Proof: Let $\{W_\alpha \mid \alpha \in A\}$ be a fundamental system of convex and symmetric nbds of the identity $e \in L$, so that defining

$E_\alpha = \{(x, y) \mid x \in L, y \in E_\alpha(x)\}$, where $E_\alpha(x) = x \oplus W_\alpha$ for each $\alpha \in A$ and $x \in L$, $\{E_\alpha \subset L \times L \mid \alpha \in A\}$ is a fundamental system of entourages of the uniform structure of L . Then, for

any $P \in \mathcal{L}(L)$, $E_\alpha(P) = P \oplus W_\alpha \subset L$ is a nbd of $P \subset L$. By

definition, the uniform structure on $\mathcal{L}(L)$ induced by $\{E_\alpha \mid \alpha \in A\}$ is the one generated by $\{\mathcal{F}_\alpha \subset \mathcal{L}(L) \times \mathcal{L}(L) \mid \alpha \in A\}$, where $\mathcal{F}_\alpha(P) = \{T \in \mathcal{L}(L) \mid P \subset E_\alpha(T) \text{ and } T \subset E_\alpha(P)\}$ ($P \in \mathcal{L}(L)$). It suffices to show that each \mathcal{F}_α is convex. To see this, fix α and note that $(P, Q) \in \mathcal{F}_\alpha$ iff $P \subset Q \oplus W_\alpha$ and $Q \subset P \oplus W_\alpha$. Let $(P, Q), (P', Q') \in \mathcal{F}_\alpha$, and consider an arbitrary convex combination $(\bar{P}, \bar{Q}) = (\lambda P \oplus \lambda' P', \lambda Q \oplus \lambda' Q')$, so that $\bar{P}, \bar{Q} \in \mathcal{L}(L)$. Now $\bar{P} = \lambda P \oplus \lambda' P' \subset \lambda(Q \oplus W_\alpha) \oplus \lambda'(Q' \oplus W_\alpha) = \bar{Q} \oplus \lambda W_\alpha \oplus \lambda' W_\alpha$. Since $W_\alpha \in \mathcal{L}(L)$ and $\mathcal{L}(L)$ is pointwise convex, we have $\lambda W_\alpha \oplus \lambda' W_\alpha = W_\alpha$, so that $\bar{P} \subset \bar{Q} \oplus W_\alpha$. Similarly, $\bar{Q} \subset \bar{P} \oplus W_\alpha$. Hence, $(\bar{P}, \bar{Q}) \in \mathcal{F}_\alpha$, so that \mathcal{F}_α is convex, as to be shown.

#

6.6 Corollary: Let L be as in 6.5. Then $\mathcal{X}\mathcal{L}(L)$, with the uniform (equivalently, the finite) topology induced by that of L , is 3° l.c.

Proof: (The parenthetically stated equivalence is directly from [Michael, 1951, Theorem 3.3].) As $\mathcal{X}\mathcal{L}(L)$ is convex in $\mathcal{L}(L)$, it inherits the local convexity of $\mathcal{L}(L)$ granted by 6.5, and is thus 3° l.c., as to be shown.

#

6.7 **Theorem:** Given a Hausdorff topological vector space L , let $X \subset L$ be non-empty and convex, and equip $\mathcal{X}(X)$ with the finite (equivalently, the uniform) topology induced by the subspace topology of X (so that $\mathcal{X}(X)$, too, is Hausdorff). Then $\mathcal{X}^2(X)$ is closed in $\mathcal{X}(X)$.

Proof: (The parenthetical assertions can be checked from [Michael, 1951, Theorems 3.3 and 4.9.3].) Clearly, $\mathcal{X}^2(X) \neq \emptyset$, as singleton sets are compact and, in a convex vector space, convex as well. Let \mathcal{B} be a filterbase on $\mathcal{X}^2(X)$ converging to some $Q \in \mathcal{X}(X)$. We show that $Q \in \mathcal{X}^2(X)$.

As $\mathcal{X}(X) = \mathcal{X}(L) \cap [X]$, the finite topology on $\mathcal{X}(X)$ induced by that of X is the same as the subspace topology on $\mathcal{X}(X) \subset \mathcal{X}(L)$ relative to the finite topology on $\mathcal{X}(L)$. As $\mathcal{X}(L)$ with the finite topology is a topological semivector space (see the comments following 6.1), the algebraic operations of $\mathcal{X}(L)$ are continuous. For each $(\lambda, \lambda') \in \Lambda_1$, define a map Ω on $\mathcal{X}(X)$ by $\Omega(P) = \lambda P \oplus \lambda' P$. As each such Ω is continuous and as X is convex, Ω is into $\mathcal{X}(X)$. As X , hence $\mathcal{X}^2(X)$, is pointwise convex, the restriction $\Omega|_{\mathcal{X}^2(X)}$ of each such Ω to $\mathcal{X}^2(X)$ is, in fact, nothing but the identity map $\iota: \mathcal{X}^2(X) \rightarrow \mathcal{X}^2(X)$.

Fix attention to any particular (λ, λ') . Let $\mathcal{V} \subset \mathcal{X}(X)$ be a nbd of $\Omega(Q)$. By continuity of Ω , there exists a nbd $\mathcal{U} \subset \mathcal{X}(X)$ of Q such that $\Omega(\mathcal{U}) \subset \mathcal{V}$. As $\mathcal{B} \rightarrow Q$, there exists an

element $\mathcal{U} \in \underline{\mathcal{B}}$ such that $\mathcal{U} \subset \mathcal{V}$. Then $\Omega(\mathcal{U}) \subset \Omega(\mathcal{V}) \subset \mathcal{V}$. But $\Omega(\mathcal{U}) = \mathcal{U}$, since $\mathcal{U} \in \mathcal{K}\mathcal{L}(X)$. This shows that $\underline{\mathcal{B}}$ converges to $\Omega(Q)$. As $\mathcal{K}(X)$ is Hausdorff, $\underline{\mathcal{B}}$ converges to a unique point. This implies that $\Omega(Q) = Q$, and this is true for any $(\lambda, \lambda') \in \Lambda_1$. Hence, $Q \subset X$ is convex and $Q \in \mathcal{K}\mathcal{L}(X)$, as to be shown.

#

6.8 Exercise: Given a (0°) locally convex Hausdorff topological vector space L , let $X \subset L$ be non-empty, compact and convex. Denote $\mathcal{K}\mathcal{L}(L)$ by S and $\mathcal{K}\mathcal{L}(X)$ by Y . Equip S with the finite topology and consider it as a (convex) topological semivector space with operations induced in the usual fashion by those of L . Then S is Hausdorff and pointwise convex having identity $\underline{e} = \{0\}$ and $0\underline{s} = \underline{e}$ for each $\underline{s} \in S$, where 0 denotes the identity of L . Furthermore, Y is a non-empty, compact, convex and 3° l.c. subset of S . Finally, $L \times S$ is a convex, pointwise convex, Hausdorff topological semivector space with identity $e = (0, \underline{e})$ and $0t = e$ for each $t \in L \times S$, while $X \times Y \in \mathcal{K}\mathcal{L}(L \times S)$ and is 3° l.c.

game theory, economic theory and other instances of social analysis.

Given topological spaces X and Y and a mapping f of X into the set of non-empty subsets of Y , when we say that f is upper semi-continuous (usc), we will mean that, for each $x \in X$, given any nbd $V \subset Y$ of $f(x)$, there exists a nbd U of X such that $f(U) \subset V$. (This definition is equivalent to the one in 6.1.)

For the composition of two binary relations $F \subset A \times B$ and $E \subset C \times D$, we will write $E \circ F$ for the set (binary relation)

$$\{(a, d) \mid \exists x \in B \cap C \text{ such that } (a, x) \in F \text{ and } (x, d) \in E\}.$$

7.1 THEOREM (Fixed Point): Given a pointwise convex topological semivector space S with Hausdorff topology, let X be the closed convex hull $X = \{\lambda_0 a_0 \oplus \dots \oplus \lambda_n a_n \mid \lambda = (\lambda_0, \dots, \lambda_n) \in \Lambda_n\}$ of some $\{a_0, \dots, a_n\} \subset S$, and let $f: X \rightarrow \mathcal{C}2(X)$ be an upper semi-continuous transformation. Then there exists a (fixed) point $x^* \in X$ such that $x^* \in f(x^*)$.

Proof: Let $\varphi: \Lambda_n \rightarrow X$ be the map defined by $\varphi(\lambda) = \lambda_0 a_0 \oplus \dots \oplus \lambda_n a_n$, and let $\tilde{\varphi}: \Lambda_n \times \Lambda_n \rightarrow X \times X$ be the map defined by $\tilde{\varphi}(\lambda, \mu) = (\varphi(\lambda), \varphi(\mu))$. Since the algebraic operations of S are continuous, so are φ and $\tilde{\varphi}$.

Let $g \subset X \times X$ be the graph of f and let $G \subset \Lambda_n \times \Lambda_n$ be the graph of the map $F: \Lambda_n \rightarrow [\Lambda_n]$ defined by $F(\lambda)$

$= \varphi^{-1}(f(\varphi(\lambda)))$. Thus, $G = \Phi^{-1}(g)$. Since Λ_n is compact, by continuity of φ , $X = \varphi(\Lambda_n)$ is compact, hence regular. Thus, g is closed, since f is usc. Hence, by continuity of Φ , G is closed, whereby F is usc by compactness of Λ_n .

Clearly, for each $\lambda \in \Lambda_n$, $F(\lambda)$ is non-empty; also, it is closed, since $f(\varphi(\lambda))$ is closed and φ is continuous. We now check that $F(\lambda)$ is convex. Let $\mu, \mu' \in F(\lambda)$ [i.e., for some $y, y' \in f(\varphi(\lambda))$, let $y = \mu_0 a_0 \oplus \dots \oplus \mu_n a_n$ and $y' = \mu'_0 a_0 \oplus \dots \oplus \mu'_n a_n$]. For arbitrary $(\beta, \beta') \in \Lambda_1$, define $\bar{\mu} = \beta\mu + \beta'\mu'$, and denote $\bar{y} = \bar{\mu}_0 a_0 \oplus \dots \oplus \bar{\mu}_n a_n$. Using pointwise convexity of S , one may compute that $\bar{y} = \beta y \oplus \beta' y'$, and convexity of $f(\varphi(\lambda))$ yields $\bar{y} \in f(\varphi(\lambda))$. Hence, $\bar{\mu} \in F(\lambda)$, showing $F(\lambda)$ to be convex.

Now applying Kakutani's fixed point theorem, there exists a $\lambda^* \in \Lambda_n$ such that $\lambda^* \in F(\lambda^*)$. Choosing $x^* = \varphi(\lambda^*)$, we see that $x^* \in f(x^*)$.

#

7.2 Corollary (Kakutani's Fixed Point Theorem [1941]): Let $f: X \rightarrow \mathcal{C}\mathcal{L}(X)$ be an upper semicontinuous transformation of an n -dimensional closed simplex $X \subset \mathbb{R}^{n+1}$ into $\mathcal{C}\mathcal{L}(X)$. Then there exists a (fixed) point $x^* \in X$ such that $x^* \in f(x^*)$.

7.3 THEOREM (Fixed Point): Let $f: X \rightarrow X$ be a continuous

transformation of a 1° l.c., non-empty, compact and convex subset X of a Hausdorff topological semivector space. Then there exists a (fixed) point $x^* \in X$ such that $x^* = f(x^*)$.

Proof: Since X is compact, there exists a unique uniformity on X compatible with its subspace topology. Since X is 1° l.c., we assume that $\{E_\alpha \subset X \times X \mid \alpha \in A\}$ is a fundamental system of closed entourages of this uniformity such that $E_\alpha(x)$ is (closed and) convex for all $x \in X$. Define $Y_\alpha = \{x \mid x \in E_\alpha(f(x))\}$. We will show that Y_α is non-empty and closed for each $\alpha \in A$. Then, as the intersection of any finite collection of Y_α 's is non-empty, compactness of X will imply that $\bigcap_{\alpha \in A} Y_\alpha \neq \emptyset$, thus proving the theorem, for $x^* \in \bigcap_{\alpha \in A} Y_\alpha$ implies $x^* = f(x^*)$.

Now first we note that, being 1° l.c. and Hausdorff, hence 0° l.c. and T_1 , X is pointwise convex (see Propositions 4.2 and 4.3). To show that Y_α is non-empty, let $\{D_\alpha \subset X \times X \mid \alpha \in A\}$ be a family of open symmetric entourages such that $D_\alpha \subset E_\alpha$ ($\alpha \in A$). Thus, for any given $\alpha \in A$, $\{D_\alpha(x) \mid x \in X\}$ is an open cover of X , so that there exist $x_0, \dots, x_n \in X$ with $X \subset \bigcup_{i=0}^n D_\alpha(x_i)$. Denote the closed convex hull of $\{x_0, \dots, x_n\}$ by $P = \{p = \lambda_0 x_0 \oplus \dots \oplus \lambda_n x_n \mid \lambda = (\lambda_0, \dots, \lambda_n) \in \Lambda_n\}$. Define the map F_α on P by $F_\alpha(p) = E_\alpha(f(p)) \cap P$. Then, by symmetry of $D_\alpha \subset E_\alpha$, for all $p \in P$, $F_\alpha(p)$ is non-empty; clearly, it is also closed and convex. Thus F_α maps P into $\mathcal{C}\mathcal{L}(P)$.

Denoting the graph of $E_\alpha \circ f$ by G_α , the graph of F_α is simply $\Gamma_\alpha = G_\alpha \cap (P \times P)$. Since E_α is usc (by the closedness of E_α in the compact $X \times X$) and since f is continuous, $E_\alpha \circ f$ is usc, i.e., G_α is closed, as X is regular (in fact, compact). Hence, Γ_α is closed and, by compactness of the range P , F_α is usc. Thus, by Theorem 7.1, there exists $p \in F_\alpha(p)$, i.e., $p \in Y_\alpha$, showing that Y_α is non-empty. Y_α is obviously closed, since it is nothing but the projection $\pi_X(G_\alpha \cap \Delta)$ of the compact set $G_\alpha \cap \Delta$, where Δ is the diagonal in $X \times X$. This completes the proof.

#

7.4 Corollary (Tychonoff's Fixed Point Theorem [1935]): Let $f: X \rightarrow X$ be a continuous transformation of a non-empty compact and convex subset X of a locally convex linear Hausdorff topological space. Then there exists a (fixed) point $x^* \in X$ such that $x^* = f(x^*)$.

7.5 THEOREM (Fixed Point): Let $\{X_\alpha \subset S_\alpha \mid \alpha \in A\}$ be a non-empty family of 1° l.c., non-empty, compact and convex subsets of Hausdorff topological semivector spaces S_α , and let $\{f_\alpha: X \rightarrow X_\alpha \mid \alpha \in A\}$ be a family of continuous functions on $X = \prod_A X_\alpha$. Define $F: X \rightarrow X$ by $F(x) = \{f_\alpha(x)\}_{\alpha \in A}$. Then there exists a (fixed) point $x^* \in X$ such that $x^* = F(x^*)$.

Proof: Clearly, the topological semivector space $S = \prod_A S_\alpha$ is Hausdorff, and $X \subset S$ is non-empty, compact and convex. Since each X_α is 1° l.c., so is X (see Lemma 5.2). Furthermore, F is continuous, as each f_α is so. Hence, the result follows readily by application of Theorem 7.3.

#

7.6 THEOREM (Minmax): Let X_1 and X_2 be 1° l.c., non-empty, compact and convex subsets, each lying in a Hausdorff topological semivector space. Let u be a continuous real-valued function on $X = X_1 \times X_2$, such that

$$\begin{aligned} f_1(x_2) &\in \{x_1 \mid u(x_1, x_2) = \max_{y \in X_1} u(y, x_2)\} \\ f_2(x_1) &\in \{x_2 \mid u(x_1, x_2) = \min_{z \in X_2} u(x_1, z)\} \end{aligned}$$

define functions $f_1: X_2 \rightarrow X_1$ and $f_2: X_1 \rightarrow X_2$. Then

$$\min_{X_2} \max_{X_1} u(x_1, x_2) = \max_{X_1} \min_{X_2} u(x_1, x_2).$$

Proof: It is obvious that, for all $(\bar{x}_1, \bar{x}_2) \in X$,

$$\begin{aligned} \max_{X_1} u(x_1, \bar{x}_2) &\geq \min_{X_2} \max_{X_1} u(x_1, x_2) \geq \max_{X_1} \min_{X_2} u(x_1, x_2) \\ &\geq \min_{X_2} u(\bar{x}_1, x_2). \end{aligned}$$

Clearly, the functions f_1 and f_2 are continuous, so that the function $F: X \rightarrow X$ defined by $F(x_1, x_2) = (f_1(x_2), f_2(x_1))$ is continuous. Then by Theorem 7.5, there exists an $x^* \in X$ such that $x^* = (x_1^*, x_2^*) = F(x^*)$. Hence, $\max_{X_1} u(x_1, x_2^*) = \min_{X_2} u(x_1^*, x_2)$, thus proving the desired equality.

#

7.7 THEOREM (Minmax): Let A_1 and A_2 be non-empty but finite sets, each lying in a pointwise convex Hausdorff topological semivector space, and let X_1 and X_2 be the closed convex hull of A_1 and A_2 , respectively. Let u be a continuous real-valued function on $X = X_1 \times X_2$, such that

$$f_1(x_2) = \{x_1 \mid u(x_1, x_2) = \max_{y \in X_1} u(y, x_2)\}$$

$$f_2(x_1) = \{x_2 \mid u(x_1, x_2) = \min_{z \in X_2} u(x_1, z)\}$$

define maps $f_1: X_2 \rightarrow \mathcal{C}\mathcal{L}(X_1)$ and $f_2: X_1 \rightarrow \mathcal{C}\mathcal{L}(X_2)$. Then

$$\min_{X_2} \max_{X_1} u(x_1, x_2) = \max_{X_1} \min_{X_2} u(x_1, x_2).$$

Proof: Use Theorem 7.1.

#

7.8 THEOREM (Fixed Point): Let $f: X \rightarrow \mathcal{C}\mathcal{L}(X)$ be an upper semi-continuous transformation of a pointwise convex, 2° l.c., non-empty, compact and convex subset X of a Hausdorff topological semivector space into $\mathcal{C}\mathcal{L}(X)$. Then there exists a (fixed) point $x^* \in X$ such that $x^* \in f(x^*)$.

Proof: As in the proof of Theorem 7.3, it suffices to show that the sets $Y_\alpha = \{x \mid x \in E_\alpha(f(x))\}$ are non-empty and closed, where, in this case, $\{E_\alpha \mid \alpha \in A\}$ is a fundamental system of closed entourages of the space X such that $E_\alpha(K)$ is (closed and) convex for each non-empty, compact

and convex subset $K \subset X$. The proof is the same as that of Theorem 7.3, except that appeal is now made to the upper semi-continuity, rather than the continuity, of f .

#

7.9 Corollary (Fan's Fixed Point Theorem [1952]): Let X be non-empty, compact and convex in a locally convex Hausdorff linear topological space, and let $f: X \rightarrow \mathcal{C}\mathcal{L}(X)$ be an upper semicontinuous transformation. Then there exists a (fixed) point $x^* \in X$ such that $x^* \in f(x^*)$.

7.10 THEOREM (Fixed Point): Let $\{X_\alpha \mid \alpha \in A\}$ be a non-empty family of pointwise convex, 2° l.c., non-empty, compact and convex subsets of Hausdorff topological semivector spaces, and let $\{f_\alpha: X \rightarrow \mathcal{C}\mathcal{L}(X_\alpha) \mid \alpha \in A\}$ be a family of upper semicontinuous transformations, where $X = \prod_A X_\alpha$. Define $F: X \rightarrow \prod_A \mathcal{C}\mathcal{L}(X_\alpha)$ by $F(x) = \prod_A f_\alpha(x)$ ($x \in X$). Then there exists a (fixed) point $x^* \in X$ such that $x^* \in F(x^*)$.

Proof: Clearly, F is an usc transformation of the pointwise convex, non-empty, compact and convex Hausdorff space X into $\mathcal{C}\mathcal{L}(X)$. Although X need not be 2° l.c., by the 2° local convexity of each X_α , the uniformity on X allows a fundamental system $\{E_i \mid i \in I\}$ of closed entourages such that, whenever K is the product $K = \prod_A K_\alpha$ of compact and

convex subsets $K_\alpha \subset X_\alpha$, each $E_i(K)$ is closed and convex (see Lemma 5.1). Notice that $F(x)$ is such a product of compact and convex sets $f_\alpha(x) \subset X_\alpha$. Thus, as in Theorem 7.8, defining $Y_i = \{x \mid x \in E_i(F(x))\}$, it is clear that Y_i is non-empty and closed for each $i \in I$, implying that $\bigcap_I Y_i \neq \emptyset$ and proving the theorem.

#

7.11 THEOREM (Minmax): Let X_1 and X_2 be pointwise convex 2° l.c., non-empty, compact and convex, each lying in some Hausdorff topological semivector space, and let u be a continuous real-valued function on $X = X_1 \times X_2$, such that

$$f_1(x_2) = \{x_1 \mid u(x_1, x_2) = \text{Max}_{y \in X_1} u(y, x_2)\}$$

$$f_2(x_1) = \{x_2 \mid u(x_1, x_2) = \text{Min}_{z \in X_2} u(x_1, z)\}$$

define maps $f_1: X_2 \rightarrow \mathcal{C}2(X_1)$ and $f_2: X_1 \rightarrow \mathcal{C}2(X_2)$, respectively. Then $\text{Min}_{X_2} \text{Max}_{X_1} u(x_1, x_2) = \text{Max}_{X_1} \text{Min}_{X_2} u(x_1, x_2)$.

Proof: Use Theorem 7.10.

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REFERENCES

1. (Bohnenblust & Karlin, 1950)

H. F. Bohnenblust and S. Karlin, "On a Theorem of Ville," Contributions to the Theory of Games, I, pp. 155-160, Annals of Mathematics Studies No. 24, Princeton University Press, Princeton, 1950.

2. (Brouwer, 1912)

L. E. J. Brouwer, "Über Abbildungen von Mannigfaltigkeiten," Mathematische Annalen, 71, 1912, pp. 97-115.

3. (Fan, 1952)

Ky Fan, "Fixed Point and Minmax Theorems in Locally Convex Topological Linear Spaces," Proceedings of the National Academy of Sciences, U. S., 38, 1952, pp. 121-126.

4. (Kakutani, 1941)

Shizuo Kakutani, "A Generalization of Brouwer's Fixed Point Theorem," Duke Mathematical Journal, 8, 1941, pp. 457-459.

5. (Kelley, 1955)

John L. Kelley, General Topology, D. Van Nostrand Company, Inc., Princeton, N. J., 1955.

6. (Michael, 1951)

Ernest Michael, "Topologies on Spaces of Subsets," Transactions of the American Mathematical Society, 71, 1951, pp. 152-182.

7. (Paalman-De Miranda, 1964)

A. B. Paalman-De Miranda, Topological Semigroups, Mathematical Centre Tract No. 11, Amsterdam, 1964.

8. (Prakash, 1971)

Prem Prakash, Foundations of Systems: For Decision, Planning, Control and Social/Economic Analysis, unpublished Ph.D. Dissertation, M. I. T., March, 1971.

9. (Prakash & Sertel, 1970)

Prem Prakash and Murat R. Sertel, "Fixed Point and Minmax Theorems in Semi-Linear Spaces," Sloan School of Management, M. I. T., Working Paper 484-70, September, 1970.

10. (Prakash & Sertel, 1971a)

Prem Prakash and Murat R. Sertel, "Semilinear (Topological) Spaces and Applications," Proceedings of the Second Florida Symposium on Automata and Semigroups, University of Florida, Gainesville, Florida, April, 1971.

11. (Prakash & Sertel, 1971b)

Prem Prakash and Murat R. Sertel, "Generalized Existence Theorems for Equilibria in Games, Economies & Social Systems," presented to the N. S. F. Conference on the Control Theory of Partial Differential Equations, University of Maryland at Baltimore, Baltimore, Maryland, August, 1971.

12. (Prakash & Sertel, 1972)

Prem Prakash and Murat R. Sertel, "Semivector Spaces: Algebraic Structure," (forthcoming).

13. (Schauder, 1930)

J. Schauder, "Der Fixpunktsatz in Funktionalräumen," Studia Mathematica, II, 1930, pp. 171-180.

14. (Sertel, 1971)

Murat R. Sertel, Elements of Equilibrium Methods for Social Analysis, unpublished Ph.D. Dissertation, M. I. T., January, 1971.

15. (Tychonoff, 1935)

A. Tychonoff, "Ein Fixpunktsatz," Mathematische Annalen, III, 1935, pp. 767-776.

