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Mirror Fermat Calabi-Yau Threefolds
and
Landau-Ginzburg Black Hole Attractors

Stefano Bellucci[♠], **Sergio Ferrara**[♢,♠], **Alessio Marrani**[♡,♠] and **Armen Yeranyan**[♣]

♠ *INFN - Laboratori Nazionali di Frascati,*
Via Enrico Fermi 40,00044 Frascati, Italy
bellucci,marrani@lnf.infn.it

♢ *Physics Department,Theory Unit, CERN,*
CH 1211, Geneva 23, Switzerland
sergio.ferrara@cern.ch

♡ *Museo Storico della Fisica e*
Centro Studi e Ricerche "Enrico Fermi"
Via Panisperna 89A, 00184 Roma, Italy

♣ *Department of Physics, Yerevan State University,*
Alex Manoogian St., 1, Yerevan, 375025, Armenia
ayeran@ysu.am

ABSTRACT

We study black hole attractor equations for one-(complex structure)modulus Calabi-Yau spaces which are the mirror dual of Fermat Calabi-Yau threefolds (CY_3s).

When exploring non-degenerate solutions near the Landau-Ginzburg point of the moduli space of such 4-dimensional compactifications, we always find two species of extremal black hole attractors, depending on the choice of the $Sp(4,\mathbb{Z})$ symplectic charge vector, one $\frac{1}{2}$-BPS (which is always stable, according to general results of special Kähler geometry) and one non-BPS. The latter turns out to be stable (local minimum of the "effective black hole potential" V_{BH}) for non-vanishing central charge, whereas it is unstable (saddle point of V_{BH}) for the case of vanishing central charge.

This is to be compared to the large volume limit of one-modulus CY_3-compactifications (of Type II A superstrings), in which the homogeneous symmetric special Kähler geometry based on cubic prepotential admits (beside the $\frac{1}{2}$-BPS ones) only non-BPS extremal black hole attractors with non-vanishing central charge, which are always stable.

Contents

1 Introduction

Extremal black hole (BH) attractors [1]-[4] have been recently widely investigated [5]- [27], especially in connection with new classes of solutions to the attractor equations corresponding to non-BPS (Bogomol'ny-Prasad-Sommerfeld) horizon geometries, supported by particular configurations of the BH electric and magnetic charges. Such geometries are *non-degenerate*, *i.e.* they have a finite, non-vanishing horizon area, and their Bekenstein-Hawking entropy [28] is obtained by extremizing an "effective BH potential".

In $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravity theories (MESGTs), non-degenerate attractor horizon geometries correspond to BH solitonic states belonging to "short massive multiplets" (for the $\frac{1}{2}$-BPS case, with $0 < |Z|_H = M_{ADM,H}$) and to "long massive multiplets", either with non-vanishing or vanishing central charge Z not saturating the BPS bound[1] [29]

$$0 \leq |Z|_H < M_{ADM,H}. \tag{1.1}$$

The Arnowitt-Deser-Misner (ADM) mass [30] at the BH horizon is obtained by extremizing a positive-definite "effective BH potential"[2] $V_{BH}\left(\phi, \widetilde{\Gamma}\right)$, where the $1 \times (2n_V + 2)$ symplectic charge vector $\widetilde{\Gamma} \equiv \left(p^\Lambda, q_\Lambda\right)_{\Lambda=0,1,\dots,n_V}$ contain both

[1]Here and in what follows, the subscript "H" will denote values at the BH event horizon.

[2]Here and below "ϕ" denotes the set of real scalars relevant for Attractor Mechanism, *i.e.* the $2n_V$ ones coming from the n_V vector supermultiplets coupled to the supergravity multiplet.

BH magnetic and electric charges, given by the asymptotical fluxes of two-form field strengths of Abelian vector fields A^{Λ}_{μ}.

The BH entropy S_{BH} is given by the Bekenstein-Hawking entropy-area formula [28, 31]

$$S_{BH}\left(\widetilde{\Gamma}\right)=\frac{A_H\left(\widetilde{\Gamma}\right)}{4}=\pi\, V_{BH}\left(\phi,\widetilde{\Gamma}\right)\Big|_{\partial V_{BH}=0}=\pi V_{BH}\left(\phi_H\left(\widetilde{\Gamma}\right)\right), \tag{1.2}$$

where A_H is the event horizon area, and the solution $\phi_H\left(\widetilde{\Gamma}\right)$ to the criticality condition

$$\partial_{\phi}V_{BH}\left(\phi,\widetilde{\Gamma}\right)=0 \tag{1.3}$$

is properly named *attractor* if the critical $(2n_V+2)\times(2n_V+2)$ real symmetric Hessian matrix

$$\left.\frac{\partial^2 V_{BH}\left(\phi,\widetilde{\Gamma}\right)}{\partial\phi\partial\phi}\right|_{\phi=\phi_H\left(\widetilde{\Gamma}\right)} \tag{1.4}$$

is a strictly positive-definite matrix[3].

Although non-supersymmetric BH attractors exist also in $\mathcal{N}>2$, $d=4$ and $d=5$ supergravities [32, 19], the most interesting examples arise in $\mathcal{N}=2$, $d=4$ MESGTs, where the scalar fluctuations relevant for the BH Attractor Mechanism parametrize a special Kähler (SK) manifold. Recently, the classification of "attractor solutions" for extremal BHs has been performed in full generality for the whole class of homogeneous symmetric SK geometries [22], and three distinct classes of extremal BH attractors (namely $\frac{1}{2}$-BPS, non-BPS $Z\neq 0$ and non-BPS $Z=0$ ones) were found as solutions to Eqs. (1.3). In such a framework, the non-BPS charge orbits have been found to depend on whether the supporting charge vector $\widetilde{\Gamma}$ is such that the $\mathcal{N}=2$ central charge vanishes or not. Moreover, the critical Hessian matrix (1.4) was usually found to exhibit zero modes (*i.e.* "flat" directions), whose attractor nature seemingly further depends on additional conditions on the charge vector $\widetilde{\Gamma}$, other than the ones given by the extremality conditions (1.3) (see *e.g.* [9]).

The aim of the present work is to study a particular class of (1-modulus) SK geometries, namely the ones underlying the complex structure moduli space of (mirror) Fermat Calabi-Yau threefolds (CY_3s) (classified by the *Fermat parameter* $k=5,6,8,10$, and firstly found in [33]). The fourth order linear Picard-Fuchs (PF) ordinary differential equations determining the holomorphic fundamental period 4×1 vector for such a class of 1-modulus CY_3s were found some time ago for $k=5$ in [34, 35] (see in particular Eq. (3.9) of [34], where $z\equiv\psi^{-5}$; see also [36]) and for $k=6,8,10$ in [37].

[3]The opposite is in general not true, *i.e.* there can be attractor points corresponding to critical Hessian matrices with "flat" directions (*i.e.* vanishing eigenvalues). In general, when a critical Hessian matrix exhibits some vanishing eigenvalues, one has to look at higher-order derivatives of V_{BH} evaluated at the considered point, and study their sign. Dependingly on the values of the supporting BH charges, one can obtains stable or unstable critical points. Examples in literature of investigations beyond the Hessian level can be found in [9, 24, 25].

In $\mathcal{N}=2$, $d=4$ MESGT the following formula holds[4] [3, 4, 38]

$$V_{BH}\left(z,\overline{z};q,p\right)=\left|Z\right|^{2}\left(z,\overline{z};q,p\right)+g^{j\overline{j}}\left(z,\overline{z}\right)D_{j}Z\left(z,\overline{z};q,p\right)\overline{D}_{\overline{j}}\overline{Z}\left(z,\overline{z};q,p\right). \tag{1.5}$$

Consequently, the criticality conditions (1.3) can be easily shown to acquire the form [31]

$$2\overline{Z}D_{i}Z+g^{j\overline{j}}\left(D_{i}D_{j}Z\right)\overline{D}_{\overline{j}}\overline{Z}=0; \tag{1.6}$$

this is what one should rigorously refer to as the $\mathcal{N}=2$, $d=4$ supergravity attractor equations (AEs). $g^{j\overline{j}}\left(z,\overline{z}\right)$ is the contravariant Kähler metric tensor, satisfying the usual orthonormality condition:

$$g^{i\overline{j}}\left(z,\overline{z}\right)\partial_{i}\overline{\partial}_{\overline{k}}K\left(z,\overline{z}\right)=\delta_{\overline{k}}^{\overline{j}}, \tag{1.7}$$

where $K\left(z,\overline{z}\right)$ is the real Kähler potential. As previously mentioned, $Z\left(z,\overline{z};q,p\right)$ is the $\mathcal{N}=2$ central charge function

$$Z\left(z,\overline{z};q,p\right)\equiv e^{\frac{1}{2}K(z,\overline{z})}\widetilde{\Gamma}\Omega\Pi\left(z\right)=e^{\frac{1}{2}K(z,\overline{z})}\left[q_{\Lambda}X^{\Lambda}\left(z\right)-p^{\Lambda}F_{\Lambda}\left(z\right)\right]\equiv e^{\frac{1}{2}K(z,\overline{z})}W\left(z;q,p\right), \tag{1.8}$$

where Ω is the $(2n_V+2)$-dim. symplectic metric (subscripts denote dimensions)

$$\Omega\equiv\left(\begin{array}{cc}0_{n_V+1} & -\mathbb{I}_{n_V+1}\\ \mathbb{I}_{n_V+1} & 0_{n_V+1}\end{array}\right), \tag{1.9}$$

and $\Pi\left(z\right)$ is the $(2n_V+2)\times 1$ holomorphic period vector in symplectic basis

$$\Pi\left(z\right)\equiv\left(\begin{array}{c}X^{\Lambda}\left(z\right)\\ F_{\Lambda}\left(z\right)\end{array}\right), \tag{1.10}$$

with $X^{\Lambda}\left(z\right)$ and $F_{\Lambda}\left(z\right)$ being the holomorphic sections of the $U(1)$ line (Hodge) bundle over the SK manifold (clearly, due to holomorphicity they do not belong to the related $U(1)$ ring). Finally, $W\left(z;q,p\right)$ is the holomorphic $\mathcal{N}=2$ central charge function, also named $\mathcal{N}=2$ superpotential.

Let us here recall that Z has Kähler weights $(p,\overline{p})=(1,-1)$; thus, its Kähler-covariant derivatives read

$$\begin{aligned}D_{i}Z&=\left(\partial_{i}+\tfrac{1}{2}\partial_{i}K\right)Z,\\ \overline{D}_{\overline{i}}Z&=\left(\overline{\partial}_{\overline{i}}-\tfrac{1}{2}\overline{\partial}_{\overline{i}}K\right)Z.\end{aligned} \tag{1.11}$$

[4] Here and below we switch to the complex parametrization of the set of scalars being considered:

$$\left\{\phi^{a}\right\}_{a=1,\ldots,2n_V}\longrightarrow\left\{z^{i},\overline{z}^{\overline{i}}\right\}_{i,\overline{i}=1,\ldots,n_V}.$$

The relation between such two equivalent parametrizations of the SK scalar manifold is given by Eq. (4.2) of [15].

The non-holomorphic basic, defining differential relations of SK geometry are[5] (see *e.g.* [38]):

$$\left\{\begin{array}{l} D_i Z = Z_i; \\ \\ D_i Z_j = iC_{ijk} g^{k\overline{k}} \overline{D}_{\overline{k}} \overline{Z} = iC_{ijk} g^{k\overline{k}} \overline{Z}_{\overline{k}}; \\ \\ D_i \overline{D}_{\overline{j}} \overline{Z} = D_i \overline{Z}_{\overline{j}} = g_{i\overline{j}} \overline{Z}; \\ \\ D_i \overline{Z} = 0, \end{array}\right. \tag{1.12}$$

where the first relation is nothing but the definition of the "matter charges" Z_is and the fourth relation expresses the Kähler-covariant holomorphicity of Z. C_{ijk} is the rank-3, completely symmetric, covariantly holomorphic tensor of SK geometry (with Kähler weights $(2,-2)$) (see *e.g.* [38]- [42]):

$$\begin{array}{l} C_{ijk} = e^K \left(\partial_i X^\Lambda\right)\left(\partial_j X^\Sigma\right)\left(\partial_k X^\Xi\right) \partial_\Xi \partial_\Sigma F_\Lambda (X) \equiv e^K W_{ijk}; \\ \\ \overline{D}_{\overline{i}} C_{jkl} = 0; \\ \\ D_{[i} C_{j]kl} = 0; \\ \\ R_{i\overline{j}k\overline{l}} = g_{i\overline{j}} g_{k\overline{l}} + g_{i\overline{l}} g_{k\overline{j}} - C_{ikp} \overline{C}_{\overline{j}\overline{l}\overline{p}} g^{p\overline{p}}, \end{array} \tag{1.13}$$

where $R_{i\overline{j}k\overline{l}}$ is the Riemann-Christoffel tensor of Kähler geometry:

$$R_{i\overline{j}k\overline{l}} = -g^{m\overline{n}} \left(\overline{\partial}_{\overline{l}} \overline{\partial}_{\overline{j}} \partial_m K\right) \partial_i \overline{\partial}_{\overline{n}} \partial_k K + \overline{\partial}_{\overline{l}} \partial_i \overline{\partial}_{\overline{j}} \partial_k K, \tag{1.14}$$

and square brackets denote antisymmetrization with respect to enclosed indices. By using the first two of relations (1.12), the $\mathcal{N}=2$ AEs (1.6) can be recast as follows [31]:

$$2\overline{Z} Z_i + iC_{ijk} g^{j\overline{j}} g^{k\overline{k}} \overline{Z}_{\overline{j}} \overline{Z}_{\overline{k}} = 0. \tag{1.15}$$

It is now worth recalling some fundamental identities defining the geometric structure of SK manifolds [43, 8, 13, 15, 16, 24]

$$\widetilde{\Gamma}^T - i\Omega \mathcal{M}(\mathcal{N}) \widetilde{\Gamma}^T = -2iZ\overline{\Pi} - 2ig^{j\overline{j}} \left(\overline{D}_{\overline{j}} \overline{Z}\right) D_j \Pi, \tag{1.16}$$

where $\mathcal{M}(\mathcal{N})$ denotes the $(2n_V+2) \times (2n_V+2)$ real symmetric matrix [38, 3, 4]

$$\mathcal{M}(\mathcal{N}) \equiv \left(\begin{array}{cc} Im(\mathcal{N}) + Re(\mathcal{N}) (Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & -Re(\mathcal{N}) (Im(\mathcal{N}))^{-1} \\ \\ -(Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & (Im(\mathcal{N}))^{-1} \end{array}\right), \tag{1.17}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is a complex symmetric matrix playing a key role in $\mathcal{N}=2$, $d=4$ MESGT (see *e.g.* the report [38]). Moreover, it should be here reminded that

$$\begin{array}{l} D_i \Pi = (\partial_i + \partial_i K) \Pi, \\ \\ \overline{D}_{\overline{i}} \Pi = \overline{\partial}_{\overline{i}} \Pi = 0, \end{array} \tag{1.18}$$

[5] Actually, there are different (equivalent) defining approaches to SK geometry. For subtleties and further elucidation concerning such an issue, see *e.g.* [39] and [40].

since Π is holomorphic with Kähler weights $(2,0)$.

The $2n_V+2$ real identities (1.16) (whose real and imaginary parts are related by a suitable "rotation" [16]) express nothing but a change of basis in the lattice $\Psi_{(p,q)}$ of BH charge configurations, between the integer symplectic (magnetic/electric) basis vector $\widetilde{\Gamma}\equiv\left(p^{\Lambda},q_{\Lambda}\right)_{\Lambda=0,1,\dots,n_V}$ and the complex "supergravity charges" vector $\mathcal{Z}\equiv\left(Z,Z_i\right)_{i=1,\dots,n_V}$. Notice that $\mathcal{Z}$ is moduli-dependent, since it refers to supermultiplet eigenstates. It is important to stress that identities (1.16) entail 2 redundant degrees of freedom, encoded in the homogeneity (of degree 1) of (1.16) under complex rescalings of $\widetilde{\Gamma}$. Indeed, by recalling the definition (1.8) it can be readily checked that the right-hand side of (1.16) acquires an overall factor λ under the rescaling

$$\widetilde{\Gamma}\longrightarrow\lambda\widetilde{\Gamma},\quad \lambda\in\mathbb{C}. \tag{1.19}$$

We will reconsider such a point in Sect. 8, when treating the 1-modulus case more in detail.

It should also be noticed that the $\mathcal{N}=2$ "effective BH potential" given by Eq. (1.5) can also be rewritten as [3, 4, 38]

$$V_{BH}\left(z,\overline{z};q,p\right)=-\frac{1}{2}\widetilde{\Gamma}\mathcal{M}\left(\mathcal{N}\right)\widetilde{\Gamma}^{T}, \tag{1.20}$$

and therefore it can be identified with the first, positive-definite real invariant of SK geometry (see *e.g.* [24, 38]). It is interesting to remark that the result (1.20) can be elegantly obtained from the SK geometry identities (1.16) by making use of the following relations [19]:

$$\frac{1}{2}\left(\mathcal{M}\left(\mathcal{N}\right)+i\Omega\right)\left(\begin{array}{c}\Pi\\ \\ \overline{D}_{\overline{j}}\overline{\Pi}\end{array}\right)=i\Omega\left(\begin{array}{c}\Pi\\ \\ \overline{D}_{\overline{j}}\overline{\Pi}\end{array}\right),\quad\forall\overline{j}, \tag{1.21}$$

which follow from the observation that

$$\mathcal{M}\left(\mathcal{N}\right)\left(\begin{array}{c}\Pi\\ \\ \overline{D}_{\overline{j}}\overline{\Pi}\end{array}\right)=i\Omega\left(\begin{array}{c}\Pi\\ \\ \overline{D}_{\overline{j}}\overline{\Pi}\end{array}\right),\quad\forall\overline{j}. \tag{1.22}$$

In the 1-modulus case a major simplification occurs, since Eqs. (1.15) and (1.5) respectively reduce to $(z^1\equiv\psi)$

$$2\overline{Z}D_{\psi}Z+iC_{\psi\psi\psi}\left(g_{\psi\overline{\psi}}\right)^{-2}\left(\overline{D}_{\overline{\psi}}\overline{Z}\right)^{2}=0; \tag{1.23}$$

$$V_{BH}\left(\psi,\overline{\psi};q,p\right)\equiv\left|Z\right|^{2}\left(\psi,\overline{\psi};q,p\right)+\left(g_{\psi\overline{\psi}}\right)^{-1}\left(\psi,\overline{\psi}\right)\left|D_{\psi}Z\right|^{2}\left(\psi,\overline{\psi};q,p\right). \tag{1.24}$$

The $\frac{1}{2}$-BPS solutions correspond to $Z\neq0$ and $D_{\psi}Z=0$, the non-BPS solutions $(D_{\psi}Z\neq0)$ can occur in two species:

1) $Z\neq0$, for which [15]

$$\left|D_{\psi}Z\right|_{non-BPS,Z\neq0}=2\left[\left(g_{\psi\overline{\psi}}\right)^{2}\frac{|Z|}{|C_{\psi\psi\psi}|}\right]_{non-BPS,Z\neq0}; \tag{1.25}$$

2) $Z=0$, in which case Eq. (1.23) yields

$$\left.C_{\psi\psi\psi}\right|_{non-BPS,Z=0}=0. \tag{1.26}$$

At such critical points, the "BH effective potential" respectively becomes (for non-BPS, $Z \neq 0$ case see [15])

$$V_{BH,non-BPS,Z\neq 0} = |Z|^2_{non-BPS,Z\neq 0}\left[1+4\frac{\left(g_{\psi\overline{\psi}}\right)^3}{|C_{\psi\psi\psi}|^2}\right]_{non-BPS,Z\neq 0}; \tag{1.27}$$

$$V_{BH,non-BPS,Z=0} = |D_\psi Z|^2_{non-BPS,Z=0}. \tag{1.28}$$

For non-BPS, $Z \neq 0$ critical points of V_{BH}, one can also define the *supersymmetry-breaking order parameter* as follows:

$$\mathcal{O}_{non-BPS,Z\neq 0} \equiv \left[\frac{\left(g_{\psi\overline{\psi}}\right)^{-1}|D_\psi Z|^2}{|Z|^2}\right]_{non-BPS,Z\neq 0} = \left[\frac{\left(g_{\psi\overline{\psi}}\right)^{-1}|D_\psi W|^2}{|W|^2}\right]_{non-BPS,Z\neq 0} = \tag{1.29}$$

$$= 4\left[\frac{\left(g_{\psi\overline{\psi}}\right)^3}{|C_{\psi\psi\psi}|^2}\right]_{non-BPS,Z\neq 0}, \tag{1.30}$$

where in the second line we used Eq. (1.25). It is worth noticing that for a cubic prepotential $\mathcal{F}(z) = \varrho z^3$ it holds that $\mathcal{O}_{non-BPS,Z\neq 0} = 3\ \forall \varrho \in \mathbb{C}$ [22]; such a result actually holds for cubic prepotentials in generic n_V-moduli SK geometries, such as the ones arising in the large volume limit of CY_3-compactifications of Type II A superstring theory (see Eq. (111) of [9]).

As we are going to compute explicitly in Sects. 4-7 for the k-parametrized class of (mirror) Fermat CY_3s, one finds that (beside the $\frac{1}{2}$-BPS solutions, existing and stable in all cases) for $k = 5, 8$ only non-BPS, $Z \neq 0$ solutions exist, and they are attractors (local minima of V_{BH}), whereas for $k = 6, 10$ only non-BPS, $Z = 0$ solutions exist, and they are not attractors in a strict sense (since they are saddle points of V_{BH}).

In the present paper we will investigate AE (1.23) near one of three typologies of regular singular points in the complex structure moduli space of (mirror) Fermat CY_3s, namely near the so-called Landau-Ginzburg (LG) point $\psi = 0$. In such a framework, the identities (1.16) of SK geometry, when considered in the 1-modulus case and in correspondence of the various above-mentioned species of critical points of V_{BH}, can be used to find the BH charge configurations supporting the LG point $\psi = 0$ to be an attractor point of the considered kind. It will be shown that, in spite of the fact that identities (1.16) give 4 real Eqs. in the 1-modulus case, only 2 of them are independent, and they are completely equivalent to the 2 real rigorously-named $\mathcal{N} = 2$, $d = 4$ supergravity AEs (1.23), which are nothing but the criticality condition $\partial_\psi V_{BH} = 0$.

The plan of the paper is as follows.

In Sect. 2 we briefly introduce the holomorphic geometry embedded in the SK geometry of the scalar manifolds of $N = 2$, $d = 4$ MESGTs. Such a geometry is relevant in order to introduce the PF differential equations. In particular, we focus on the 1-modulus case.

Then, in Sect. 3 we give a sketchy presentation of the formalism of the (mirror) Fermat CY_3s (classified by the Fermat parameter $k = 5, 6, 8, 10$), in particular near the LG point $\psi = 0$ of their (complex structure deformation) moduli space. The general analysis of Sect. 3 is consequently specialized to the study of non-degenerate extremal BH LG attractors in the complex structure moduli space of the four mirror Fermat CY_3s, corresponding to $k = 5$ (Sect. 4), $k = 6$ (Sect. 5), $k = 8$ (Sect. 6), and $k = 10$ (Sect. 7).

In Sect. 8, in order to study the extremal BH LG attractors for the above-mentioned class of CY_3s, we exploit the so-called *"SK geometrical identities" approach*. This amounts to evaluating near $\psi = 0$ the 4 real fundamental identities of 1-modulus SK geometry at the geometrical *loci* corresponding to the various species of critical points of the relevant "effective BH potential". We obtain results perfectly coinciding with the ones we got in Sects. 4-7 by exploiting the so-called *"criticality condition" approach*, corresponding to solve near the LG point the 2 real *criticality conditions* of V_{BH}, corresponding in the 1-modulus case to the real and imaginary part of the so-called $\mathcal{N} = 2$, $d = 4$ supergravity AEs.

Then, in Sect. 9 we face the problem of the consistent normalization of the PF ordinary differential equation obeyed by the vector of fundamental periods of the holomorphic 3-form defined on the above-mentioned Fermat CY_3s.

Concluding remarks, summarizing observations and outlooking comments are the contents of the final Sect. 10.

2 Holomorphic Geometry

In this Section we will present a summary of the holomorphic geometry embedded in the SK geometry of the scalar manifolds of $N = 2$, $d = 4$ MESGTs. The main references for such an issue are [44] and [45], to which we will refer at the relevant points of the treatment.

The PF Equations, satisfied in SK geometry by the holomorphic period vector (in a suitable basis, named *PF basis*) are a consequence of SK geometry and of the underlying symplectic structure of the *flat symplectic bundle* [46], which encodes the differential relations obeyed by the covariantly holomorphic sections and their covariant derivatives.

Let us start by considering the Kähler-covariantly holomorphic, symplectic $1 \times (2n_V + 2)$ vector[6]

$$V(z,\overline{z}) \equiv \left(L^{\Lambda}(z,\overline{z}), M_{\Lambda}(z,\overline{z}) \right) = e^{\frac{1}{2}K(z,\overline{z})} \Pi^{T}(z). \tag{2.1}$$

Flatness of the symplectic connection entails the following relations [46]:

$$\begin{cases} D_{\alpha}V = U_{\alpha}; \\ D_{\alpha}U_{\beta} = iC_{\alpha\beta\gamma}g^{\gamma\overline{\gamma}}\overline{D}_{\overline{\gamma}}\overline{V} = iC_{\alpha\beta\gamma}g^{\gamma\overline{\gamma}}\overline{U}_{\overline{\gamma}}; \\ D_{\alpha}\overline{D}_{\overline{\beta}}\overline{V} = D_{\alpha}\overline{U}_{\overline{\beta}} = g_{\alpha\overline{\beta}}\overline{V}; \\ D_{\alpha}\overline{V} = 0. \end{cases} \tag{2.2}$$

[6] In order to make the contact with the relevant literature easier, in this Section, as well in the next one, we will change some notations with respect to the previous treatment.

Firstly, we will consider row (*i.e.* $1 \times (2n_V + 2)$), instead of column (*i.e.* $(2n_V + 2) \times 1$), period vectors.

Secondly, we will use lowercase Greek indices to denote homogeneous coordinates (instead of lowercase Latin indices, as done in the previous Section). Lowercase Latin indices will rather be used to denote indices pertaining to the so-called *holomorphic geometry* we are going to discuss.

Notice that, by the definition (2.1), the $\mathcal{N}=2$ central charge function (defined by Eq. (1.8)) can be rewritten (in the notation for period vectors used in the present Section) as $Z=\widetilde{\Gamma}\Omega V^T$, and the defining relations (1.12) of SK geometry can thus be obtained by transposing the relations (2.2) and by further left-multiplying them by the $1\times(2n_V+2)$ vector $\widetilde{\Gamma}\Omega$.

Let us now consider a new $1\times(2n_V+2)$ vector of holomorphic sections[7] $(a=1,...,n_V)$

$$V_h\left(X(z)\right)\equiv\left(X^0\left(z\right),X^a\left(z\right),F_a\left(X(z)\right),-F_0\left(X(z)\right)\right). \tag{2.3}$$

We notice that, while $V\left(z,\overline{z}\right)$ defined in Eq. (2.1) is symplectic with respect to the symplectic metric Ω, this does not hold for $V_h\left(X(z)\right)$ defined in Eq. (2.3), which is instead symplectic with respect to a newly defined anti-diagonal symplectic metric ($Q^T=-Q$, $Q^2=-\mathbb{I}_{2n_V+2}$):

$$Q\equiv\begin{pmatrix} & & & 1\\ & & -\mathbb{I}_{n_V} & \\ & \mathbb{I}_{n_V} & & \\ -1 & & & \end{pmatrix}, \tag{2.4}$$

where unwritten elements vanish.

In the treatment which follows we will assume the existence of an holomorphic prepotential $F\left(X\left(z\right)\right)$ of $\mathcal{N}=2$, $d=4$ vector multiplet couplings such that $F_\Lambda\left(z\right)=\partial_\Lambda F\left(X\left(z\right)\right)$, which is in turn implied by the assumption that the holomorphic square matrix

$$e^a_\alpha\left(z\right)\equiv\frac{\partial\left[\frac{X^a(z)}{X^0(z)}\right]}{\partial z^\alpha}\equiv\frac{\partial t^a\left(z\right)}{\partial z^\alpha} \tag{2.5}$$

is invertible (non-singular), where in the last step we introduced the *homogeneous* (Kähler-invariant) coordinates $t^a\left(z\right)\equiv\frac{X^a(z)}{X^0(z)}$ (see *e.g.* [38]). The matrix $e^a_\alpha\left(z\right)$ expresses nothing but the change of basis between the $t^a\left(z\right)$s and the z^αs. *Special (symplectic) coordinates* correspond to the case $e^a_\alpha\left(z\right)=\delta^a_\alpha$, implying that $t^a\left(z\right)\equiv\frac{X^a(z)}{X^0(z)}=z^a$ (in such a case a-indices and α-indinces do coincide). By further fixing the Kähler gauge such that $X^0=1$, one finally gets $t^a\left(z\right)=X^a\left(z\right)=z^a$ and $X^0=1$, which is the usual definition of special coordinates (yielding $\partial_\alpha X^\Lambda=\delta^a_\alpha$).

The holomorphic period vector $V_h\left(X(z)\right)$ in special coordinates (Kähler gauge $X^0=1$ fixed understood throughout, unless otherwise noted) reads as follows:

$$V_{h,special}\left(z\right)\equiv\left(1,z^a,\partial_a\mathcal{F}\left(z\right),-\mathcal{F}_0\left(z\right)\right)=\left(1,z^a,\partial_a\mathcal{F}\left(z\right),z^a\partial_a\mathcal{F}\left(z\right)-2\mathcal{F}\left(z\right)\right), \tag{2.6}$$

where $\mathcal{F}\left(z\right)$ is the holomorphic prepotential in special coordinates (and for $X^0=1$), and in the second step the homogeneity of degree 2 of the prepotential was used; for general symplectic and special ($X^0=1$) coordinates it respectively reads

$$\begin{aligned} &X^\Lambda\partial_\Lambda F(X)=X^0\partial_0F\left(X\right)+X^a\partial_aF\left(X\right)=2F(X);\\ &\mathcal{F}_0\left(z\right)+z^a\partial_a\mathcal{F}\left(z\right)=2\mathcal{F}(X). \end{aligned} \tag{2.7}$$

[7]The subscript "h" stands for "holomorphic".

By starting from Eq. (2.6) and by differentiating once and twice $V_{h,special}(z)$, one respectively achieves

$$\partial_b V_{h,special}(z) = \left(0, \delta_b^a, \partial_a\partial_b\mathcal{F}(z), -\partial_b\mathcal{F}(z) + z^a\partial_a\partial_b\mathcal{F}(z)\right); \tag{2.8}$$

$$\partial_b\partial_c V_{h,special}(z) = \left(0, 0, \partial_a\partial_b\partial_c\mathcal{F}(z), z^a\partial_a\partial_b\partial_c\mathcal{F}(z)\right), \tag{2.9}$$

implying that

$$\begin{aligned} &\partial_b\partial_c V_{h,special}(z) = W_{abc}(z)\, V^a_{h,special}(z); \\ &\partial_a V^b_{h,special}(z) = \delta^b_a V^0_{h,special}, \end{aligned} \tag{2.10}$$

where $W_{abc}(z) \equiv \partial_a\partial_b\partial_c\mathcal{F}(z)$ is the holomorphic part of $C_{\alpha\beta\gamma}$ in special coordinates and for $X^0 = 1$ (see first of relations (1.13)) and

$$V^a_{h,special}(z) \equiv \left(0, 0, \delta^a_d, z^a\right), \tag{2.11}$$

$$V^0_{h,special}(z) \equiv (0, 0, 0, 1). \tag{2.12}$$

By adding the definition $V_{h,special,a}(z) \equiv \partial_a V_{h,special}(z)$ and the trivial result $\partial_a V^0_{h,special}(z) = 0$ to Eqs. (2.10), one finally gets the set of differential relations [45]

$$\begin{aligned} &\partial_a V_{h,special}(z) = V_{h,special,a}(z), \\ &\partial_a\partial_b V_{h,special}(z) = \partial_a V_{h,special,b}(z) = W_{abc}(z)\, V^c_{h,special}(z), \\ &\partial_a V^b_{h,special}(z) = \delta^b_a V^0_{h,special}, \\ &\partial_a V^0_{h,special}(z) = 0, \end{aligned} \tag{2.13}$$

which are the holomorphic counterparts of SK relations (2.2), written in special coordinates and for $X^0 = 1$.

By "holomorphically covariantizing" the relations (2.13), *i.e.* by writing them in a generic system of homogeneous coordinates, one obtains (notice that here a-indices and α-indices in general do not coincide) [45]

$$\begin{aligned} &\widehat{D}_\alpha V_h(z) = V_{h,\alpha}(z), \\ &\widehat{D}_\alpha\widehat{D}_\beta V_h(z) = \widehat{D}_\alpha V_{h,\beta}(z) = W_{\alpha\beta\gamma}(z))\, V^\gamma_h(z), \\ &\widehat{D}_\alpha V^\beta_h(z) = \delta^\beta_\alpha V^0_h(z), \\ &\widehat{D}_\alpha V^0_h(z) = 0, \end{aligned} \tag{2.14}$$

where $V_h(z)$, $V_{h,\beta}(z)$, $V_h^{\beta}(z)$ and $V_h^0(z)$ are respectively given by the following formulæ[8] [45]:

$$V_h(z) = \left(X^0(z), X^a(z), X^0(z)e_a^{\alpha}(z)\partial_\alpha F(z), X^a(z)e_a^{\alpha}(z)\partial_\alpha F(z) - 2X^0(z)F(z)\right);$$

$$V_{h,\beta}(z) = \widehat{D}_\beta V_h(X(z)) = \Big(0, X^0(z)e_\beta^a(z), X^0(z)e_\alpha^a(z)\widehat{D}_\alpha\partial_\beta F(z), -X^0(z)\partial_\beta F(z) + X^a(z)e_a^{\alpha}(z)\widehat{D}_\alpha\partial_\beta F(z)\Big);$$

$$V_h^{\beta}(z) = \Big(0, 0, \left(X^0(z)\right)^{-1} e_a^{\beta}(z), \left(X^0(z)\right)^{-2} X^a(z)e_a^{\beta}(z)\Big);$$

$$V_h^0(z) = \Big(0, 0, 0, \left(X^0(z)\right)^{-1}\Big), \tag{2.15}$$

which correspond to the "holomorphically covariantized" counterparts of Eqs. (2.6), (2.8), (2.11) and (2.12), respectively.

Notice that a new holomorphic covariant derivative $\widehat{D}_\alpha$ has been introduced. In analogy with the usual covariant derivative in Kähler-Hodge manifold, the action of $\widehat{D}_\alpha$ on a vector ϕ_β with Kähler weight p reads [44, 45]

$$\widehat{D}_\alpha\phi_\beta(z,\overline{z}) = \Big(\partial_a + \frac{p}{2}\widehat{K}_\alpha(z)\Big)\phi_\beta(z,\overline{z}) - \widehat{\Gamma}_{\alpha\beta}{}^{\gamma}(z)\,\phi_\gamma(z,\overline{z}), \tag{2.16}$$

where $\widehat{\Gamma}_{\alpha\beta}{}^{\gamma}(z)$ is the holomorphic part of the Christoffel connection $\Gamma_{\alpha\beta}{}^{\gamma}(z,\overline{z})$ of the SK manifold being considered [44, 45] ($e_\alpha^a(z)\,e_a^{\gamma}(z) = \delta_\alpha^{\gamma}$, $e_\alpha^a(z)\,e_b^{\alpha}(z) = \delta_b^a$):

$$\widehat{\Gamma}_{\alpha\beta}{}^{\gamma}(z) \equiv (\partial_\beta e_\alpha^a(z))\,e_a^{\gamma}(z) = \Gamma_{\alpha\beta}{}^{\gamma}(z,\overline{z}) - T_{\alpha\beta}{}^{\gamma}(z,\overline{z}) = \tag{2.17}$$

$$= g^{\gamma\overline{\gamma}}(z,\overline{z})\,\partial_\alpha\partial_\beta\overline{\partial}_{\overline{\gamma}}K(z,\overline{z}) - e_\alpha^a(z)e_\beta^b(z)\left[\frac{\partial^3 K\left(t(z),\overline{t}(\overline{z})\right)}{\partial t^b\partial t^a\overline{\partial}\overline{t}^{\overline{d}}}\right]g^{c\overline{d}}(z,\overline{z})\,e_c^{\gamma}(z). \tag{2.18}$$

It can be checked that $\widehat{\Gamma}_{\alpha\beta}{}^{\gamma}(z)$ transforms as a connection under holomorphic reparametrizations. Moreover, since $X^0(z)$ has Kähler weights $(2,0)$, the quantity

$$\widehat{K}_\alpha(z) \equiv -\partial_\alpha\left[ln\left(X^0(z)\right)\right] \tag{2.19}$$

transforms as a connection under Kähler gauge transformations:

$$K(z,\overline{z}) \longrightarrow K(z,\overline{z}) + f(z) + \overline{f}(\overline{z}) \Longrightarrow \widehat{K}_\alpha(z) \longrightarrow \widehat{K}_\alpha(z) + \partial_\alpha f(z). \tag{2.20}$$

It is worth pointing out that the $\widehat{\Gamma}_{\alpha\beta}{}^{\gamma}$s are the Christoffel symbols of the second kind of an holomorphic Riemann metric

$$\widehat{g}_{\alpha\beta}(z) \equiv e_\alpha^a(z)e_\beta^b(z)\eta_{ab} = \frac{\partial\left[\frac{X^a(z)}{X^0(z)}\right]}{\partial z^\alpha}\frac{\partial\left[\frac{X^b(z)}{X^0(z)}\right]}{\partial z^\beta}\eta_{ab}, \tag{2.21}$$

where η_{ab} is constant (invertible) symmetric matrix (note that $\widehat{g}_{\alpha\beta}(z)$ has two holomorphic indices, in contrast to the Kähler metric $g_{\alpha\overline{\beta}}(z,\overline{z}) = \partial_\alpha\overline{\partial}_{\overline{\beta}}K(z,\overline{z})$). $\widehat{g}_{\alpha\beta}(z)$ is the metric tensor of the so-called *holomorphic geometry*

[8] The first of relations (2.15) corresponds to Eq. (2.3) by using the definition of $t^a(z)$s and the homogeneity of degree 2 of the prepotential F.

"embedded" in the considered SK geometry. Due to Eq. (2.5), it can be checked that $\widehat{\Gamma}_{\alpha\beta}{}^{\gamma}(z)$ is actually a Riemann-flat connection, since it holds that

$$\widehat{R}_{\delta\alpha\beta}{}^{\gamma}(z) \equiv \partial_{\delta}\widehat{\Gamma}_{\alpha\beta}{}^{\gamma}(z) - \partial_{\alpha}\widehat{\Gamma}_{\delta\beta}{}^{\gamma}(z) + \widehat{\Gamma}_{\alpha\beta}{}^{\zeta}(z)\widehat{\Gamma}_{\zeta\delta}{}^{\gamma}(z) - \widehat{\Gamma}_{\delta\beta}{}^{\zeta}(z)\widehat{\Gamma}_{\zeta\alpha}{}^{\gamma}(z) = 0. \tag{2.22}$$

Finally, it should observed that special coordinates are flat coordinates for the holomorphic geometry, because for special coordinates ($e^{a}_{\alpha}(z) = \delta^{a}_{\alpha}$) (in the Kähler gauge $X^0 = 1$) Eqs. (2.17), (2.19) and (2.21) respectively reduce to

$$\widehat{\Gamma}_{\alpha\beta}{}^{\gamma}(z) = 0; \tag{2.23}$$

$$\widehat{K}_{\alpha}(z) = 0; \tag{2.24}$$

$$\widehat{g}_{\alpha\beta}(z) = \eta_{\alpha\beta}. \tag{2.25}$$

It is worth pointing out that the system (2.14) is the holomorphic counterpart of the system (2.2), and it is manifestly covariant with respect to the holomorphic geometry defined by $\widehat{g}_{\alpha\beta}(z)$ and $\widehat{K}_{\alpha}(z)$. By breaking the "holomorphic covariance" and choosing special coordinates (and fixing Kähler gauge such that $X^0 = 1$), the system (2.14) reduces to the system (2.13). The system of holomorphic differential relations (2.14) is usually referred to as the (holomorphic) *Picard-Fuchs (PF) system.*

Let us now specialize the treatment to the 1-modulus case. Once again, such a case is peculiarly simple, since the Vielbein is nothing but an holomorphic function (we denote $z^1 \equiv \psi$):

$$e^{1}_{\psi}(\psi) \equiv \frac{\partial\left[\frac{X^1(\psi)}{X^0(\psi)}\right]}{\partial\psi} \equiv \frac{\partial t^1(\psi)}{\partial\psi} \equiv e(\psi), \tag{2.26}$$

and the connections and metric of holomorphic geometry reduce to

$$\begin{aligned} &\widehat{\Gamma}_{\psi\psi}{}^{\psi}(\psi) = \partial_{\psi}\left[ln\left(e(\psi)\right)\right]; \\ &\widehat{K}_{\psi}(\psi) = -\partial_{\psi}\left[ln\left(X^0(\psi)\right)\right]; \\ &\widehat{g}_{\psi\psi}(\psi) \equiv \left[e(\psi)\right]^2\eta_{\psi\psi}. \end{aligned} \tag{2.27}$$

Consequently, the action of $\widehat{D}_{\psi}$ on a 1-vector (function) $\phi\left(\psi,\overline{\psi}\right)$ with Kähler weights $(p,\overline{p})$ reads

$$\begin{aligned} \widehat{D}_{\psi}\phi\left(\psi,\overline{\psi}\right) &= \left(\partial_{\psi} + \frac{p}{2}\widehat{K}_{\psi}(\psi) - \widehat{\Gamma}_{\psi\psi}{}^{\psi}(\psi)\right)\phi\left(\psi,\overline{\psi}\right) = \\ &= \left\{\partial_{\psi} - \frac{p}{2}\partial_{\psi}\left[ln\left(X^0(\psi)\right)\right] - \partial_{\psi}\left[ln\left(e(\psi)\right)\right]\right\}\phi\left(\psi,\overline{\psi}\right). \end{aligned} \tag{2.28}$$

It can be shown [44, 45] that in the 1-modulus case the PF system (2.14) is equivalent to the following complex differential relation:

$$\widehat{D}\widehat{D}\left[W^{-1}(\psi)\right]\widehat{D}\widehat{D}V_h(\psi) = 0, \tag{2.29}$$

where $\widehat{D} \equiv \widehat{D}_\psi$,

$$
\begin{aligned}
W(\psi) &\equiv W_{\psi\psi\psi}(\psi) = e^{-K\left(\psi,\overline{\psi}\right)} C_{\psi\psi\psi}\left(\psi,\overline{\psi}\right) = \\
&= \left[\partial_\psi X^0(\psi)\right]^3 F_{000}(\psi) + 3\left[\partial_\psi X^0(\psi)\right]^2 \left[\partial_\psi X^1(\psi)\right] F_{001}(\psi) + \\
&+3\left[\partial_\psi X^1(\psi)\right]^2 \left[\partial_\psi X^0(\psi)\right] F_{011}(\psi) + \left[\partial_\psi X^1(\psi)\right]^3 F_{111}(\psi),
\end{aligned} \tag{2.30}
$$

and (see Eq. (2.3) and the first of relations (2.15))

$$
V_h(\psi) = \left(X^0(\psi), X^1(\psi), F_1(X(\psi)), -F_0(X(\psi))\right) = \tag{2.31}
$$

$$
= \left(X^0(\psi), X^1(\psi), X^0(\psi)e(\psi)\partial_\psi F(\psi), X^1(\psi)e(\psi)\partial_\psi F(\psi) - 2X^0(\psi)F(\psi)\right). \tag{2.32}
$$

Eq. (2.29) can also be rewritten as a fourth order linear ordinary differential equation in $V_h(\psi)$ (*1-modulus PF Eq.*) [44, 45]:

$$
\sum_{n=0}^{4} a_n(\psi)\,\partial^n V_h(\psi) = 0, \tag{2.33}
$$

where $\partial^n \equiv \frac{\partial^n}{(\partial\psi)^n}$ ($n = 0$ corresponds to the identity operator)[9]. The functional coefficients $a_n(\psi)$s can be obtained by comparing Eqs. (2.29) and (2.33) [44] ($\partial \equiv \partial_\psi$):

$$
\begin{aligned}
a_4 &\equiv W^{-1}; \\
a_3 &\equiv 2\partial\left(W^{-1}\right); \\
a_2 &\equiv W^{-1}\left(\partial\widehat{\Lambda} - \widehat{\Lambda}^2 + 2\widehat{\Sigma}\right) + \left[\partial\left(W^{-1}\right)\right]\widehat{\Lambda} + \partial^2\left(W^{-1}\right); \\
a_1 &\equiv W^{-1}\left(\partial^2\widehat{\Lambda} + 2\partial\widehat{\Sigma} - 2\widehat{\Lambda}\partial\widehat{\Lambda}\right) + \left[\partial^2\left(W^{-1}\right)\right]\widehat{\Lambda} + \\
&+ \left[\partial\left(W^{-1}\right)\right]\left(2\widehat{\Sigma} + 2\partial\widehat{\Lambda} - \widehat{\Lambda}^2\right); \\
a_0 &\equiv W^{-1}\left(\widehat{\Sigma}^2 - \widehat{\Sigma}\partial\widehat{\Lambda} - \widehat{\Lambda}\partial\widehat{\Sigma} + \partial^2\widehat{\Sigma}\right) + \left[\partial^2\left(W^{-1}\right)\right]\widehat{\Sigma} + \\
&+ \left[\partial\left(W^{-1}\right)\right]\left(2\partial\widehat{\Sigma} - \widehat{\Lambda}\widehat{\Sigma}\right),
\end{aligned} \tag{2.34}
$$

[9] For a general treatment of the n_V-moduli case, see [45].

where the following holomorphic functions have been introduced (recall the first two Eqs. of (2.27)):

$$\widehat{\Lambda}(\psi) \equiv 2\partial_\psi \widehat{K}_\psi(\psi) - \widehat{\Gamma}_{\psi\psi}{}^{\psi}(\psi) = -2\partial_\psi^2\left[ln\left(X^0(\psi)\right)\right] - \partial_\psi\left[ln\left(e(\psi)\right)\right]; \tag{2.35}$$

$$\begin{aligned} \widehat{\Sigma}(\psi) &\equiv \partial_\psi^2 \widehat{K}_\psi(\psi) + \left[\partial_\psi \widehat{K}_\psi(\psi)\right]^2 - \widehat{\Gamma}_{\psi\psi}{}^{\psi}(\psi)\,\partial_\psi \widehat{K}_\psi(\psi) = \\ &= -\partial_\psi^3\left[ln\left(X^0(\psi)\right)\right] + \left\{\partial_\psi^2\left[ln\left(X^0(\psi)\right)\right]\right\}^2 + \partial_\psi\left[ln\left(e(\psi)\right)\right]\partial_\psi^2\left[ln\left(X^0(\psi)\right)\right]. \end{aligned} \tag{2.36}$$

The definitions (2.34) entail the following differential relations between the functional coeffiecients of 1-modulus PF Eq. (2.33):

$$\begin{aligned} a_3(\psi) &= 2\partial_\psi a_4(\psi); \\ a_1(\psi) &= \partial_\psi\left[a_2(\psi) - \tfrac{1}{2}\partial_\psi a_3(\psi)\right]. \end{aligned} \tag{2.37}$$

In special coordinates (with $X^0 = 1$) one has $X^1(\psi) = t^1(\psi) = \psi$ and $e(\psi) = 1$, and the $a_n(\psi)$s simplify drastically[10]:

$$\begin{aligned} a_{4,special}(\psi) &\equiv W^{-1}(\psi); \\ a_{3,special}(\psi) &\equiv 2\partial_\psi\left[W^{-1}(\psi)\right]; \\ a_{2,special}(\psi) &\equiv \partial_\psi^2\left[W^{-1}(\psi)\right]; \\ a_{1,special}(\psi) &\equiv a_{0,special}(\psi) = 0. \end{aligned} \tag{2.38}$$

It is interesting to notice that not all the $a_n(\psi)$s are actually relevant. Firstly, one can scale $a_4(\psi)$ out from the 1-modulus PF Eq. (2.33), and furthermore drop the coefficient proportional to $a_3(\psi)$ by performing the following rescaling redefinition of $V_h(\psi)$ [45]:

$$V_h(\psi) \longrightarrow V_h(\psi)\, exp\left[-\frac{1}{4}\int^{\psi} d\psi' \frac{a_3(\psi')}{a_4(\psi')}\right]. \tag{2.39}$$

By doing this, the PF Eq. (2.33) can be recast in the following form:

$$\mathcal{D}_\psi V_h(\psi) \equiv \left[\partial_\psi^4 + c_2(\psi)\,\partial_\psi^2 + c_1(\psi)\,\partial_\psi + c_0(\psi)\right] V_h(\psi) = 0, \tag{2.40}$$

where the new functional coefficients $c_n(\psi)$s are (rather complicated) combinations of the $a_n(\psi)$s and their derivatives. Notice that, due to the redefinition (2.39), $c_3(\psi) = 0$.

As shown in [46], the basic, defining differential relations (2.2) of SK geometry can be recast as a vanishing condition for a suitably defined flat symplectic non-holomorphic connection. Analogously, the holomorphic differential Eqs. (2.14) can be rewritten as a vanishing condition for a suitably defined flat holomorphic connection, *i.e.* as [45]

$$\left(\mathbb{I}_{2n_V+2}\partial_\alpha - \mathbf{A}_\alpha(z)\right)\mathbf{V}_h(z) = 0, \tag{2.41}$$

[10] However, in the following treatment of Fermat CY_3s $t^1(\psi)$ is *not* a special coordinate, *i.e.* $t^1(\psi) \neq \psi$.

where $\mathbf{V}_h(z)$ is a $(2n_V+2)\times(2n_V+2)$ holomorphic matrix ($(2n_V+2)\times 1$ vector with $1\times(2n_V+2)$ vector entries) defined as follows [45]:

$$\mathbf{V}_h(z) \equiv \begin{pmatrix} V_h(z) \\ V_{h,\beta}(z) \\ V_h^{\beta}(z) \\ V_h^0(z) \end{pmatrix}, \tag{2.42}$$

where the entries are defined in Eqs. (2.15). On the other hand, $\mathbf{A}_\alpha(z)$ is the following $(2n_V+2)\times(2n_V+2)$ holomorphic connection matrix:

$$\mathbf{A}_\alpha(z) \equiv \begin{pmatrix} -\widehat{K}_\alpha(z) & \delta_\alpha^\gamma & 0 & 0 \\ 0 & \left(\widehat{\Gamma}_\alpha(z) - \widehat{K}_\alpha(z)\,\mathbb{I}_{n_V}\right)_\beta^{\ \gamma} & (W_\alpha)_{\gamma\beta}(z) & 0 \\ 0 & 0 & \left(\widehat{K}_\alpha(z)\,\mathbb{I}_{n_V} - \widehat{\Gamma}_\alpha(z)\right)_\gamma^{\ \beta} & \delta_\alpha^\beta \\ 0 & 0 & 0 & \widehat{K}_\alpha(z) \end{pmatrix}. \tag{2.43}$$

It should be noticed that $\mathbf{A}_\alpha(z)$ is Lie-algebra valued in $\mathfrak{sp}(2n_V+2)$, *i.e.* it satisfies the infinitesimal symplecticity condition [45]

$$\mathbf{A}_\alpha^T(z)\,Q + Q\mathbf{A}_\alpha(z) = 0, \tag{2.44}$$

where Q is the symplectic metric defined in Eq. (2.4).

Put another way, it can be stated that the PF Eqs. (2.14) are equivalent to the matrix system (2.41), with $\mathbf{A}_\alpha(z)$ defined by Eq. (2.43). The general solution of such an holomorphic matrix system is given by Eqs. (2.15) arranged as a vector as given by Eq. (2.42).

As expected, by specializing the holomorphic matrix system (2.41) in special coordinates and choosing the Kähler gauge to be such that $X^0=1$, one gets the following holomorphic matrix system:

$$\left(\mathbb{I}_{2n_V+2}\partial_a - \mathbf{A}_{a,special}(z)\right)\mathbf{V}_{h,special}(z) = 0, \tag{2.45}$$

which is equivalent to the holomorphic system (2.13).

$\mathbf{V}_{h,special}(z)$ is a $(2n_V+2)\times(2n_V+2)$ holomorphic matrix ($(2n_V+2)\times 1$ vector with $1\times(2n_V+2)$ vector entries) defined as follows [45]:

$$\mathbf{V}_{h,special}(z) \equiv \begin{pmatrix} V_{h,special}(z) \\ V_{h,special,b}(z) \\ V_{h,special}^{b}(z) \\ V_{h,special}^0(z) \end{pmatrix}, \tag{2.46}$$

where the entries are defined in Eqs. (2.6), (2.8), (2.11) and (2.12). It is worth mentioning that the matrices $\mathbf{V}_{h,special}(z)$ and $\mathbf{V}_h(z)$ have a symplectic structure with respect to the symplectic metric relevant for holomorphic geometry, *i.e.* with respect to Q defined in Eq. (2.4):

$$\begin{aligned}\mathbf{V}^T_{h,special}(z)\,Q\mathbf{V}_{h,special}(z) &= Q;\\ \mathbf{V}^T_h(z)\,Q\mathbf{V}_h(z) &= Q.\end{aligned} \tag{2.47}$$

$\mathbf{A}_{a,special}(z)$ (named $\mathbb{C}_a$ in Eq. (3.6) of the first Ref. of [45]) is the $(2n_V+2)\times(2n_V+2)$ holomorphic connection matrix obtained by $\mathbf{A}_\alpha(z)$ (given by Eq. (2.43)) by putting $\widehat{\Gamma}_\alpha(z) = 0 = \widehat{K}_\alpha(z)$ (also recalling that in special coordinates a-indices and α-indinces coincide). Clearly, as its "holomorphically covariant" counterpart $\mathbf{A}_\alpha(z)$, clearly also $\mathbf{A}_{a,special}(z)$ is Lie-algebra valued in $\mathfrak{sp}(2n_V+2)$, and therefore it satisfies a corresponding infinitesimal symplecticity condition.

In other words, it can be stated that the the holomorphic system (2.13) can be recast in the matrix form (2.45), with $\mathbf{A}_{a,special}(z)$ defined by Eq. (3.6) of the first Ref. of [45]. The general solution of such an holomorphic matrix system is given by Eqs. (2.6), (2.8), (2.11) and (2.12) arranged as a vector as given by Eq. (2.46).

Once again, by considering the 1-modulus case more in detail, one obtains a major simplification. The 1-modulus PF Eq. (2.33) can be rewritten in matrix form as follows:

$$(\mathbb{I}_4\partial_\psi - \mathbf{A}_\psi(\psi))\,\mathbf{V}_h(\psi) = 0, \tag{2.48}$$

where $\mathbf{V}_h(\psi)$ is a 4×4 holomorphic matrix, corresponding to $n_V = 1$ in Eq. (2.42):

$$\mathbf{V}_h(\psi) \equiv \begin{pmatrix} V_h(\psi) \\ V_{h,\psi}(\psi) \\ V_h^{\psi}\psi \\ V_h^0(\psi) \end{pmatrix} = \tag{2.49}$$

$$= \begin{pmatrix} X^0(\psi) & X^1(\psi) & X^0(\psi)e(\psi)\partial_\psi F(\psi) & X^1(\psi)e(\psi)\partial_\psi F(\psi) - 2X^0(\psi)F(\psi) \\ 0 & X^0(\psi)e(\psi) & X^0(\psi)e(\psi)\widehat{D}_\psi\partial_\psi F(\psi) & -X^0(\psi)\partial_\psi F(\psi) + X^1(\psi)e(\psi)\widehat{D}_\psi\partial_\psi F(\psi) \\ 0 & 0 & \left(X^0(\psi)\right)^{-1}e(\psi) & \left(X^0(\psi)\right)^{-2}X^1(\psi)e(\psi) \\ 0 & 0 & 0 & \left(X^0(\psi)\right)^{-1} \end{pmatrix}, \tag{2.50}$$

where the second row can be further elaborated by making use of Eq (2.28) (recall that the holomorphic prepotential F has Kähler weights $(4,0)$):

$$\widehat{D}_\psi\partial_\psi F(\psi) = \left\{\partial_\psi - 2\partial_\psi\left[ln\left(X^0(\psi)\right)\right] - \partial_\psi\left[ln\left(e(\psi)\right)\right]\right\}\partial_\psi F(\psi). \tag{2.51}$$

On the other hand, $\mathbf{A}_{\psi}(\psi)$ is a 4×4 holomorphic connection matrix, which is Lie-algebra valued in $\mathfrak{sp}(4)$ and corresponds to $n_V = 1$ in Eq. (2.43):

$$\mathbf{A}_{\psi}(\psi) = \\ = \begin{pmatrix} \partial_{\psi}\left[ln\left(X^{0}(\psi)\right)\right] & 1 & 0 & 0 \\ 0 & \begin{array}{c}\partial_{\psi}\left[ln\left(e(\psi)\right)\right]+ \\ +\partial_{\psi}\left[ln\left(X^{0}(\psi)\right)\right]\end{array} & W_{\psi\psi\psi}(\psi) & 0 \\ 0 & 0 & \begin{array}{c}-\partial_{\psi}\left[ln\left(e(\psi)\right)\right]+ \\ -\partial_{\psi}\left[ln\left(X^{0}(\psi)\right)\right]\end{array} & 1 \\ 0 & 0 & 0 & -\partial_{\psi}\left[ln\left(X^{0}(\psi)\right)\right] \end{pmatrix}, \tag{2.52}$$

where use of Eqs. (2.27) has been made (see also Eq. (2.30)).

It can be stated that the 1-modulus PF Eq. (2.33) is equivalent to the matrix system (2.48), with $\mathbf{A}_{\psi}(\psi)$ defined by Eq. (2.52). The general solution of such an holomorphic matrix system (which corresponds to the most general solution of the fourth order linear PF Eq. (2.52)) is given by Eqs. (2.49)-(2.50) (implemented by Eq. (2.51)).

Let us now further specialize our treatment to the 1-modulus SK geometries endowing the moduli space of Fermat CY_3s. As previously mentioned, the fourth order linear PF ordinary differential equation for each of the four threefolds (classified by the index $k = 5, 6, 8, 10$: see next Section) of such a class of CY_3s has been obtained for $k = 5$ in [34, 35] (see in particular Eq. (3.9) of [34], where $z \equiv \psi^{-5}$; see also [36]), and for $k = 6, 8, 10$ in [37] (see Eq. (3.1) of such a Ref., with notation $\alpha \equiv \psi$), where a unified, k-parametrized treatment was exploited. We will shortly review it in the next Section.

In order to recast the 1-modulus PF Eqs. given by Eq. (3.1) of [37] in the form (2.33) with the differential relations (2.37) between the $a_n(\psi)$s holding, one must multiply them by the function $\psi^{-\xi_k}$, with $\xi_k = 0, 3, 6, 8$ for $k = 5, 6, 8, 10$ respectively. By doing this, one achieves the result that for Fermat CY_3s the fourth order linear PF Eqs. (2.33) can be recast in the following k-parametrized form[11]:

$$\begin{gathered} \sum_{n=0}^{4} a_{n,k}(\psi)\,\partial^{n} V_h(\psi) = 0, \\ a_{n,k}(\psi) \equiv -\sigma_n \psi^{n+1} + (-1)^{n}\,\tau_{n,k}\psi^{n+1-k}, \end{gathered} \tag{2.53}$$

with the constants σ_ns and $\tau_{n,k}$s given by Tables 1 and 2:

3 General Analysis

In the present Section we will briefly present the formalism of one-modulus (mirror) Fermat Calabi-Yau threefolds (CY_3s). We will mainly follow [37], and cite where appropriate other relevant works. We also derive original formal

[11] As we will see in Sect. 9, for self-consistency reasons the 1-modulus PF Eqs. (2.53) (with Tables 1 and 2) (which are "corrected" by an overall factor $\psi^{-\xi_k}$ with respect to the ones given in Eq. (3.1) of [37]) need also to be further multiplied by a suitable "normalization" constant (see Eq. (9.9)).

n	σ_n
0	1
1	15
2	25
3	10
4	1

Table 1: Values of the integer constants σ_n

$k \longrightarrow$	5	6	8	10
$\tau_{0,k}$	0	0	0	0
$\tau_{1,k}$	0	0	15	35
$\tau_{2,k}$	0	2	15	35
$\tau_{3,k}$	0	2	6	10
$\tau_{4,k}$	1	1	1	1

Table 2: Values of the integer constants $\tau_{n,k}$

results, which will be then used in the case-by-case analysis of extremal BH LG attractors performed in next Sections.

Fermat CY_3s can be defined as the vanishing *locus* of quasi-homogeneous polynomials in 5 complex variables, of the general form:

$$\mathcal{W}_0 = \sum_{i=0}^{4} \nu_i \left(x^i\right)^{n_i} = 0; \tag{3.1}$$

such a *locus* gives the embedding of the considered CY_3 in a suitably weighted complex projective space $\mathbb{WCP}^4_{\nu_0,\nu_1,\nu_2,\nu_3,\nu_4}$. By imposing the defining conditions of vanishing first Chern class and of absence of singularities, it is possible to show that only four possible sets of $\{\nu_i, n_i\}_{i=0,1,2,3,4}$ exist, *all* corresponding to CY_3s with $h(1,1) = dim\left(H^{1,1}(CY_3)\right) = 1$ (*i.e.* only one Kähler modulus). The four existing Fermat CY_3s can be classified by introducing the *Fermat parameter* k, defined as the smallest common multiple of the n_is[12]; the only allowed values of k turn out to be $k = 5, 6, 8, 10$.

[12] k can equivalently be defined as the degree of $\mathcal{W}_0$. It turns out that $k = n_i\nu_i$ (no summation on i) $\forall i = 0,1,2,3,4$. Moreover, it also holds that $k = \sum_{i=0}^{4} \nu_i$.

k $\downarrow$	G	$Ord\,(G)$	$h\,(1,1)$	$h\,(2,1)$	$\varkappa \equiv 2\,[h\,(1,1) - h\,(2,1)]$
5	$(\mathbb{Z}_5)^3$	5^3	1	101	-200
6	$\mathbb{Z}_3 \otimes (\mathbb{Z}_6)^2$	$3 \cdot 6^2$	1	103	-204
8	$(\mathbb{Z}_8)^2 \otimes \mathbb{Z}_2$	$2 \cdot 8^2$	1	149	-296
10	$(\mathbb{Z}_{10})^2$	$1 \cdot 10^2$	1	145	-288

Table 3: **Basic topological data of Fermat CY_3s $\mathcal{M}_k$s**

Thus, the four existing Fermat CY_3s $\mathcal{M}_k$ are given by the following geometrical *loci*[13] [33, 34, 35, 37]:

$$
\begin{aligned}
&k=5: \textit{quintic}\ \mathcal{M}_5 = \left\{ x^i \in (\mathbb{W})\,\mathbb{CP}^4_{1,1,1,1,1} : \mathcal{W}_{0,5} = \textstyle\sum_{i=0}^4 \left(x^i\right)^5 = 0 \right\};\\
&k=6: \textit{sextic}\ \mathcal{M}_6 = \left\{ x^i \in \mathbb{WCP}^4_{2,1,1,1,1} : \mathcal{W}_{0,6} = 2\left(x^0\right)^3 + \textstyle\sum_{i=1}^4 \left(x^i\right)^6 = 0 \right\};\\
&k=8: \textit{octic}\ \mathcal{M}_8 = \left\{ x^i \in \mathbb{WCP}^4_{4,1,1,1,1} : \mathcal{W}_{0,8} = 4\left(x^0\right)^2 + \textstyle\sum_{i=1}^4 \left(x^i\right)^8 = 0 \right\};\\
&k=10: \textit{dectic}\ \mathcal{M}_{10} = \left\{ x^i \in \mathbb{WCP}^4_{5,2,1,1,1} : \mathcal{W}_{0,10} = 5\left(x^0\right)^2 + 2\left(x^1\right)^5 + \textstyle\sum_{i=2}^4 \left(x^i\right)^{10} = 0 \right\}.
\end{aligned} \tag{3.2}
$$

By orbifolding the $\mathcal{M}_k$s and quotienting by the full phase symmetry group G (see [37] and Refs. therein), one obtains a pair of Fermat CY_3s[14] $(\mathcal{M}_k, \mathcal{M}'_k)$ related by the so-called *mirror symmetry* [47, 48, 49, 34, 35], with $h(1,1)$ and $h(2,1) = dim\left(H^{2,1}\left(CY_3\right)\right)$ interchanged (and therefore opposite Euler number $\varkappa$). Correspondingly, the defining vanishing geometrical *loci* will be "deformed" as follows:

$$
\mathcal{W}_0 \longrightarrow \mathcal{W} \equiv \mathcal{W}_0 - k\psi \prod_{i=0}^4 x^i. \tag{3.3}
$$

All the relevant topological data of Fermat CY_3s $\mathcal{M}_k$s are given in Table 3.

[13]Here and below, we give a name to the Fermat CY_3s corresponding to the various possible values of the Fermat parameter. The Fermat CY_3 with $k=5$ has been named *quintic* some time ago (see *e.g.* [33, 36, 34, 35, 37]). In a similar fashion, by using the corresponding Latin cardinal adjectives, we name *sextic*, *octic*, and *dectic* the Fermat CY_3s with $k = 6, 8, 10$, respectively.

[14]For simplicity's sake, we denote in the same way the starting Fermat CY_3 and the one obtained by orbifolding and then quotienting by G.

ψ is the *Kähler deformation modulus* for Fermat CY_3s $\mathcal{M}_k$s (all having $h(1,1) = 1$) and the *complex structure deformation modulus* for the corresponding *mirror* Fermat CY_3s $\mathcal{M}'_k$s (all having $h(2,1) = 1$). Since in the treatment and computations performed below we will consider ψ as a complex structure modulus, we will be actually working in the *mirror description* of the considered CY_3s, *i.e.* we will be considering the *mirror* Fermat CY_3s $\mathcal{M}'_k$s ($k = 5, 6, 8, 10$).

In such a framework, the relevant quantities for the $d = 4$ low-energy effective Lagrangian of the $d = 10$ superstring theory compactified on $\mathcal{M}_k$ are given (within the complex structure moduli space ($dim_{\mathbb{C}} = 1$)) by the Kähler metric and Yukawa couplings on $\mathcal{M}'_k$ (related to $\mathcal{M}_k$ by mirror symmetry). All such quantities will be obtained by the solutions of the fourth order linear PF ordinary differential Eqs. (2.33).

Near the LG point $\psi = 0$, the 4×1 period vector[15] in the PF basis $\varpi_k(\psi)$ is obtained by solving the PF Eqs.[16]

$$\sum_{n=0}^{4} a_{n,k}(\psi)\,\partial^n \varpi_k(\psi) = 0. \tag{3.4}$$

Here we choose the normalization and the gauge of the holomorphic 3-form defined on $\mathcal{M}'_k$ such that[17]

$$\varpi_k(\psi) \equiv -\frac{1}{\psi}\frac{(2\pi i)^3}{Ord(G_k)} \begin{pmatrix} \omega_{2,k}(\psi) \\ \omega_{1,k}(\psi) \\ \omega_{0,k}(\psi) \\ \omega_{k-1,k}(\psi) \end{pmatrix}, \tag{3.5}$$

with

$$\begin{aligned} &\omega_{j,k}(\psi) \equiv \omega_{0,k}(\beta_k^{2j}\psi), \\ &\beta_k \equiv exp(\tfrac{\pi i}{k}) \end{aligned} \tag{3.6}$$

[15] Once again, in order to make the contact with the relevant literature easier, in this Section as well as in the next ones, we will reconsider column (*i.e.* 4×1), rather than row (*i.e.* 1×4), period vectors. Moreover, the holomorphic period vector in the symplectic basis (hitherto named $V_h(\psi)$) will be henceforth denoted by $\Pi(\psi)$.

[16] When comparing Eq. (3.4) to Eq. (2.33) (and, more in general, considering the treatment given in Sect. 2), the 4×1 symplectic holomorphic period vector $V_h(\psi) \equiv \Pi(\psi)$ and the 4×1 PF holomorphic period vector $\varpi(\psi)$ turn out to satisfy the same fourth order linear ordinary differential equation.

Consequently, they necessarily have to be related by a *global* (*i.e.* ψ*-independent*) "rotation" in the moduli space. This is precisely what happens, with such a "rotation" in the moduli space expressed by the 4×4 real matrices M_ks given in Eqs. (3.15)-(3.16) (see Sect. 4 of [37]).

[17] The normalization of $\varpi_k(\psi)$ adopted in the present work is the same of [34, 35, 37], and it differs from the one adopted in (a part of the) literature on flux compactifications (see *e.g.* Subsect. 3.2 of [50]) by a factor $\frac{1}{Ord(G_k)}$ (the reason is that we are interested in the mirror manifolds $\mathcal{M}'_k$s, not in $\mathcal{M}_k$s).

On the other hand, it is easy to realize that the gauge of the holomorphic 3-form adopted in [34, 35, 37] is mostly convenient in order to study the large complex structure modulus limit. Since we will investigate the LG limit, for our purposes it is better to adopt the gauge of [50], which amounts to performing the following gauge transformation on the holomorphic 3-form:

$$\Xi(\psi) \longrightarrow \frac{1}{\psi}\Xi(\psi),$$

in turn corresponding to the following transformation of the Kähler potential:

$$\partial_\psi K \longrightarrow \partial_\psi K + ln\left(|\psi|^2\right).$$

all being solutions of Eq. (3.4) ($j = 0, 1, ..., k-1$). However, since Eq. (3.4) is a fourth order (linear) differential equation, only 4 linearly independent solutions $\omega_{j,k}(\psi)$s exist. Thus, $\forall k = 5, 6, 8, 10$, $k-4$ linear relations between the $\omega_{j,k}(\psi)$s hold. One possible choice is the following one [34, 35, 37]:

$$k = 5 : \ \sum_{j=0}^{4} \omega_{j,5}(\psi) = 0. \tag{3.7}$$

$$k = 6 : \begin{cases} \omega_{0,6}(\psi) + \omega_{2,6}(\psi) + \omega_{4,6}(\psi) = 0; \\ \\ \omega_{1,6}(\psi) + \omega_{3,6}(\psi) + \omega_{5,6}(\psi) = 0. \end{cases} \tag{3.8}$$

$$k = 8 : \omega_{i,8}(\psi) + \omega_{i+4,8}(\psi) = 0, \quad i = 0, 1, 2, 3. \tag{3.9}$$

$$k = 10 : \begin{cases} \omega_{i,10}(\psi) + \omega_{i+5,10}(\psi) = 0, \quad i = 0, 1, 2, 3, 4; \\ \\ \omega_{0,10}(\psi) + \omega_{2,10}(\psi) + \omega_{3,10}(\psi) + \omega_{4,10}(\psi) + \omega_{5,10}(\psi) = 0. \end{cases} \tag{3.10}$$

The defining Eq. (3.5) expresses the usual conventions, in which one takes $\omega_{0,k}(\psi)$, $\omega_{1,k}(\psi)$, $\omega_{2,k}(\psi)$ and $\omega_{k-1,k}(\psi)$ as basis for $\varpi_k(\psi)$. Therefore, due to relations (3.6), the key quantity turns out to be the holomorphic function, whose series expansion (convergent for $|\psi| < 1$, with the *fundamental region* [34, 35, 37] selected by $0 \leqslant arg(\psi) < \frac{2\pi}{k}$) reads [37]

$$\omega_{0,k}(\psi) = -\sum_{m=1}^{\infty} C_{k,m-1} \beta_k^{(k-1)m} \psi^m, \tag{3.11}$$

with

$$\begin{aligned} C_{k,m-1} &\equiv \frac{\Gamma(\frac{m}{k})\Gamma(1-\frac{m}{k})k^{m-1}}{\Gamma(m)\Pi_{i=0}^{4}\Gamma(1-\frac{m}{k}\nu_{i,k})} \gamma_k^m, \\ \\ \gamma_k &\equiv \Pi_{i=0}^{4} (\nu_{i,k})^{-\nu_{i,k}/k}, \end{aligned} \tag{3.12}$$

where in $C_{k,m-1}$ Γ denotes the Euler gamma function $\Gamma(s) \equiv \int_0^\infty t^{s-1} e^{-t} dt$ (with $Re(s) > 0$). By using Eqs. (3.11)-(3.12), the series expansion (convergent for $|\psi| < 1$, $0 \leqslant arg(\psi) < \frac{2\pi}{k}$) of $\varpi_k(\psi)$ can be written as follows:

$$\varpi_k(\psi) = -\frac{(2\pi i)^3}{Ord(G)} \sum_{m=1}^{\infty} (-1)^m C_{k,m-1} \psi^{m-1} \begin{pmatrix} \beta_k^{3m} \\ \\ \beta_k^{m} \\ \\ \beta_k^{-m} \\ \\ \beta_k^{-3m} \end{pmatrix}. \tag{3.13}$$

The change between the PF basis and the symplectic basis for holomorphic 4×1 period vector is given by:

$$\Pi_k(\psi) = M_k \varpi_k(\psi). \tag{3.14}$$

where the 4×4 constant matrices M_k read [34, 35, 37]

$$M_5 = \begin{pmatrix} -\frac{3}{5} & -\frac{1}{5} & \frac{21}{5} & \frac{8}{5} \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 8 & 3 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{3.15}$$

$$M_8 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad M_{10} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}. \tag{3.16}$$

The Kähler potential is given by:

$$K_k\left(\psi, \overline{\psi}\right) = -ln\left[-i\,\Pi_k^{\dagger}\left(\overline{\psi}\right)\Sigma\Pi_k\left(\psi\right)\right] = -ln\left[-i\,\overline{\varpi}_k^{\dagger}\left(\overline{\psi}\right)m_k\varpi_k\left(\psi\right)\right], \tag{3.17}$$

where[18]

$$\Sigma \equiv \begin{pmatrix} 0_2 & \mathbb{I}_2 \\ & \\ -\mathbb{I}_2 & 0_2 \end{pmatrix}; \tag{3.18}$$

$$m_k \equiv M_k^{\dagger}\Sigma M_k = \frac{1}{\chi_k}\begin{pmatrix} 0 & -1 & -\lambda_k & -1 \\ 1 & 0 & -\varsigma_k & -\lambda_k \\ \lambda_k & \varsigma_k & 0 & -1 \\ 1 & \lambda_k & 1 & 0 \end{pmatrix}, \tag{3.19}$$

with the values of χ_k, λ_k and ς_k given in Table 4. By recalling the third column from the left of Table 3, one can observe that $Ord\left(G_k\right) = \chi_k k^2$. Substituting Eq. (3.13) and definition (3.19) into Eq. (3.17), one obtains the series expansion (converging for $|\psi| < 1$, $0 \leqslant arg\left(\psi\right) < \frac{2\pi}{k}$) of the Kähler potential:

$$K_k\left(\psi, \overline{\psi}\right) = -ln\left(\frac{(2\pi)^6}{(Ord(G_k))^2}\sum_{m,n=1}^{\infty} C_{k,m-1}C_{k,n-1}\psi^{m-1}\overline{\psi}^{n-1}F_{k,mn}\right), \tag{3.20}$$

where the following infinite rank-2 tensor has been introduced:

$$\begin{aligned} F_{k,m\,n} &\equiv i(-1)^{m+n+1}\begin{pmatrix} \beta_k^{-3n}, & \beta_k^{-n}, & \beta_k^{n}, & \beta_k^{3n} \end{pmatrix} m_k \begin{pmatrix} \beta_k^{3m} \\ \beta_k^{m} \\ \beta_k^{-m} \\ \beta_k^{-3m} \end{pmatrix} = \\ &= \frac{2}{\chi_k}e^{i(m+n)\pi}\left\{sin(\tfrac{3n-m}{k}\pi) + sin(\tfrac{3m-n}{k}\pi) + sin(\tfrac{3(n+m)}{k}\pi) + \right. \\ &\left. + \varsigma_k sin(\tfrac{n+m}{k}\pi) + \lambda_k\left[sin(\tfrac{3n+m}{k}\pi) + sin(\tfrac{3m+n}{k}\pi)\right]\right\}. \end{aligned} \tag{3.21}$$

[18] Note the change of convention with respect to (the case $n_V = 1$ of) the defining Eq. (1.9): $\Sigma = -\left.\Omega\right|_{n_V=1}$.

$k \downarrow$	χ_k	λ_k	ς_k
5	5	3	3
6	3	2	0
8	2	2	1
10	1	1	-1

Table 4: **Values of the integer constants χ_k, λ_k and ς_k**

From such a definition, $F_{k,mn}$ turns out to have the following relevant properties:

$$\begin{aligned}
&\overline{F_{k,m\,n}} = F_{k,m\,n};\\
&F_{k,m\,n} = F_{k,n\,m};\\
&F_{k,m+k\ \ n} = F_{k,m\ \ n+k} = (-1)^{k+1}F_{k,m\,n};\\
&F_{k,m\,n} = 0 \text{ if } n+m=k;\\
&F_{k,kk} = 0.
\end{aligned} \tag{3.22}$$

Consequently, at most only $\frac{k(k+1)}{2} - \left[\frac{k}{2}\right] - 1$ (real) non-vanishing independent elements of $F_{k,m\,n}$ exist (where $\left[\frac{k}{2}\right]$ denotes the integer part of $\frac{k}{2}$), even though, as evident from Eqs. (4.2), (5.2), (6.2) and (7.2) below, actually such an upper bound is never reached for the allowed values of the Fermat parameter $k = 5, 6, 8, 10$.

The holomorphic superpotential (also named $\mathcal{N}=2$ holomorphic central charge function) is given by:

$$W_k\left(\psi; q, p\right) = \Gamma\Pi_k\left(\psi\right), \tag{3.23}$$

where the 1×4 BH charge vector in the symplectic basis is here defined as[19]

$$\begin{aligned}
&\Gamma \equiv \left(-p^0,\ -p^1,\ q_0,\ q_1\right) = \widetilde{\Gamma}\Sigma,\\
&\widetilde{\Gamma} \equiv \left(q_0,\ q_1, p^0,\ p^1\right).
\end{aligned} \tag{3.24}$$

Using Eqs. (3.13), (3.14) and (3.19), one can obtain the following series expansion (convergent for $|\psi| < 1$, $0 \leqslant$

[19]Notice the change in the notation of the symplectic charge vectors with respect to the notation used in Sects. 1 and 2.

$arg\left(\psi\right) < \frac{2\pi}{k}$) of the holomorphic superpotential:

$$W_k\left(\psi; q, p\right) = A_k \sum_{m=1}^{\infty} C_{k,m-1}\psi^{m-1} N_{k,m}\left(q, p\right), \tag{3.25}$$

where the following quantities have been introduced:

$$A_k \equiv -\tfrac{1}{\chi_k}\tfrac{(2\pi i)^3}{Ord(G_k)}; \tag{3.26}$$

$$N_{k,m}\left(q,p\right) \equiv (-1)^m \left[n_{k,1}\left(q,p\right)\beta_k^{3m} + n_{k,2}\left(q,p\right)\beta_k^{m} + n_{k,3}\left(q,p\right)\beta_k^{-m} + n_{k,4}\left(q,p\right)\beta_k^{-3m}\right], \tag{3.27}$$

where

$$n_k\left(q,p\right) \equiv \chi_k \Gamma M_k \in \mathbb{Z}^4 \tag{3.28}$$

is the 1×4 BH charge vector in the PF basis.

By inverting the definition (3.28), one obtains

$$\Gamma = \frac{1}{\chi_k} n_k\left(q,p\right) M_k^{-1} \in \mathbb{Z}^4, \tag{3.29}$$

where Γ is defined in Eq. (3.24). Eqs. (3.28)-(3.29) express the change between the symplectic and PF basis of BH charges.

By recalling Eq. (1.24) and using Eqs. (3.20) and (3.25), the general form of the "effective BH potential" function $V_{BH,k}\left(\psi,\overline{\psi};q,p\right)$ for the Calabi-Yau threefolds $\mathcal{M}_k'$s reads

$$\begin{aligned} V_{BH,k}\left(\psi,\overline{\psi};q,p\right) &= \frac{1}{\chi_k^2 F_{k,11}}\left[exp\left[\widetilde{K}_k\left(\psi,\overline{\psi}\right)\right]\right]\left[\left|\widetilde{W}_k\right|^2\left(\psi,\overline{\psi};q,p\right) + \left(g_{\psi\bar{\psi},k}\left(\psi,\overline{\psi}\right)\right)^{-1}\left|D_\psi \widetilde{W}_k\right|^2\left(\psi,\overline{\psi};q,p\right)\right] \equiv \\ &\equiv \frac{1}{\chi_k^2 F_{k,11}}\widetilde{V}_{BH,k}\left(\psi,\overline{\psi};q,p\right), \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} &\widetilde{K}_k\left(\psi,\overline{\psi}\right) \equiv K_k\left(\psi,\overline{\psi}\right) + ln\left[\tfrac{(2\pi)^6}{(Ord(G_k))^2} C_{k,0}^2 F_{k,11}\right]; \\ &\widetilde{W}_k\left(\psi;q,p\right) \equiv \tfrac{W_k(\psi;q,p)}{A_k C_{k,0}}. \end{aligned} \tag{3.31}$$

A remark worth making concerns the holomorphic prepotential $F\left(X\left(\psi\right)\right)$. In the treatment of 1-modulus SK geometry underlying the moduli space of Fermat CY_3-compactifications, we will assume it to exist. By specializing Eq. (2.7) for $n_V = 1$, one achieves:

$$F\left(X\left(\psi\right)\right) = \frac{1}{2}\left[F_0\left(\psi\right)X^0\left(\psi\right) + F_1\left(\psi\right)X^1\left(\psi\right)\right] = \frac{1}{2}\left[\Pi^1\left(\psi\right)\Pi^3\left(\psi\right) + \Pi^2\left(\psi\right)\Pi^4\left(\psi\right)\right], \tag{3.32}$$

where $\Pi^i\left(\psi\right)$ denotes the i-th component ($i = 1,2,3,4$) of the 4×1 symplectic holomorphic period vector $\Pi_{(k)}\left(\psi\right)$. Consequently, by using Eqs. (3.13) and (3.14), $F\left(X\left(\psi\right)\right)$ can be explicitly computed in power series expansion (convergent for $|\psi| < 1$, $0 \leqslant arg\left(\psi\right) < \frac{2\pi}{k}$) for the k-parametrized class of Fermat CY_3s.

Let us now consider the (k-indexed) $n_V = 1$ case of Eqs. (1.6), corresponding to the (k-indexed) 1-modulus AEs (without explicit use of $C_{\psi\psi\psi}$, in which case one would obtain Eq. (1.23)). By recalling the last step in Eq. (1.8), and considering that the SK geometry is assumed to be *regular* (*i.e.* with $\left|K_k\left(\psi,\overline{\psi}\right)\right| < \infty$ everywhere), one obtains the 1-modulus AEs in terms of the superpotential and its covariant derivatives:

$$2\overline{W}_k D_\psi W_k + \left(g_{\psi\bar{\psi},k}\right)^{-1}\left(D_\psi D_\psi W_k\right)\overline{D}_{\overline{\psi}}\overline{W}_k = 0, \tag{3.33}$$

where ($\Gamma_{\psi\psi,k}^{\ \ \ \ \psi} = g_k^{\psi\overline{\psi}}\partial_\psi g_{\psi\overline{\psi},k} = \partial_\psi ln\left(g_{\psi\overline{\psi},k}\right)$, and recall that W_k has Kähler weights $(2,0)$)

$$D_\psi W_k = \left(\partial_\psi + \partial_\psi K_k\right)W_k; \tag{3.34}$$

$$\begin{aligned}
&D_\psi D_\psi W_k = \left(\partial_\psi + \partial_\psi K_k\right)D_\psi W_k - \Gamma_{\psi\psi,k}^{\ \ \ \ \psi}D_\psi W_k = \\
&= \left(\partial_\psi + \partial_\psi K_k\right)\left(\partial_\psi + \partial_\psi K_k\right)W_k - \partial_\psi ln\left(g_{\psi\overline{\psi},k}\right)D_\psi W_k = \\
&= \left[\partial_\psi^2 + \partial_\psi^2 K_k + 2\partial_\psi K_k\partial_\psi + \left(\partial_\psi K_k\right)^2 - \partial_\psi ln\left(\partial_\psi\overline{\partial}_{\overline{\psi}}K_k\right)\left(\partial_\psi + \partial_\psi K_k\right)\right]W_k.
\end{aligned} \tag{3.35}$$

In the next Sects. 4-7 we will consistently solve the 1-modulus AEs (3.33) (with covariant derivatives given by Eqs. (3.34)-(3.35)) near the LG point $\psi = 0$, using all the formal machinery elaborated above in the framework of 1-modulus SK geometry underlying the moduli space of Fermat CY_3-compactifications. In other words, we will solve the criticality condition for the "effective BH potential" (3.30) near the LG point $\psi = 0$, obtaining the constraints which define the BH charge configurations supporting the LG point to be a critical point of $V_{BH,k}$ given by Eq. (3.30). Furthermore, we will address the issue of the stability, by inspecting the real form of the 2×2 Hessian matrix.

We will exploit such a procedure for each of the mirror Fermat CY_3s $\mathcal{M}_k'$s, classified by the values of the Fermat parameter $k = 5, 6, 8, 10$.

4 $k = 5$: Mirror *Quintic*

In the case of mirror *quintic* $\mathcal{M}_5'$ it is easy to realize that one has to consider the "effective BH potential" (3.30) (at least) up to $\mathcal{O}\left(\psi^3\right)$ (or, as always understood below, $\mathcal{O}\left(\overline{\psi}^3\right)$). As a consequence, the AEs (3.33) and the Hessian matrix will be known up to $\mathcal{O}\left(\psi\right)$.

For $k = 5$ the definitions (3.12) yield

$$C_{5,5l-1} = 0, \quad l \in \mathbb{N}; \tag{4.1}$$

moreover, since $F_{5,m+5,\,n} = F_{5,m\,,n+5} = F_{5,m\,n}$ (see the third of properties (3.22)), the only independent elements of the rank-2 tensor F_5 belong to the 5×5 matrix

$$F_{5,mn} = \left(\begin{array}{ccccc} \sqrt{5+2\sqrt{5}} & 0 & 0 & 0 & -\sqrt{5+2\sqrt{5}} \\ 0 & -\sqrt{5-2\sqrt{5}} & 0 & 0 & \sqrt{5-2\sqrt{5}} \\ 0 & 0 & \sqrt{5-2\sqrt{5}} & 0 & -\sqrt{5-2\sqrt{5}} \\ 0 & 0 & 0 & -\sqrt{5+2\sqrt{5}} & \sqrt{5+2\sqrt{5}} \\ -\sqrt{5+2\sqrt{5}} & \sqrt{5-2\sqrt{5}} & -\sqrt{5-2\sqrt{5}} & \sqrt{5+2\sqrt{5}} & 0 \end{array} \right). \tag{4.2}$$

Let us now write down all the relevant quantities up to the needed order (here and below, unless otherwise specified, we omit the Fermat parameter $k = 5$):

$$\widetilde{K} \approx (\sqrt{5}-2)\tfrac{C_1^2}{C_0^2}\left[\psi\bar{\psi} - \left(\tfrac{C_2^2}{C_1^2} - \tfrac{(\sqrt{5}-2)}{2}\tfrac{C_1^2}{C_0^2}\right)(\psi\bar{\psi})^2 + \tfrac{C_5 C_0}{C_1^2}(\sqrt{5}+2)(\psi^5+\bar{\psi}^5)\right] + \mathcal{O}(\psi^6); \tag{4.3}$$

$$g_{\psi\bar{\psi}} \approx (\sqrt{5}-2)\tfrac{C_1^2}{C_0^2}\left[1 - 4\left(\tfrac{C_2^2}{C_1^2} - \tfrac{(\sqrt{5}-2)}{2}\tfrac{C_1^2}{C_0^2}\right)\psi\bar{\psi}\right] + \mathcal{O}(\psi^4); \tag{4.4}$$

$$\widetilde{W} \approx N_1 + \tfrac{C_1}{C_0}N_2\psi + \tfrac{C_2}{C_0}\bar{N}_2\psi^2 + \tfrac{C_3}{C_0}\bar{N}_1\psi^3 + \mathcal{O}(\psi^5). \tag{4.5}$$

Now, by using the formulæ of the general analysis exploited in Sect. 3, we can get the "effective BH potential" and the holomorphic superpotential, as well as their (covariant) derivatives, up to $\mathcal{O}(\psi)$ (notice that in all the treatments of Sects. 4-7 we are interested only in ordinary derivatives of $\widetilde{V}_{BH}$, since they coincide with the covariant ones at the critical points of $\widetilde{V}_{BH}$):

$$\widetilde{W} = N_1 + \tfrac{C_1}{C_0}N_2\psi; \tag{4.6}$$

$$D_\psi\widetilde{W} = \tfrac{C_1}{C_0}\left[N_2 + 2\tfrac{C_2}{C_1}\bar{N}_2\psi + \tfrac{C_1}{C_0}(\sqrt{5}-2)N_1\bar{\psi}\right]; \tag{4.7}$$

$$D_\psi D_\psi\widetilde{W} = 2\tfrac{C_2}{C_0}\bar{N}_2 + 6\tfrac{C_3}{C_0}\bar{N}_1\psi + 4\tfrac{C_2^2}{C_0 C_1}N_2\bar{\psi}, \tag{4.8}$$

$$\begin{aligned}\widetilde{V}_{BH} &= |N_1|^2 + (\sqrt{5}+2)|N_2|^2 + 2\tfrac{C_1}{C_0}\left[N_2\bar{N}_1 + (\sqrt{5}+2)\tfrac{C_2C_0}{C_1^2}(\bar{N}_2)^2\right]\psi + \\ &\quad +2\tfrac{C_1}{C_0}\left[\bar{N}_2 N_1 + (\sqrt{5}+2)\tfrac{C_2C_0}{C_1^2}(N_2)^2\right]\bar{\psi};\end{aligned} \tag{4.9}$$

$$\begin{aligned}\partial_\psi\widetilde{V}_{BH} &= 2\tfrac{C_1}{C_0}\left[N_2\bar{N}_1 + (\sqrt{5}+2)\tfrac{C_2C_0}{C_1^2}(\bar{N}_2)^2\right] + 6\tfrac{C_2}{C_0}\left[1 + (\sqrt{5}+2)\tfrac{C_3C_0}{C_2C_1}\right]\bar{N}_1\bar{N}_2\psi + \\ &\quad +2\tfrac{C_1^2}{C_0^2}\left[|N_1|^2(\sqrt{5}-2) + |N_2|^2\left(1 + 4(\sqrt{5}+2)\tfrac{C_0^2C_2^2}{C_1^4}\right)\right]\bar{\psi};\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
&\partial_\psi \partial_\psi \widetilde{V}_{BH} = 6\tfrac{C_2}{C_0}\left[1+(\sqrt{5}+2)\tfrac{C_3 C_0}{C_2 C_1}\right]\bar{N}_1\bar{N}_2 + \\
&\quad +24\tfrac{C_3}{C_0}(\bar{N}_1)^2\psi + 4\tfrac{C_1}{C_0}\left[\tfrac{C_2}{C_0}\left(1+4(\sqrt{5}+2)\tfrac{C_0^2 C_2^2}{C_1^4}\right)(\bar{N}_2)^2 + \right. \\
&\quad \left. +(\sqrt{5}-2)\tfrac{C_1^2}{C_0^2}\left(1+(\sqrt{5}+2)\tfrac{C_0^2C_2^2}{C_1^4}+3(\sqrt{5}+2)^2\tfrac{C_0^3 C_2 C_3}{C_1^5}\right)\bar{N}_1 N_2\right]\bar{\psi};
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
&\partial_\psi \overline{\partial}_{\bar{\psi}} \widetilde{V}_{BH} = 2\tfrac{C_1^2}{C_0^2}\left[|N_1|^2(\sqrt{5}-2)+|N_2|^2\left(1+4(\sqrt{5}+2)\tfrac{C_0^2C_2^2}{C_1^4}\right)\right] + \\
&\quad +4\tfrac{C_1}{C_0}\left[\tfrac{C_2}{C_0}\left(1+4(\sqrt{5}+2)\tfrac{C_0^2C_2^2}{C_1^4}\right)(\bar{N}_2)^2 + \right. \\
&\quad \left. +(\sqrt{5}-2)\tfrac{C_1^2}{C_0^2}\left(1+(\sqrt{5}+2)\tfrac{C_0^2C_2^2}{C_1^4}+3(\sqrt{5}+2)^2\tfrac{C_0^3C_2C_3}{C_1^5}\right)\bar{N}_1 N_2\right]\psi + \\
&\quad +4\tfrac{C_1}{C_0}\left[\tfrac{C_2}{C_0}\left(1+4(\sqrt{5}+2)\tfrac{C_0^2C_2^2}{C_1^4}\right)(N_2)^2 + \right. \\
&\left. +(\sqrt{5}-2)\tfrac{C_1^2}{C_0^2}\left(1+(\sqrt{5}+2)\tfrac{C_0^2C_2^2}{C_1^4}+3(\sqrt{5}+2)^2\tfrac{C_0^3C_2C_3}{C_1^5}\right)N_1\bar{N}_2\right]\bar{\psi}.
\end{aligned} \tag{4.12}
$$

Let us now find the solutions of the AE $\partial_\psi \widetilde{V}_{BH}\left(\psi,\overline{\psi};q,p\right)=0$, and check their stability. Since we are working near the LG point, by using Eq. (4.9) we can rewrite the AE for $\mathcal{M}_5'$ as follows:

$$
N_2\bar{N}_1+(\sqrt{5}+2)\frac{C_2C_0}{C_1^2}(\bar{N}_2)^2 \approx 0. \tag{4.13}
$$

Here we simply put $\psi=0$. Solving Eq. (4.13), we will find one (or more) set(s) of BH charges supporting $\psi\approx 0$ to be a critical point of V_{BH}. As understood throughout all our treatment of Sects. 4-7, *ça va sans dire* that actual BH charges are very close to the found one, and also that the critical value of ψ is not zero, but it belongs to a suitable neighbourhood of the LG point.

The stability of the critical point $\psi\approx 0$ of V_{BH} is governed by the symmetric, real form of Hessian 2×2 matrix of V_{BH} evaluated at the considered extremum; in general, it reads

$$
H^{V_{BH}}_{\text{real form}} \equiv \left(\begin{array}{cc} \mathcal{A} & \mathcal{C} \\ \mathcal{C} & \mathcal{B}\end{array}\right), \tag{4.14}
$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}\in\mathbb{R}$ are given in terms of $\partial_\psi\partial_\psi V_{BH}$, $\partial_\psi\overline{\partial}_{\overline{\psi}}V_{BH}$, $\overline{\partial}_{\overline{\psi}}\overline{\partial}_{\overline{\psi}}V_{BH}\in\mathbb{C}$ by Eqs. (4.7)-(4.12) of [15]. By using such Eqs., and also Eqs. (4.11)-(4.12) evaluated along the criticality condition (4.13), it can be computed that the components of $H^{\widetilde{V}_{BH}}_{\text{real form}}$ constrained by the AE (4.13) read as follows:

$$\begin{aligned}\mathcal{A} \approx& \tfrac{C_1^2}{C_0^2}\left[|N_1|^2(\sqrt{5}-2)+|N_2|^2\left(1+4(\sqrt{5}+2)\tfrac{C_0^2C_2^2}{C_1^4}\right)\right]+\\ &+\tfrac{3}{2}\tfrac{C_2}{C_0}\left[1+(\sqrt{5}+2)\tfrac{C_3C_0}{C_2C_1}\right](\bar{N}_1\bar{N}_2+N_1N_2);\end{aligned} \tag{4.15}$$

$$\begin{aligned}\mathcal{B} \approx& \tfrac{C_1^2}{C_0^2}\left[|N_1|^2(\sqrt{5}-2)+|N_2|^2\left(1+4(\sqrt{5}+2)\tfrac{C_0^2C_2^2}{C_1^4}\right)\right]+\\ &-\tfrac{3}{2}\tfrac{C_2}{C_0}\left[1+(\sqrt{5}+2)\tfrac{C_3C_0}{C_2C_1}\right](\bar{N}_1\bar{N}_2+N_1N_2);\end{aligned} \tag{4.16}$$

$$\mathcal{C} \approx -\tfrac{3}{2}i\tfrac{C_2}{C_0}\left[1+(\sqrt{5}+2)\tfrac{C_3C_0}{C_2C_1}\right](\bar{N}_1\bar{N}_2-N_1N_2). \tag{4.17}$$

The resulting real eigenvalues of $H^{\widetilde{V}_{BH}}_{\text{real form}}$ constrained by the AE (4.13) read:

$$\begin{aligned}\lambda_\pm \quad \approx \quad & \frac{C_1^2}{C_0^2}\left[|N_1|^2(\sqrt{5}-2)+|N_2|^2\left(1+4(\sqrt{5}+2)\frac{C_0^2C_2^2}{C_1^4}\right)\right]+\\ &\pm 3\frac{C_2}{C_0}\left[1+(\sqrt{5}+2)\frac{C_3C_0}{C_2C_1}\right]|N_1||N_2|.\end{aligned} \tag{4.18}$$

By recalling Eq. (3.30) and using Eq. (4.9) with $\psi \approx 0$ and constrained by the AE (4.13), one obtains that the purely charge-dependent LG critical values of the "effective BH potential" for the mirror *quintic* $\mathcal{M}'_5$ are

$$V_{BH,LG-critical,k=5} \approx \frac{1}{25\sqrt{5+2\sqrt{5}}}\left[|N_1|^2+(\sqrt{5}+2)|N_2|^2\right]; \tag{4.19}$$

by recalling formula (1.2), this directly yields the following purely charge-dependent values of the BH entropy at the LG critical points of $V_{BH,5}$ in the moduli space of $\mathcal{M}'_5$:

$$S_{BH,LG-critical,k=5} \approx \frac{\pi}{25\sqrt{5+2\sqrt{5}}}\left[|N_1|^2+(\sqrt{5}+2)|N_2|^2\right]. \tag{4.20}$$

Let us write down here the numerical values of constants relevant to our treatment:

$$C_0 \approx 2.5, \quad C_1 \approx 2.25, \quad C_2 \approx 0.77, \quad C_3 \approx 0.054. \tag{4.21}$$

Let us now analyze more in depth the species of LG attractor points arising from the AE (4.13). As it can be easily seen, the AE (4.13) has two *non-degenerate* solutions (*i.e.* with non-vanishing V_{BH} and therefore with non-vanishing BH entropy, see Eq. (1.2)):

I. The first non-degenerate solution to AE (4.13) is

$$N_2 \approx 0. \tag{4.22}$$

As one can see from Eq. (4.6)-(4.7), such a solution corresponds to a $\frac{1}{2}$-BPS LG critical point of V_{BH} ($\widetilde{W} \neq 0$, $D_\psi \widetilde{W} = 0$). From the definitions (3.27) and (3.28), in order to get the solution (4.22), we have to fine-tune 2 PF BH charges out of 4 in the following way:

$$n_3 \approx \frac{1}{2}(1+\sqrt{5})(n_2 - n_1), \quad n_4 \approx -\frac{1}{2}(1+\sqrt{5})n_1 + n_2. \tag{4.23}$$

The charges n_1, n_2 are not fixed; they only satisfy the non-degeneration condition $N_1 \neq 0$. The real eigenvalues (4.18) for the $\frac{1}{2}$-BPS critical solution coincide and, as it is well known [31, 15, 16], are strictly positive:

$$\lambda_{+,\frac{1}{2}-BPS} = \lambda_{-,\frac{1}{2}-BPS} \approx (\sqrt{5}-2)\frac{C_1^2}{C_0^2}|N_1|^2 > 0. \tag{4.24}$$

Consequently, the $\frac{1}{2}$-BPS LG critical point $\psi \approx 0$ supported by the PF BH charge configuration (4.23) is a stable extremum, since it is a (local) minimum of V_{BH}, and it is therefore an attractor in a strict sense. The "effective BH potential" and BH entropy at such a (class of) $\frac{1}{2}$-BPS LG attractor(s) take the values

$$V_{BH,\frac{1}{2}-BPS} \approx 0.013|N_1|^2, \quad S_{BH,\frac{1}{2}-BPS} \approx 0.013\pi|N_1|^2, \tag{4.25}$$

where

$$N_1 \approx \frac{n_1}{8}\left[4\sqrt{5} - i\sqrt{2(5+\sqrt{5})^3}\right] - \frac{n_2}{4}\left[5+\sqrt{5} - i\sqrt{10(5-\sqrt{5})}\right]. \tag{4.26}$$

II. The second non-degenerate solution to AE (4.13) is

$$\begin{aligned} &|N_1| \approx \xi|N_2|, \\ &\xi \equiv (\sqrt{5}+2)\tfrac{C_0C_2}{C_1^2} \approx 1,6; \\ &arg(N_1) - 3arg(N_2) \approx \pi, \end{aligned} \tag{4.27}$$

where

$$N_1 \approx \frac{\sqrt{5}-1}{4}\left(n_1 + n_4 - \frac{(3+\sqrt{5})}{2}(n_2+n_3)\right) + \frac{i}{2}\sqrt{\frac{(5+\sqrt{5})}{2}}\left(n_4 - n_1 + \frac{(\sqrt{5}-1)}{2}(n_3 - n_2)\right), \tag{4.28}$$

$$N_2 \approx -\frac{\sqrt{5}+1}{4}\left(n_1 + n_4 - \frac{(3-\sqrt{5})}{2}(n_2+n_3)\right) + \frac{i}{2}\sqrt{\frac{(5-\sqrt{5})}{2}}\left(n_4 - n_1 - \frac{(\sqrt{5}+1)}{2}(n_3 - n_2)\right). \tag{4.29}$$

As one can see from Eq. (4.6)-(4.7), such a solution corresponds to a non-BPS, $Z \neq 0$ LG critical point of V_{BH} ($\widetilde{W} \neq 0$, $D_\psi \widetilde{W} \neq 0$). The real eigenvalues (4.18) for such a non-BPS, $Z \neq 0$ critical solution read

$$\lambda_{\pm,non-BPS,Z\neq 0} \approx \frac{C_1^2}{C_0^2}|N_2|^2\left[1 + 5(\sqrt{5}-2)\xi^2 \pm 3\left(\xi(\sqrt{5}-2) + (\sqrt{5}+2)\frac{C_0^2C_3}{C_1^3}\right)\xi\right]. \tag{4.30}$$

Substituting the numerical values (4.21) of the involved constants in Eq. (4.30), one reaches the conclusion that both $\lambda_{\pm,non-BPS,Z\neq 0}$ are strictly positive:

$$\lambda_{\pm,non-BPS,Z\neq 0} \approx |N_2|^2 \left[3.277 \pm 1.97\right] > 0. \tag{4.31}$$

Thus, the non-BPS, $Z \neq 0$ LG critical point $\psi \approx 0$ supported by the PF BH charge configuration (4.27)-(4.29) is a (local) minimum of V_{BH} and consequently an attractor in a strict sense.

Let us now find the fine-tuning conditions for PF BH charges supporting the considered non-BPS, $Z \neq 0$ LG attractor for the mirror *quintic* $\mathcal{M}'_5$. This amounts to solving Eqs. (4.27)-(4.29) by recalling the definitions (3.27) and (3.28). By doing so, one gets the following three different sets of constraining relations on the PF BH charges:

$$\begin{aligned} &\mathbf{II.1}) \qquad n_2 = n_1 \tfrac{a_2 - a_{1,\pm} - 1}{a_{1,\pm}+2}, \quad n_3 = -n_1 \tfrac{a_2 + a_{1,\pm} - 1}{a_{1,\pm}+2}, \quad n_4 = n_1 \tfrac{a_{1,\pm}-2}{a_{1,\pm}+2}, \\ &a_{1,\pm}(\xi) \equiv \pm\sqrt{\tfrac{20(1+3\xi)}{-5+2\sqrt{5}+\xi(-5-4\sqrt{5}\xi+2(5+\sqrt{5})\xi^2)}}, \quad a_2(\xi) \equiv \tfrac{\sqrt{5(5+2\sqrt{5})}+\sqrt{10(5+2\sqrt{5})}\xi}{\sqrt{5-2\sqrt{5}}+\sqrt{2(5+2\sqrt{5})}\xi}; \end{aligned} \tag{4.32}$$

$$\mathbf{II.2}) \qquad n_2 = n_1 \tfrac{1-\sqrt{5}+\xi(1+\sqrt{5})}{\xi(\sqrt{5}-1)-\sqrt{5}-1}, \quad n_3 = n_2, \quad n_4 = n_1; \tag{4.33}$$

$$\begin{aligned} &\mathbf{II.3}) \qquad n_2+n_3-n_1-n_4 = a, \quad n_1+n_2+n_3+n_4 = b, \quad 2n_3-2n_2+d = c, \quad n_4-n_1 = d, \\ &a(\xi;b,c,d) \equiv -\tfrac{\sqrt{5+2\sqrt{5}}c-\sqrt{5(5-2\sqrt{5})}d+\sqrt{2(5+\sqrt{5})}\xi(-c+\sqrt{5}d)}{\sqrt{5(5-2\sqrt{5})}c-\sqrt{5(5+2\sqrt{5})}d+\sqrt{2(5+\sqrt{5})}\xi(\sqrt{5}c-5d)} b, \\ &b(\xi;c,d) \equiv \frac{\sqrt{(2\sqrt{5}+(5+\sqrt{5})\xi)c^2-10(1+\sqrt{5})\xi cd+5(-2\sqrt{5}+(5+\sqrt{5})\xi)d^2}}{2\sqrt{\frac{(-2\sqrt{5}+3(5+\sqrt{5})\xi)c^2-30(1+\sqrt{5})\xi cd+5(2\sqrt{5}+3(5+\sqrt{5})\xi)d^2}{(5-2\sqrt{5}+2\xi(5-\sqrt{5}+(5+\sqrt{5})\xi))c^2-10(1+\xi(\xi+1)(1+\sqrt{5}))cd+5(5+2\sqrt{5}+2\xi(5+3\sqrt{5}+(5+\sqrt{5})\xi))d^2}}}. \end{aligned} \tag{4.34}$$

Notice that the typology **II.2** of fine-tuning conditions for PF BH charges given by Eq. (4.33) is the one adopted in [9] (see in particular Sect. 4 and App. C of such a Ref.).

By recalling that $\xi \approx 1,6$ (see Eq. (4.27)), the typology **II.1** of fine-tuning conditions for PF BH charges yields

$$n_2/n_1 \approx 0.342(-138.3), \quad n_3/n_1 \approx -1.352(35), \quad n_4/n_1 \approx 0.009(102.3), \tag{4.35}$$

where, here and below, the numbers in round brackets correspond to considering $a_{1,-}$, rather than $a_{1,+}$, in Eqs. (4.32). Since the PF BH charges are integers as are the symplectic BH charges (see definition (3.28)), the numerical conditions (4.35) can approximately be met by taking *e.g.*

$$n_1 = 1000(10), \quad n_2 = 342(-1383), \quad n_3 = -1352(350), \quad n_4 = 9(1023). \tag{4.36}$$

Switching to the symplectic (electric/magnetic) basis for BH charges by using Eq. (3.29), one finally gets

$$\begin{aligned} &p^0 = 3n_1 + n_4 = 3009(1053), \quad p^1 = \tfrac{1}{5}\left[-4(n_1+n_4)+n_2+n_3\right] = -1009(-1033), \\ &q_0 = \tfrac{1}{5}(8n_1+3n_4) = 1605(630), \quad q_1 = \tfrac{1}{5}(-3n_1+n_2-n_4) = -533(-487). \end{aligned} \tag{4.37}$$

By repeating the same procedure for the typology **II.2** of fine-tuning conditions for PF BH charges given by Eq. (4.33), one achieves the same results obtained at the end of Sect. 4 of [9].

It is worth remarking that *all* three distinct sets of fine-tuning conditions for PF BH charges (4.32)-(4.34) do support a non-BPS, $Z \neq 0$ LG attractor in a strict sense.

The "effective BH potential" and BH entropy at such a (class of) non-BPS, $Z \neq 0$ LG attractor(s) take the values

$$V_{BH,non-BPS,Z\neq0} \approx 0.055|N_2|^2, \quad S_{BH,non-BPS,Z\neq0} \approx 0.055\pi|N_2|^2, \tag{4.38}$$

where N_2 is given by Eq.(4.29), implemented by one of the fine-tuning conditions (4.32)-(4.34).

Finally, by recalling the definition (1.29), one can compute the supersymmetry-breaking order parameter for the non-BPS, $Z \neq 0$ LG attractor in the mirror *quintic* $\mathcal{M}'_5$; by using Eqs. (4.4), (4.6), (4.7) and (4.27), one gets

$$\mathcal{O}_{non-BPS,Z\neq0} \equiv \left[\frac{\left(g_{\psi\overline{\psi}}\right)^{-1}|D_\psi W|^2}{|W|^2}\right]_{non-BPS,Z\neq0} = \left[\frac{\left(g_{\psi\overline{\psi}}\right)^{-1}|D_\psi \widetilde{W}|^2}{|\widetilde{W}|^2}\right]_{non-BPS,Z\neq0} \approx \frac{(\sqrt{5}+2)}{\xi^2} \approx 1.65, \tag{4.39}$$

which is consistent with the result obtained at the end of Sect. 4 of [9].

5 $k = 6$: Mirror *Sextic*

For the mirror *sextic* $\mathcal{M}'_6$ the computations (*but not the results!*) go the same way as for the mirror *quintic* $\mathcal{M}'_5$. Thus, also for $k = 6$ it is easy to realize that one has to consider the "effective BH potential" (3.30) (at least) up to $\mathcal{O}\left(\psi^3\right)$ in order to get the AEs (3.33) and the Hessian matrix up to $\mathcal{O}\left(\psi\right)$.

For $k = 6$ the definitions (3.12) yield

$$C_{6,3l-1} = 0, \quad l \in \mathbb{N} \tag{5.1}$$

moreover, since $F_{6,m+6,n} = F_{6,m,n+6} = -F_{6,m\,n}$ (see the third of properties (3.22)), the only independent elements of the rank-2 tensor F_6 belong to the 6×6 matrix

$$F_{6,m\,n} = \begin{pmatrix} 2\sqrt{3} & 0 & -\sqrt{3} & 0 & 0 & 3 \\ 0 & -\frac{2}{\sqrt{3}} & 1 & 0 & 0 & -\frac{1}{\sqrt{3}} \\ -\sqrt{3} & 1 & 0 & -1 & \sqrt{3} & -2 \\ 0 & 0 & -1 & \frac{2}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \sqrt{3} & 0 & -2\sqrt{3} & 3 \\ 3 & -\frac{1}{\sqrt{3}} & -2 & -\frac{1}{\sqrt{3}} & 3 & 0 \end{pmatrix}. \tag{5.2}$$

Let us now write down all the relevant quantities up to the needed order (here and below, unless otherwise specified, we omit the Fermat parameter $k=6$):

$$\widetilde{K} \approx \frac{1}{3}\frac{C_1^2}{C_0^2}\left[\psi\bar{\psi} + \frac{1}{6}\frac{C_1^2}{C_0^2}(\psi\bar{\psi})^2\right] + \mathcal{O}(\psi^6); \tag{5.3}$$

$$g_{\psi\bar{\psi}} \approx \frac{1}{3}\frac{C_1^2}{C_0^2}\left[1 + \frac{2}{3}\frac{C_1^2}{C_0^2}\psi\bar{\psi}\right] + \mathcal{O}(\psi^4); \tag{5.4}$$

$$\widetilde{W} \approx N_1 + \frac{C_1}{C_0}N_2\psi - \frac{C_3}{C_0}\bar{N}_2\psi^3 - \frac{C_4}{C_0}\bar{N}_1\psi^4 + \mathcal{O}(\psi^6). \tag{5.5}$$

Now, by using the formulæ of the general analysis exploited in Sect. 3, we can get the "effective BH potential" and the holomorphic superpotential, as well as their (covariant) derivatives, up to $\mathcal{O}(\psi)$:

$$\widetilde{W} = N_1 + \frac{C_1}{C_0}N_2\psi; \tag{5.6}$$

$$D_\psi\widetilde{W} = \frac{C_1}{C_0}\left[N_2 + \frac{1}{3}\frac{C_1}{C_0}N_1\bar{\psi}\right]; \tag{5.7}$$

$$D_\psi D_\psi\widetilde{W} = -6\frac{C_3}{C_0}\bar{N}_2\psi - 12\frac{C_4}{C_0}\bar{N}_1\psi^2; \tag{5.8}$$

$$\widetilde{V}_{BH} = |N_1|^2 + 3|N_2|^2 + 2\frac{C_1}{C_0}N_2\bar{N}_1\psi + 2\frac{C_1}{C_0}\bar{N}_2N_1\bar{\psi}; \tag{5.9}$$

$$\partial_\psi\widetilde{V}_{BH} = 2\frac{C_1}{C_0}N_2\bar{N}_1 - 18\frac{C_3}{C_1}(\bar{N}_2)^2\psi + \frac{2}{3}\frac{C_1^2}{C_0^2}\left(|N_1|^2 + 3|N_2|^2\right)\bar{\psi}; \tag{5.10}$$

$$\partial_\psi\partial_\psi\widetilde{V}_{BH} = -18\frac{C_3}{C_1}(\bar{N}_2)^2 - 24\frac{C_3}{C_0}\left(1 + \frac{C_4C_0}{C_3C_1}\right)\bar{N}_1\bar{N}_2\psi + \frac{4}{3}\frac{C_1^3}{C_0^3}\bar{N}_1N_2\bar{\psi}; \tag{5.11}$$

$$\partial_\psi\overline{\partial}_{\bar{\psi}}\widetilde{V}_{BH} = \frac{2}{3}\frac{C_1^2}{C_0^2}\left(|N_1|^2 + 3|N_2|^2\right) + \frac{4}{3}\frac{C_1^3}{C_0^3}\bar{N}_1N_2\psi + \frac{4}{3}\frac{C_1^3}{C_0^3}N_1\bar{N}_2\bar{\psi}. \tag{5.12}$$

From the definitions (3.27) and (3.28), for $k=6$ one gets that

$$N_1 = -\frac{\sqrt{3}}{2}(n_2+n_3) + \frac{i}{2}(2n_4 - 2n_1 + n_3 - n_2), \tag{5.13}$$

$$N_2 = \frac{1}{2}(n_3 + n_2 - 2n_4 - 2n_1) - \frac{\sqrt{3}i}{2}(n_3 - n_2). \tag{5.14}$$

Let us now find the solutions of the AE $\partial_\psi\widetilde{V}_{BH}\left(\psi,\overline{\psi};q,p\right) = 0$, and check their stability. Since we are working near the LG point, by using Eq. (5.9) we can rewrite the AE for $\mathcal{M}'_6$ as follows:

$$N_2\bar{N}_1 \approx 0. \tag{5.15}$$

Here we simply put $\psi = 0$. Solving Eq. (5.15), we will find one (or more) set(s) of BH charges supporting $\psi \approx 0$ to be a critical point of V_{BH}.

By using Eqs. (4.7)-(4.12) of [15] and Eqs. (5.11)-(5.12) evaluated along the criticality condition (5.15), it can be computed that the components of $H^{\widetilde{V}_{BH}}_{\text{real form}}$ (given by Eq. (4.14)) constrained by the AE (5.15) read as follows:

$$\mathcal{A} = \frac{1}{3}\frac{C_1^2}{C_0^2}\left(|N_1|^2 + 3|N_2|^2\right) - \frac{9}{2}\frac{C_3}{C_1}\left((\bar{N}_2)^2 + (N_2)^2\right); \tag{5.16}$$

$$\mathcal{B} = \frac{1}{3}\frac{C_1^2}{C_0^2}\left(|N_1|^2 + 3|N_2|^2\right) + \frac{9}{2}\frac{C_3}{C_1}\left((\bar{N}_2)^2 + (N_2)^2\right); \tag{5.17}$$

$$\mathcal{C} = \frac{9}{2}i\frac{C_3}{C_1}\left((\bar{N}_2)^2 - (N_2)^2\right). \tag{5.18}$$

The resulting real eigenvalues of $H^{\widetilde{V}_{BH}}_{\text{real form}}$ constrained by the AE (5.15) read:

$$\lambda_\pm \approx \frac{C_1^2}{C_0^2}\left[\frac{1}{3}|N_1|^2 + |N_2|^2\left(1 \pm 9\frac{C_0^2 C_3}{C_1^3}\right)\right]. \tag{5.19}$$

By recalling Eq. (3.30) and using Eq. (5.9) with $\psi \approx 0$ and constrained by the AE (5.15), one obtains that the purely charge-dependent LG critical values of the "effective BH potential" for the mirror *sextic* $\mathcal{M}_6'$ are

$$V_{BH,LG-critical,k=6} \approx \frac{1}{18\sqrt{3}}\left[|N_1|^2 + 3|N_2|^2\right]; \tag{5.20}$$

by recalling formula (1.2), this directly yields the following purely charge-dependent values of the BH entropy at the LG critical points of $V_{BH,6}$ in the moduli space of $\mathcal{M}_6'$:

$$S_{BH,LG-critical,k=6} \approx \frac{\pi}{18\sqrt{3}}\left[|N_1|^2 + 3|N_2|^2\right]. \tag{5.21}$$

Let us write down here the numerical values of constants relevant to our treatment:

$$C_0 \approx 2.27, \quad C_1 \approx 1.52, \quad C_3 \approx -0.247, \quad C_4 \approx 0.054; \quad 9\frac{C_0^2 C_3}{C_1^3} \approx -3.25. \tag{5.22}$$

Let us now analyze more in depth the species of LG attractor points arising from the AE (5.15). As it can be easily seen, the AE (5.15) has two *non-degenerate* solutions:

I. The first non-degenerate solution to AE (5.15) is

$$N_2 \approx 0. \tag{5.23}$$

This is nothing but the $k = 5$ solution (4.22). As one can see from Eq. (5.6)-(5.7), also for $k = 6$ such a solution corresponds to a $\frac{1}{2}$-BPS LG critical point of V_{BH} ($\widetilde{W} \neq 0$, $D_\psi \widetilde{W} = 0$). From the definitions (3.27) and (3.28), in order to get the solution (5.23), we have to fine-tune 2 PF BH charges out of 4 in the following way:

$$n_3 \approx n_2, \quad n_4 \approx n_2 - n_1. \tag{5.24}$$

The charges n_1, n_2 are not fixed; they only satisfy the non-degeneration condition $N_1 \neq 0$. As it was for $k = 5$, also the real eigenvalues (5.19) for the $\frac{1}{2}$-BPS critical solution coincide and, as it is well known [31, 15, 16], are strictly positive:

$$\lambda_{+,\frac{1}{2}-BPS} = \lambda_{-,\frac{1}{2}-BPS} \approx \frac{1}{3}\frac{C_1^2}{C_0^2}|N_1|^2 > 0. \tag{5.25}$$

Consequently, the $\frac{1}{2}$-BPS LG critical point $\psi \approx 0$ supported by the PF BH charge configuration (5.24) is a stable extremum, since it is a (local) minimum of V_{BH}, and it is therefore an attractor in a strict sense. The "effective BH potential" and BH entropy at such a (class of) $\frac{1}{2}$-BPS LG attractor(s) take the values

$$V_{BH,\frac{1}{2}-BPS} \approx 0.032|N_1|^2, \quad S_{BH,\frac{1}{2}-BPS} \approx 0.032\pi|N_1|^2, \tag{5.26}$$

where

$$N_1 \approx -\sqrt{3}n_2 + i(n_2 - 2n_1). \tag{5.27}$$

II. The second non-degenerate solution to AE (5.15) is

$$N_1 \approx 0. \tag{5.28}$$

As one can see from Eq. (5.6)-(5.7), such a solution corresponds to a non-BPS, $Z = 0$ LG critical point of V_{BH} ($\widetilde{W} = 0$, $D_\psi \widetilde{W} \neq 0$). The real eigenvalues (5.19) for such a non-BPS, $Z = 0$ critical solution read

$$\lambda_{\pm,non-BPS,Z=0} \approx \frac{C_1^2}{C_0^2}\left(1 \pm 9\frac{C_0^2 C_3}{C_1^3}\right)|N_2|^2. \tag{5.29}$$

Substituting the numerical values (5.22) of the involved constants in Eq. (5.29), one reaches the conclusion that one eigenvalue is positive and the other one is negative:

$$\lambda_{\pm,non-BPS,Z=0} \approx |N_2|^2\left[0.45 \mp 1.46\right] \lessgtr 0. \tag{5.30}$$

Let us now find the fine-tuning conditions for PF BH charges supporting the considered non-BPS, $Z = 0$ LG attractor for the mirror *sextic* $\mathcal{M}_6'$. This amounts to solving Eq. (5.28) by recalling the definitions (3.27) and (3.28). By doing so, one gets the following unique set of constraining relations on PF BH charges:

$$n_3 \approx -n_2, \quad n_4 \approx n_2 + n_1. \tag{5.31}$$

Thus, the non-BPS, $Z = 0$ LG critical point $\psi \approx 0$ supported by the PF BH charge configuration (5.31) is a saddle point of V_{BH} and consequently it is *not* an attractor in a strict sense.

The "effective BH potential" and BH entropy at such a (class of) non-BPS, $Z = 0$ LG saddle point(s) take the values

$$V_{BH,non-BPS,Z=0} \approx 0.096|N_2|^2, \quad S_{BH,non-BPS,Z=0} \approx 0.096\pi|N_2|^2, \tag{5.32}$$

where

$$N_2 \approx -2n_1 - n_2 + i\sqrt{3}n_2. \tag{5.33}$$

Switching to the symplectic (electric/magnetic) basis for BH charges by using Eq. (3.29), one gets

$$\begin{aligned} n_1 &= p^0 - 3q_0; \\ n_2 &= p^0 + 3q_1; \\ n_3 &= -p^0 + 3p^1 + 9q_0 - 3q_1; \\ n_4 &= -p^0 + 6q_0, \end{aligned} \tag{5.34}$$

one can easily show that fine-tuning condition (5.31) can be rewritten in terms of symplectic BH charges as follows:

$$p^0 = 3q_0 - q_1, \quad p^1 = -3q_0. \tag{5.35}$$

It is worth remarking that such a critical solution for the mirror *sextic* $\mathcal{M}'_6$ had been previously investigated in Sect. 7 of [10]. Up to irrelevant changes of notation, Eq. (5.35) coincides with Eq. (7.8) of [10]. By considering the second derivatives (5.11)-(5.12) of the "effective BH potential" constrained by Eq. (5.28) and comparing them with Eq. (7.9) of [10], one can state that the crucial difference between the results of [10] and ours lies in the critical value of the second holomorphic derivative of V_{BH}. Indeed, Eq. (7.9) of [10] reads

$$\left(\partial_\psi \partial_\psi V_{BH}\right)_{non-BPS, Z=0} = 0. \tag{5.36}$$

From our previous computations, consistently taking into account the needed orders in ψ to get the series expansion for V_{BH} up to $\mathcal{O}\left(\psi^2\right)$ (or $\mathcal{O}\left(\psi^3\right)$), we disagree with the critical value of the second holomorphic derivative of V_{BH} at the considered non-BPS, $Z = 0$ critical point given by Eq. (5.36). According to our results, the statement made in [10] that the considered non-BPS, $Z = 0$ LG critical point of V_{BH} is actually an attractor in a strict sense for all possible supporting symplectic BH charge configurations (5.35) does *not* hold. Instead, as correctly stated above, the non-BPS, $Z = 0$ LG critical point $\psi \approx 0$ supported by the BH charge configurations (5.31) (PF) and (5.35) (symplectic) is a saddle point of V_{BH} and consequently it is *not* an attractor in a strict sense.

Also, by recalling Eq. (5.32) and using Eqs. (5.33) and (5.34)-(5.35), our analysis yields that the "effective BH potential" and BH entropy at the considered (class of) non-BPS, $Z = 0$ LG saddle point(s) take the purely charge-dependent values:

$$V_{BH,non-BPS,Z=0} = \frac{1}{2\sqrt{3}}(3q_0^2 + q_1^2), \quad S_{BH,non-BPS,Z=0} = \frac{\pi}{2\sqrt{3}}(3q_0^2 + q_1^2). \tag{5.37}$$

As one can see, the value of $V_{BH,non-BPS,Z=0}$ given by Eq. (5.37) does *not* coincide with the one given by Eq. (7.10) of [10].

6 $k = 8$: Mirror *Octic*

The case of mirror *octic* $\mathcal{M}'_8$ (as well as the one of mirror *dectic* $\mathcal{M}'_{10}$ treated in Sect. 7) needs a different approach with respect to the cases of mirror *quintic* $\mathcal{M}'_5$ and mirror sextic $\mathcal{M}'_6$, respectively treated in Sects. 4 and 5.

Indeed, contrary to what happens for $k = 5, 6$ (see Eqs. (4.4) and (5.4), respectively), for $k = 8, 10$ the series expansion of the Kähler metric $g_{\psi\overline{\psi}}\left(\psi, \overline{\psi}\right)$ near the LG point starts with *no* constant term (namely, it is *not* regular at $\psi = 0$). As a consequence, one has to consider the series expansion of the "effective BH potential" V_{BH} up to $\mathcal{O}\left(\psi^4\right)$ (rather than up to $\mathcal{O}\left(\psi^3\right)$, as it is for $k = 5, 6$), in order to obtain all the relevant quantities up to $\mathcal{O}\left(\psi^2\right)$ (rather than up to $\mathcal{O}\left(\psi\right)$, as it is for $k = 5, 6$).

For $k = 8$ the definitions (3.12) yield

$$C_{8,2l-1} = 0, \quad l \in \mathbb{N}; \tag{6.1}$$

moreover, since $F_{8,m+8,\,n} = F_{8,m\,,n+8} = -F_{8,m\,n}$ (see the third of properties (3.22)), the only independent elements of the rank-2 tensor F_8 belong to the 8×8 matrix

$$F_{8,mn} \quad = \quad \begin{pmatrix} 2(2+\sqrt{2}) & -\sqrt{10+7\sqrt{2}} & 0 & \sqrt{2+\sqrt{2}} & 0 & -\sqrt{2+\sqrt{2}} & 0 & \sqrt{10+7\sqrt{2}} \\ -\sqrt{10+7\sqrt{2}} & 2 & \sqrt{2-\sqrt{2}} & -\sqrt{2} & \sqrt{10-7\sqrt{2}} & 0 & \sqrt{2+\sqrt{2}} & -3\sqrt{2} \\ 0 & \sqrt{2-\sqrt{2}} & 2(-2+\sqrt{2}) & \sqrt{2-\sqrt{2}} & 0 & -\sqrt{10-7\sqrt{2}} & 0 & \sqrt{10-7\sqrt{2}} \\ \sqrt{2+\sqrt{2}} & -\sqrt{2} & \sqrt{2-\sqrt{2}} & 0 & -\sqrt{2-\sqrt{2}} & \sqrt{2} & -\sqrt{2+\sqrt{2}} & 2 \\ 0 & \sqrt{10-7\sqrt{2}} & 0 & -\sqrt{2-\sqrt{2}} & 4-2\sqrt{2} & -\sqrt{2-\sqrt{2}} & 0 & \sqrt{10-7\sqrt{2}} \\ -\sqrt{2+\sqrt{2}} & 0 & -\sqrt{10-7\sqrt{2}} & \sqrt{2} & -\sqrt{2-\sqrt{2}} & -2 & \sqrt{10+7\sqrt{2}} & -3\sqrt{2} \\ 0 & \sqrt{2+\sqrt{2}} & 0 & -\sqrt{2+\sqrt{2}} & 0 & \sqrt{10+7\sqrt{2}} & -4-2\sqrt{2} & \sqrt{10+7\sqrt{2}} \\ \sqrt{10+7\sqrt{2}} & -3\sqrt{2} & \sqrt{10-7\sqrt{2}} & 2 & \sqrt{10-7\sqrt{2}} & -3\sqrt{2} & \sqrt{10+7\sqrt{2}} & 0 \end{pmatrix}. \tag{6.2}$$

Let us now write down all the relevant quantities up to the needed order (here and below, unless otherwise specified, we omit the Fermat parameter $k = 8$):

$$\widetilde{K} \approx (3-2\sqrt{2})\tfrac{C_2^2}{C_0^2}(\psi\bar{\psi})^2\left[1-\left(\tfrac{C_4^2}{C_2^2}-\tfrac{1}{2}(3-2\sqrt{2})\tfrac{C_2^2}{C_0^2}\right)(\psi\bar{\psi})^2\right]+\tfrac{C_8}{C_0}(\psi^8+\bar{\psi}^8)+\mathcal{O}(\psi^9); \tag{6.3}$$

$$g_{\psi\bar{\psi}} \approx 4(3-2\sqrt{2})\tfrac{C_2^2}{C_0^2}\psi\bar{\psi}\left[1-4\left(\tfrac{C_4^2}{C_2^2}-\tfrac{1}{2}(3-2\sqrt{2})\tfrac{C_2^2}{C_0^2}\right)(\psi\bar{\psi})^2\right]+\mathcal{O}(\psi^7); \tag{6.4}$$

$$\widetilde{W} \approx N_1+\tfrac{C_2}{C_0}N_3\psi^2-\tfrac{C_4}{C_0}\bar{N}_3\psi^4-\tfrac{C_6}{C_0}\bar{N}_1\psi^6+\mathcal{O}(\psi^8). \tag{6.5}$$

Now, by using the formulæ of the general analysis exploited in Sect. 3, we can get the "effective BH potential" and the holomorphic superpotential, as well as their (covariant) derivatives, up to $\mathcal{O}\left(\psi^2\right)$:

$$\widetilde{W} = N_1 + \tfrac{C_2}{C_0} N_3 \psi^2; \tag{6.6}$$

$$D_\psi \widetilde{W} = 2\tfrac{C_2}{C_0} N_3 \psi - 4\tfrac{C_4}{C_0} \bar{N}_3 \psi^3 + 2(3-2\sqrt{2})\tfrac{C_2^2}{C_0^2} N_1 \psi \bar{\psi}^2; \tag{6.7}$$

$$D_\psi D_\psi \widetilde{W} = -8\tfrac{C_4}{C_0} \bar{N}_3 \psi^2; \tag{6.8}$$

$$\begin{aligned} \widetilde{V}_{BH} = {} & |N_1|^2 + (3+2\sqrt{2})|N_3|^2 + 2\tfrac{C_2}{C_0}\left(N_3\bar{N}_1 - (3+2\sqrt{2})\tfrac{C_4C_0}{C_2^2}(\bar{N}_3)^2\right)\psi^2 + \\ & +2\tfrac{C_2}{C_0}\left(\bar{N}_3 N_1 - (3+2\sqrt{2})\tfrac{C_4C_0}{C_2^2}(N_3)^2\right)\bar{\psi}^2; \end{aligned} \tag{6.9}$$

$$\begin{aligned} \partial_\psi \widetilde{V}_{BH} = 4\psi \Big[& \tfrac{C_2}{C_0}\left(N_3\bar{N}_1 - (3+2\sqrt{2})\tfrac{C_4C_0}{C_2^2}(\bar{N}_3)^2\right) - 3\tfrac{C_4}{C_0}\left(1 + (3+2\sqrt{2})\tfrac{C_6C_0}{C_4C_2}\right)\bar{N}_1\bar{N}_3\psi^2 + \\ & +\tfrac{C_2^2}{C_0^2}\left(|N_1|^2(3-2\sqrt{2}) + |N_3|^2\left(1 + 4(3+2\sqrt{2})\tfrac{C_0^2C_4^2}{C_2^4}\right)\right)\bar{\psi}^2\Big]; \end{aligned} \tag{6.10}$$

$$\begin{aligned} \partial_\psi\partial_\psi \widetilde{V}_{BH} = {} & 4\tfrac{C_2}{C_0}\left(N_3\bar{N}_1 - (3+2\sqrt{2})\tfrac{C_4C_0}{C_2^2}(\bar{N}_3)^2\right) - 36\tfrac{C_4}{C_0}\left(1 + (3+2\sqrt{2})\tfrac{C_6C_0}{C_4C_2}\right)\bar{N}_1\bar{N}_3\psi^2 + \\ & +4\tfrac{C_2^2}{C_0^2}\left(|N_1|^2(3-2\sqrt{2}) + |N_3|^2\left(1 + 4(3+2\sqrt{2})\tfrac{C_0^2C_4^2}{C_2^4}\right)\right)\bar{\psi}^2; \end{aligned} \tag{6.11}$$

$$\partial_\psi \overline{\partial}_{\bar{\psi}} \widetilde{V}_{BH} = 8\tfrac{C_2^2}{C_0^2}\left(|N_1|^2(3-2\sqrt{2}) + |N_3|^2\left(1 + 4(3+2\sqrt{2})\tfrac{C_0^2C_4^2}{C_2^4}\right)\right)\psi\bar{\psi}. \tag{6.12}$$

Let us stress once again that, contrary to the treatment of Sects. 4 and 5, and as evident from Eqs. (6.9)-(6.12), for the case of mirror *octic* we truncate the series expansion of the "effective BH potential" and of its second derivatives around the LG point up to $\mathcal{O}\left(\psi^2\right)$ included, and the series expansion of its first derivative around the LG point up to $\mathcal{O}\left(\psi^3\right)$ included. This is due to the absence of an $\mathcal{O}\left(\psi\right)$ term in expression of $\widetilde{V}_{BH}$ given by Eq. (6.9). As mentioned at the start of the present Section, such a fact can be traced back to the non-regularity of $g_{\psi\bar{\psi}}$ at $\psi = 0$ (see Eq. (6.4)).

Let us now find the solutions of the AE $\partial_\psi \widetilde{V}_{BH}\left(\psi, \overline{\psi}; q, p\right) = 0$, and check their stability. Since we are working near the LG point, by using Eq. (6.9) we can rewrite the AE for $\mathcal{M}'_8$ as follows:

$$\begin{aligned} & \tfrac{C_2}{C_0}\left(N_3\bar{N}_1 - (3+2\sqrt{2})\tfrac{C_4C_0}{C_2^2}(\bar{N}_3)^2\right) + \\ & +\tfrac{C_2^2}{C_0^2}\left(|N_1|^2(3-2\sqrt{2}) + |N_3|^2\left(1 + 4(3+2\sqrt{2})\tfrac{C_0^2C_4^2}{C_2^4}\right)\right)\bar{\psi}^2 \approx \\ & \approx 3\tfrac{C_4}{C_0}\left(1 + (3+2\sqrt{2})\tfrac{C_6C_0}{C_4C_2}\right)\bar{N}_1\bar{N}_3\psi^2. \end{aligned} \tag{6.13}$$

Solving Eq. (6.13), we will find one (or more) set(s) of BH charges supporting $\psi \approx 0$ to be a critical point of V_{BH}. Since we are working near the LG point, it is clear that the first term in the left-hand side (l.h.s.) of Eq. (6.13) must be small enough. This implies the following fine-tuning condition:

$$\begin{aligned} & N_3\bar{N}_1 - \vartheta(\bar{N}_3)^2 \approx 0, \\ & \vartheta \equiv (3+2\sqrt{2})\tfrac{C_4 C_0}{C_2^2}. \end{aligned} \tag{6.14}$$

By using Eqs. (4.7)-(4.12) of [15] and Eqs. (6.11)-(6.12) evaluated along the criticality condition (6.13)-(6.14), it can be computed that the components of $H^{\widetilde{V}_{BH}}_{\text{real form}}$ (given by Eq. (4.14)) constrained by Eqs. (6.13)-(6.14) read as follows:

$$\begin{aligned} \mathcal{A} = & -6\tfrac{C_4}{C_0}\left(1+(3+2\sqrt{2})\tfrac{C_6 C_0}{C_4 C_2}\right)\left(\bar{N}_1\bar{N}_3\psi^2 + N_1 N_3\bar{\psi}^2\right) + \\ & +4\tfrac{C_2^2}{C_0^2}\psi\bar{\psi}\left(|N_1|^2(3-2\sqrt{2}) + |N_3|^2\left(1+4(3+2\sqrt{2})\tfrac{C_0^2 C_4^2}{C_2^4}\right)\right); \end{aligned} \tag{6.15}$$

$$\begin{aligned} \mathcal{B} = & \; 6\tfrac{C_4}{C_0}\left(1+(3+2\sqrt{2})\tfrac{C_6 C_0}{C_4 C_2}\right)\left(\bar{N}_1\bar{N}_3\psi^2 + N_1 N_3\bar{\psi}^2\right) + \\ & +4\tfrac{C_2^2}{C_0^2}\psi\bar{\psi}\left(|N_1|^2(3-2\sqrt{2}) + |N_3|^2\left(1+4(3+2\sqrt{2})\tfrac{C_0^2 C_4^2}{C_2^4}\right)\right); \end{aligned} \tag{6.16}$$

$$\mathcal{C} = -6i\tfrac{C_4}{C_0}\left(1+(3+2\sqrt{2})\tfrac{C_6 C_0}{C_4 C_2}\right)\left(N_1 N_3\bar{\psi}^2 - \bar{N}_1\bar{N}_3\psi^2\right). \tag{6.17}$$

The resulting real eigenvalues of $H^{\widetilde{V}_{BH}}_{\text{real form}}$ constrained by Eqs. (6.13)-(6.14) read:

$$\begin{aligned} \lambda_{\pm} \approx 4\psi\bar{\psi}\Bigg[& \frac{C_2^2}{C_0^2}\left(|N_1|^2(3-2\sqrt{2}) + |N_3|^2\left(1+4(3+2\sqrt{2})\frac{C_0^2 C_4^2}{C_2^4}\right)\right) \pm \\ & \pm 3\frac{C_4}{C_0}\left(1+(3+2\sqrt{2})\frac{C_6 C_0}{C_4 C_2}\right)|N_1||N_3|\Bigg]. \end{aligned} \tag{6.18}$$

By recalling Eq. (3.30) and using Eq. (6.9) with $\psi \approx 0$ and constrained by Eqs. (6.13)-(6.14), one obtains that the purely charge-dependent LG critical values of the "effective BH potential" for the mirror *octic* $\mathcal{M}'_8$ are

$$V_{BH,LG-critical,k=8} \approx \frac{1}{8(2+\sqrt{2})}\left[|N_1|^2 + (3+2\sqrt{2})|N_3|^2\right]; \tag{6.19}$$

by recalling formula (1.2), this directly yields the following purely charge-dependent values of the BH entropy at the LG critical points of $V_{BH,8}$ in the moduli space of $\mathcal{M}'_8$:

$$S_{BH,LG-critical,k=8} \approx \frac{\pi}{8(2+\sqrt{2})}\left[|N_1|^2 + (3+2\sqrt{2})|N_3|^2\right]. \tag{6.20}$$

Let us write down here the numerical values of constants relevant to our treatment:

$$C_0 \approx 1.64,\ C_2 \approx -0.9,\ C_4 \approx 0.24,\ C_6 \approx -0.007,\ C_8 \approx -0.004; \quad \vartheta \approx 2.83. \tag{6.21}$$

Let us now analyze more in depth the species of LG attractor points arising from the AE (6.14). As it can be easily seen, the AE (6.14) has two *non-degenerate* solutions:

I. The first non-degenerate solution to AE (6.14) is

$$N_3 \approx 0. \tag{6.22}$$

As one can see from Eq. (6.6)-(6.7), such a solution corresponds to a $\frac{1}{2}$-BPS LG critical point of V_{BH} ($\widetilde{W} \neq 0$, $D_\psi \widetilde{W} = 0$). From the definitions (3.27) and (3.28), in order to get the solution (6.22), we have to fine-tune 2 PF BH charges out of 4 in the following way:

$$n_3 \approx -n_1 + \sqrt{2} n_2, \quad n_4 \approx -\sqrt{2} n_1 + n_2, \tag{6.23}$$

The charges n_1, n_2 are not fixed; they only satisfy the non-degeneration condition $N_1 \neq 0$. The real eigenvalues (6.18) for the $\frac{1}{2}$-BPS critical solution coincide and, as it is well known [31, 15, 16], are strictly positive:

$$\lambda_{+,\frac{1}{2}-BPS} = \lambda_{-,\frac{1}{2}-BPS} \approx 4\psi\bar{\psi}\frac{C_2^2}{C_0^2}|N_1|^2(3 - 2\sqrt{2}) > 0.$$

Consequently, the $\frac{1}{2}$-BPS LG critical point $\psi \approx 0$ supported by the PF BH charge configuration (6.23) is a stable extremum, since it is a (local) minimum of V_{BH}, and it is therefore an attractor in a strict sense. The "effective BH potential" and BH entropy at such a (class of) $\frac{1}{2}$-BPS LG attractor (s) take the values

$$V_{BH,\frac{1}{2}-BPS} \approx 0.0366|N_1|^2, \quad S_{BH,\frac{1}{2}-BPS} \approx 0.0366\pi|N_1|^2, \tag{6.24}$$

where

$$N_1 \approx \sqrt{4 - 2\sqrt{2}}[n_1(1 - i(1 + \sqrt{2})) - n_2(1 + \sqrt{2} - i)]. \tag{6.25}$$

II. The second non-degenerate solution to AE (6.14) reads (from Eqs. (6.14) and (6.21): $\vartheta \equiv (3+2\sqrt{2})\frac{C_4 C_0}{C_2^2} \approx 2.83$):

$$\begin{aligned} &|N_1| \approx \vartheta |N_3|, \\ &arg(N_1) \approx 3arg(N_3), \end{aligned} \tag{6.26}$$

where

$$N_1 \approx -\frac{\sqrt{2-\sqrt{2}}}{2}\left(n_1 + n_4 + (1 + \sqrt{2})(n_2 + n_3) - i((n_4 - n_1)(1 + \sqrt{2}) + n_3 - n_2)\right); \tag{6.27}$$

$$N_3 \approx \frac{\sqrt{2+\sqrt{2}}}{2}\left(n_1 + n_4 + (1 - \sqrt{2})(n_2 + n_3) + i((n_4 - n_1)(1 - \sqrt{2}) + n_3 - n_2)\right). \tag{6.28}$$

As one can see from Eq. (6.6)-(6.7), such a solution corresponds to a non-BPS, $Z \neq 0$ LG critical point of V_{BH} ($\widetilde{W} \neq 0$, $D_\psi \widetilde{W} \neq 0$). The real eigenvalues (6.18) for such a non-BPS, $Z \neq 0$ critical solution read

$$\begin{aligned} \lambda_{\pm,non-BPS,Z\neq 0} &\approx 4\psi\bar{\psi}|N_3|^2\frac{C_2^2}{C_0^2}\left[(1 + 5(3 - 2\sqrt{2})\vartheta^2) \pm 3(3 - 2\sqrt{2})\vartheta^2\left(1 + (3 + 2\sqrt{2})\frac{C_0 C_6}{C_2 C_4}\right)\right] \approx \\ &\approx 4\psi\bar{\psi}|N_3|^2\frac{C_2^2}{C_0^2}\left[7.9 \pm 5.5\right] > 0, \end{aligned} \tag{6.29}$$

where in the second line we replaced some constants with their numerical values by using Eq. (6.21).

Thus, the non-BPS, $Z \neq 0$ LG critical point $\psi \approx 0$ supported by the PF BH charge configuration (6.26)-(6.28) is a (local) minimum of V_{BH} and consequently an attractor in a strict sense.

Let us now find the fine-tuning conditions for PF BH charges supporting the considered non-BPS, $Z \neq 0$ LG attractor for the mirror *octic* $\mathcal{M}'_8$. This amounts to solving Eqs. (6.26)-(6.28) by recalling the definitions (3.27) and (3.28). By doing so, one obtains the following three different sets of constraining relations on PF BH charges:

$$\begin{array}{l} \mathbf{II.1}) \qquad n_2 = (a_{1,\pm} + a_2) n_1, \quad n_3 = n_1 (a_{1,\pm} - a_2), \quad n_4 = -n_1, \\ a_{1,\pm}(\vartheta) \equiv \pm \frac{\sqrt{2}}{\vartheta+1} \sqrt{\frac{3(2-\sqrt{2})\vartheta+\sqrt{2}}{(2-\sqrt{2})\vartheta-\sqrt{2}}}, \quad a_2(\vartheta) \equiv \frac{\sqrt{2}+\vartheta(2-\sqrt{2})}{\sqrt{2}(1+\vartheta)}; \end{array} \tag{6.30}$$

$$\mathbf{II.2}) \qquad n_2 = n_1 \frac{2+\sqrt{2}(\vartheta-1)}{(2-\sqrt{2})\vartheta-\sqrt{2}}, \quad n_3 = n_2, \quad n_4 = n_1; \tag{6.31}$$

$$\begin{array}{l} \mathbf{II.3}) \qquad n_2 + n_3 = a, \quad n_1 + n_4 = b_\pm, \quad n_2 - n_3 = c, \quad n_1 - n_4 = d, \\ a(\vartheta; b_\pm, c, d) \equiv -\frac{(1-(\sqrt{2}+1)\vartheta)c+(1+\vartheta)d}{(1+\vartheta)c-(1+(\sqrt{2}-1)\vartheta)d} b_\pm, \\ b_\pm(\vartheta; c, d) \equiv \pm \frac{\sqrt{-(\sqrt{2}+(2+\sqrt{2})\vartheta)c^2+2\sqrt{2}(\vartheta-1)cd+(\sqrt{2}+(-2+\sqrt{2})\vartheta)d^2}}{\sqrt{\frac{2(\sqrt{2}-3(2+\sqrt{2})\vartheta)c^2+4\sqrt{2}(1+3\vartheta)cd+2(-\sqrt{2}+3(-2+\sqrt{2})\vartheta)d^2}{(1+\vartheta)2c^2-2(1+\vartheta)(1+\vartheta(\sqrt{2}-1))cd+(1+\vartheta(-2+2\sqrt{2}+(3-2\sqrt{2})\vartheta))d^2}}}. \end{array} \tag{6.32}$$

By recalling that $\vartheta \approx 2.83$ (see Eq. (6.21)), the typology **II.1** of fine-tuning conditions for PF BH charges yields

$$n_2/n_1 \approx 2.44(-1.31), \quad n_3/n_1 \approx 1.31(-2.44), \tag{6.33}$$

where, here and below, the numbers in round brackets correspond to consider $a_{1,-}$, rather than $a_{1,+}$, in Eqs. (6.30). Since the PF BH charges are integers as are the symplectic BH charges (see definition (3.28)), the numerical conditions (6.33) can approximately be met by taking *e.g.*

$$n_1 = 100, \quad n_2 = 244(-131), \quad n_3 = 131(-244), \quad n_4 = -100. \tag{6.34}$$

Switching to the symplectic (electric/magnetic) basis for BH charges by using Eq. (3.29), one gets

$$\begin{array}{l} p^0 = 2n_1 + n_4 = 100(100), \quad p^1 = \frac{1}{2}\left[-3(n_1+n_4) + n_2 + n_3\right] = 187(-187), \\ q_0 = \frac{1}{2}(n_1 + n_4) = 0(0), \quad q_1 = \frac{1}{2}(-2n_1 + n_2 - n_4) = 72(-115). \end{array} \tag{6.35}$$

For what concerns the typology **II.2** of fine-tuning conditions for PF BH charges, it is worth remarking that Eq. (6.31) is the analogue for the mirror *octic* $\mathcal{M}'_8$ of the fine-tuning condition (4.33) adopted for the mirror *quintic* $\mathcal{M}'_5$

in [9] (in particular, see Sect. 4 and App. C of such a Ref.). By recalling that $\vartheta \approx 2.83$ (see Eq. (6.21)), the typology **II.2** of fine-tuning conditions for PF BH charges yields

$$n_2/n_1 \approx 18.6. \tag{6.36}$$

Once again, since the PF BH charges are integers as are the symplectic BH charges, the numerical conditions (6.36) can approximately be met by taking *e.g.*

$$n_1 = 10, \quad n_2 = 186, \quad n_3 = 186, \quad n_4 = 10. \tag{6.37}$$

Switching to the symplectic (electric/magnetic) basis for BH charges by using Eq. (3.29), one obtains

$$\begin{aligned} &p^0 = 2n_1 + n_4 = 30, \quad p^1 = \tfrac{1}{2}\left[-3(n_1+n_4) + n_2 + n_3\right] = 156, \\ &q_0 = \tfrac{1}{2}(n_1+n_4) = 10, \quad q_1 = \tfrac{1}{2}(-2n_1 + n_2 - n_4) = 78. \end{aligned} \tag{6.38}$$

Once again, it is worth remarking that *all* three distinct sets of fine-tuning conditions for PF BH charges (6.30)-(6.32) do support a non-BPS, $Z \neq 0$ LG attractor in a strict sense.

The "effective BH potential" and BH entropy at such a (class of) non-BPS, $Z \neq 0$ LG attractor(s) take the values

$$V_{BH,non-BPS,Z\neq 0} \approx 0.211|N_3|^2, \quad S_{BH,non-BPS,Z\neq 0} \approx 0.211\pi|N_3|^2, \tag{6.39}$$

where N_3 is given by Eq.(6.28), implemented by one of the fine-tuning conditions (6.30)-(6.32).

Finally, by recalling the definition (1.29), one can compute the supersymmetry-breaking order parameter for the non-BPS, $Z \neq 0$ LG attractor in the mirror *octic* $\mathcal{M}'_8$; by using Eqs. (6.4), (6.6), (6.7) and (6.26), one gets

$$\mathcal{O}_{non-BPS,Z\neq 0} \equiv \left[\frac{\left(g_{\psi\overline{\psi}}\right)^{-1}|D_\psi W|^2}{|W|^2}\right]_{non-BPS,Z\neq 0} = \left[\frac{\left(g_{\psi\overline{\psi}}\right)^{-1}|D_\psi \widetilde{W}|^2}{|\widetilde{W}|^2}\right]_{non-BPS,Z\neq 0} \approx \frac{3+2\sqrt{2}}{\vartheta^2} \approx 0.72. \tag{6.40}$$

7 $k = 10$: Mirror *Dectic*

For the mirror *dectic* $\mathcal{M}'_{10}$ the computations (*but not the results!*) go the same way as for the mirror *octic* $\mathcal{M}'_8$.

For $k = 10$ the definitions (3.12) yield

$$C_{2l-1} = C_{5l-1} = 0, \quad l \in \mathbb{N}; \tag{7.1}$$

moreover, since $F_{10,m+10,n} = F_{10,m,n+10} = -F_{10,mn}$ (see the third of properties (3.22)), the only independent elements of the rank-2 tensor F_{10} belong to the 10×10 matrix

$$F_{10,m\,n} =$$

$$\begin{pmatrix}
\sqrt{5(5+2\sqrt{5})} & -3-\sqrt{5} & 0 & 3+\sqrt{5} & -\sqrt{5(5+2\sqrt{5})} & 2+\sqrt{5} & 0 & -2 & 0 & 2+\sqrt{5} \\
-3-\sqrt{5} & \sqrt{5+2\sqrt{5}} & 3-\sqrt{5} & -\sqrt{2(5+\sqrt{5})} & 2+\sqrt{5} & -\sqrt{10-2\sqrt{5}} & -2+\sqrt{5} & 0 & 2 & -\sqrt{25-2\sqrt{5}} \\
0 & 3-\sqrt{5} & -\sqrt{5(5-2\sqrt{5})} & 2 & -\sqrt{5(5-2\sqrt{5})} & 3-\sqrt{5} & 0 & 2-\sqrt{5} & 0 & -2+\sqrt{5} \\
3+\sqrt{5} & -\sqrt{2(5+\sqrt{5})} & 2 & -\sqrt{5-2\sqrt{5}} & -2+\sqrt{5} & 0 & -3+\sqrt{5} & \sqrt{10-2\sqrt{5}} & -2-\sqrt{5} & \sqrt{25+2\sqrt{5}} \\
-\sqrt{5(5+2\sqrt{5})} & 2+\sqrt{5} & -\sqrt{5(5-2\sqrt{5})} & -2+\sqrt{5} & 0 & 2-\sqrt{5} & \sqrt{5(5-2\sqrt{5})} & -2-\sqrt{5} & \sqrt{5(5+2\sqrt{5})} & 8 \\
2+\sqrt{5} & -\sqrt{10-2\sqrt{5}} & 3-\sqrt{5} & 0 & 2-\sqrt{5} & \sqrt{5-2\sqrt{5}} & -2 & \sqrt{2(5+\sqrt{5})} & -3-\sqrt{5} & \sqrt{25+2\sqrt{5}} \\
0 & -2+\sqrt{5} & 0 & -3+\sqrt{5} & \sqrt{5(5-2\sqrt{5})} & -2 & \sqrt{5(5-2\sqrt{5})} & -3+\sqrt{5} & 0 & -2+\sqrt{5} \\
-2 & 0 & 2-\sqrt{5} & \sqrt{10-2\sqrt{5}} & -2-\sqrt{5} & \sqrt{2(5+\sqrt{5})} & -3+\sqrt{5} & -\sqrt{5+2\sqrt{5}} & 3+\sqrt{5} & -\sqrt{25-2\sqrt{5}} \\
0 & 2 & 0 & 2-\sqrt{5} & \sqrt{5(5+2\sqrt{5})} & -3-\sqrt{5} & 0 & 3+\sqrt{5} & -\sqrt{5(5+2\sqrt{5})} & 2+\sqrt{5} \\
2+\sqrt{5} & -\sqrt{25-2\sqrt{5}} & -2+\sqrt{5} & \sqrt{25+2\sqrt{5}} & 8 & \sqrt{25+2\sqrt{5}} & -2+\sqrt{5} & -\sqrt{25-2\sqrt{5}} & 2+\sqrt{5} & 0
\end{pmatrix} \tag{7.2}$$

Let us now write down all the relevant quantities up to the needed order (here and below, unless otherwise specified, we omit the Fermat parameter $k = 10$):

$$\widetilde{K} \approx (5-\sqrt{2})\tfrac{C_2^2}{C_0^2}(\psi\bar{\psi})^2\left[1+\tfrac{(5-\sqrt{2})}{2}\tfrac{C_2^2}{C_0^2}(\psi\bar{\psi})^2\right]+\mathcal{O}(\psi^9); \tag{7.3}$$

$$g_{\psi\bar{\psi}} \approx 4(5-\sqrt{2})\tfrac{C_2^2}{C_0^2}\psi\bar{\psi}\left[1+2(5-\sqrt{2})\tfrac{C_2^2}{C_0^2}(\psi\bar{\psi})^2\right]+\mathcal{O}(\psi^7); \tag{7.4}$$

$$\widetilde{W} \approx N_1+\tfrac{C_2}{C_0}N_3\psi^2-\tfrac{C_6}{C_0}\bar{N}_3\psi^6+\mathcal{O}(\psi^8). \tag{7.5}$$

Now, by using the formulæ of the general analysis exploited in Sect. 3, we can get the "effective BH potential" and the holomorphic superpotential, as well as their (covariant) derivatives, up to $\mathcal{O}\left(\psi^2\right)$:

$$\widetilde{W} = N_1+\tfrac{C_2}{C_0}N_3\psi^2; \tag{7.6}$$

$$D_\psi\widetilde{W} = 2\tfrac{C_2}{C_0}N_3\psi+2(\sqrt{5}-2)\tfrac{C_2^2}{C_0^2}N_1\psi\bar{\psi}^2; \tag{7.7}$$

$$D_\psi D_\psi\widetilde{W} = -24\tfrac{C_6}{C_0}\bar{N}_3\psi^4; \tag{7.8}$$

$$\widetilde{V}_{BH} = |N_1|^2+(\sqrt{5}+2)|N_3|^2+2\tfrac{C_2}{C_0}N_3\bar{N}_1\psi^2+2\tfrac{C_2}{C_0}\bar{N}_3N_1\bar{\psi}^2; \tag{7.9}$$

$$\partial_\psi \widetilde{V}_{BH} = 4\psi \left(\tfrac{C_2}{C_0} N_3 \bar{N}_1 - 3(\sqrt{5}+2) \tfrac{C_6}{C_2} (\bar{N}_3)^2 \psi^2 + \tfrac{C_2^2}{C_0^2} \left(|N_1|^2 (\sqrt{5}-2) + |N_3|^2 \right) \bar{\psi}^2 \right); \tag{7.10}$$

$$\partial_\psi \partial_\psi \widetilde{V}_{BH} = 4 \left(\tfrac{C_2}{C_0} N_3 \bar{N}_1 - 9(\sqrt{5}+2) \tfrac{C_6}{C_2} (\bar{N}_3)^2 \psi^2 + \tfrac{C_2^2}{C_0^2} \left(|N_1|^2 (\sqrt{5}-2) + |N_3|^2 \right) \bar{\psi}^2 \right); \tag{7.11}$$

$$\partial_\psi \overline{\partial}_{\bar{\psi}} \widetilde{V}_{BH} = 8 \tfrac{C_2^2}{C_0^2} \left(|N_1|^2 (\sqrt{5}-2) + |N_3|^2 \right) \psi \bar{\psi}. \tag{7.12}$$

Contrary to the cases of mirror *quintic* and *sextic* (see Sects. 4 and 5, respectively), and similarly to the case of mirror *octic* (see Sect. 6), for the case of mirror *dectic* it is evident from Eqs. (7.9)-(7.12) that we truncate the series expansion of the "effective BH potential" and of its second derivatives around the LG point up to $\mathcal{O}\left(\psi^2\right)$ included, and the series expansion of its first derivative around the LG point up to $\mathcal{O}\left(\psi^3\right)$ included. This is due to the absence of an $\mathcal{O}(\psi)$ term in expression of $\widetilde{V}_{BH}$ given by Eq. (7.9). As mentioned in Sect. 6, such a fact can be traced back to the non-regularity of $g_{\psi\bar{\psi}}$ at $\psi = 0$ (see Eq. (7.4)).

Let us now find the solutions of the AE $\partial_\psi \widetilde{V}_{BH}\left(\psi, \overline{\psi}; q, p\right) = 0$, and check their stability. Since we are working near the LG point, by using Eq. (7.9) we can rewrite the AE for $\mathcal{M}'_{10}$ as follows:

$$\frac{C_2}{C_0} N_3 \bar{N}_1 + \frac{C_2^2}{C_0^2} \left(|N_1|^2 (\sqrt{5}-2) + |N_3|^2 \right) \bar{\psi}^2 \approx 3(\sqrt{5}+2) \frac{C_6}{C_2} (\bar{N}_3)^2 \psi^2 \tag{7.13}$$

Solving Eq. (6.13), we will find one (or more) set(s) of BH charges supporting $\psi \approx 0$ to be a critical point of V_{BH}. Since we are working near the LG point, it is clear that the first term in the l.h.s. of Eq. (7.13) must be small enough. This implies the following fine-tuning condition:

$$N_3 \bar{N}_1 \approx 0. \tag{7.14}$$

By using Eqs. (4.7)-(4.12) of [15] and Eqs. (7.11)-(7.12) evaluated along the criticality condition (7.13)-(7.14), it can be computed that the components of $H^{\widetilde{V}_{BH}}_{\text{real form}}$ (given by Eq. (4.14)) constrained by the Eqs. (7.13)-(7.14) read as follows:

$$\mathcal{A} = -6(\sqrt{5}+2) \frac{C_6}{C_2} \left((\bar{N}_3)^2 \psi^2 + (N_3)^2 \bar{\psi}^2 \right) + 4 \frac{C_2^2}{C_0^2} \psi \bar{\psi} \left(|N_1|^2 (\sqrt{5}-2) + |N_3|^2 \right); \tag{7.15}$$

$$\mathcal{B} = 6(\sqrt{5}+2) \frac{C_6}{C_2} \left((\bar{N}_3)^2 \psi^2 + (N_3)^2 \bar{\psi}^2 \right) + 4 \frac{C_2^2}{C_0^2} \psi \bar{\psi} \left(|N_1|^2 (\sqrt{5}-2) + |N_3|^2 \right); \tag{7.16}$$

$$\mathcal{C} = -6i(\sqrt{5}+2) \frac{C_6}{C_2} \left((N_3)^2 \bar{\psi}^2 - (\bar{N}_3)^2 \psi^2 \right). \tag{7.17}$$

The resulting real eigenvalues of $H^{\widetilde{V}_{BH}}_{\text{real form}}$ constrained by Eqs. (7.13)-(7.14) read:

$$\lambda_\pm \approx 4\psi\bar{\psi} \left[\frac{C_2^2}{C_0^2} \left(|N_1|^2 (\sqrt{5}-2) + |N_3|^2 \right) \pm 3(\sqrt{5}+2) \frac{C_6}{C_2} |N_3|^2 \right]. \tag{7.18}$$

By recalling Eq. (3.30) and using Eq. (7.9) with $\psi \approx 0$ and constrained by by Eqs. (7.13)-(7.14), one obtains that the purely charge-dependent LG critical values of the "effective BH potential" for the mirror *dectic* $\mathcal{M}'_{10}$ are

$$V_{BH,LG-critical,k=10} \approx \frac{1}{\sqrt{5(5+2\sqrt{5})}}\left[|N_1|^2 + (\sqrt{5}+2)|N_3|^2\right]; \tag{7.19}$$

by recalling formula (1.2), this directly yields the following purely charge-dependent values of the BH entropy at the LG critical points of $V_{BH,10}$ in the moduli space of $\mathcal{M}'_{10}$:

$$S_{BH,LG-critical,k=10} \approx \frac{\pi}{\sqrt{5(5+2\sqrt{5})}}\left[|N_1|^2 + (\sqrt{5}+2)|N_3|^2\right]. \tag{7.20}$$

In both Eqs. (7.19) and (7.20) N_1 and N_3 are given by

$$N_1 = -\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}\left(n_1 + n_4 + \frac{(1+\sqrt{5})}{2}(n_2+n_3)\right) + i\frac{\sqrt{5}+1}{4}\left(n_4 - n_1 + \frac{3-\sqrt{5}}{2}(n_3-n_2)\right); \tag{7.21}$$

$$N_3 = \frac{1}{2}\sqrt{\frac{(5-\sqrt{5})}{2}}\left(n_1 + n_4 - \frac{\sqrt{5}-1}{2}(n_2+n_3)\right) + i\frac{\sqrt{5}-1}{4}\left(n_4 - n_1 + \frac{3+\sqrt{5}}{2}(n_3-n_2)\right). \tag{7.22}$$

Let us write down here the numerical values of constants relevant to our treatment:

$$C_0 \approx 1.57, \quad C_2 \approx -0.66, \quad C_6 \approx 0.077; \quad 3(\sqrt{5}+2)\frac{C_6}{C_2} \approx -1.48. \tag{7.23}$$

Let us now analyze more in depth the species of LG attractor points arising from the AE (7.14). As it can be easily seen, the AE (7.14) has two *non-degenerate* solutions:

As it can be easily seen, also in this case the attractor equation (7.14) has two (non-degenerate) solutions:

I. The first non-degenerate solution to AE (7.14) is

$$N_3 \approx 0. \tag{7.24}$$

This is nothing but the $k = 8$ solution (6.22). As one can see from Eq. (7.6)-(7.7), also for $k = 10$ such a solution corresponds to a $\frac{1}{2}$-BPS LG critical point of V_{BH} ($\widetilde{W} \neq 0$, $D_\psi \widetilde{W} = 0$). From the definitions (3.27) and (3.28), in order to get the solution (7.24), we have to fine-tune 2 PF BH charges out of 4 in the following way:

$$n_3 \approx \frac{1}{2}(\sqrt{5}-1)(n_1+n_2), \quad n_4 \approx -\frac{1}{2}(\sqrt{5}-1)n_1 + n_2. \tag{7.25}$$

The charges n_1, n_2 are not fixed; they only satisfy the non-degeneration condition $N_1 \neq 0$. As it was for $k = 5, 6, 8$, also the real eigenvalues (7.18) for the $\frac{1}{2}$-BPS critical solution coincide and, as it is well known [31, 15, 16], are strictly positive:

$$\lambda_{+,\frac{1}{2}-BPS} = \lambda_{-,\frac{1}{2}-BPS} \approx 4(\sqrt{5}-2)\psi\bar{\psi}\frac{C_2^2}{C_0^2}|N_1|^2 > 0. \tag{7.26}$$

Consequently, the $\frac{1}{2}$-BPS LG critical point $\psi \approx 0$ supported by the PF BH charge configuration (7.25) is a stable extremum, since it is a (local) minimum of V_{BH}, and it is therefore an attractor in a strict sense. The "effective BH potential" and BH entropy at such a (class of) $\frac{1}{2}$-BPS LG attractor(s) take the values

$$V_{BH,\frac{1}{2}-BPS} \approx 0.166|N_1|^2, \quad S_{BH,\frac{1}{2}-BPS} \approx 0.166\pi|N_1|^2, \tag{7.27}$$

where

$$N_1 \approx -\frac{\sqrt{5}}{2}\left(\sqrt{5-2\sqrt{5}}n_1 + \sqrt{\frac{5+\sqrt{5}}{2}}n_2 + i\left(n_1 - \frac{\sqrt{5}-1}{2}n_2\right)\right). \tag{7.28}$$

II. The second non-degenerate solution to AE (7.14) is

$$N_1 \approx 0. \tag{7.29}$$

Interestingly, this is nothing but the $k = 6$ solution (5.28). As one can see from Eq. (7.6)-(7.7), also for $k = 10$ such a solution corresponds to a non-BPS, $Z = 0$ LG critical point of V_{BH} ($\widetilde{W} = 0$, $D_\psi \widetilde{W} \neq 0$). The real eigenvalues (7.18) for such a non-BPS, $Z = 0$ critical solution read

$$\lambda_{\pm,non-BPS,Z=0} \approx 4\psi\bar{\psi}|N_3|^2 \left[\frac{C_2^2}{C_0^2} \pm 3(\sqrt{5}+2)\frac{C_6}{C_2}\right]. \tag{7.30}$$

Substituting the numerical values (7.23) of the involved constants in Eq. (7.30), one reaches the conclusion that one eigenvalue is positive and the other one is negative:

$$\lambda_{\pm,non-BPS,Z=0} \approx 4\psi\bar{\psi}|N_3|^2 \left[0.18 \mp 1.48\right] \lessgtr 0. \tag{7.31}$$

Let us now find the fine-tuning conditions for PF BH charges supporting the considered non-BPS, $Z = 0$ LG attractor for the mirror *dectic* $\mathcal{M}'_{10}$. This amounts to solving Eq. (7.29) by recalling the definitions (3.27) and (3.28). By doing so, one gets the following unique set of constraining relations on PF BH charges:

$$n_3 \approx -\frac{1}{2}(1+\sqrt{5})(n_1+n_2), \quad n_4 \approx \frac{1}{2}(1+\sqrt{5})n_1 + n_2. \tag{7.32}$$

Thus, the non-BPS, $Z = 0$ LG critical point $\psi \approx 0$ supported by the PF BH charge configuration (7.32) is a saddle point of V_{BH} and consequently it is *not* an attractor in a strict sense.

The "effective BH potential" and BH entropy at such a (class of) non-BPS, $Z = 0$ LG saddle point(s) take the values

$$V_{BH,non-BPS,Z=0} \approx 0.7|N_3|^2, \quad S_{BH,non-BPS,Z=0} \approx 0.7\pi|N_3|^2, \tag{7.33}$$

where

$$N_3 \approx \frac{\sqrt{5}}{2}\left(\sqrt{5+2\sqrt{5}}n_1 + \sqrt{\frac{5-\sqrt{5}}{2}}n_2 - i\left(n_1 + \frac{\sqrt{5}+1}{2}n_2\right)\right). \tag{7.34}$$

Similarly to the treatment of Sects. 4-6, one can also switch to the symplectic (electric/magnetic) basis for BH charges by using Eq. (3.29), re-expressing the fine-tuning condition (7.32) in terms of the symplectic BH charges $\Gamma \equiv (-p^0, -p^1, q_0, q_1)$ (see definitions (3.24)).

8 Special Kähler Geometrical Identities and Fermat CY_3s

Let us now consider the real part of the $n_V = 1$ case of SK geometry identities (1.16) [43, 8, 13, 15, 16, 24]; by taking into account the change in the notation of the symplectic charge vectors with respect to the notation used in Sects. 1 and 2 (see Footnote before Eq. (3.24)), one achieves:

$$\widetilde{\Gamma}^T = 2e^K Im \left[W\overline{\Pi} + \left(g_{\psi\overline{\psi}} \right)^{-1} \bar{D}_{\bar{\psi}} \overline{W} D_\psi \Pi \right], \tag{8.1}$$

where the 1×4 BH charge vector in the symplectic basis $\widetilde{\Gamma}$ is defined in Eq. (3.24).

Next, let us switch to more convenient variables for the treatment of 1-modulus SK geometries endowing the moduli space of Fermat CY_3s. By recalling the definition (3.28) of the 1×4 PF BH charge vector n, we can rewrite Eq. (8.1) as follows (here and below, unless otherwise specified, we omit the classifying Fermat parameter $k = 5, 6, 8, 10$):

$$n^T = \frac{2}{\chi F_{1\,1}} e^{\widetilde{K}} Im \left[\widetilde{W} \overline{\widetilde{\Phi}} + \left(g_{\psi\overline{\psi}} \right)^{-1} \bar{D}_{\bar{\psi}} \overline{\widetilde{W}} D_\psi \widetilde{\Phi} \right], \tag{8.2}$$

where the notations introduced in Sect. (3) have been used. Furthermore, we defined the 4×1 holomorphic vector

$$\widetilde{\Phi} \equiv \frac{1}{AC_0} M^T \Sigma^T \Pi = \frac{1}{AC_0} m^T \varpi, \tag{8.3}$$

where Eqs. (3.14) and (3.19) have been used in the second step.

Firstly, let us investigate Eq.(8.2) in a certain neighbourhood of the LG point $\psi = 0$. The treatment given in Sects.

4-7 yields that, by its very definition (8.3), $\widetilde{\Phi}$ has the following series expansion near the LG point[20]:

$$k=5: \quad \left\{ \begin{array}{l} \widetilde{\Phi}\left(\psi\right) = \varphi^1 + \frac{C_1}{C_0}\varphi^2\psi, \\ \\ D_\psi\widetilde{\Phi} = \frac{C_1}{C_0}\left[\varphi^2 + 2\frac{C_2}{C_1}\overline{\varphi}^2\psi + \frac{C_1}{C_0}(\sqrt{5}-2)\varphi^1\bar{\psi}\right]; \end{array} \right. \tag{8.4}$$

$$k=6: \left\{ \begin{array}{l} \widetilde{\Phi}\left(\psi\right) = \varphi^1 + \frac{C_1}{C_0}\varphi^2\psi, \\ \\ D_\psi\widetilde{\Phi} = \frac{C_1}{C_0}\left[\varphi^2 + \frac{1}{3}\frac{C_1}{C_0}\varphi^1\bar{\psi}\right]; \end{array} \right. \tag{8.5}$$

$$k=8: \left\{ \begin{array}{l} \widetilde{\Phi}\left(\psi\right) = \varphi^1 + \frac{C_2}{C_0}\varphi^3\psi^2, \\ \\ D_\psi\widetilde{\Phi} = 2\frac{C_2}{C_0}\psi\left[\varphi^3 - 2\frac{C_4}{C_2}\bar{\varphi}^3\psi^2 + \frac{C_2}{C_0}(3-2\sqrt{2})\varphi^1\bar{\psi}^2\right]; \end{array} \right. \tag{8.6}$$

$$k=10: \quad \left\{ \begin{array}{l} \widetilde{\Phi}\left(\psi\right) = \varphi^1 + \frac{C_2}{C_0}\varphi^3\psi^2, \\ \\ D_\psi\widetilde{\Phi} = 2\frac{C_2}{C_0}\psi\left[\varphi^3 + \frac{C_2}{C_0}(\sqrt{5}-2)\varphi^1\bar{\psi}^2\right]. \end{array} \right. \tag{8.7}$$

We defined the complex 4×1 vector

$$\varphi_k^m \equiv m_k \begin{pmatrix} \beta_k^{3m} \\ \\ \beta_k^m \\ \\ \beta_k^{-m} \\ \\ \beta_k^{-3m} \end{pmatrix}, \quad (m=1,2,3), \tag{8.8}$$

[20]Notice that, consistently with the approach to the truncation of series expansions near the LG point performed in Sects. 4-7, for $k=5,6$ we truncate up to $\mathcal{O}\left(\psi\right)$ included, whereas for $k=8,10$ we truncate up to $\mathcal{O}\left(\psi^2\right)$ included.

whose explicit forms relevant for the series expansion (8.4)-(8.7) read as follows:

$$k=5:\ \ \varphi^1=\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}\begin{pmatrix}-\sqrt{5+2\sqrt{5}}+i\sqrt{5}\\ -\sqrt{\frac{5+\sqrt{5}}{2}}+\frac{1}{2}i(5+3\sqrt{5})\\ \sqrt{\frac{5+\sqrt{5}}{2}}+\frac{1}{2}i(5+3\sqrt{5})\\ \sqrt{5+2\sqrt{5}}+i\sqrt{5}\end{pmatrix},\qquad \varphi^2=\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}\begin{pmatrix}\sqrt{5-2\sqrt{5}}-i\sqrt{5}\\ -\sqrt{\frac{5-\sqrt{5}}{2}}+\frac{1}{2}i(5-3\sqrt{5})\\ \sqrt{\frac{5-\sqrt{5}}{2}}+\frac{1}{2}i(5-3\sqrt{5})\\ -\sqrt{5-2\sqrt{5}}-i\sqrt{5}\end{pmatrix};\ \ (8.9)$$

$$k=6:\qquad \varphi^1=\frac{3}{2}\begin{pmatrix}-\sqrt{3}+i\\ 2i\\ 2i\\ \sqrt{3}+i\end{pmatrix},\qquad \varphi^2=\frac{1}{2}\begin{pmatrix}1-i\sqrt{3}\\ -1\\ 1\\ -1-i\sqrt{3}\end{pmatrix};\tag{8.10}$$

$$k=8:\ \ \varphi^1=\frac{1}{2}\sqrt{2+\sqrt{2}}\begin{pmatrix}-2-\sqrt{2}+i\sqrt{2}\\ -\sqrt{2}+i(2+\sqrt{2})\\ \sqrt{2}+i(2+\sqrt{2})\\ 2+\sqrt{2}+i\sqrt{2}\end{pmatrix},\qquad \varphi^3=\frac{1}{2}\sqrt{2-\sqrt{2}}\begin{pmatrix}-2+\sqrt{2}+i\sqrt{2}\\ \sqrt{2}+i(-2+\sqrt{2})\\ -\sqrt{2}+i(-2+\sqrt{2})\\ 2-\sqrt{2}+i\sqrt{2}\end{pmatrix};\tag{8.11}$$

$$k=10:\ \ \varphi^1=\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}\begin{pmatrix}-\frac{1}{2}\sqrt{3+\sqrt{5}}+i\sqrt{\frac{5+\sqrt{5}}{10}}\\ 1+i\sqrt{1+\frac{2}{\sqrt{5}}}\\ -1+i\sqrt{1+\frac{2}{\sqrt{5}}}\\ \frac{1}{2}\sqrt{3+\sqrt{5}}+i\sqrt{\frac{5+\sqrt{5}}{10}}\end{pmatrix},\qquad \varphi^3=\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}\begin{pmatrix}\frac{1}{2}\sqrt{-3+\sqrt{5}}+i\sqrt{\frac{5-\sqrt{5}}{10}}\\ 1-i\sqrt{1-\frac{2}{\sqrt{5}}}\\ -1-i\sqrt{1-\frac{2}{\sqrt{5}}}\\ \frac{1}{2}\sqrt{3-\sqrt{5}}+i\sqrt{\frac{5-\sqrt{5}}{10}}\end{pmatrix}.\tag{8.12}$$

Using Eqs. (8.3)-(8.12), one obtains that Eq. (8.2) near the LG point of the moduli space of Fermat CY_3-compactifications (when consistently truncated up to the order in ψ considered above) reads

$$k\ =\ 5,\, k=6:\quad n^T=\frac{2}{\chi F_{1\,1}}Im\left[N_1\bar{\varphi}^1-\frac{F_{1\,1}}{F_{2\,2}}\bar{N}_2\varphi^2\right];\tag{8.13}$$

$$k\ =\ 8,\, k=10:\quad n^T=\frac{2}{\chi F_{1\,1}}Im\left[N_1\bar{\varphi}^1-\frac{F_{1\,1}}{F_{3\,3}}\bar{N}_3\varphi^3\right].\tag{8.14}$$

Thence, it is easy to check that, substituting the explicit forms of $N_{k,m}(q,p)$, φ_k^m and $F_{k,m\,n}$ (see Eqs. (3.27), (8.8) and (3.21), respectively) in Eqs. (8.13)-(8.14), they become trivial identities, yielding nothing but

$$n_1=n_1,\quad n_2=n_2,\quad n_3=n_3,\quad n_4=n_4.\tag{8.15}$$

In other words, the 4 real Eqs. (8.2) are not equations, but rather they are identities. Therefore, they are satisfied at every point in the moduli space and for every BH charge configuration. Therefore, it is no surprise if, when evaluating them in a certain neighbourhood of the LG point as we did, one finds the identical relations (8.15). Thus, we found nothing new but another confirmation of a well known fact of SK geometry [43, 8, 13, 15, 16, 24].

However, the 4 real identities (8.2) can still be used to find extremal BH attractors, when properly evaluated along the constraints defining the various species of such attractors satisfying the criticality condition of the "effective BH potential" V_{BH}. Put another way, in the 1-modulus case with which we are concerned, when evaluated at the geometrical *loci* in the moduli space defining the various tipologies (*i.e.* $\frac{1}{2}$-BPS, non-BPS $Z\neq 0$ and non-BPS $Z=0$)

of attractors, the 4 real identities (8.2) become 4 real equations. These are equivalent to the 2 real equations given by the real and imaginary part of the criticality condition $\partial_\psi V_{BH} = 0$. This approach has recently been used in [8] for the general n_V-moduli case, and then further investigated in [13]. The SK geometrical identities in the general n_V-moduli case had previously been formulated in [43] in terms of the decomposition of the third real cohomology $H^3(CY_3;\mathbb{R})$ of the CY_3 in the Dolbeaut cohomology basis (see [43], and [38] for further Refs.).

In the remaining part of the present Section we will focus on the 1-modulus case related to Fermat CY_3-compactifications, and we will evaluate the 4 real identities (8.2) at the geometrical *loci* in the moduli space defining the various classes of extremal BH attractors. We will consequently show that solving the obtained 4 real equations is equivalent to solving the 2 real equations corresponding to the real and imaginary parts of the criticality condition (3.33). Thus, it follows that only 2 equations are independent out of the starting 4 ones. From a computational point of view, one can realize that, at least in the framework we are considering, the *"criticality condition" approach* is simpler than the *"SK geometrical identities" approach*, at least for the non-BPS, $Z \neq 0$ case.

Let us now evaluate the 4 real SK identities (8.2) along the 3 geometrical *loci* defining the 3 species of critical points of V_{BH} arising in SK geometry.

$\frac{1}{2}$-BPS critical points. The corresponding geometrical *locus* in the moduli space is given by the constraints $\widetilde{W} \neq 0$, $D_\psi \widetilde{W} = 0$, which directly solve the criticality condition (*i.e.* the 1-modulus AE) (3.33). By evaluating the 4 real SK identities (8.2) along such critical constraints, one gets

$$n^T = \frac{2}{\chi F_{1\,1}} \left\{ e^{\widetilde{K}} Im \left[\widetilde{W} \overline{\widetilde{\Phi}} \right] \right\}_{\frac{1}{2}-BPS}. \tag{8.16}$$

Such 4 real equations constrain the PF BH charge configurations along the *locus* $\widetilde{W} \neq 0$, $D_\psi \widetilde{W} = 0$ of $\frac{1}{2}$-BPS critical points of V_{BH} in the moduli space ($dim_{\mathbb{C}} = 1$) of Fermat CY_3s. One can explicitly check that for all Fermat CY_3s the solutions of the 4 real Eqs. (8.16) near the LG point give nothing but the $\frac{1}{2}$-BPS-supporting PF BH charge configurations previously computed in Sects. 4-7 exploiting the so-called *"criticality condition" approach*.

non-BPS, $Z \neq 0$ critical points. The corresponding geometrical *locus* in the moduli space is given by the constraints $\widetilde{W} \neq 0$, $D_\psi \widetilde{W} \neq 0$, which, by the criticality condition (3.33) and the definition (3.31), yield

$$\left(\overline{D}_{\overline{\psi}} \overline{\widetilde{W}} \right)_{non-BPS,Z\neq 0} = - \left[\frac{(g_{\psi\bar{\psi}})^{-1} \left(\overline{D}_{\overline{\psi}} \overline{D}_{\overline{\psi}} \overline{\widetilde{W}} \right) D_\psi \widetilde{W}}{2 \widetilde{W}} \right]_{non-BPS,Z\neq 0}. \tag{8.17}$$

By inserting Eq. (8.17) in the 4 real SK identities (8.2), one obtains [8, 13, 16]

$$\begin{aligned} n^T &= \frac{2}{\chi F_{1\,1}} \left\{ e^{\widetilde{K}} Im \left[\widetilde{W} \overline{\widetilde{\Phi}} - \frac{(g_{\psi\bar{\psi}})^{-2} \left(\overline{D}_{\overline{\psi}} \overline{D}_{\overline{\psi}} \overline{\widetilde{W}} \right) D_\psi \widetilde{W}}{2 \widetilde{W}} D_\psi \widetilde{\Phi} \right] \right\}_{non-BPS,Z\neq 0} = \qquad (8.18) \\ &= \frac{2}{\chi F_{1\,1}} \left\{ e^{\widetilde{K}} Im \left[\widetilde{W} \overline{\widetilde{\Phi}} + i \frac{(g_{\psi\bar{\psi}})^{-3} C_{\psi\psi\psi} \left(\overline{D}_{\overline{\psi}} \overline{\widetilde{W}} \right)^2}{2 \overline{\widetilde{W}}} \overline{D}_{\overline{\psi}} \overline{\widetilde{\Phi}} \right] \right\}_{non-BPS,Z\neq 0}, \qquad (8.19) \end{aligned}$$

where, in the second line, we used the $n_V = 1$ case of the second SK differential relation of (1.12), yielding

$$D_\psi D_\psi \widetilde{W} = iC_{\psi\psi\psi}(g_{\psi\bar{\psi}})^{-1}\overline{D}_{\overline{\psi}}\overline{\widetilde{W}} \tag{8.20}$$

at every point in the moduli space. The 4 real Eqs. (8.18)-(8.19) constrain the PF BH charge configurations along the *locus* (8.17) of non-BPS $Z \neq 0$ critical points of V_{BH} in the moduli space ($dim_{\mathbb{C}} = 1$) of Fermat CY_3s.

Let us for example consider the mirror *quintic* $\mathcal{M}'_5$. From the treatment given in Sect. 4 and above, the 4 real Eqs. (8.18) take the following form near the LG point:

$$n^T = \frac{2}{\chi F_{1\,1}} Im\left[N_1\bar{\varphi}^1 - (\sqrt{5}+2)\xi\frac{N_2 N_2}{N_1}\varphi^2\right], \tag{8.21}$$

where ξ is defined in Eq. (4.27). Substituting into Eqs. (8.21) the explicit expressions for the Ns (see Sect. 4) and the φs (see Eq. (8.9)) and performing long but straightforward computations, it can be shown that one generally recovers all the three distinct sets of BH charge configurations (4.32)-(4.34) supporting the considered non-BPS $Z \neq 0$ LG attractor. The same can be explicitly checked for the mirror *octic* $\mathcal{M}'_8$.

non-BPS, $Z = 0$ critical points. The corresponding geometrical *locus* in the moduli space is given by the constraints $\widetilde{W} = 0$, $D_\psi \widetilde{W} \neq 0$, which, by the criticality condition (3.33) and the definition (3.31), yield

$$\left(D_\psi D_\psi \widetilde{W}\right)_{non-BPS, Z=0} = 0. \tag{8.22}$$

By recalling Eq. (3.35), the replacement of $\widetilde{W} = 0$ and of the condition (8.22) into the 4 real SK geometrical identities (8.2) yields the following 4 real equations:

$$n^T = -\frac{2}{\chi F_{1\,1}}\left\{\left(g_{\psi\overline{\psi}}\right)^{-1} e^{\widetilde{K}} Im\left[\frac{\partial_\psi^2 \widetilde{W} + \left(\partial_\psi \widetilde{K}\right)\partial_\psi \widetilde{W}}{\partial_\psi\left[ln\left(g_{\psi\bar{\psi}}\right) - \widetilde{K}\right]}\overline{D}_{\overline{\psi}}\overline{\widetilde{\Phi}}\right]\right\}_{non-BPS, Z=0}. \tag{8.23}$$

Such 4 real equations constrain the PF BH charge configurations along the *locus* (8.22) of non-BPS $Z = 0$ critical points of V_{BH} in the moduli space ($dim_{\mathbb{C}} = 1$) of Fermat CY_3s.

Let us for example consider the mirror *sextic* $\mathcal{M}'_6$. As one can easily check by using Eqs. (3.35) and (5.6)-(5.8), in this case $\widetilde{W} = 0$ directly satisfies the criticality condition (3.33). Consequently, rather than Eqs. (8.23), in order to exploit the so-called *"SK geometrical identities" approach*, one can consider the 4 real equations

$$n^T = \frac{2}{\chi F_{1\,1}}\left\{\left(g_{\psi\overline{\psi}}\right)^{-1} e^{\widetilde{K}} Im\left[\overline{\partial}_{\bar{\psi}}\overline{\widetilde{W}} D_\psi \widetilde{\Phi}\right]\right\}_{non-BPS, Z=0}, \tag{8.24}$$

obtained from identities (8.2) by simply putting $\widetilde{W} = 0$ and by replacing $\overline{D}_{\bar{\psi}}\overline{\widetilde{W}}$ with $\overline{\partial}_{\bar{\psi}}\overline{\widetilde{W}}$, as implied by $\widetilde{W} = 0$. From the treatment of Sect. 5 and above, the 4 real Eqs. (8.24) take the following form near the LG point:

$$n^T = -\frac{2}{\chi F_{2\,2}} Im\left[\bar{N}_2\varphi^2\right]. \tag{8.25}$$

Substituting the explicit expressions for N_2 (see Eq. (5.33)) and for φ^2 (see Eq. (8.10)) into Eqs. (8.25), it can be shown that one obtains nothing but the fine-tuning conditions (5.31) for PF BH charges supporting the considered non-BPS $Z = 0$ LG attractor. The same can be explicitly checked for the mirror *dectic* $\mathcal{M}'_{10}$.

9 Consistent Normalization of Picard-Fuchs Equations for Fermat CY_3s

It is interesting to notice that Eq. (8.20) yields a way to compute the covariantly-holomorphic Yukawa coupling function $C_{\psi\psi\psi}\left(\psi,\overline{\psi}\right)$ along the *locus* $D_\psi\widetilde{W}\neq 0$ of $n_V=1$ SK manifolds, such as the moduli spaces of Fermat CY_3s. Indeed, Eq. (8.20) implies

$$iC_{\psi\psi\psi}=g_{\psi\bar{\psi}}\frac{D_\psi D_\psi\widetilde{W}}{\bar{D}_{\bar{\psi}}\overline{\widetilde{W}}},\ \left(D_\psi\widetilde{W}\neq 0\right). \tag{9.1}$$

Therefore, by employing the formulæ and the treatment given in Sects. 4-7, one can use Eq. (9.1) to compute $iC_{\psi\psi\psi}$ for all the Fermat CY_3s near the LG point $\psi=0$. Keeping only the first orders, the *LG limit* for $iC_{\psi\psi\psi}$ or all the Fermat CY_3s reads:

$$k=5:\quad iC_{\psi\psi\psi}=2(\sqrt{5}-2)\frac{C_1C_2}{C_0^2}; \tag{9.2}$$

$$k=6:\quad iC_{\psi\psi\psi}=-2\frac{C_1C_3}{C_0^2}\psi; \tag{9.3}$$

$$k=8:\quad iC_{\psi\psi\psi}=-16(3-2\sqrt{2})\frac{C_2C_4}{C_0^2}\psi^3; \tag{9.4}$$

$$k=10:\quad iC_{\psi\psi\psi}=-48(\sqrt{5}-2)\frac{C_2C_6}{C_0^2}\psi^5. \tag{9.5}$$

Now, by recalling the first of defining relations (2.34) and Eq. (2.30), one arrives at

$$a_{4,k}\left(\psi\right)\equiv\frac{1}{W_{\psi\psi\psi,k}}=i\frac{e^{K_k}}{iC_{\psi\psi\psi,k}}=i\frac{(Ord(G_k))^2}{(2\pi)^6C_{k,0}^2F_{k,11}}\frac{e^{\widetilde{K}_k}}{(iC_{\psi\psi\psi,k})}, \tag{9.6}$$

where the first definition of (3.31) has been used in the last step, and classifiying Fermat parameter k has been restored in the notation. By substituting Eqs. (9.2)-(9.5) in Eq. (9.6) and recalling the treatment of Sects. 4-7, one gets nothing but the LG limit of the holomorphic function $a_{4,k}\left(\psi\right)$ for Fermat CY_3s:

$$lim_{\psi\longrightarrow 0}a_{4,k}\left(\psi\right)=lim_{\psi\longrightarrow 0}i\frac{(Ord(G_k))^2}{(2\pi)^6C_{k,0}^2F_{k,11}}\frac{e^{\widetilde{K}_k}}{(iC_{\psi\psi\psi,k})}=i\frac{k\chi_k}{(2\pi)^3}\psi^{5-k}, \tag{9.7}$$

where the constants χ_ks are given in Table 4.

In order for the general treatment reported in Sects. 1 and 2 to be consistent with the general analysis and explicit computations for Fermat CY_3s performed in Sects. 3-8, the LG limit given by Eq. (9.7) should coincide with the LG limit of $a_{4,k}\left(\psi\right)$ as given by Eq. (2.53). By putting $n=4$ in such an equation, and recalling that $\tau_{4,k}=1$ $\forall k=5,6,8,10$ (see Table 2), one achieves that

$$lim_{\psi\longrightarrow 0}a_{4,k}\left(\psi\right)=lim_{\psi\longrightarrow 0}\left[-\sigma_4\psi^5+\tau_{4,k}\psi^{5-k}\right]=\psi^{5-k}. \tag{9.8}$$

By comparing Eq. (9.7) with Eq. (9.8), one notices that they differ by the factor $i\frac{k\chi_k}{(2\pi)^3}$.

The factor "i" can be explained simply: the definitions (2.34) are consistent with a notation in which the C_{ijk} tensor of our treatment actually is "$-iC_{ijk}$" (compare *e.g.* the second of Eqs. (8) of [44] with the second of Eqs. (2.2)); thus, one can get rid of the "i" without any problem.

Concerning the k-dependent real factor "$\frac{k\chi_k}{(2\pi)^3}$", it simply means that, in order to make our treatment of 1-modulus SK geometry of the moduli space of Fermat CY_3s consistent with the general theory exposed in Sects. 1 and 2, one has to multiply the l.h.s. of 1-modulus PF Eq. (2.53) for Fermat CY_3s by the k-dependent real factor $\frac{k\chi_k}{(2\pi)^3}$. Of course, such an overall multiplication by a constant factor will not affect the differential relations (2.37), nor will it change the solutions of 1-modulus PF Eq. (2.53).

In other words, the consistent normalization of 1-modulus PF Eq. (2.53) for Fermat CY_3s implies Eq. (2.53) to be further "corrected" as

$$\begin{aligned} &\sum_{n=0}^{4} a_{n,k}(\psi)\,\partial^n V_h(\psi) = 0, \\ &a_{n,k}(\psi) \equiv \tfrac{k\chi_k}{(2\pi)^3}\left[-\sigma_n \psi^{n+1} + (-1)^n \tau_{n,k}\psi^{n+1-k}\right]. \end{aligned} \tag{9.9}$$

Having obtained the matching in the LG limit, we can now reconsider the first of the defining relations (2.34); from the correctly normalized definition of the $a_{n,k}(\psi)$s given in the second line of Eq. (9.9), one can achieve the exact, k-parametrized formula for the holomorphic part $W_{\psi\psi\psi,k}(\psi)$ of $C_{\psi\psi\psi,k}\left(\psi,\overline{\psi}\right)$ for Fermat CY_3s:

$$W_{\psi\psi\psi,k}(\psi) \equiv W_k(\psi) = \left[a_{4,k}(\psi)\right]^{-1} = \frac{(2\pi)^3}{k\chi_k}\frac{1}{\left(\psi^{5-k}-\psi^5\right)}, \tag{9.10}$$

where Tables 1 and 2 have been used.

By using the exact formula (9.10), the evaluation of such an holomorphic Yukawa coupling function near the three species of regular singular of points of PF ordinary differential Eqs. for Fermat CY_3s yields:

$$\textit{LG limit}:\ lim_{\psi\longrightarrow 0} W_k(\psi) = lim_{\psi\longrightarrow 0}\tfrac{(2\pi)^3}{k\chi}\psi^{k-5} = \tfrac{(2\pi)^3}{k\chi}\delta_{k,5}; \tag{9.11}$$

$$\textit{Conifold limit}:\ \left|lim_{\psi^k\longrightarrow 1} W_k(\psi)\right| = \infty; \tag{9.12}$$

$$\textit{Large complex structure modulus limit}:\ lim_{\psi\longrightarrow\infty} W_k(\psi) = -\tfrac{(2\pi)^3}{k\chi}\psi^{-5}. \tag{9.13}$$

Finally, by multiplying $W_k(\psi)$ by $e^{K_k\left(\psi,\overline{\psi}\right)}$ and recalling the first definition of (3.31), one can obtain the k-parametrized formula for $C_{\psi\psi\psi,k}\left(\psi,\overline{\psi}\right)$ for the class of Fermat CY_3s:

$$C_{\psi\psi\psi,k}\left(\psi,\overline{\psi}\right) = e^{K_k\left(\psi,\overline{\psi}\right)}\left[a_{4,k}(\psi)\right]^{-1} = \frac{(Ord(G_k))^2}{(2\pi)^3 C_{k,0}^2 F_{k,11} k\chi_k}\frac{e^{\widetilde{K}_k\left(\psi,\overline{\psi}\right)}}{\left(\psi^{5-k}-\psi^5\right)} = \frac{k^3\chi_k}{(2\pi)^3 C_{k,0}^2 F_{k,11}}\frac{e^{\widetilde{K}_k\left(\psi,\overline{\psi}\right)}}{\left(\psi^{5-k}-\psi^5\right)}, \tag{9.14}$$

where in the last step we used the relation $Ord(G_k) = \chi_k k^2$ (see Sect. 3). If the real function $\widetilde{K}_k\left(\psi,\overline{\psi}\right)$ is left as a generic one, Eq. (9.14) can be considered as the exact formula for $C_{\psi\psi\psi,k}\left(\psi,\overline{\psi}\right)$, holding true at every point in the moduli space of the class of Fermat CY_3s. On the other hand, if $\widetilde{K}_k\left(\psi,\overline{\psi}\right)$ is given, through the first definition of (3.31), by Eq. (3.20), Eq. (9.14) gives the series expansion of the covariantly-holomorphic Yukawa coupling function $C_{\psi\psi\psi,k}\left(\psi,\overline{\psi}\right)$ near the LG point of the moduli space of Fermat CY_3s. In such a case, by performing the LG limit $\psi\longrightarrow 0$ of Eq. (9.14) and considering the treatment and the formulæ from Sects. 3-7, one finally achieves:

$$lim_{\psi\longrightarrow 0} C_{\psi\psi\psi,k}\left(\psi,\overline{\psi}\right) \approx \frac{k^3\chi_k}{(2\pi)^3 C_{k,0}^2 F_{k,11}}\psi^{k-5}. \tag{9.15}$$

As expected, this is nothing but the k-parametrized formula summarizing Eqs. (9.2)-(9.5).

10 Conclusions and Outlook

In the present work we investigated non-degenerate extremal BH attractors near the so-called LG point $\psi = 0$ (herein named *LG attractors*) of the moduli space ($dim_{\mathbb{C}} = 1$) of the class of Fermat Calabi-Yau threefolds. We found the BH charge configurations supporting $\psi = 0$ to be a critical point of the real, positive-definite "effective BH potential" V_{BH} defined in Eq. (1.24).

In order to do this, we exploited two different approaches:

1) *"criticality condition" approach*: we solved at $\psi \approx 0$ the 2 real *criticality conditions* of V_{BH}, corresponding in the 1-modulus case to the real and imaginary part of the *Attractor Eqs.* (1.23) (see Sects. 4-7);

2) *"SK geometrical identities" approach*: we evaluated at $\psi \approx 0$ the 4 real fundamental identities (8.1) of 1-modulus SK geometry at the geometrical *loci* corresponding to the various species of critical points of V_{BH} (see Sect. 8).

We found that the results of two such solving approaches do coincide, in spite of the different number of real Eqs. involved in approaches 1 and 2. The equivalence of the above-mentioned approaches to find the critical points of V_{BH} (and the BH charge configurations supporting them) is explicit proof of the fact that the relations (8.1) actually are *identities* and not equations, *i.e.* that, for any point of the moduli space at which we evaluate them, they do not give any constraint on the charge configuration.

It is worth pointing out that the *"criticality condition" approach* had been previously exploited in literature only for the following cases:

a) *mirror quintic* ($k = 5$) in [9], where however peculiar *Ansätze* (on the BH charge configuration and on ψ in the neighbourhood of the LG point) were used, implying a certain loss of generality;

b) *mirror sextic* ($k = 6$) in [10].

On the other hand, the *"SK geometrical identities" approach* (and its equivalence with the *"criticality condition"* one) had been hitherto exploited only in [13]; in such a Ref., the mirror *quintic* was considered within the same simplifying *Ansätze* formulated in [9], obtaining a complete agreement with the results of [9].

As a by-product of our computations, we extended the results of [9] and [13] to full generality (see Sect. 4). Moreover, we found that the analysis of the stability of $\psi = 0$ as a non-BPS, $Z = 0$ critical point of V_{BH} in the mirror *sextic*, performed in Sect. 7 of [10], suffers from some problems of inconsistency. Indeed, in [10] it was found that the LG point (supported by a certain BH charge configuration characterizing it as a non-BPS, $Z = 0$ critical point of V_{BH}) is *stable* (minimum of V_{BH}). Instead, our computations (see Sect. 5), which carefully took into account the relevant orders in ψ and $\overline{\psi}$ in the truncation of the series expansion around $\psi = 0$, allow us to conclude that, *for the same supporting BH charge configuration*, the LG point is *unstable* (namely, a saddle point of V_{BH}).

We also checked the stability of $\psi = 0$ by inspecting the Hessian matrix of V_{BH} in correspondence to the various BH charge configurations supporting the LG point to be a critical point of V_{BH}. A sketchy summary of our results is given by the following Table:

$k \longrightarrow$	5	6	8	10
$\frac{1}{2}$-BPS	*stable*, 1 charge config.	*stable*, 1 charge config.	*stable*, 1 charge config.	*stable*, 1 charge config.
non-BPS, $Z \neq 0$	*stable*, 3 charge configs.	–	*stable*, 3 charge configs.	–
non-BPS, $Z = 0$	–	*unstable*, 1 charge config.	–	*unstable*, 1 charge config.

Table 5: **Species and stability of the critical points of V_{BH} in the moduli space of Fermat CY_3s**

The stability of $\psi = 0$ as a $\frac{1}{2}$-BPS attractor agrees with the known results from general analysis of SK geometry of scalar manifolds in $\mathcal{N} = 2$, $d = 4$ supergravity coupled to n_V Abelian vector multiplets [31, 15, 16].

Regardless of the kind of BH charge configuration supporting them, the non-BPS, $Z \neq 0$ LG attractors, when they exist, are found to be stable (local minima of V_{BH}). This means that, for all the configurations of supporting BH charges, these non-BPS, $Z \neq 0$ LG attractors satisfy the general condition of stability in 1-modulus SK geometry, given by Eq. (4.27) of [15].

It is interesting to compare such a result to what happens in the large volume limit of CY_3-compactifications of Type II A superstring theory. Indeed, in such a framework (with a generic number n_V of complex structure moduli) in [9] it was shown that the stability of non-BPS, $Z \neq 0$ critical points of V_{BH} (and therefore their actual attractor behaviour) within a certain supporting BH charge configuration, crucially depends on the possible vanishing of p^0, *i.e.* of the asymptotical magnetic flux of the graviphoton field strength, whose microscopical interpretation corresponds to a $D6$-brane wrapping p^0 times a 3-cycle of the considered CY_3. Nevertheless, also in such a context in the 1-modulus case ($n_V = 1$) the non-BPS, $Z \neq 0$ critical points of V_{BH} are always stable, and therefore they are attractors in a strict sense.

Furthermore, one can also observe that all Fermat CY_3s admit *only one* kind of non-BPS LG attractors, either with $Z \neq 0$ or with $Z = 0$; for the allowed values of the classifying Fermat parameter $k = 5, 6, 8, 10$, one gets the "pattern" shown in Table 5 above.

Once again, such a feature is exhibited also by the large volume limit of of CY_3-compactifications, whose related SK geometry is characterized by cubic holomorphic prepotentials; indeed, it can be explicitly computed that the 1-modulus prepotential $\mathcal{F}(z) = \frac{1}{3}z^3$, corresponding to the homogeneous symmetric SK manifold $\frac{SU(1,1)}{U(1)}$ (see [22] and Refs. therein), admits $\frac{1}{2}$-BPS and non-BPS, $Z \neq 0$ attractors only [22].

It is worth mentioning that the fourth order linear ordinary differential Picard-Fuchs equations of Fermat CY_3s (2.53) (specified by Tables 1 and 2) exhibit other two species of regular singular points, namely the k-th roots of unity ($\psi^k = 1$, the so-called *conifold points*) and the *point at infinity* $\psi \longrightarrow \infty$ in the moduli space, corresponding to the so-called *large complex structure modulus limit*. It would be interesting to solve criticality conditions for V_{BH} near such regular singular points, *i.e.* to investigate *extremal BH conifold attractors* and *extremal BH large complex* *structure attractors* in the moduli space of 1-modulus (Fermat) CY_3s, also in view of recent investigations of extremal BH attractors in specific examples of 2-moduli CY_3-compactifications [25].

When CY_3-compactifications with more than one complex structure deformation modulus are considered, it is clear that interesting situations might arise other than the ones present at 1-modulus level. Indeed, differently from what has been studied so far [25], in such frameworks all three species of extremal BH (LG) attractors (namely $\frac{1}{2}$-BPS, non-BPS $Z \neq 0$ and non-BPS $Z = 0$) should exist, each typology being supported by distinct, zero-overlapping BH charge configurations. *Ça va sans dire* that such an issue deserves more investigation and analyzing efforts.

Finally, it is worth spending a few words concerning the instability of non-BPS, $Z = 0$ (LG) attractors in the 1-modulus case. It would be intriguing to extend to such a framework the same conjecture formulated in [24]. In Sect. 5 of such a Ref., in the framework of (the large volume limit of CY_3-compactifications leading to) the peculiarly symmetric case of cubic *stu* model, it was argued that the instability of the considered non-BPS attractors might be only apparent, since such attractors might correspond to multi-centre stable attractor solutions, whose stable nature should be "resolved" only at sufficiently small distances. As mentioned, it would be interesting to extend such a conjecture to the non-BPS, $Z = 0$ (LG) attractors, also in relation to the possible existence of *non-BPS lines of marginal stability* [51, 52].

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